engineering - a context for learning mathematics: the case of guitar fret spacing

By Joe Garofalo, Kimberly Corum, and James Rutter

Unlike their typical classroom tasks, the students found the fret spacing challenge to be an authentic problem worth solving.

The expression "STEM Education" is widely present in today's educational literature, often just as a listing of subjects, without a definition. The International Technology and Engineering Education Association (ITEEA) defines Integrative STEM Education as "the application of technological/engineering design based pedagogical approaches to *intentionally* teach content and practices of science and mathematics education through the content and practices of technology/engineering education" (Wells & Ernst, 2012/2015). This definition, which positions engineering as a context for learning mathematics and science, is supported by the National Academy of Engineering's (NAE) conclusion that "…limited but intriguing evidence suggests that engineering education can stimulate interest and improve learning in mathematics and science as well as improve understanding of engineering and technology" (2010, p. 10). While these groups see engineering as providing opportunities for students to learn mathematics, Carr, Bennet, and Strobel (2012) reported that of 41 states including "engineering content in their educational standards," only one referred to engineering in their mathematics standards.

In conjunction with local middle schools, the Make to Learn Lab in the School of Education and Human Development at the University of Virginia offered a two-week Summer Engineering Academy for students who completed an introductory engineering course during the previous school year. In that course, students worked with circuits and advanced manufacturing equipment to build artifacts, such as solenoids, motors, and speakers. One example of how engineering projects from the Academy were used as contexts for students' exploration of previous and new mathematics topics is shared here. In this example, students algebraically determined where to place frets on the fretboard of a stringed instrument they were designing and along the way made use of a fractional exponent that was beyond what they experienced in prior courses.

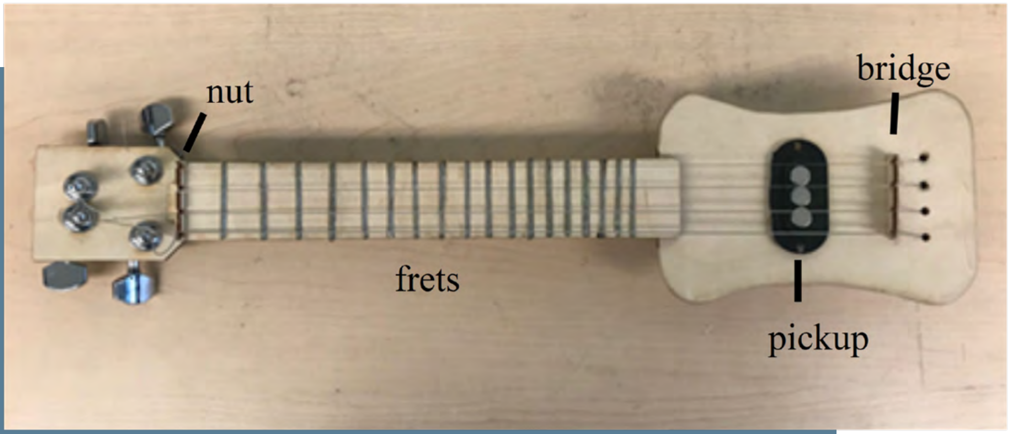

Figure 1. Instrument designed and constructed by students

Fret Spacing Challenge Description

Academy participants were given several options for a design project, one of which was to design and construct an electric stringed instrument and an electric pickup to amplify the signal from the instrument. Two rising eighth-grade students, who had just completed an algebra course, chose to build a four-string electric instrument (Figure 1).

Students had access to a laser cutter, 3D printer, various tools and materials (e.g., magnets for the pickups, wire, tuning machines, strings), and an input jack. One challenge was to determine where to put the frets on the fretboard so that the change in pitch (frequency) between notes played from fret to fret was uniform along the fretboard. Although these students had some background with music, one of the authors played successive notes on one guitar string all the way down the fretboard to remind students that there are twelve semitones in an octave and that the note played by fretting a string at its center (i.e., pressing the string at the 12th fret) is an octave higher than the note played by plucking the string open (without fretting it). The students noticed during this demonstration that the pitch of a fretted note is related to the length of string from the fret to the bridge. After the challenge was posed, the students suspected that there must be online fret spacing calculators and asked if they could use one. They were told to figure out the spacing for themselves and later compare their results with those of an online fret calculator. Since proper fret spacing has been daunting for some instrument makers, the challenge was broken into several tasks, first asking students to determine fret spacing for a 2-fret instrument, then for a 3-fret instrument, and finally create an algorithm for fret spacing for a regular guitar.

Task 1: 2-Fret Instrument. The 2-fret instrument was to include one fret midway between the nut and bridge and another fret placed between the nut and the midpoint fret. Each string would then be able to produce three notes—one when the string was played open, a note an octave higher when the string was fretted at the center, and a third note when the instrument was fretted at the second fret to be placed. The students understood that the change in pitch from the open note to the non-octave note was to be the same as the change in pitch from the new note to the octave note. They began by drawing a string with a nut (left), bridge (right), and string center (the fret position for the octave note) marked, along with a mark to represent where the new fret should be placed (separating the X and Y on the diagram in Figure 2).

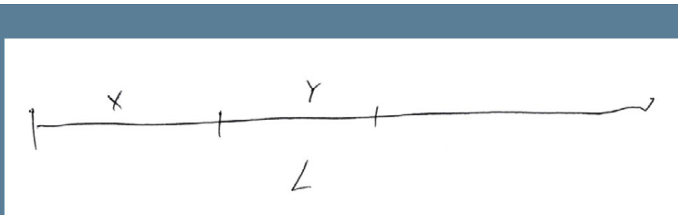

Figure 2. Students' initial diagram.

They initially used X and Y to denote segment lengths, but soon changed their meanings to those described below. They surmised from the demonstration that the ratios of each fretted string length to the next fretted string length should be equal for the instrument to properly intonate. *Without redrawing it*, they reinterpreted their initial diagram as shown in Figure 3, where L represents the full length of the string (nut to bridge), X represents the length of string from the fret to be placed (point A) to the bridge, and L/2 represents the distance from the *midpoint fret* of the string (point B) to the bridge.

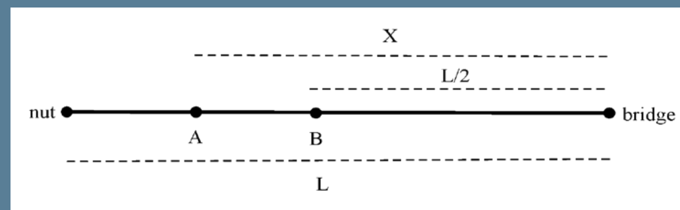

Figure 3. How the students reinterpreted their diagram.

After reinterpreting their initial diagram, the students set up the proportion L : X = X : L/2 (shown in the center of Figure 4). They rewrote the proportion as L/X = X/L/2 and then solved for X.

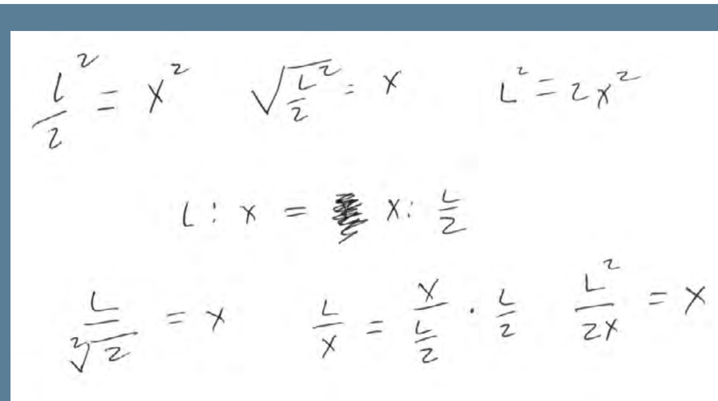

Figure 4. The 2-fret solution.

Their work was not written in an orderly manner and their writing did not always represent all of their steps clearly. For example, they multiplied both sides of the equation L/X = X/L/2 by L/2, but you can see they only wrote L/2 on the right side and they forgot to square an X term after cross multiplying. Ultimately, they calculated the expression for X, the distance from the bridge to the fret:

$$\frac{L}{\sqrt[2]{2}} = x$$

The students determined that the position of the (non-midpoint) fret can be found by dividing the string length L by $\sqrt[2]{2}$ The authors then discussed the musical limitations of an instrument with 3 notes and extended the task by asking them to add another note.

Task 2: The 3-Fret Instrument. The students continued their reasoning from the 2-fret task to set up proportions involving three

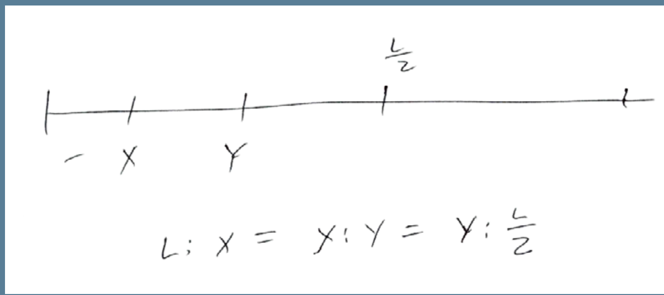

Figure 5. Diagram and proportions for the 3-fret instrument.

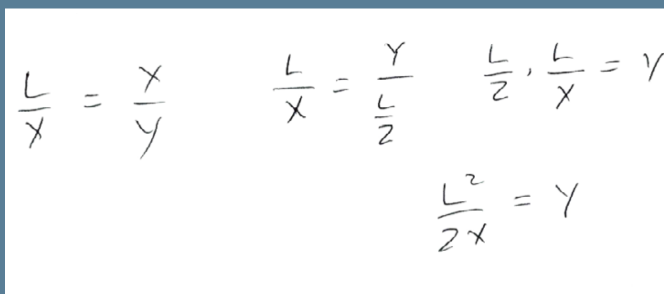

Figure 6. First attempt at solving the proportions.

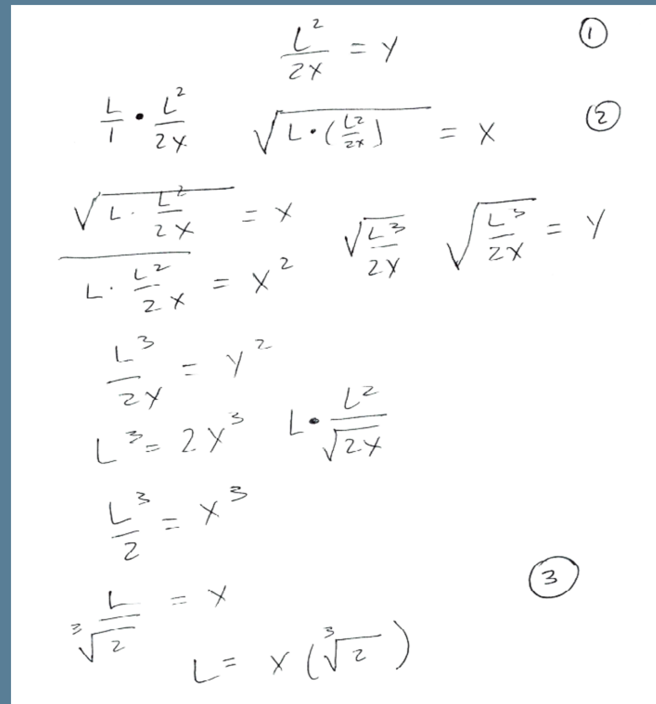

Figure 7. Solving the proportions.

frets: open string to fret position X, fret position X to fret position Y, and fret position Y to the midpoint fret (L/2) of the string (Figure 5). Similar to their work on the 2-fret solution, X and Y in their diagram and proportions represented distances from the marked fret positions to the bridge (far right).

Unlike the previous task, which involved two proportions, this task involved three proportions. However, the students recognized the similarities between the two tasks and revisited their problem-solving approach from the first task. They translated their proportion statement into equations (Figure 6) and then attempted to solve for X.

This first attempt was cut short by the end of the day, but the next morning they continued where they left off, as shown in Figure 7.

The students started with the last derived equation shown in Figure 6, solving for Y in terms of L and X (see equation marked 1 in Figure 7), then they substituted that expression for Y into another equation derived from the proportions to solve for X (equation marked 2 in Figure 7). Near the end of their work they explained:

> Archie: Yeah, so you would have L times L squared over 2X equals X squared, so divide by 2…then you take the cube, so L over the cube of 2 equals X.

They concluded that the location of fret position X is found by dividing the whole string length L by $\sqrt[3]{2}$ (equation marked 3 in Figure 7).

They were then asked to determine the placement of the next fret. The students returned to the first equation shown in Figure 6 and substituted the value for X they just determined (shown in the next to last line in Figure 7); this yielded the top left equation in Figure 8. They struggled with simplifying ratios over ratios using their school-taught strategy of "Keep Change Flip," but persisted and de-

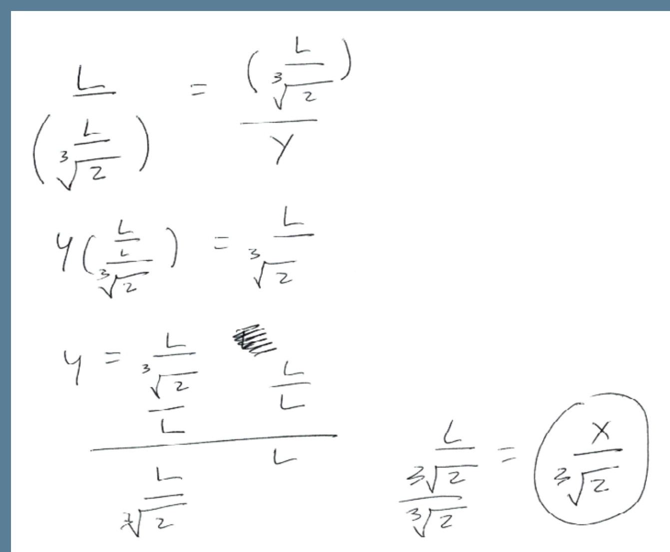

Figure 8. Determining the scale factor for the 3-fret solution.

termined that after one fret is placed at position X (i.e., $L/\sqrt[3]{2}$), the next fret (at position Y) should be located by dividing that resultant string length (i.e., *x*) again by $\sqrt[3]{2}$. See the work shown in the lower right of Figure 8.

Task 3: Developing an Algorithm.

After these students solved the 3-fret task, they were asked to write an algorithm for placing frets on a regular guitar. Revisiting their solutions to the first two tasks, the students noticed that for a 2-fret instrument, the string length was divided by the square root of 2 and for a three-fret instrument, the string length was divided by the cube root of 2. They noticed a pattern and generalized their solution to a stringed instrument with twelve semitones. They then

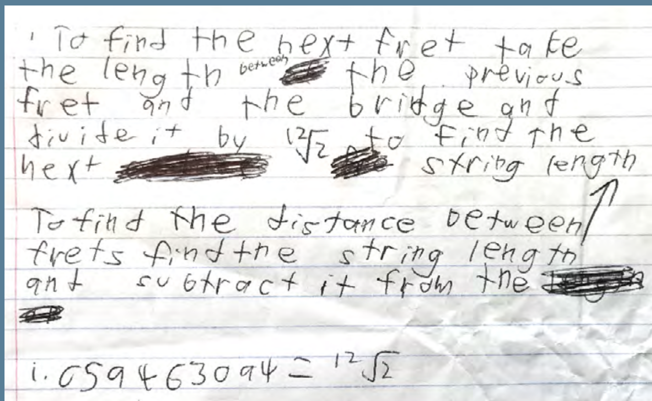

Figure 9. Mike and Archie's fret-spacing algorithm.

verbalized their algorithm for placing each fret (see Figure 9).

> Archie: So, what we're going to need to do is just every time measure the length and divide it by the twelfth root of 2. So we just have to find the twelfth root of 2
>
> Mike: My phone might do it…my phone can't do it. What we need is like a scientific calculator.
>
> Archie: Go steal one from one of the math rooms.

After using a graphing calulator to find the value they wrote out their algorithm, shown in Figure 9.

The students later reported that some online fret spacing calculators "round their numbers to three places."

Implications

The students' work on this fret-spacing challenge and their comments, both along the way and after, revealed much about their mathematical behavior, thoughts about school mathematics, and the importance of motivation when working through non-trivial tasks.

Students' Mathematical Behavior. Archie and Mike used, and then extended, the mathematics they were taught in school to solve an authentic problem arising outside of their mathematics class by creating an effective algorithm. They then summarized their results. These students used knowledge of an octave and the need for a constant ratio between successive string lengths to set up diagrams and several proportions. In determining the fret spacing these students worked through some complicated algebraic steps and monitored and checked their work. Mike commented that this was the only time he solved a proportion problem that was truly authentic for him. And, although they had no prior exposure to fractional exponents beyond square and cube roots, these students deduced that the solution to this challenge involved $\sqrt[12]{2}$ by detecting and generalizing a pattern based on 12 semitones. They also surmised that the way to enter this number into a graphing calculator was to input 2 ^ (1/12), generalizing from their experience with cube root calculations.

Importance of Motivation. These students worked tirelessly on this series of tasks for over half an hour one afternoon and then for over an hour the next morning. When asked about the fret-spacing challenge, they shared the following thoughts.

> Mike: It was pretty hard, but solvable. I don't like them easy.
>
> I: Why not?
>
> Mike: Because that's school.
>
> Archie: It gets boring.
>
> Mike: Doing the same stuff you know how to do over and over.

Unlike their typical classroom tasks, the students found the fret-spacing challenge to be an authentic problem worth solving. The nature of the challenge necessitated a solution, encouraging the students to persevere. As Mike shared, "Like we had to get it. 'Cause building a guitar is cool…It actually means something. It had a purpose." The students' stick-to-it-ness was critical; they were motivated to figure out where to appropriately place the frets on the instrument they were building.

Implementation in Other Settings.

The Summer Engineering Academy provided a unique setting with sufficient time and resources available for middle school students to work through this challenge. And, more importantly, the students who participated in the challenge were motivated by the instrument-building project that led to the challenge. Admittedly, implementing this challenge in a typical school setting would not be as easy. There are several ways fret spacing tasks can be used in other settings.

Attending to student background and classroom constraints. Teachers can break up the challenge into a series of subtasks, similarly to how the challenge was implemented during the Academy, but spread the work over several periods, posing them one at a time and providing information and suggestions as needed. Alternatively, they can pose only those tasks that are appropriate for their specific context and constraints. For example, a teacher might present the overall challenge but only ask students to solve the 2-fret task. This task could be followed by giving a partial solution (or maybe even the result) for the 3-fret task, and then asking students to come up with a general solution.

Posing simpler technical versions of the challenge. A technically simpler variant of the challenge can be posed using a monochord (or any stretched string) and having students continuously pluck the string while depressing it with a hard object that is being slowly moved down the string to approximate places where the played notes seem to change in a consistent way. In a different workshop, a student observed that pressing the string on a monochord she built at both 1/4 of the way and 1/3 of the way from the nut to the bridge yielded notes that sounded reasonably good with each other and with the open string note. Although this adaptation does not satisfy the equal temperament condition originally specified (i.e., constant change in pitch between frets), it can be useful as an introductory task, especially as a springboard for a discussion of the Pythagoreans. See Figure 10 for a monochord similar to the one the student constructed.

An operationally simpler version is for a teacher to bring in a guitar or other fretted instrument when presenting the challenge. Some

Figure 10.

students will need to be told or reminded that a chromatic scale consists of twelve semitones and that when a string is halved and plucked the resultant tone will be an octave higher than that of the full string played open. This can be done by playing a chromatic scale on a string so students can hear the relative change in pitch between fretted notes and observe that playing down to the twelfth fret on a string essentially cuts the string in half and gives a note an octave higher than plucking the string open. Prior implementations of the fret-spacing challenge without breaking the challenge into subtasks have shown that, even after given these bits of information, high school students, college students, and even some teachers, are "befuddled" at first, but subsequently determine the solution by deriving and solving the algebraic equation: $L/r^{12}=2$ for r.

Team teaching. It could also be worthwhile for a team of teachers to implement this challenge in a variety of ways and explore the interesting science, technology, engineering, mathematics, and music history connected with electric stringed instruments.

Conclusion

The activities used in the Summer Engineering Academy, such as this fretboard challenge, are a proof of concept, confirming the positions of ITEEA and NAE that engineering projects can provide authentic contexts for mathematics learning. Indeed, other Academy project artifacts have been used to present mathematical challenges to participants. For one of these challenges, students measured the strength of the magnetic fields generated in a series of solenoids to successfully develop a mathematical model of Ampere's Law: $B=k\,(N{\cdot}I/L)$ (Corum and Garofalo, 2018). For another challenge, students measured the sound pressure level coming from a speaker they constructed and simultaneously measured the voltage coming from a sound source as they increased the volume to develop graphical and verbal representations of a logarithmic relationship between these two variables (Rutter and Garofalo, 2021).

These challenges support *Standards for Technological and Engineering Literacy*, specifically Standard 3: Integration of Knowledge, Technologies, and Practices (ITEEA, 2020) because they show that "technology and engineering are interdisciplinary, relating to more than one content area" (p. 36). The posing of each of these challenges also promotes ambitious mathematics teaching, as characterized by the National Council of Teachers of Mathematics (NCTM, 2014). Students engaged in reasoning and problem solving, had extended meaningful discourse with each other, stayed motivated through productive struggle, and actively built new mathematical understandings from new experiences and prior mathematical knowledge.

Furthermore, these projects were engaging to a range of students, not just those enrolled in honors level mathematics courses (Corum and Garofalo, 2018; Rutter and Garofalo, 2021).

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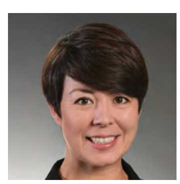
Joe Garofalo, Ph.D., is Co-Director of the Make to Learn Lab at the School of Education and Human Development at the University of Virginia. He can be reached at jg2e@virginia.edu.



Kimberly Corum, Ph.D., is an assistant professor of mathematics education at the Fisher College of Science and Mathematics at Towson University. She can be reached at kcorum@towson.edu.

James Rutter is Technology Director at the Haystack Mountain School of Crafts. He can be reached at fablab@haystack-mtn.org.

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