SYZYGIES OF $\mathbb{P}^1 \times \mathbb{P}^1$: DATA AND CONJECTURES

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Abstract. We provide a number of new conjectures and questions concerning the syzygies of $\mathbb{P}^1 \times \mathbb{P}^1$. The conjectures are based on computing the graded Betti tables and related data for large number of different embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$. These computations utilize linear algebra over finite fields and high-performance computing.

1. Introduction

While syzygies are a much-studied topic in algebraic geometry and commutative algebra, the Betti tables for varieties of dimension ≥ 2 remain largely mysterious. For instance, the Betti table of $\mathbb{P}^2$ under the d-uple Veronese embedding is only fully understood for $d \leq 6$ [BEGY18, CCDL19], and there is not yet even a conjectural picture for the values of such Betti tables. One obstacle to developing such a conjecture is a lack of data: for the d-uple embedding of $\mathbb{P}^2$, the required number of variables grows like d^2, and so free resolution computations tend to overflow memory.

In [BEGY18], the computation of syzygies was approached via an alternate method. Instead of using symbolic Gröbner basis methods to compute a minimal free resolution, we computed the Betti numbers via the cohomology of the Koszul complex. In essence, this swapped a symbolic computation for a massive linear algebra computation. (See §2 for the theoretical background on this approach.) This reduced the computation to a number of individual rank computations, one for each multigraded Betti number, and then we performed those computations using high-throughput computations.

The present work has three foci: we improve the framework for this alternate approach to Betti numbers; we apply it to the case of $\mathbb{P}^1 \times \mathbb{P}^1$ to generate a wealth of new data; and we use that data to offer new conjectures and questions about the syzygies of $\mathbb{P}^1 \times \mathbb{P}^1$.

1.1. **Overview of the computation.** For any $\mathbf{d} = (d_1, d_2) \in \mathbb{Z}^2_{>0}$, we can embed $\iota_{\mathbf{d}} \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^{(d_1+1)(d_2+1)-1}$ by the complete linear series for $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathbf{d})$, and we want to understand the syzygies of this image. Following a philosophy implicit in Green's foundational work on syzygies [Gre84b], and echoed in later results on asymptotic syzygies [EL12, EEL16], we will

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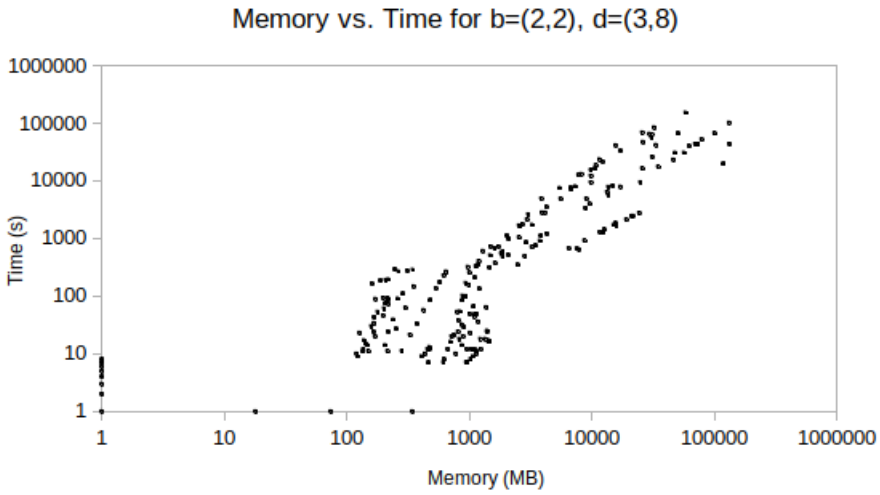

FIGURE 1. Memory vs. time to compute ranks of matrices for $\mathbf{b} = (2, 2)$, $\mathbf{d} = (3, 8)$

study the syzygies of not only the structure sheaf, but also of the pushforward of various line bundles. In particular, our goal is to compute the syzygies of $\iota_{\mathbf{d}*}\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbf{b})$ for as many choices of $\mathbf{d}$ and $\mathbf{b}$ as possible. Note very few cases were previously known [BM81, Lem21]

Depending on the grading group or equivariant structure under consideration, we can represent these Betti numbers in a multitude of ways. See §2 for a summary of notation.

Our main computation involves the $\mathbb{Z}^4$-graded Betti numbers. There are $\approx d_1 d_2$ entries of the Betti table which could be nonzero, and each of those entries will involve at most $\approx d_1^3 d_2^3$ distinct $\mathbb{Z}^4$ multidegrees. However, by using known vanishing and duality results, accounting for symmetry, and applying elementary results on the relationship between Betti numbers and Hilbert function, we can shrink down to a much smaller number of matrices, which we refer to as the **relevant range**, and which are sufficient to determine all of the Betti numbers. (See §4 for details on the relevant range.)

The main computation involves computing the ranks of all of the matrices from the relevant range. The rank of each matrix can be computed in parallel, allowing us to leverage high throughput computational resources. In addition, some of the matrices are quite massive, and we thus require huge amounts of memory for those particular matrices.

For concreteness, let us consider our largest complete computation, which is the case $\mathbf{b} = (2, 2)$ and $\mathbf{d} = (3, 8)$. The relevant range involves 1130 matrices, the largest of which is $2,124,896 \times 3,719,448$, and Figure 1 provides a scatterplot of the time and memory involved in computing the ranks of those matrices. Only a handful of cases took over a day.

1.2. **Computational improvements.** Our current work improves on the method of [BEGY18] in a number of ways. Most notably, [BEGY18] relied on floating-point rank computations of sparse real matrices, using a MATLAB implementation of the LU-algorithm; by contrast, our current work simply performs the computations over finite fields in MAGMA. MAGMA

Method	Average time per job (secs)	Max time (secs)
MatLab LU-algorithm over $\mathbb{R}$	220	4735
MAGMA rank algorithm over $\mathbb{F}_{32003}$	7	99

FIGURE 2. We compared floating-point LU-algorithm computations in MatLab with rank computations in MAGMA over the finite field $\mathbb{F}_{32003}$, for all of the multigraded matrices related to one individual Betti number. This anecdotally suggests that MAGMA computations over finite fields are significantly faster, though we did not do any comprehensive testing.

recently introduced major improvements in their linear algebra of finite fields [Ste], which seemed to make these rank computations much faster than our previous method; see Figure 2.

Moreover, this switch to working over finite fields enabled us to use exact calculations, eliminating the need for floating-point approximations. While an exact computation over a finite field will not necessarily agree with the exact computation over $\mathbb{Q}$, there are only finitely many primes where the computations could disagree, and these discrepancies seem to rarely arise for reasonably large primes. This switch to working over finite fields thus had a significant downstream effect: the main computations in [BEGY18] introduced some numerical errors as $\mathbf{d}$ grew larger, requiring the use of representation theoretic techniques to detect these errors. By contrast, our finite field computations produced no such numerical errors, and we were able to produce Schur functor decompositions without the need for the sort of "error correction" from [BEGY18, §5].

1.3. **New Data.** After computing the multigraded ranks for the relevant range, we process the data into usable formats. The rank computations quickly yield $\mathbb{Z}^4$-multigraded Betti numbers, but most mathematical conjectures focus on either the standard $\mathbb{Z}$-graded Betti numbers or on the underlying $\mathbf{GL}_2 \times \mathbf{GL}_2$-Schur modules. We convert into those formats and encode all of the results into a Macaulay2 package for ease of use. A preliminary version of this package is available at:

https://github.com/julietteBruce/P1P1syzygies/

In total, we compute complete Betti tables for just shy of 200 total pairs of $\mathbf{b}$ and $\mathbf{d}$. See §3 and Table 1 for more details on the data.

1.4. **Conjectures.** Based on the data we computed, we develop a number of new conjectures, and we provided evidence in support of some previous conjectures.

We first examine the quantitative behavior of the standard grade Betti numbers, with conjectures in §5 that address unimodality properties of the Betti numbers and various statistics. In addition, we consider our data in relation to a conjecture of Ein, Erman, and Lazarsfeld that, for large values of $\mathbf{d}$, the Betti numbers in any given row of the Betti table should behave like a binomial distribution [EEL15, Conjecture B]. A theorem of Bruce [Bru20b, Theorem A] implies that the first row of the Betti table for $\mathbb{P}^1 \times \mathbb{P}^1$ and line bundles $(2, d_2)$ satisfies

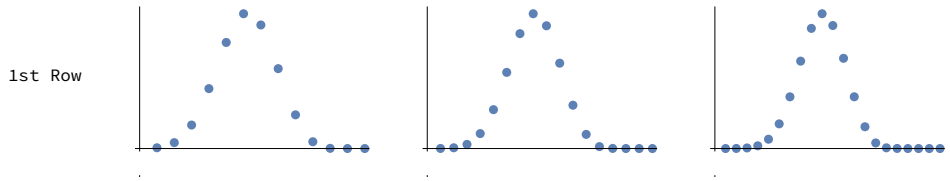

FIGURE 3. Here we plot the Betti numbers of the first row of the Betti tables of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $(d, 3)$, for $d = 3, 4$ and 5. They appear to resemble a normal distribution, as predicted by a conjecture of [EEL15].

exactly this behavior as $d_2 \to \infty$. Our data provides further support for the normal distribution behavior suggested by the conjecture, and seems to show this behavior even for the low values of d_2 for which we have data. See Figure 3.

In §6, we consider several conjectures related to the $\mathbf{GL}_2 \times \mathbf{GL}_2$ structure of these syzygies. This includes precise conjectures on the Schur functor decomposition of certain entries; an analysis of the shapes of partitions that arise; and a discussion of "redundant" representations.

In §7, we present a collection of conjectures involving the Boij-Söderberg decompositions of these Betti tables. In particular, we provide a complete conjectural description of the Boij-Söderberg coefficients of the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, d_2)$

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2. BACKGROUND AND NOTATION

Throughout this section, we work over an arbitrary field $\mathbb{K}$. Our convention will be to write integer vectors using boldface, as in $\mathbf{d} \in \mathbb{Z}^2$, and to specify the coordinates as $\mathbf{d} = (d_1, d_2)$. We let $\mathbf{0} = (0, 0) \in \mathbb{Z}^2$.

As we are interested in the syzygies of $\mathbb{P}^1 \times \mathbb{P}^1$ throughout we let $S = \mathbb{K}[x_0, x_1, y_0, y_1]$ be the corresponding polynomial of over a field $\mathbb{K}$. When viewed as the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ [Cox95], the ring S inherits a $\mathbb{Z}^2$-bigrading given by $\deg x_0 = \deg x_1 = (1, 0) \in \mathbb{Z}^2$ and $\deg y_0 = \deg y_1 = (0, 1) \in \mathbb{Z}^2$. The ring S also admits a $\mathbb{Z}^4$-multigrading given by setting the degree of each variable to be a generator of $\mathbb{Z}^4$, e.g. $\deg(x_0) = (1, 0, 0, 0)$ and $\deg(x_1) = (0, 1, 0, 0)$ and so on.

2.1. **Standard graded Betti numbers.** The syzygies of $\mathbb{P}^1 \times \mathbb{P}^1$ under various embeddings come from studying Segre-Veronese modules of S. Given $\mathbf{d} \in \mathbb{Z}^2_{>0}$ and $\mathbf{b} \in \mathbb{Z}^2$ the Segre-Veronese module is

$$S(\mathbf{b};\mathbf{d}) \coloneqq \bigoplus_{k\in\mathbb{Z}} S_{k\mathbf{d}+\mathbf{b}}.$$

Since $k\mathbf{d}+\mathbf{b}$ determines a ray in $\mathbb{Z}^2$ as k varies in $\mathbb{Z}$, $S(\mathbf{b};\mathbf{d})$ is naturally a $\mathbb{Z}$-graded module over the polynomial $R = \operatorname{Sym} S_{\mathbf{d}}$. When $\mathbf{b} = \mathbf{0}$ the module $S(\mathbf{0};\mathbf{d})$ is isomorphic to the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbf{d})$ into the projective space $\mathbb{P}^{(\mathbf{d}_1+1)(\mathbf{d}_2+1)-1} = \operatorname{Proj} R$. If $\mathbf{b} \neq \mathbf{0}$, then $S(\mathbf{b};\mathbf{d})$ is naturally isomorphic to the section module of a pushforward of a line bundle; specifically, $S(\mathbf{b};\mathbf{d})$ is the R-module associated to the sheaf $(\iota_{\mathbf{d}})_*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbf{b})$. As noted in the introduction, while our primary interest is in the syzygies of the homogeneous coordinate rings $S(\mathbf{0};\mathbf{d})$, past work shows that studying the syzygies of other line bundles is often helpful in providing a more uniform picture [EL12, EEL16, Gre84b].

The Betti numbers of a graded R-module M are defined as $\beta_{i,j}(M) = \dim_{\mathbb{K}} \operatorname{Tor}_i^R(M,\mathbb{K})_j$, which denotes the degree j part of the Tor_i-module. For convenience, when studying the Betti numbers $S(\mathbf{b};\mathbf{d})$, we will omit reference to the ambient polynomial ring R, and write $\beta_{i,j}(S(\mathbf{b};\mathbf{d})) = \beta_{i,j}(\mathbb{P}^1\times\mathbb{P}^1,\mathbf{b};\mathbf{d})$. The Betti numbers of a graded module are often computed using a minimal free resolution [Eis05, M2]. However, an alternate characterization of the Betti numbers, via Koszul cohomology, is more relevant for our computational approach.

The Koszul complex of $S(\mathbf{b};\mathbf{d})$ over the ring R is the complex:

$$\cdots \longrightarrow \bigwedge^1 R^{(\mathbf{b}_1+1)(\mathbf{b}_2+1)} \otimes S(\mathbf{b};\mathbf{d}) \longrightarrow \bigwedge^0 R^{(\mathbf{b}_1+1)(\mathbf{b}_2+1)} \otimes S(\mathbf{b};\mathbf{d}),$$

which is naturally $\mathbb{Z}$-graded since since R is $\mathbb{Z}$-graded. Given a pair of integers (p,q), we can analyze the cohomology of the degree $p+q$ strand of this complex, in homological degree p. This will be denoted by $K_{p,q}(\mathbb{P}^1\times\mathbb{P}^1,\mathbf{b};\mathbf{d})$.[1] It can be computed explicitly as the middle cohomology of the following complex:

(2.1)

$$\cdots \longrightarrow \bigwedge^{p+1} S_{\mathbf{d}} \otimes S_{(q-1)\mathbf{d}+\mathbf{b}} \xrightarrow{\partial_{p+1,q-1}} \bigwedge^{p} S_{\mathbf{d}} \otimes S_{q\mathbf{d}+\mathbf{b}} \xrightarrow{\partial_{p,q}} \bigwedge^{p-1} S_{\mathbf{d}} \otimes S_{(q+1)\mathbf{d}+\mathbf{b}} \longrightarrow \cdots$$

where the differentials are given by

$$\partial_{p+1,q-1}\left(m_0\wedge m_1\wedge\cdots\wedge m_p\otimes f\right) = \sum_{i=0}^{p}(-1)^i m_0\wedge m_1\wedge\cdots\wedge\hat{m}_i\wedge\cdots\wedge m_p\otimes m_i f$$

$$\partial_{p,q}\left(m_1\wedge m_2\wedge\cdots\wedge m_p\otimes f\right) = \sum_{i=1}^{p}(-1)^i m_1\wedge m_2\wedge\cdots\wedge\hat{m}_i\wedge\cdots\wedge m_p\otimes m_i f.$$

In other words, instead of computing all of the Betti numbers simultaneously via a minimal free resolution, we can compute each Betti number individually using the complex of vector spaces in (2.1). This, in essence, turns a problem of symbolic algebra into a (massive but largely distributable) problem in linear algebra.

[1] We remark that $K_{p,q}$ and $\beta_{i,j}$ provide two different notations for similar invariants, though $K_{p,q}$ is a vector space whereas $\beta_{i,j}$ is an integer; both are commonly used in the literature. We will primarily use the $K_{p,q}$-notation, however the conversion between the two notations is given by the simple rule $\dim K_{p,q} \leftrightarrow \beta_{p,p+q}$.

2.2. **Multigraded Betti numbers.** By incorporating the $\mathbb{Z}^4$-grading on S, we can subdivide the problem even further and obtain the $\mathbb{Z}^4$-graded Betti numbers. For a multidegree $\mathbf{e} \in \mathbb{Z}^4$, we define $\beta_{i,\mathbf{e}}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d}) = \beta_{i,\mathbf{e}}(S(\mathbf{b}; \mathbf{d})) = \dim_{\mathbb{K}} \operatorname{Tor}_i^R(S(\mathbf{b}; \mathbf{d}), \mathbb{K})_{\mathbf{e}}$. This is well defined because both R and $S(\mathbf{b}; \mathbf{d})$ inherit $\mathbb{Z}^4$-multigradings from S. From the Koszul cohomology perspective, the Koszul complex of $S(\mathbf{b}; \mathbf{d})$ over R is also homogeneous with respect to the $\mathbb{Z}^4$-grading. Thus, we can analyze the cohomology of the degree $\mathbf{e}$-strand, which provides our method for computing $\beta_{i,\mathbf{e}}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$.

2.3. **Schur functor decomposition.** The action of $\mathbf{GL}_2 \times \mathbf{GL}_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ turns the vector space $K_{p,q}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$ into a $\mathbf{GL}_2 \times \mathbf{GL}_2$-representation. We can therefore decompose $K_{p,q}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$ into a direct sum of irreducible $\mathbf{GL}_2 \times \mathbf{GL}_2$-representations. These irreducible representations have the form $\mathbf{S}_\lambda \otimes \mathbf{S}_\mu$, where λ, μ are partitions with length ≤ 2. See [FH91, Exercise 2.36] for background. For brevity, we write $\mathbf{S}_{(a,b,c,d)}$ for the Schur module $\mathbf{S}_{(a,b)} \otimes \mathbf{S}_{(c,d)}$.

Example 2.2. Let $\mathbf{b} = (2, 2)$ and $\mathbf{d} = (3, 3)$. The Betti table for $S(\mathbf{b}; \mathbf{d})$ is

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0 :	9	108	585	1872	3861	5148	4026	1080	**165**	·	·	·	·	·
1 :	·	·	·	·	·	165	1080	4026	5148	3861	1872	585	108	9

The bold entry in the Betti table tells us that $\dim K_{8,0}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d}) = \beta_{8,8}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d}) = 165$. Viewed as $\mathbf{GL}_2 \times \mathbf{GL}_2$-representation, $K_{8,0}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$ decomposes as

$$K_{8,0}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d}) \cong \mathbf{S}_{(17,9,17,9)} \oplus \mathbf{S}_{(16,10,16,10)} \oplus \mathbf{S}_{(15,11,15,11)} \oplus \mathbf{S}_{(14,12,14,12)} \oplus \mathbf{S}_{(13,13,13,13)}.$$

The dimensions of these Schur modules are $81, 49, 25, 9$ and 1, respectively.

2.4. **Koszul Duality.** Using duality of Koszul cohomology groups (see, for instance [Gre84a, Duality Theorem (2.c.9)]), we can derive data for more values of $\mathbf{b}$ and $\mathbf{d}$, as we now explain. Given $\mathbf{b}$ we define its *Koszul dual* as $\mathbf{b}' := \mathbf{d} - \mathbf{b} - (2, 2)$. We have

$$K_{p,q}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}'; \mathbf{d}) \cong K_{(d_1+1)(d_2+1)-3-p,2-q}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$$

as vector spaces. Visually, this means that the Betti table for $(\mathbf{b}'; \mathbf{d})$ is obtained by rotating the Betti table for $(\mathbf{b}; \mathbf{d})$ by 180°. We will illustrate this phenomenon in Example 2.3. Note that $(d_1 + 1)(d_2 + 1) - 3$ is the codimension of $\mathbb{P}^1 \times \mathbb{P}^1$ in the embedding by $\mathbf{d}$. The duality also applies to the Schur functor decomposition via the following formula. To phrase this, we need some more notation. Let

$$\alpha := \left(\tfrac{(d_1+1)(d_2+1)d_1-2}{2}, \tfrac{(d_1+1)(d_2+1)d_1-2}{2}, \tfrac{(d_1+1)(d_2+1)d_2-2}{2}, \tfrac{(d_1+1)(d_2+1)d_2-2}{2} \right).$$

Given any $w = (w_0, w_1, w_2, w_3) \in \mathbb{Z}^4$ we write $w^{\text{opp}} = (w_1, w_0, w_3, w_2)$ and we choose w' so that $w + (w')^{\text{opp}} = \alpha$. The multiplicity of the Schur functor S_w in $K_{p,q}(\mathbf{b}; \mathbf{d})$ equals the multiplicity of the Schur functor $S_{w'}$ in the dual Koszul cohomology group $K_{(d_1+1)(d_2+1)-3-p,2-q}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}'; \mathbf{d})$, where $\mathbf{b}'$ is defined as above.

Example 2.3. Let $\mathbf{b} = (0, 0)$ and $\mathbf{d} = (3, 3)$. The Betti table for $S(\mathbf{b}; \mathbf{d})$ is

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0 :	1	·	·	·	·	·	·	·	·	·	·	·	·	·
1 :	·	87	676	2691	6864	12155	15444	14157	9152	3861	780	**22**	·	·
2 :	·	·	·	·	·	·	·	·	·	·	165	144	39	4

The bold entry in the Betti table tells us that $\dim K_{11,1}(\mathbb{P}^1\times\mathbb{P}^1,\mathbf{b};\mathbf{d}) = \beta_{11,12}(\mathbb{P}^1\times\mathbb{P}^1,\mathbf{b};\mathbf{d}) = 22$. Viewed as $\mathbf{GL}_2\times\mathbf{GL}_2$-representation, $K_{11,1}(\mathbb{P}^1\times\mathbb{P}^1,\mathbf{b};\mathbf{d})$ decomposes as

$$K_{11,1}(\mathbb{P}^1\times\mathbb{P}^1,\mathbf{b};\mathbf{d}) \cong \mathbf{S}_{(23,13,18,18)} \oplus \mathbf{S}_{(18,18,23,13)}.$$

The Koszul dual pair to $(\mathbf{b};\mathbf{d})$ is $\mathbf{b}' = (1,1)$ and $\mathbf{d}' = (3,3)$. The Betti table for $S(\mathbf{b}';\mathbf{d}')$ is

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0 :	4	39	144	165	·	·	·	·	·	·	·	·	·	·
1 :	·	·	**22**	780	3861	9152	14157	15444	12155	6864	2691	676	87	·
2 :	·	·	·	·	·	·	·	·	·	·	·	·	·	1

We see that this Betti table is exactly that corresponding to $S(\mathbf{b};\mathbf{d})$ rotated by 180°. The $(11,1)$ entry for $S(\mathbf{b};\mathbf{d})$ corresponds to $(2,1)$ for $S(\mathbf{b}';\mathbf{d}')$. Viewed as $\mathbf{GL}_2\times\mathbf{GL}_2$-representation, $K_{2,1}(\mathbb{P}^1\times\mathbb{P}^1,\mathbf{b};\mathbf{d})$ decomposes as

$$K_{2,1}(\mathbb{P}^1\times\mathbb{P}^1,\mathbf{b};\mathbf{d}) \cong \mathbf{S}_{(10,0,5,5)} \oplus \mathbf{S}_{(5,5,10,0)}$$

3. Computed Data

Using the algorithms outlined in Section 4 we computed the Betti tables, $\mathbb{Z}^4$-multigraded Betti numbers, and Schur functor decompositions for over 150 distinct pairs $(\mathbf{b};\mathbf{d})$, including 27 distinct $\mathbf{d}$-values. In Table 1, we list, for each $\mathbf{d}$, the number of $\mathbf{b}$'s for which we have complete data. For comparison: [BEGY18] computed similar data for $\mathbb{P}^2$ for about 15 distinct pairs $(\mathbf{b};\mathbf{d})$, which included 5 distinct $\mathbf{b}$ values; and [CCDL19], which only considered the case $\mathbf{b} = \mathbf{0}$, computed data for $\mathbb{P}^2$ for 5 distinct $\mathbf{d}$ values. There appears to be no significant computational work on syzygies for $\mathbb{P}^1\times\mathbb{P}^1$, although [Lem21] does construct a non-minimal resolution. In other words, these computations represent a significant contribution to the available syzygy data for $\mathbb{P}^1\times\mathbb{P}^1$ specifically, as well as for toric surfaces more generally.

		d_2								
		2	3	4	5	6	7	8	9	10
d_1	2	3	6	8	10	12	14	13	6	6
	3	·	6	12	15	13	12	8	4	2
	4	·	·	9	14	9	5	1	1	0
	5	·	·	·	1	1	1	1	0	0

Table 1. For each $\mathbf{d}$, the number of $\mathbf{b}$ for which we compute the Betti tables, $\mathbb{Z}^4$-multigraded Betti numbers, and Schur decompositions

Remark 3.1. In Table 1, for the symmetric cases $\mathbf{d} = (d,d)$, we only record $\mathbf{b} = (b_1,b_2)$ with $b_1 \le b_2$ for which we have data. For example, when $\mathbf{d} = (2,2)$, we only count the cases $\mathbf{b} = (0,0)$, $(0,1)$, and $(1,1)$; we do not include $(1,0)$.

4. Main Computation

Broadly speaking, our approach to computing the Betti table, $\mathbb{Z}^4$-multigraded Betti numbers, and Schur functor decompositions for a given pair $(\mathbf{b};\mathbf{d})$ proceeds as follows:

(1) **Reduction to the relevant range:** By combining a computation of the multigraded Hilbert series with known vanishing results for syzygies (relying primarily on Castelnuovo-Mumford regularity), we conclude that a small subset of the Betti numbers determines all of the Betti numbers. This smaller subset is the relevant range, and is the focus of our computations.

(2) **Constructing the matrices in the relevant range:** We follow the ideas in [BEGY18] to efficiently construct and store the matrices from the relevant range.

(3) **High throughput rank computations:** We use distributed high throughput computation to find the ranks of all the matrices in the relevant range. These computations are done via linear algebra over the finite field $\mathbb{F}_{32003}$ in MAGMA. This is by far the most computationally intensive aspect.

(4) **Post-processing:** Using standard ideas from representation theory, we convert the multigraded Betti number into Schur functor decompositions.

While the techniques here are broadly similar to those in [BEGY18], which focused on computing syzygies of Veronese embeddings of $\mathbb{P}^2$, the passage from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$ requires new code in each step and we further refine this implementation and approach. The most significant distinction is in the third step abvoe: the core algorithm in the current work uses linear algebra over finite fields, whereas in [BEGY18] it used floating-point computations.

4.1. **Relevant Range.** We expedite our computations significantly by utilizing the fact that for many values of p and many multidegrees $\mathbf{a}$, the multigraded Betti number $\beta_{p,\mathbf{a}}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$ is determined entirely by the $\mathbb{Z}^4$-multigraded Hilbert series of $S(\mathbf{b}; \mathbf{d})$. In the following lemma, we use vector notation $t^{\mathbf{a}} := t_0^{a_0} t_1^{a_1} t_2^{a_2} t_3^{a_3}$ if $\mathbf{a} = (a_0, a_1, a_2, a_3)$.

Lemma 4.1. *The $\mathbb{Z}^4$-multigraded Hilbert series of $S(\mathbf{b}; \mathbf{d})$ is a rational function of the form: $A(t_0, t_1, t_2, t_3)/B(t_0, t_1, t_2, t_3)$ where*

$$A = \sum_{p,\mathbf{a}} \beta_{p,\mathbf{a}}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d}) t^{\mathbf{a}} \text{ and } B = \prod_{\mathbf{b} \in \mathbb{N}^4, b_0+b_1=d_1, b_2+b_3=d_2} (1 - t^{\mathbf{b}})$$

The proof is nearly identical to that of [BEGY18, Lemma 3.1], so we omit it.

With this in mind, our main computations reduce to determining the ranks $\partial_{p,q}$ for p, q in what we call the relevant range.

Definition 4.2. *Fixing $\mathbf{b}$ and $\mathbf{d}$ we define the relevant range to be the set of pairs (p, q) such that $K_{p,q}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d}) \neq 0$ and either $K_{p-1,q+1}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d}) \neq 0$ or $K_{p+1,q-1}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d}) \neq 0$.*

In general we determine the relevant range by finding the smallest p such that $K_{p,q}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d}) \neq 0$ and then applying duality (see [EL12, Proposition 3.5]). When $\mathbf{b} = \mathbf{0}$ the only case of interest is $q = 1$, and we find the smallest p such that $K_{p,1}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d}) \neq 0$ via [CCDL19, Theorem 1.4]. When $\mathbf{b} \neq \mathbf{0}$ we determine the relevant range using the fairly coarse vanishing bounds from [EL12, Proposition 5.1]. While a sharper bound on the relevant range would allow us to compute ranks for many fewer matrices, we found that in practice, these potentially extraneous matrices did not cause any bottlenecks in the actual computation.

An algorithm entirely analogous to [BEGY18, Algorithm 3.3] enables us to efficiently compute the multigraded Betti numbers outside of the relevant range.

4.2. **Constructing the matrices in the relevant range.** After computing the relevant range and the relevant multidegrees, this data is fed to the code to compute the matrices representing the differentials in the relevant range. We first use the $\mathfrak{S}_2 \times \mathfrak{S}_2$-symmetries of the multidegrees to restrict to those multidegrees (a, b, c, d) where $a \geq b$ and $c \geq d$. As in [BEGY18] we use duality for Koszul cohomology groups to reduce the number of matrices we compute [Gre84a, Theorem 2.c.6]. Unfortunately unlike in the case of the Veronese, the bi-graded structure means that it is not possible to use this duality to reduce to a finite set of non-redundant Betti tables.

When constructing the matrices, we use the fact that all of the maps $(\partial_{p,q})_{\mathbf{a}}$ correspond to submatrices of the boundary map $d_p : \bigwedge^p S_{\mathbf{d}} \to \bigwedge^{p-1} S_{\mathbf{d}}$. In particular, $(\partial_{p,q})_{\mathbf{a}}$ is given by restricting to the submatrix $d_{p,\leq \mathbf{a}}$ given by those entries in degrees $\leq \mathbf{a}$. However, instead of storing the map d_p we simply use this fact to compute all of the various $(\partial_{p,q})_{\mathbf{a}}$ for all multidegrees at once. This was implemented as it was found that as the degrees got larger, more of the entries in the d_p matrix correspond to multidegrees that are not in the relevant range. This is entirely analogous to [BEGY18, §4.1], which provides further details. In Appendix A, we list the number of matrices we must compute and the largest such matrix.

Example 4.3. For $\mathbf{d} = (3, 8)$, $\mathbf{b} = (2, 2)$, the full computation of which is discussed in more detail in Example 4.4, it took a modern laptop computer, 5min 25sec to compute all the relevant matrices, entailing a total of 1130 matrices, taking a total of 13GB of space. The single largest matrix had 16,999,168 non-zero entries.

4.3. **High Throughput Computations.** The rank computations can be efficiently distributed over numerous different computers. We implemented these computations using high throughput computing via HTCondor on the University of Wisconsin–Madison Mathematics department computer servers. Many of the matrices are small, and hence do not require much memory to compute the rank. Because our hardware grid has fewer nodes with large amounts of available RAM, the initial submissions are allocated a small amount of RAM (e.g. 2GB). For the jobs that fail, we resubmit with a larger memory allocation, and repeat this process until the computation terminates.

Example 4.4. In this example, we provide a detailed analysis of how we determine the Betti table for $\mathbf{d} = (3, 8)$ and $\mathbf{b} = (2, 2)$, one of our larger computations. There are only two rows $q = 0, 1$, and 34 columns; we display the first several columns below.

	0	1	2	3	4	5	6	7	8	9	
0 :	9	258	3465	28512	156546	568620	1210506	697680	203490	·	⋯
1 :	·	·	·	1050	28476	498498	5444400	41855958	194378184	671067540	

The relevant range is $(p, 0)$ for $4 \leq p \leq 8$ and $(p, 1)$ for $3 \leq p \leq 7$. Because $K_{p,0} - K_{p-1,1}$ is determined by the Hilbert function of the module, we need only compute one of $K_{p,0}$ or $K_{p-1,1}$, and we compute the former. To that end, we form the matrices $(\partial_{p,0})_{\mathbf{a}}$ and $(\partial_{p+1,-1})_{\mathbf{a}}$ for $4 \leq p \leq 8$ and compute their ranks. Fortunately, $(\partial_{p+1,-1})_{\mathbf{a}} = 0$. After accounting for

$\mathfrak{S}_2 \times \mathfrak{S}_2$-symmetry, we are left to compute ranks of 1130 matrices, the largest of which is 2,124,896 × 3,719,448. In this case, up to symmetry there were 39788 multidegrees with non-zero entries in the Betti table. For these entries, in absence of the consideration about relevant ranges, to compute these entries would have required the computation of at least 81,437 matrices.

The amount of RAM and time used in the rank calculation is recorded in Figure 1. The vast majority of matrices require less than 1MB of RAM and 10 seconds. Figure 4 has two plots displaying the average and maximum memory, resp. time, needed to compute the ranks of the matrices $(\partial_{p,0})_{\mathbf{a}}$ as a function of p.

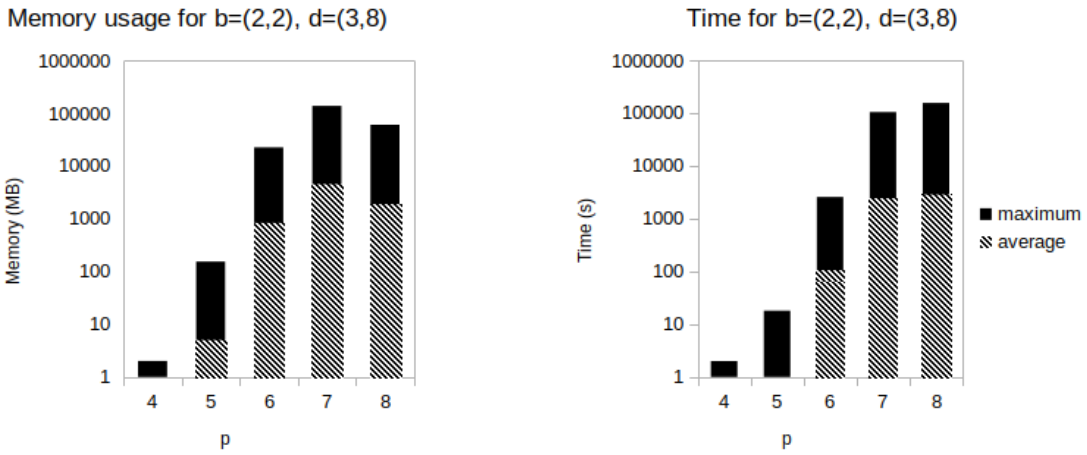

FIGURE 4. Memory and time to compute ranks of matrices for $\mathbf{b} = (2, 2)$, $\mathbf{d} = (3, 8)$ and $q = 0$

Figure 5 illustrates how memory usage varies with multidegree for each $(p, 0)$. The plots are arranged left to right $(p, q) = (4, 0)$ through $(8, 0)$. Here is how to interpret these plots. Within each plot, each square represents a multidegree, and its color measures the memory usage: light gray is 0 GB and black reaches the maximum of 132 GB of RAM. Because of the $\mathfrak{S}_2 \times \mathfrak{S}_2$-symmetry, we need only consider the multidegrees (a, b, c, d) satisfying $a + b = 26$, $a \leq b$ and $c + d = 66$, $c \leq d$. Each row has (a, b) constant, each column has (c, d) constant, and a, resp. c, increases in the downward, resp. left, direction.

Here are some take-aways from this example. We see that the amount of memory and time needed to compute ranks of matrices comprising the differential $\partial_{p,q}$ grows as p moves towards the center of the Betti table. Nevertheless, for a fixed (p, q), nearly all of the matrices $(\partial_{p,q})_{\mathbf{a}}$ require minimal memory and time. The $(\partial_{p,q})_{\mathbf{a}}$ that require the most resources are those for which $\mathbf{a} = (a, b, c, d)$ are balanced, i.e., for which $|a - b|$ and $|c - d|$ are minimized.

Remark 4.5. The fact that the most computationally intensive are those for which $\mathbf{a}$ is balanced could allow one to potentially dig deeper into conjectures related to Schur functors. Namely, the highest weight of a given Schur module tends to be quite unbalanced. Given the parallel nature of these computations, one could potentially rule out the presence of certain Schur modules for many values of $\mathbf{b}$ and $\mathbf{d}$ for which a full computation would be impossible.

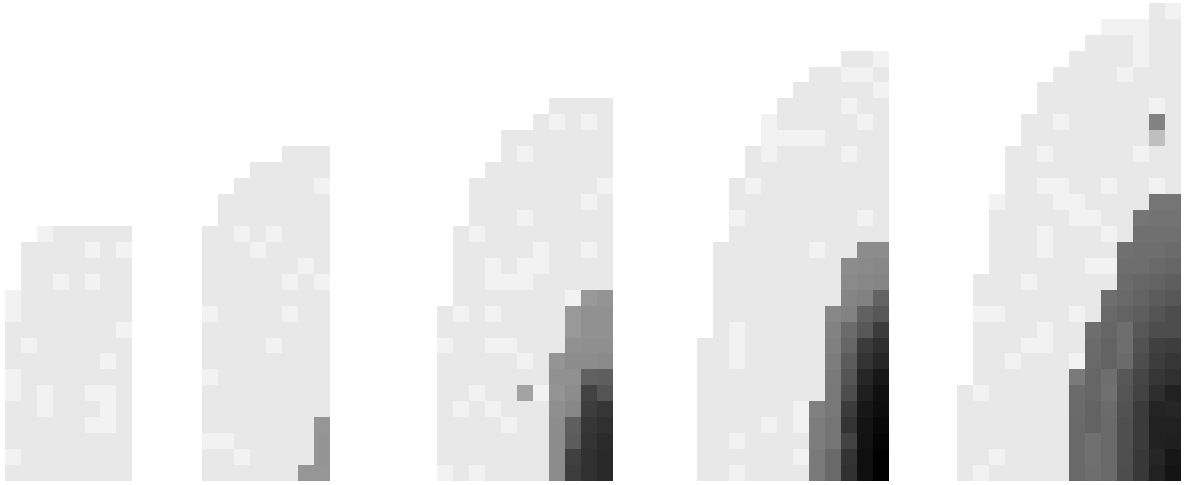

FIGURE 5. Memory usage to compute ranks for each multidegree for $\mathbf{b} = (2, 2)$, $\mathbf{d} = (3, 8)$. Plots are arranged left to right $(p, q) = (4, 0)$ through $(8, 0)$.

4.4. **Post-processing.** Having computed all of the multigraded Betti numbers, we can easily combine the values to obtain the standard graded Betti numbers. Obtaining the Schur functor decompositions is a bit more involved, though it is nearly identical to the process in [BEGY18, §5.1]. The main idea is once again a highest weight greedy algorithm. In the $\mathbb{P}^2$ case, the authors were considering the decomposition as a $\mathbf{GL}_3$-module. In our case, we are considering the decomposition as a $\mathbf{GL}_2 \times \mathbf{GL}_2$-module. The irreducible polynomial representations of $\mathbf{GL}_2 \times \mathbf{GL}_2$ are products $\mathbf{S}_\lambda \otimes \mathbf{S}_\mu$ of Schur functors where λ, μ are partitions with length ≤ 2. For further details, see [FH91, Chapter 6, Exercise 2.36].

We order bi-partitions using the standard Lex order on $\mathbb{Z}^4$. That is, for two bi-partitions $(\lambda, \mu), (\nu, \eta)$ we say that $(\lambda, \mu) \leq (\nu, \eta)$ if $(\lambda_1, \lambda_2, \mu_1, \mu_2) \leq (\nu_1, \nu_2, \eta_1, \eta_2)$ in the standard Lex order on $\mathbb{Z}^4$. This gives us a well order on bi-partitions. In particular, we can select a largest element.

To decompose $K_{p,q}(\mathbb{P}^1 \times \mathbb{P}^1; \mathbf{b}; \mathbf{d})$ into Schur functors we apply the Algorithm for Schur Functor Decomposition (see below). The algorithm terminates due to the semi-simplicity of $\mathbf{GL}_2 \times \mathbf{GL}_2$. More specifically, semi-implicitly implies that there is a finite decomposition $K_{p,q}(\mathbb{P}^1 \times \mathbb{P}^1; \mathbf{b}; \mathbf{d}) \cong \bigoplus_{\lambda,\mu} (\mathbf{S}_\lambda(\mathbb{C}^2) \otimes \mathbf{S}_\mu(\mathbb{C}^2))^{\oplus c_{\lambda,\mu}}$ for some constants $c_{\lambda,\mu}$ only finitely many of which are nonzero. This means the multigraded Hilbert series H in the algorithm above is a sum of Hilbert series corresponding to $(\mathbf{S}_\lambda(\mathbb{C}^2) \otimes \mathbf{S}_\mu(\mathbb{C}^2))$ which is just the product of the Hilbert series for $\mathbf{S}_\lambda$ in the variables t_0, t_1 and the Hilbert series for $\mathbf{S}_\mu$ in a second set of variables t_2, t_3. The weight of the lex-leading monomial of the Hilbert series will always be a bi-partition, i.e. $\lambda_1 \geq \lambda_2$ and $\mu_1 \geq \mu_2$; and that monomial will correspond to the highest weight of some Schur modules appearing in the decomposition. Thus, the algorithm uses the lex-leading monomial of the Hilbert series to iteratively pick off summands in the decomposition of $K_{p,q}(\mathbb{P}^1 \times \mathbb{P}^1; \mathbf{b}; \mathbf{d})$.

Algorithm for Schur Functor Decomposition

Input : $\beta_{p,\mathbf{a}}(\mathbf{b};\mathbf{d})$ for fixed $\mathbf{b};\mathbf{d},p$ and all $\mathbf{a}\in\mathbb{Z}^4_{\geq 0}$ with $|\mathbf{a}|=(p+q)(d_1+d_2)+(b_1+b_2)$

Output : A list K of bi-partitions appearing in the Schur module decomposition of $K_{p,q}(\mathbb{P}^1\times\mathbb{P}^1;\mathbf{b};\mathbf{d})$, with multiplicity.

Steps : $L \coloneqq \{\mathbf{a} \mid |\mathbf{a}|=(p+q)(d_1+d_2)+(b_1+b_2)\}$ and $H=\sum_{\mathbf{a}\in L}\beta_{p,\mathbf{a}}(\mathbf{b};\mathbf{d})\cdot t^{\mathbf{a}}$

$K=\{\}$

While the coefficient of $\mathrm{lex}(H)>0$ do:

Let $(\lambda,\mu)=(\lambda_1,\lambda_2,\mu_1,\mu_2)$ be the weight of the lex-leading monomial in H

Let $K=K\cup\{(\lambda,\mu)\}$

Let H equal H minus the multigraded Hilbert series of $\mathbf{S}_\lambda(\mathbb{C}^2)\otimes\mathbf{S}_\mu(\mathbb{C}^2)$.

Return K.

5. Qualitative Aspects of the Computed Data

5.1. **Unimodality.** Our data strongly suggests that several statistics associated with the syzygies of $\mathbb{P}^1\times\mathbb{P}^1$ are unimodal. More specifically, our data leads to the following conjecture.

Conjecture 5.1. *For any* $\mathbf{b}$*, if either* d_1 *or* d_2 *is sufficiently large, then each of the following functions is unimodal:*

(1) The standard graded Betti numbers in a single row: $i\mapsto\beta_{i,i+k}(\mathbb{P}^1\times\mathbb{P}^1,\mathbf{b};\mathbf{d})$ *for any fixed* k.[2]

(2) The number of Schur functors with multiplicity appearing in a given row: fix some q *and consider* $p\mapsto$ *the total number of Schur functors, counted with multiplicity, appearing in* $K_{p,q}(\mathbb{P}^1\times\mathbb{P}^1,\mathbf{b};\mathbf{d})$.

(3) The largest multiplicity of Schur functors appearing in a given row: fix some q *and consider* $p\mapsto$ *the largest multiplicity of a Schur functors appearing in* $K_{p,q}(\mathbb{P}^1\times\mathbb{P}^1,\mathbf{b};\mathbf{d})$.

Remark 5.2. Our data also suggests that even the multigraded Betti numbers exhibit unimodality in certain ways, although in the multigraded setting there is no canonical choice for what one might expect to be unimodal. For example, fixing any multidegree $\mathbf{e}$ and our data suggests that $i\mapsto\beta_{i,i\mathbf{e}}(\mathbb{P}^1\times\mathbb{P}^1,\mathbf{b};\mathbf{d})$. It would be interesting to explore other ways in which the multigraded Betti numbers might satisfy some sort of unimodality or concavity properties. Given the large number of possible multidegrees, such questions can be somewhat complex.

Patterns similar to Conjecture 5.1 were observed for the Veronese syzygies of $\mathbb{P}^2$ in [BEGY18, Section 6.4]. Interestingly in this setting the authors observed that the function $p\mapsto$ the

[2]In the range of $\mathbf{b}$ we have considered in this paper, these functions are only interesting for $k=0,1$ or 2.

number of distinct Schur functors appearing in $K_{p,q}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b); \mathcal{O}_{\mathbb{P}^2}(d))$ appears to be unimodal (see [BEGY18, Question 6.11.(2)]). By contrast, our data provides a large number of counterexamples to that for $\mathbb{P}^1 \times \mathbb{P}^1$. More specifically, out of the rough 90 pairs of $\mathbf{b}$ and $\mathbf{d}$ that we tested, the number of distinct Schur functors appearing was not unimodal.

Example 5.3. Letting $\mathbf{b} = \mathbf{0}$, $\mathbf{d} = (3, 4)$, and considering $q = 1$ we see that the number of distinct Schur functors appearing in the decompositions of $K_{p,1}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$ is

$$(9, 26, 42, 52, 67, 71, 82, \mathbf{80}, 87, \mathbf{78}, 79, 63, 49, 5, 1),$$

which is not unimodal. We see a similar failure of the number of distinct Schur functors appearing in the decompositions of $K_{p,1}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$ when $\mathbf{b} = \mathbf{0}$ and $\mathbf{d} = (3, 5)$:

$$(11, 32, 56, 67, 96, 101, 127, \mathbf{125}, 146, \mathbf{137}, 154, \mathbf{135}, 141, 118, 116, 81, 33, 5, 1).$$

5.2. **Normality.** Ein, Erman, and Lazarsfeld have conjectured that, for large values of $\mathbf{d}$, the Betti numbers in any given row $\beta(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d})$ should look approximately like a normal distribution [EEL15, Conjecture B]. Bruce proved that a similar phenomena holds for the first row when $\mathbf{d} = (2, d_2)$ and $d_2 \to \infty$ in [Bru20a, Theorem A], but that it fails for the second row under the same hypotheses [Bru20a, Theorem B]. See also [Lem21, EY18] for related results.

Our data, while somewhat limited, suggests that results similar to [Bru20a, Theorem A, Theorem B] also hold for $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $(3, d_2)$ as $d_2 \to \infty$. In particular, as $d_2 \to \infty$ the Betti numbers in the $q = 1$ row of $\beta(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; (3, d_2))$ approach a normal distribution, while Betti numbers in the $q = 2$ row do not. Figure 3 highlights this for the $q = 1$ row.

It would be interesting to better understand what happens for the $q = 2$ row and a fixed d_1. This is likely related to the phenomenon of asymptotic non-vanishing of syzygies in the semi-ample setting as discussed in [Bru20b]. Concretely, we ask:

Question 5.4. *Does there exist $d_1 \in \mathbb{Z}_{\geq 2}$ such that the Betti numbers in the $q = 2$ row of $\beta(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; (d_1, d_2))$ approach a normal distribution as $d_2 \to \infty$?*

6. Representation Theoretic Conjectures

Utilizing the representation theory of $\mathbf{GL}_2 \times \mathbf{GL}_2$ provides the most concise way to express the syzygies of $\mathbb{P}^1 \times \mathbb{P}^1$. Our Schur functor data enabled us to make conjectures related to specific entries of the Betti tables. Additionally, our data raises questions regarding the ubiquity of redundant Schur functors.

6.1. **Specific Entries.** We first consider conjectures on specific $K_{p,q}$ groups. As noted earlier, the case when $\mathbf{b} = \mathbf{0}$ is of particular interest, as this case corresponds to the syzygies of the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ under the embedding by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathbf{d})$. Moreover, based on our data and the unimodality conjectures from the previous section, we expect the extremal entries in a row to involve the fewest Schur functors.

We thus are most interested in extremal entires in a row in the case $\mathbf{b} = 0$. We first offer a conjecture about the last entry of the $q = 1$ row:

Conjecture 6.1 (Row $q = 1$)**.** *Let* $\mathbf{d} \in \mathbb{Z}^2_{\geq 1}$ *and* $p = (d_1+1)(d_2-1)+d_1$*. (This is the largest value of* p *such that* $K_{p,1}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d}) \neq 0$ *in this case.) Let*

$$\mathbf{a} \coloneqq \left(\tbinom{d_1+1}{2}\tbinom{d_2}{1}\ ,\ \tbinom{d_1+1}{2}\tbinom{d_2}{1}\ ,\ \tbinom{d_1+1}{1}\tbinom{d_2+1}{2}-1\ ,\ \tbinom{d_1+1}{1}\tbinom{d_2}{2}+1\right) \in \mathbb{Z}^4.$$

(1) **Last entry:** *Assume* $d_2 > d_1$*. Then* $K_{p,1}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d})$ *is an irreducible Schur module. Specifically, if* $d_2 > d_1$ *then*

$$K_{p,1}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d}) \cong \mathbf{S}_{\mathbf{a}+(0,0,-1,1)}.$$

(2) **Second-to-last entry:** *Assume* $d_2 > d_1+1$*. Then* $K_{p-1,1}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d})$ *is the direct sum of* d_2 *distinct irreducible Schur modules. Specifically, if* $d_2 > d_1+1$ *then*

$$K_{p-1,1}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d}) \cong \bigoplus_{i=0}^{d_2-1} \mathbf{S}_{\mathbf{a}+(0,-d_1,-2-i,-d_2+2+i)}.$$

Our next conjectures focus on the last entries in the $q = 2$ row. In particular, the following conjecture describes the Schur functor decomposition for the last entry in the $q = 2$ row for all $\mathbf{d}$, as well as the decomposition for the second to last entry in the $q = 2$ row in the special cases when $\mathbf{d} = (2, d)$ and $\mathbf{d} = (3, d)$.

Conjecture 6.2 (Row $q = 2$)**.** *Let* $\mathbf{d} \in \mathbb{Z}^2_{\geq 1}$ *and let* $p = (d_1+1)(d_2+1)-3$*. (This is the largest value of* p *such that* $K_{p,2}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d}) \neq 0$*.)*

(1) **Last entry:** *The space* $K_{p,2}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d})$ *is a unique irreducible Schur module. Specifically,* $K_{p,2}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d}) \cong \mathbf{S}_{\mathbf{a}}$*, where*

$$\mathbf{a} \coloneqq \left(\tbinom{d_1+1}{2}\tbinom{d_2+1}{1}-1\ ,\ \tbinom{d_1+1}{2}\tbinom{d_2+1}{1}-d_1+1\ ,\ \tbinom{d_1+1}{1}\tbinom{d_2+1}{2}-1\ ,\ \tbinom{d_1+1}{1}\tbinom{d_2+1}{2}-d_2+1\right) \in \mathbb{Z}^4.$$

(2) **Second-to-last entry, $\mathbf{d} = (2, d)$:** *Assume that* $\mathbf{d} = (2, d)$*. The space* $K_{p,2}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d})$ *is the direct sum of* $d-2$ *Schur modules. Specifically,* $K_{p-1,2}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d}) \cong \oplus_{i=0}^{d-3} \mathbf{S}_{\mathbf{a}+(0,0,-i,+i)}$*, where*

$$\mathbf{a} \coloneqq \left(3d+2\ ,\ 3d\ ,\ \tfrac{1}{2}(3d^2+3d-2)-1\ ,\ \tfrac{1}{2}(3d^2+3d-2)-2(d_2-d_1)-3\right) \in \mathbb{Z}^4.$$

(3) **Second-to-last entry, $\mathbf{d} = (3, d)$:** *Assume that* $\mathbf{d} = (3, d)$*. The space* $K_{p,2}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d})$ *is the direct sum of* $2d-3$ *irreducible Schur module. More specifically, if*

$$\mathbf{a} \coloneqq \left(6d+5\ ,\ 6d+1\ ,\ 2d^2+2d-2\ ,\ 2d^2+2d-2d+3\right) \in \mathbb{Z}^4,$$
$$\mathbf{b} \coloneqq \left(6d+4\ ,\ 6d+2\ ,\ 2d^2+2d-2\ ,\ 2d^2+2d-2d+1\right) \in \mathbb{Z}^4,$$

then $K_{p-1,2}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d}) \cong \oplus_{i=0}^{d-3} \mathbf{S}_{\mathbf{a}+(0,0,-i,i)} \oplus \oplus_{j=0}^{d_2-2} \mathbf{S}_{\mathbf{b}+(0,0,-i,i)}$*.*

As we have only computed the full Betti table $\beta(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; (3, d_2))$ for four values of d_2, the evidence for part (3) of Conjecture 6.2 is admittedly scant. That said, the $\mathbf{a}$'s in both parts (2) and (3) of Conjecture 6.2, seem to fit into a potentially more general pattern. This leads us to ask the following question concerning the Schur functor decomposition for the second to last entry in the $q = 2$ row in general.

Question 6.3. *Let* $\mathbf{d} \in \mathbb{Z}^2_{\geq 1}$ *and let* $p = (d_1+1)(d_2+1)-3$. *(This is the largest value of* p *such that* $K_{p,2}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d}) \neq 0$.*) If*

$$\mathbf{a} \coloneqq \left(\tbinom{d_1+1}{2}\tbinom{d_2+1}{1} - 1\ ,\ \tbinom{d_1+1}{2}\tbinom{d_2+1}{1} - 2d_1 + 1\ ,\ \tbinom{d_1+1}{1}\tbinom{d_2+1}{2} - 2\ ,\ \tbinom{d_1+1}{1}\tbinom{d_2+1}{2} - 2d_2 + 2\right).$$

then is it the case that as representations of $\mathbf{GL}_2 \times \mathbf{GL}_2$*:*

$$\bigoplus_{i=0}^{d_2-3} \mathbf{S}_{\mathbf{a}+(0,0,-i,i)} \subset K_{p-1,2}\left(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d}\right)?$$

6.2. **Redundant Schur Functors.** The central result of [EL12] shows that asymptotically, Betti tables have numerous "redundant" entries. That is, it is very often the case that both $K_{p,q}$ and $K_{p-1,q+1}$ will be nonzero. These entries are "redundant" in the sense that they could not be predicted by the Hilbert function of the module.

A folklore question asks to find similar "redundant" representation in the Schur functor decomposition of $K_{p,q}$ and $K_{p-1,q+1}$. More specifically, we consider examples of a Schur functor $\mathbf{S}_\mu \otimes \mathbf{S}_\lambda$ that appears in the Schur functor decomposition of both $K_{p,q}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$ and $K_{p-1,q+1}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$. In [BEGY18, Example 6.17 and Question 6.16], the authors give examples of redundant Schur functors for $\mathbb{P}^2$ and ask whether redundant Schur functors occur frequently or sporadically. Based upon our data, redundant Schur functors seem quite common for $\mathbb{P}^1 \times \mathbb{P}^1$. For example, out of the approximately 200 pairs of $(\mathbf{b}; \mathbf{d})$ for which we computed Schur functor computations rough two-thirds contained redundant Schur functors.

While we did not find much of a pattern for when and where redundant Schur functors might occur, it would be interesting to explore that question further. We did observe, anecdotally, that redundant Schur functors were more likely to occur if one of d_1, d_2, b_1 or b_2 is sufficiently large. Focusing on the case when $\mathbf{b} = \mathbf{0}$ our data suggests the following conjecture.

Conjecture 6.4. *If either* d_1 *or* d_2 *is sufficiently large, then there exists* p, q *such that* $K_{p,q}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; \mathbf{d})$ *has redundant Schur functors.*

Furthermore, within each example, the number of redundant Schur functors seems able to be quite large both in terms of the total number and in terms of percentage of total Schur functors. For example, the largest total number of redundant Schur functors we observed is when $\mathbf{d} = (2, 10)$ and $\mathbf{b} = (0, 8)$; in this case, there are 596 redundant Schur functors out of 7135 total Schur functors (without multiplicity). The redundant Schur functors makes up the largest percentage of total Schur functors (counted without multiplicity) occurs when $\mathbf{d} = (3, 5)$ and $\mathbf{b} = (2, 4)$ where approximately 22.9% of Schur functors are redundant.

In addition, our data shows a number of examples where for a particular p and q all of the Schur functors appearing in the decomposition of $K_{p,q}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$ are redundant. For example, when $\mathbf{d} = (2, 3)$ and $\mathbf{b} = (1, 2)$ both $K_{5,0}(\mathbb{P}^1 \times \mathbb{P}^1, (1, 2); (2, 3))$ and $K_{4,1}(\mathbb{P}^1 \times \mathbb{P}^1, (1, 2); (2, 3))$ are isomorphic to $\mathbf{S}_{(8,3,11,6)} \oplus \mathbf{S}_{(7,4,10,7)} \oplus \mathbf{S}_{(6,5,9,8)}$ implying all of these Schur functors are redundant. Appendix C includes the Schur functor decompositions of $K_{p,q}(\mathbb{P}^1 \times \mathbb{P}^1, (1, 2); (2, 3))$ for all p and q.

7. Boij-Söderberg Theory Conjectures and Questions

7.1. **Background on Boij–Söderberg Theory.** Boij–Söderberg theory provides a way to decompose a Betti table as a positive rational sum of certain atomic building blocks called pure diagrams. The theory was conjectured by [BS08] and the main results were proven in [ES09]. See also [Fløø12, FMP16] for expository treatments of the theory or [BS12, ES10, EE17, Fløø10, BEKS12, BZEKS13, Erm09, GJM+15, GHS19] for more details on various aspects of the theory.

Having computed an array of Betti tables for embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$, we can analyze the pure diagrams and coefficients that arise in corresponding Boij–Söderberg decompositions. In order to get well-defined coefficients, we need to choose a specific set of representatives for the pure diagrams $\pi_{\boldsymbol{\delta}}$.

Set $[n] = \{0, 1, ..., n-1\}$. Given a sequence of integers $\boldsymbol{\delta} = (\delta_0, \ldots, \delta_r)$, called a *degree sequence*, let $\pi_{\boldsymbol{\delta}}$ be the Betti table with entries

$$\beta_{i,j}(\pi_{\boldsymbol{\delta}}) = \begin{cases} \prod_{i \neq j} \frac{1}{|\delta_i - \delta_j|} & \text{if } j = \delta_i \\ 0 & \text{if } j \neq \delta_i. \end{cases}$$

For instance

$$\pi_{(0,1,3,4)} = \begin{pmatrix} \frac{1}{12} & \frac{1}{6} & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{6} & \frac{1}{12} \end{pmatrix}$$

Note in particular, that $\pi_{\boldsymbol{\delta}}$ will often have entries in $\mathbb{Q}$, not in $\mathbb{Z}$.

For any graded Cohen-Macaulay module M over a polynomial ring, there exists a unique set of degree sequences C_M such that

$$\beta(M) = \sum_{\boldsymbol{\delta} \in C_M} a_{\boldsymbol{\delta}} \pi_{\boldsymbol{\delta}} \qquad \text{with } a_{\boldsymbol{\delta}} \in \mathbb{Q}.$$

This is called the *Boij-Söderberg decomposition* of M, and the rational numbers $\{a_{\boldsymbol{\delta}} \mid \boldsymbol{\delta} \in C_M\}$ are called the *Boij-Söderberg coefficients* of M.

7.2. **Conjectures on Boij-Söderberg coefficients.** Formulas for the coefficients have been found in certain cases where M has a well-understood algebraic or combinatorial structure [EJO20, MT19, NS13, GJM+15, GHS19, AGHS18, EES13, ES16]. In this section, we aim to provide conjectures on Boij-Söderberg coefficients for the Betti tables of $\mathbb{P}^1 \times \mathbb{P}^1$.

One common feature of Boij-Söderberg decompositions, exhibited in many of the examples referenced above, is that they rarely "skip over" potential degree sequences. For instance, in the case $\mathbf{b} = (0, 0)$ and $\mathbf{d} = (2, 5)$, the shape of the Betti table is:

$$\begin{pmatrix} * & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & * & * & * & * & * & * & * & * & * & * & * & * & * & * & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * & * \end{pmatrix}$$

where the zero entries are marked with $\cdot$ and nonzero entries are marked with $*$. Based on this shape, there are only 4 pure diagrams which could potentially arise in the Boij-Söderberg decomposition, depending on where you choose to shift from the 1st row to the 2nd row. (See also Example 7.3 below, which specifies the corresponding degree sequences.)

In this example, the coefficients of each such potential pure diagram turn out to be nonzero, although there is no obvious reason why this ought to be true. Conjecture 7.1 posits that this phenomenon occurs whenever $\mathbf{b} = (0,0)$. More precisely, when $\mathbf{b} = (0,0)$ and $\mathbf{d} = (d_1, d_2)$ where $d_1 \leq d_2$, the degree sequences which could possibly occur are given by

$$\boldsymbol{\delta}_j = [(d_1+1)(d_2+1)] \setminus \{1, (d_1+1)(d_2+1) - d_1 - j\} \quad \text{for} \quad 0 \leq j \leq (d_1-1)(d_2-2)$$

and we conjecture the following.

Conjecture 7.1. *The Boij-Söderberg coefficient $a_{\boldsymbol{\delta}_j}$ is nonzero for each j.*

We now attempt to better understand the values of the nonzero coefficients. Our first such conjecture, provides a complete description of the Boij-Söderberg coefficients in the case where $\mathbf{d} = (2, d_2)$ and $\mathbf{b} = (0, b_2)$ for $0 \leq b_2 \leq d_2 - 2$ and $d_2 \geq 3$. In particular, taking $b_2 = 0$, this provides a complete conjectural description of the Boij-Söderberg coefficients of the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, d_2)$.

Conjecture 7.2. *Let $\mathbf{d} = (2, d_2)$ and $\mathbf{b} = (0, b_2)$ for some $0 \leq b_2 \leq d_2 - 2$. Assume $d_2 \geq 3$. The Boij-Söderberg decomposition will involve the degree sequences $\boldsymbol{\delta}_j$ for $0 \leq j \leq d_2 - 2$ where $\boldsymbol{\delta}_j$ is defined as*

$$\boldsymbol{\delta}_j = \begin{cases} [3(d_2+1)] \setminus \{b_2+1, 3d_2+1-j\} & 0 \leq j \leq d_2 - b_2 - 2 \\ [3(d_2+1)] \setminus \{d_2 - j - 1, 2d_2 + b_2 + 3\} & d_2 - b_2 - 1 \leq j \leq d_2 - 2. \end{cases}$$

Moreover, the Boij-Söderberg coefficients corresponding to $\boldsymbol{\delta}_j$ will be given by the formula

$$a_{\boldsymbol{\delta}_j} = \begin{cases} 2(3d_2)! & j \neq d_2 - b_2 - 2 \\ 2(d_2+2)(3d_2)! & j = d_2 - b_2 - 2. \end{cases}$$

In particular, all of the coefficients, except for the last one, will be identical, and as $d_2 \to \infty$, the last coefficient will dominate.

To prove Conjecture 7.2, one might be able to use [Lem18, Corollary 5], which provides an explicit formula for the Betti numbers in this case.

Example 7.3. As noted above, if we take $b_2 = 0$ and $d_2 \geq 3$, then Conjecture 7.2 implies that the Boij-Söderberg decomposition for the the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, d_2)$ is:

$$\beta\left(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; (2, d_2)\right) = 2(3d_2)! \left(\pi_{\boldsymbol{\delta}_0} + \pi_{\boldsymbol{\delta}_1} + \cdots + \pi_{\boldsymbol{\delta}_{d_2-3}}\right) + 2(d_2+2)(3d_2)! \pi_{\boldsymbol{\delta}_{d_2-2}}$$

where $\boldsymbol{\delta}_j$ is the degree sequence $(0, 2 \ldots, 3d_2 - j, \widehat{3d_2 + 1 - j}, 3d_2 + 2 - j \ldots, 3d_2 + 2)$. For example, if $d_2 = 5$ then we have:

$$\begin{cases} \boldsymbol{\delta}_0 &= (0,2,3,4,5,6,7,8,9,10,11,12,13,14,15,\widehat{16},17) \\ \boldsymbol{\delta}_1 &= (0,2,3,4,5,6,7,8,9,10,11,12,13,14,\widehat{15},16,17) \\ \boldsymbol{\delta}_2 &= (0,2,3,4,5,6,7,8,9,10,11,12,13,\widehat{14},15,16,17) \\ \boldsymbol{\delta}_3 &= (0,2,3,4,5,6,7,8,9,10,11,12,\widehat{13},14,15,16,17) \end{cases}$$

and Conjecture 7.2 states that

$$\beta\left(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{0}; (2,5)\right) = 2(15!)\left(\pi_{\boldsymbol{\delta}_0} + \pi_{\boldsymbol{\delta}_1} + \pi_{\boldsymbol{\delta}_2}\right) + 14(15!)\pi_{\boldsymbol{\delta}_3}.$$

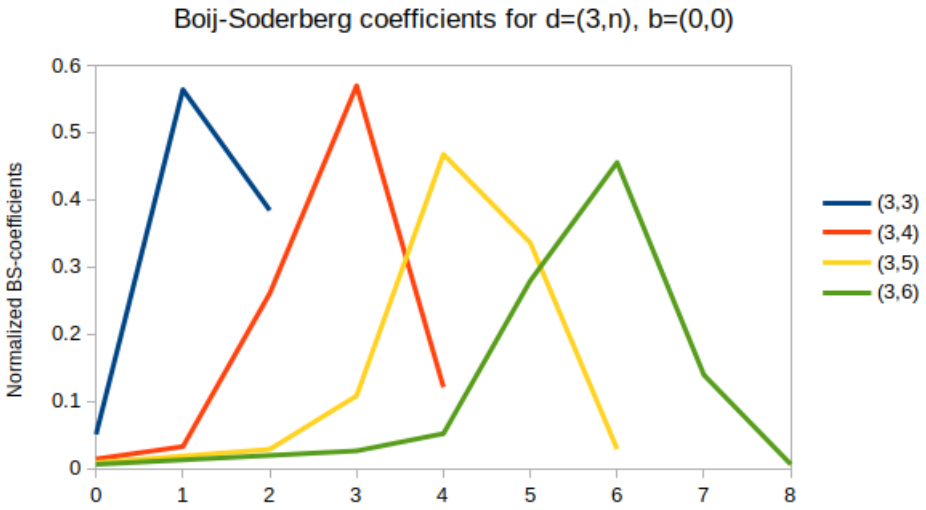

FIGURE 6. The Boij-Soderberg coefficients for $\mathbf{d} = (3, n)$ and $\mathbf{b} = (0, 0)$

Remark 7.4. Conjecture 7.2 would imply the following curious fact: consider the Betti table of the homogeneous coordinate ring (that is, with $\mathbf{b} = \mathbf{0}$) of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $(2, d_2)$. As $d_2 \to \infty$, these Betti tables will be "asymptotically pure" in a sense that parallels the main result of [Erm15], where these Betti tables are asymptotically dominated by the contributions from a single pure diagram. See also [Tay21, ES17]. It would be very interesting to better understand the limits under which such Betti tables are "asymptotically pure"; this question is wide open for $\mathbb{P}^2$ as well, as discussed in [BEGY18, §6.3]

When $\mathbf{d} = (3, d_2)$ and $\mathbf{b} = (0, 0)$, we have a conjecture for roughly the first half of the coefficients. Figure 6 displays these coefficients, rescaled by a factor of $6d_2(3d_2)!$ (so that these numbers sum to 1) to allow for a better comparison as d_2 grows. Notice that in each case, there is a set of small values followed by a peak.

Conjecture 7.5. *For* $\mathbf{b} = (0, 0)$ *and* $\mathbf{d} = (3, d_2)$*, with* $d_2 \geq 4$*, the Boij-Söderberg coefficients for* $j = 0, \ldots, d_2 - 4$ *are*

$$a_{\delta_j} = \frac{(j+1)(4d_2+4)!}{4\binom{4d_2+4}{4}}.$$

7.3. **More questions.** Our data on Boij-Söderberg coefficients also illuminated some fascinating patterns which we were not able to convert into precise conjectures. We conclude by drawing attention to a couple of these phenomena for curious readers.

As we saw in the previous conjectures, there are various situations where, if we fix some of the variables b_1, b_2, d_1, or d_2, then the number of Boij-Söderberg coefficients remains fixed. When this happens, it is natural to understand how the individual coefficients depend on the remaining variables.

d_1	$\mathbf{b}$	Normalized BS-coefficients as $d_2 \to \infty$
2	$(1,0)$	$(1/2, 1/2)$
	$(1,1)$	$(2/9, 5/9, 2/9)$
	$(1,2)$	$(8/81, 32/81, 67/162, 5/54)$
3	$(2,0)$	$(3/8, 1/4, 3/8)$
4	$(3,0)$	$(8/25, 4/25, 11/50, 3/10)$

TABLE 2. Asymptotic values of BS-coefficients as $d_2 \to \infty$, normalized so that the numbers sum to 1

The sum of the Boij-Söderberg coefficients of a module can be determined by the multiplicity of that module, and in the $\mathbb{P}^1 \times \mathbb{P}^1$ case, this sum is

$$\sum_{\boldsymbol{\delta} \in C_{S(\mathbf{b};\mathbf{d})}} a_{\boldsymbol{\delta}} = \frac{2d_1d_2}{((d_1+1)(d_2+1))^{\underline{3}}} \cdot ((d_1+1)(d_2+1))!$$

where $x^{\underline{n}}$ denotes the *falling factorial*:

$$x^{\underline{n}} = \prod_{k=0}^{n-1}(x+k).$$

To better analyze the coefficients, we rescale:

$$b_{\boldsymbol{\delta}} = \frac{a_{\boldsymbol{\delta}}}{((d_1+1)(d_2+1))!} \text{ and note that } \sum_{\boldsymbol{\delta} \in C_{S(\mathbf{b};\mathbf{d})}} b_{\boldsymbol{\delta}} = \frac{2d_1d_2}{((d_1+1)(d_2+1))^{\underline{3}}}.$$

While it appears difficult to give concrete conjectural formulae for the Boij-Söderberg coefficients for larger values of $\mathbf{d}$ than those studied in the previous subsection, the above equation suggests something about the behavior of the $b_{\boldsymbol{\delta}}$ as a rational function of d_1 and/or of d_2.

The following is a concrete conjecture in this direction:

Conjecture 7.6. *For $\mathbf{b} = (d_1 - 1, b_2)$ and $\mathbf{d} = (d_1, d_2)$, with $d_1 \leq d_2$, $0 \leq b_2 \leq d_2 - 2$, the degree sequences appearing in the Boij-Söderberg decomposition of $\beta(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$ are precisely:*

$$\delta_j = [(d_1+1)(d_2+1) - 1] \setminus \{(b_2+1)d_1 - j\} \qquad 0 \leq j \leq (b_2+1)(d_1-1).$$

For any fixed b_2 and d_1, the coefficient a_{δ_j} has the form

$$a_{\delta_j} = p_j(d_2)((d_1+1)(d_2+1))!$$

where p_j is some degree -2 rational function in d_2.

Evidence for this conjecture is provided in Appendix D. Moreover, the above discussion and Conjecture 7.6 suggest the following question.

Question 7.7. *What is the value of*

$$\lim_{d_2 \to \infty} a_{\delta_j} \frac{((d_1+1)(d_2+1))^{\underline{3}}}{2d_1d_2((d_1+1)(d_2+1))!}?$$

Most of the conjectures considered in this section can be understood as being motivated by the following overarching but vague question:

Question 7.8. *To what extent, and under what additional restrictions, can the Boij-Söderberg coefficients of* $\beta(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$ *be understood as rational functions in* b_1, b_2, d_1 *and/or* d_2*?*

We end with a mystery. In Appendix D, we plotted the Boij-Söderberg coefficients of $\beta(\mathbb{P}^1 \times \mathbb{P}^1, \mathbf{b}; \mathbf{d})$, after rescaling so that the sum of the coefficients is 1, for various natural families depending on $\mathbf{b}$ and/or $\mathbf{d}$. We simply note that the graphics suggest a remarkable uniformity among these families as one varies the parameters. Can one explain, or even precisely describe, this phenomenon?

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Appendix A. Number and size of matrices computed

We record, for nearly all $(\mathbf{b};\mathbf{d})$ pairs for which we have complete data, the number of matrices in the relevant range and the size of the largest matrix.

d	**b**	Number of matrices	Largest matrix	**b**	Number of matrices	Largest matrix
$(2,4)$	$(0,0)$	75	625×2431	$(1,0)$	0	N/A
	$(0,1)$	0	N/A	$(1,1)$	23	73×81
	$(0,2)$	17	19×15	$(1,2)$	109	554×909
	$(0,3)$	44	77×82	$(1,3)$	212	1387×3171
$(2,5)$	$(0,0)$	216	3386×13946	$(1,0)$	0	N/A
	$(0,1)$	101	1508×6988	$(1,1)$	31	108×116
	$(0,2)$	20	23×18	$(1,2)$	135	1245×1911
	$(0,3)$	55	116×117	$(1,3)$	297	5302×10822
	$(0,4)$	110	434×552	$(1,4)$	486	9432×25262
$(2,6)$	$(0,0)$	466	$18,902 \times 81,386$	$(1,0)$	0	N/A
	$(0,1)$	273	$8547 \times 40,922$	$(1,1)$	35	148×156
	$(0,2)$	150	$3075 \times 16,649$	$(1,2)$	171	2476×3607
	$(0,3)$	62	159×155	$(1,3)$	367	$15,588 \times 29,403$
	$(0,4)$	131	723×868	$(1,4)$	651	$44,886 \times 107,138$
	$(0,5)$	212	2512×3580	$(1,5)$	919	$62,250 \times 187,699$
$(2,7)$	$(0,0)$	831	$108,060 \times 482,053$	$(1,0)$	0	NA
	$(0,1)$	573	$49,808 \times 243,840$	$(1,1)$	43	196×204
	$(0,2)$	368	$18,682 \times 102,154$	$(1,2)$	197	4392×6171
	$(0,3)$	226	$5600 \times 34,800$	$(1,3)$	456	$39,140 \times 69,452$
	$(0,4)$	148	1118×1286	$(1,4)$	795	$163,325 \times 358,383$
	$(0,5)$	251	4562×6132	$(1,5)$	1198	$352,746 \times 949,098$
	$(0,6)$	385	$14,782 \times 22,836$	$(1,6)$	1619	$436,912 \times 1,248,208$
$(2,8)$	$(0,0)$	1391	$627,537 \times 2,886,389$	$(0,7)$	622	$87,266 \times 144,514$
	$(0,1)$	995	$291,943 \times 1,460,756$	$(1,0)$	0	NA
	$(0,2)$	721	$113,886 \times 627,766$	$(1,1)$	47	249×255
	$(0,3)$	479	$36,350 \times 224,623$	$(1,2)$	233	7310×9966
	$(0,4)$	348	$9408 \times 66,110$	$(1,3)$	527	$86,245 \times 146,042$
	$(0,5)$	280	7594×9764	$(1,4)$	968	$498,024 \times 1,022,361$
	$(0,6)$	445	$28,470 \times 41,648$			
$(2,9)$	$(0,7)$	809	$177,658 \times 278,759$	$(1,2)$	259	$11,415 \times 15,196$
	$(1,0)$	0	N/A	$(1,3)$	616	$174,144 \times 283,727$
	$(1,1)$	55	310×314			
$(2,10)$	$(0,8)$	1106	$1,111,726 \times 1,843,366$	$(1,2)$	295	$17,132 \times 22,350$
	$(1,0)$	0	N/A	$(1,3)$	687	$325,114 \times 513,364$
	$(1,1)$	59	376×378			
$(2,11)$	$(1,0)$	0	N/A	$(1,2)$	321	$24,649 \times 31,638$
	$(1,1)$	67	450×450	$(1,3)$	776	$574,112 \times 882,626$

Table 3. Matrix data

d	**b**	Number of matrices	Largest matrix	**b**	Number of matrices	Largest matrix
$(3,3)$	$(0,0)$	104	1772×6180	$(1,1)$	31	88×96
	$(0,1)$	0	NA	$(1,2)$	125	740×1204
	$(0,2)$	19	20×16	$(2,2)$	308	2838×7308
$(3,4)$	$(0,0)$	521	$25,320 \times 87,114$	$(1,2)$	177	2038×3023
	$(0,1)$	148	$6678 \times 29,840$	$(1,3)$	368	$11,086 \times 21,945$
	$(0,2)$	23	26×20	$(2,0)$	24	24×20
	$(1,3)$	58	130×140	$(2,1)$	164	1956×2975
	$(1,0)$	122	$2407 \times 12,740$	$(2,2)$	468	$19,478 \times 43,618$
	$(1,1)$	37	138×144	$(2,3)$	836	$35,556 \times 96,730$
$(3,5)$	$(0,0)$	1344	$361,276 \times 1,231,276$	$(1,3)$	495	$39,424 \times 70,894$
	$(0,1)$	711	$119,254 \times 505,443$	$(1,4)$	858	$162,286 \times 367,093$
	$(0,2)$	236	19168×104246	$(2,0)$	29	30×24
	$(0,3)$	72	196×200	$(2,1)$	219	4350×6320
	$(0,4)$	153	1078×1328	$(2,2)$	618	$87,401 \times 178,536$
	$(1,0)$	338	$18,014 \times 101,895$	$(2,3)$	1217	$348,702 \times 971,100$
	$(1,1)$	210	$5758 \times 35,668$	$(2,4)$	1891	$467,124 \times 1,322,104$
	$(1,2)$	215	4470×6248			
$(3,6)$	$(0,3)$	334	$45,094 \times 290,746$	$(1,3)$	598	$110,702 \times 186,050$
	$(0,4)$	181	1774×2076	$(1,4)$	1106	$697,950 \times 1,436,165$
	$(0,5)$	296	$8224 \times 11,390$	$(2,0)$	34	34×28
	$(1,0)$	740	$142,906 \times 845,408$	$(2,1)$	259	$8347 \times 11,760$
	$(1,1)$	476	$44,876 \times 290,369$	$(2,2)$	793	$300,091 \times 573,890$
	$(1,2)$	468	$11,665 \times 83,466$			
$(3,7)$	$(0,4)$	518	$94,088 \times 700,128$	$(1,2)$	827	$97,064 \times 709,416$
	$(0,5)$	348	$14,768 \times 19,378$	$(1,3)$	968	268940×428636
	$(0,6)$	554	$68,616 \times 102,344$	$(2,0)$	39	40×32
	$(1,0)$	1130	$1,128,854 \times 6,980,468$	$(2,1)$	314	$14,766 \times 20,308$
	$(1,1)$	965	$369,576 \times 2,450,184$	$(2,2)$	950	$855,136 \times 1,556,128$
$(3,8)$	$(0,5)$	745	$177,432 \times 1,500,926$	$(2,0)$	44	44×36
	$(0,6)$	636	$130,144 \times 184,592$	$(2,1)$	354	$24,136 \times 32,574$
	$(0,7)$	900	$553,291 \times 879,321$	$(2,2)$	1130	$2,124,896 \times 3,719,448$
$(3,9)$	$(0,7)$	1026	$1,105,918 \times 1,673,092$	$(2,1)$	409	$37,620 \times 49,992$
	$(2,0)$	49	50×40			
$(4,4)$	$(0,0)$	1715	$853,068 \times 2,722,820$	$(1,2)$	228	5269×7364
	$(0,1)$	764	$165,929 \times 743,227$	$(1,3)$	501	$50,156 \times 91,458$
	$(0,2)$	198	$6518 \times 43,768$	$(2,2)$	682	$121,747 \times 241,924$
	$(0,3)$	80	207×222	$(2,3)$	1321	$581,410 \times 1,582,730$
	$(1,1)$	249	$24,765 \times 138,553$			
$(4,5)$	$(0,2)$	1102	$731,824 \times 4,033,789$	$(2,0)$	525	$63,634 \times 456,031$
	$(0,3)$	315	$14,409 \times 117,520$	$(2,1)$	521	$15,511 \times 120,826$
	$(0,4)$	207	2121×2596	$(2,2)$	879	$517,511 \times 948,896$
	$(1,1)$	1075	$755,881 \times 4,074,383$	$(2,3)$		
	$(1,2)$	559	$70,246 \times 471,986$	$(3,0)$	94	300×318
	$(1,3)$	661	$171,904 \times 287,389$	$(3,1)$	673	$169,940 \times 292,984$
$(4,6)$	$(0,4)$	507	$27,864 \times 267,592$	$(2,2)$	1381	1713790×2964636
	$(0,5)$	411	$21,318 \times 28,941$	$(3,0)$	116	417×438
	$(1,3)$	1132	$471,259 \times 740,692$	$(3,1)$	813	$462,729 \times 767,366$
	$(2,1)$	956	$157,164 \times 1,277,412$			
$(4,7)$	$(0,5)$	791	$49,046 \times 542,194$	$(3,0)$	130	540×568
	$(0,6)$	762	$221,972 \times 324,448$	$(3,1)$	991	$1,100,334 \times 1,771,080$

TABLE 4. Matrix data

Appendix B. Total Betti Numbers

$\beta(\mathbf{0};(2,2)) =$

	0	1	2	3	4	5	6	7
0:	1	.	.	.	.	.	.	.
1:	.	20	64	90	64	20	.	.
2:	.	.	.	.	.	.	1	.

$\beta(\mathbf{0};(2,3)) =$

	0	1	2	3	4	5	6	7	8	9	10
0:	1	.	.	.	.	.	.	.	.	.	.
1:	.	43	222	558	840	798	468	147	8	.	.
2:	.	.	.	.	.	.	.	.	9	2	.

$\beta(\mathbf{0};(2,4)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	75	536	1947	4488	7095	7920	6237	3344	1089	120	11	.	.
2:	.	.	.	.	.	.	.	.	.	.	66	24	3	.

$\beta(\mathbf{0};(2,5)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	116	1060	5040	15652	34580	56628	70070	65780	46332	23660	8008	1260	195	14	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	455	210	45	4	.

$\beta(\mathbf{0};(2,6)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	166	1848	10863	42432	120360	259488	436254	579904	612612	512720	335478	166464	58344	11424	2295	288	17	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3060	1632	459	72	5	.

$\beta(\mathbf{0};(2,7)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	225	2954	20685	97356	337155	901170	1912806	3281680	4598874	5290740	4996810	3852744	2393430	1166676	421515	95760	22610	3780	399	20	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	20349	11970	3990	840	105	6	.

$\beta(\mathbf{0};(2,8)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	293	4432	36018	198352	811118	2586672	6628853	13921072	24270543	35421472	43474508	44930592	39017108	28289632	16915833	8152672	3023603	765072	201894	40480	5796	528	23	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	134596	85008	31878	8096	1380	144	7	.

$\beta(\mathbf{0};(3,3)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	87	676	2691	6864	12155	15444	14157	9152	3861	780	22	.	.	.
2:	.	.	.	.	.	.	.	.	.	.	165	144	39	4	.

$\beta(\mathbf{0};(3,4)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	147	1530	8364	30192	78540	153816	232050	272272	247962	172380	87516	28560	3939	238	15	.	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	1287	3094	1800	528	85	6	.

$\beta(\mathbf{0};(3,5)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
0:	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	223	2912	20265	94696	325185	860472	1804278	3049120	4191894	4702880	4291378	3147312	1805570	759696	195390	25088	3247	360	19	.	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	6435	37856	41684	20520	6270	1240	147	8	.

$\beta((1,1);(2,3)) =$

	0	1	2	3	4	5	6	7	8	9	10	11
0:	4	28	72	56	.	.	.	.	.	.	.	.
1:	.	.	8	168	504	672	528	252	68	8	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.

$\beta((1,1);(2,4)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0:	4	36	120	120	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	32	660	2772	5808	7920	7524	5060	2376	744	140	12	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

$\beta((1,1);(2,5)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0:	4	44	180	220	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	80	1820	9828	28028	54340	77220	82940	68068	42588	20020	6860	1620	236	16	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

$\beta((1,1);(2,6)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0:	4	52	252	364	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	160	4080	26928	97104	243984	461448	680680	797368	747864	562224	337008	159120	57936	15708	2988	356	20	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

$\beta((1,1);(2,7)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
0:	4	60	336	560	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	280	7980	62244	271320	837216	1976760	3708040	5643456	7054320	7289464	6240360	4418640	2573664	1220940	464436	138320	31080	4956	500	24	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

$\beta((1,1);(2,8)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
0:	4	68	432	816	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	448	14168	127512	651728	2384272	6749028	15363172	28765088	44930592	59075408	65731792	62047008	49685152	33668228	19208772	9152528	3598672	1147608	289432	55568	7632	668	28	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

$\beta((1,1);(2,9)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
0:	4	76	540	1140	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	672	23400	238680	1399320	5920200	19536660	52295100	116233260	217809540	347677200	476050320	561632400	572330160	504131940	383467500	251213820	141098100	67490280	27232920	9149400	2513160	549900	92196	11124	860	32	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

$\beta((1,1);(2,10)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0:	4	84	660	1540	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	960	39540	454956	3795116	23332760	120077160	490777340	1515479760	3788077340	8023111540	15630000900	26018758760	37598833800	45114746320	45025148520	37695130760	25072133820	14140149140	6516467760	2502562160	799112760	208605876	39765180	6444420	1191480	159140	15876	84	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

$\beta((1,1);(2,11)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
0:	4	92	792	2044	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	1320	54908	687456	5865196	37351456	191554396	781748196	2503628716	6332130916	13399932716	24651148956	36911299956	44420979236	44420979236	36167738956	24144149156	13471471476	5586270916	1666351916	433553076	83328756	10114076	1141236	89996	5516	60	.	.	.	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

$\beta((1,1);(2,12)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
0:	4	100	936	2600	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	1760	79040	1063652	9756460	74301120	446373560	2146977320	8145975320	24144975320	55966955320	104457956520	154957956520	185949547820	185949547820	154957956520	104457956520	55966955320	24144975320	8145975320	2146977320	446373560	74301120	9756460	1063652	79040	1760	.	.	.	.	.	.	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

$B((1,1);(3,3)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0:	4	39	144	165	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	22	780	3861	9152	14157	15444	12155	6864	2691	676	87	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.

$B((1,1);(3,4)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0:	4	50	240	364	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	58	2448	16728	57120	132600	228072	301444	311168	251940	159120	77112	27744	6936	1020	30	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	16	2	.

$B((1,1);(3,5)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
0:	4	61	360	680	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	127	6020	52269	233016	721905	1697688	3155710	4730056	5819814	5878600	4888282	3333360	1846914	817836	281295	71288	11795	756	39	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	190	40	3	.

$B((1,1);(3,6)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
0:	4	72	504	1140	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	240	12600	132480	728640	2823480	8424900	20189400	39801900	65523780	90930960	106977600	106977600	90930960	65523780	39801960	20189400	8424900	2823480	728640	132480	12600	1124	48	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3024	552	72	4	.

$B((1,1);(3,7)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0:	4	83	672	1771	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	408	23548	290403	1900080	8838765	31962060	93776865	228914400	472526145	833976780	1268123745	1669794480	1909884465	1900189620	1644131655	1234857120	802110855	447943860	213176535	85322640	28144935	7396740	1448811	175392	20678	1564	57	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	20475	6552	1134	112	5	.

$B((1,1);(4,4)) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
0:	4	64	400	816	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	.	112	6776	64064	304304	1003200	2515524	5002624	8072416	10708096	11757200	10708096	8072416	5002624	2515524	1003200	304304	64064	6776	112	.	.	.
2:	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	816	400	64	4	.

Appendix C. Example of Schur Functor Decomposition

$$K_{0,0}(\mathbb{P}^1\times\mathbb{P}^1,(1,2);(2,3)) = \mathbf{S}_{(1,0,2,0)}$$

$$K_{1,0}(\mathbb{P}^1\times\mathbb{P}^1,(1,2);(2,3)) = \mathbf{S}_{(2,1,3,2)}\oplus\mathbf{S}_{(2,1,4,1)}\oplus\mathbf{S}_{(2,1,5,0)}\oplus\mathbf{S}_{(3,0,3,2)}\oplus\mathbf{S}_{(3,0,4,1)}$$

$$\begin{aligned}K_{2,0}(\mathbb{P}^1\times\mathbb{P}^1,(1,2);(2,3)) &= \mathbf{S}_{(3,2,4,4)}\oplus\mathbf{S}_{(3,2,5,3)}^{\oplus 2}\oplus\mathbf{S}_{(3,2,6,2)}^{\oplus 2}\oplus\mathbf{S}_{(3,2,7,1)}\oplus\mathbf{S}_{(4,1,4,4)}\oplus\mathbf{S}_{(4,1,5,3)}^{\oplus 2}\\ &\quad\oplus\mathbf{S}_{(4,1,6,2)}^{\oplus 2}\oplus\mathbf{S}_{(4,1,7,1)}\oplus\mathbf{S}_{(5,0,5,3)}\end{aligned}$$

$$\begin{aligned}K_{3,0}(\mathbb{P}^1\times\mathbb{P}^1,(1,2);(2,3)) &= \mathbf{S}_{(4,3,6,5)}^{\oplus 2}\oplus\mathbf{S}_{(4,3,7,4)}^{\oplus 3}\oplus\mathbf{S}_{(4,3,8,3)}^{\oplus 2}\oplus\mathbf{S}_{(4,3,9,2)}\oplus\mathbf{S}_{(5,2,6,5)}^{\oplus 2}\oplus\mathbf{S}_{(5,2,7,4)}^{\oplus 3}\\ &\quad\oplus\mathbf{S}_{(5,2,8,3)}^{\oplus 2}\oplus\mathbf{S}_{(5,2,9,2)}\oplus\mathbf{S}_{(6,1,6,5)}\oplus\mathbf{S}_{(6,1,7,4)}\oplus\mathbf{S}_{(6,1,8,3)}\end{aligned}$$

$$\begin{aligned}K_{4,0}(\mathbb{P}^1\times\mathbb{P}^1,(1,2);(2,3)) &= \mathbf{S}_{(5,4,7,7)}\oplus\mathbf{S}_{(5,4,8,6)}^{\oplus 2}\oplus\mathbf{S}_{(5,4,9,5)}^{\oplus 2}\oplus\mathbf{S}_{(5,4,10,4)}\oplus\mathbf{S}_{(5,4,11,3)}\oplus\mathbf{S}_{(6,3,7,7)}\\ &\quad\oplus\mathbf{S}_{(6,3,8,6)}^{\oplus 2}\oplus\mathbf{S}_{(6,3,9,5)}^{\oplus 2}\oplus\mathbf{S}_{(6,3,10,4)}\oplus\mathbf{S}_{(7,2,8,6)}\oplus\mathbf{S}_{(7,2,9,5)}\oplus\mathbf{S}_{(7,2,10,4)}\end{aligned}$$

$$K_{4,1}(\mathbb{P}^1\times\mathbb{P}^1,(1,2);(2,3)) = \mathbf{S}_{(6,5,9,8)}\oplus\mathbf{S}_{(7,4,10,7)}\oplus\mathbf{S}_{(8,3,11,6)}$$

$$K_{5,0}(\mathbb{P}^1\times\mathbb{P}^1,(1,2);(2,3)) = \mathbf{S}_{(6,5,9,8)}\oplus\mathbf{S}_{(7,4,10,7)}\oplus\mathbf{S}_{(8,3,11,6)}$$

$$\begin{aligned}K_{5,1}(\mathbb{P}^1\times\mathbb{P}^1,(1,2);(2,3)) &= \mathbf{S}_{(7,6,10,10)}\oplus\mathbf{S}_{(7,6,11,9)}^{\oplus 2}\oplus\mathbf{S}_{(7,6,12,8)}^{\oplus 2}\oplus\mathbf{S}_{(7,6,13,7)}\oplus\mathbf{S}_{(7,6,14,6)}\oplus\mathbf{S}_{(8,5,10,10)}\\ &\quad\oplus\mathbf{S}_{(8,5,11,9)}^{\oplus 2}\oplus\mathbf{S}_{(8,5,12,8)}^{\oplus 2}\oplus\mathbf{S}_{(8,5,13,7)}\oplus\mathbf{S}_{(9,4,11,9)}\oplus\mathbf{S}_{(9,4,12,8)}\oplus\mathbf{S}_{(9,4,13,7)}\end{aligned}$$

$$\begin{aligned}K_{6,1}(\mathbb{P}^1\times\mathbb{P}^1,(1,2);(2,3)) &= \mathbf{S}_{(8,7,12,11)}^{\oplus 2}\oplus\mathbf{S}_{(8,7,13,10)}^{\oplus 3}\oplus\mathbf{S}_{(8,7,14,9)}^{\oplus 2}\oplus\mathbf{S}_{(8,7,15,8)}\oplus\mathbf{S}_{(9,6,12,11)}^{\oplus 2}\oplus\mathbf{S}_{(9,6,13,10)}^{\oplus 3}\\ &\quad\oplus\mathbf{S}_{(9,6,14,9)}^{\oplus 2}\oplus\mathbf{S}_{(9,6,15,8)}\oplus\mathbf{S}_{(10,5,12,11)}\oplus\mathbf{S}_{(10,5,13,10)}\oplus\mathbf{S}_{(10,5,14,9)}\end{aligned}$$

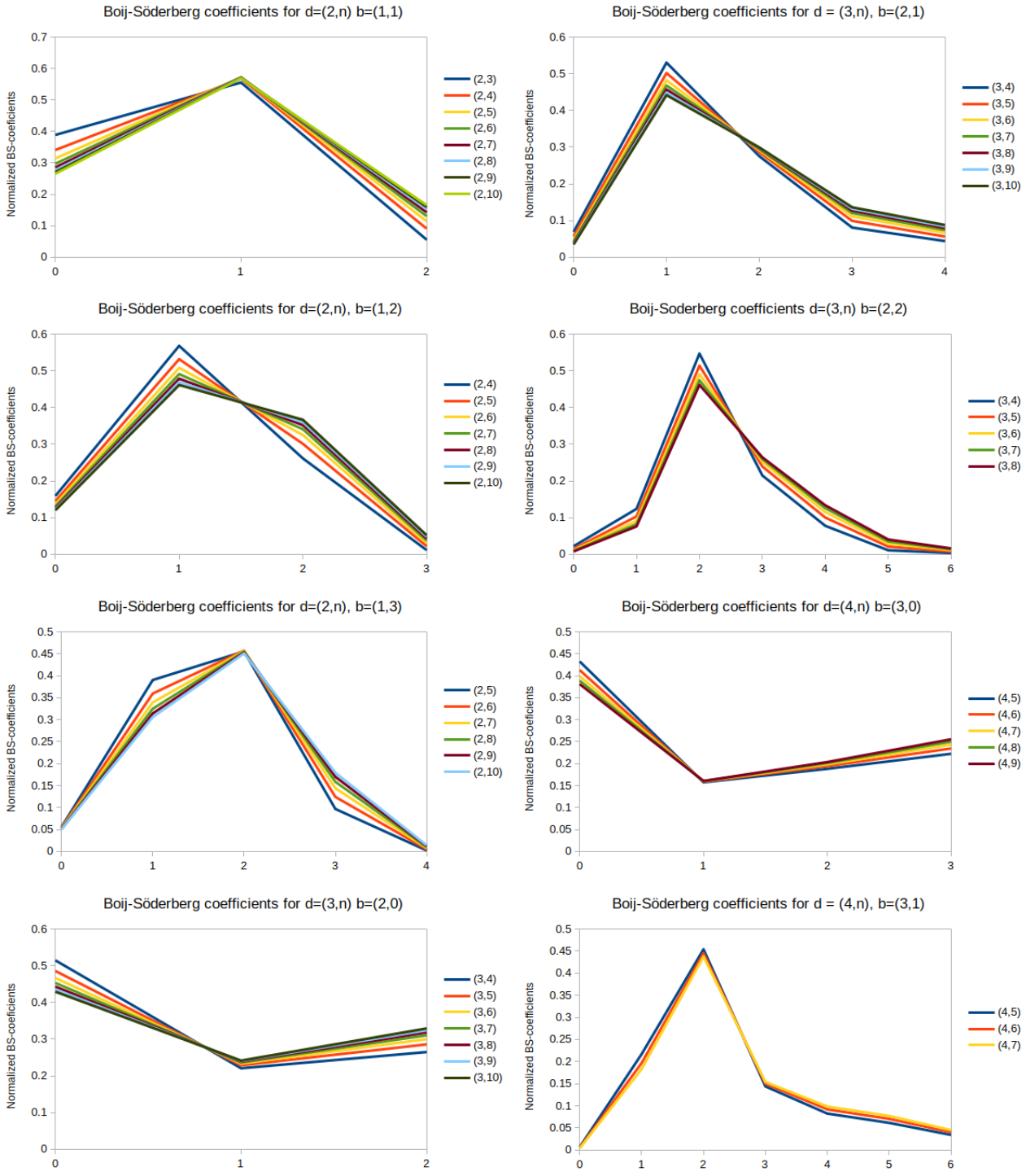
$$\begin{aligned}K_{7,1}(\mathbb{P}^1\times\mathbb{P}^1,(1,2);(2,3)) &= \mathbf{S}_{(9,8,13,13)}\oplus\mathbf{S}_{(9,8,14,12)}^{\oplus 2}\oplus\mathbf{S}_{(9,8,15,11)}^{\oplus 2}\oplus\mathbf{S}_{(9,8,16,10)}\oplus\mathbf{S}_{(10,7,13,13)}\\ &\quad\oplus\mathbf{S}_{(10,7,14,12)}^{\oplus 2}\oplus\mathbf{S}_{(10,7,15,11)}^{\oplus 2}\oplus\mathbf{S}_{(10,7,16,10)}\oplus\mathbf{S}_{(11,6,14,12)}\end{aligned}$$

$$K_{8,1}(\mathbb{P}^1\times\mathbb{P}^1,(1,2);(2,3)) = \mathbf{S}_{(10,9,15,14)}\oplus\mathbf{S}_{(10,9,16,13)}\oplus\mathbf{S}_{(10,9,17,12)}\oplus\mathbf{S}_{(11,8,15,14)}\oplus\mathbf{S}_{(11,8,16,13)}$$

$$K_{9,1}(\mathbb{P}^1\times\mathbb{P}^1,(1,2);(2,3)) = \mathbf{S}_{(11,10,17,15)}$$

Appendix D. Boij-Söderberg coefficients for $\mathbf{d} = (d_1, d_2)$, $\mathbf{b} = (d_1 - 1, b_2)$

We record the Boij-Söderberg coefficients for $\mathbf{d} = (d_1, d_2)$, $\mathbf{b} = (d_1 - 1, b_2)$, normalized so that the coefficients sum to 1. This provides evidence for Conjecture 7.6 and illustrates the asymptotic behavior of the Boij-Söderberg coefficients in 1-parameter families of fixed degree sequence length.



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