Analytical Approximate Solution to Mathieu's Equation Enables Real-Time Motion Planning for Legged Robot Walking on a Vertically Moving Surface

Amir Iqbal, Sushant Veer**, and Yan Gu**

BACKGROUND

Linear inverted pendulum (LIP) models [1] capture the essential locomotion dynamics of legged robots with heavy trunks and lightweight legs. Thanks to its linearity and low dimensionality, the LIP models could be used to reduce computational cost of motion planning [1]. Yet, the existing LIP models assume the surface is static and thus may not be valid for locomotion on nonstationary surfaces (e.g., vessels, aircraft, and trains). Our previous study shows that dynamics of LIP walking on a periodically moving surface are described by Hill's equation [3], [4], or Mathieu's equation if the surface motion is sinusoidal. Still, the analytical solution to the LIP model and motion planning based on the model remain to be studied. This study aims to: (a) derive the analytical solution to a LIP model representing legged walking on a nonstationary surface, (b) exploit the solution to enable online motion planning, and (c) validate the planner on the Laikago quadrupedal robot.

METHOD

LIP Model. This study assumes the dynamic rigid surface (DRS) has a periodic vertical acceleration and a constant horizontal velocity. Under these assumptions, the horizontal components of the LIP model (which we call DRS-LIP) are Hill's equations (see Fig. 1). For a DRS with a sinusoidal vertical movement, the DRS-LIP becomes the standard Mathieu's equation. We choose an existing analytical approximate solution of Mathieu's equation [2] to solve the DRS-LIP because of its convergence and computational efficiency.

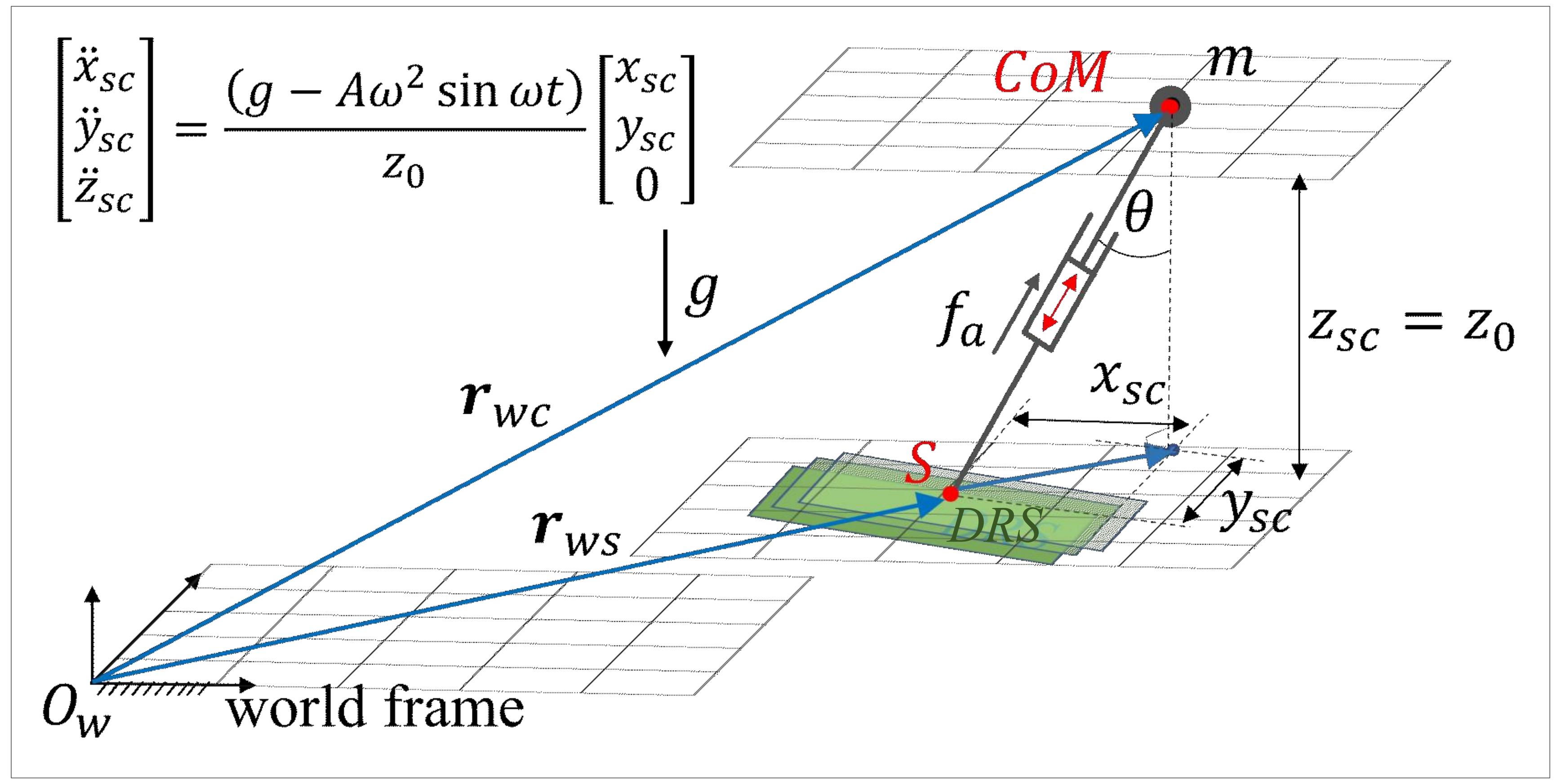


Fig. 1: Illustration of the proposed DRS-LIP and its dynamics under a vertical, sinusoidal DRS motion. $(*)_{sc}$ denotes the coordinate of CoM with respect to the support point S in the world frame.

Analytical Approximate Solution. The horizontal (x and y) components of the DRS-LIP dynamics share the same form. The solution to x-direction dynamics is given as [4], [5]:

$$x_{sc}(t) \coloneqq \alpha_1 e^{\mu t} \sum_{n=-N}^{N} C_{2n} e^{i2nt} + \alpha_2 e^{-\mu t} \sum_{n=-N}^{N} C_{2n} e^{-i2nt},$$

where $\mu \in \mathbb{R}$ is the characteristic exponent of Mathieu's equation, $C_{2n} \in \mathbb{C}$ denotes the solution coefficients, $\alpha_1, \alpha_2 \in \mathbb{R}$ are constants dependent on initial conditions, and $N \in \mathbb{N}$ determines the number of terms to keep. The constants μ and C_{2n} can be approximately solved based on closed-form equations. The value of N is determined by convergence analysis for a user-specified tolerance. The approximate solution obtained using this method is 99.98% accurate and approximately **sixteen times faster** than the numerical solution [4]. With the analytical solution, the solution time is 0.16 ms for 500 equally distributed points in [0, 0.5 s], whereas the time cost is 2.61 ms using the numerical solution in MATLAB [4].

Motion Planning based on DRS-LIP. The approximate analytical solution is utilized in a hierarchical planner [4], as shown in Fig. 2. With nonlinear programming in IPOPT, the proposed planner takes 51 ms to generate the desired full-body robot motion of 2 s, which is **seven to ten times faster** than the numerical solution based planner (388 ms) [4]. The speed of the proposed planner is sufficient for real-time planning for a real-world DRS, such as vessels in regular sea waves, with moderate and varying motion.

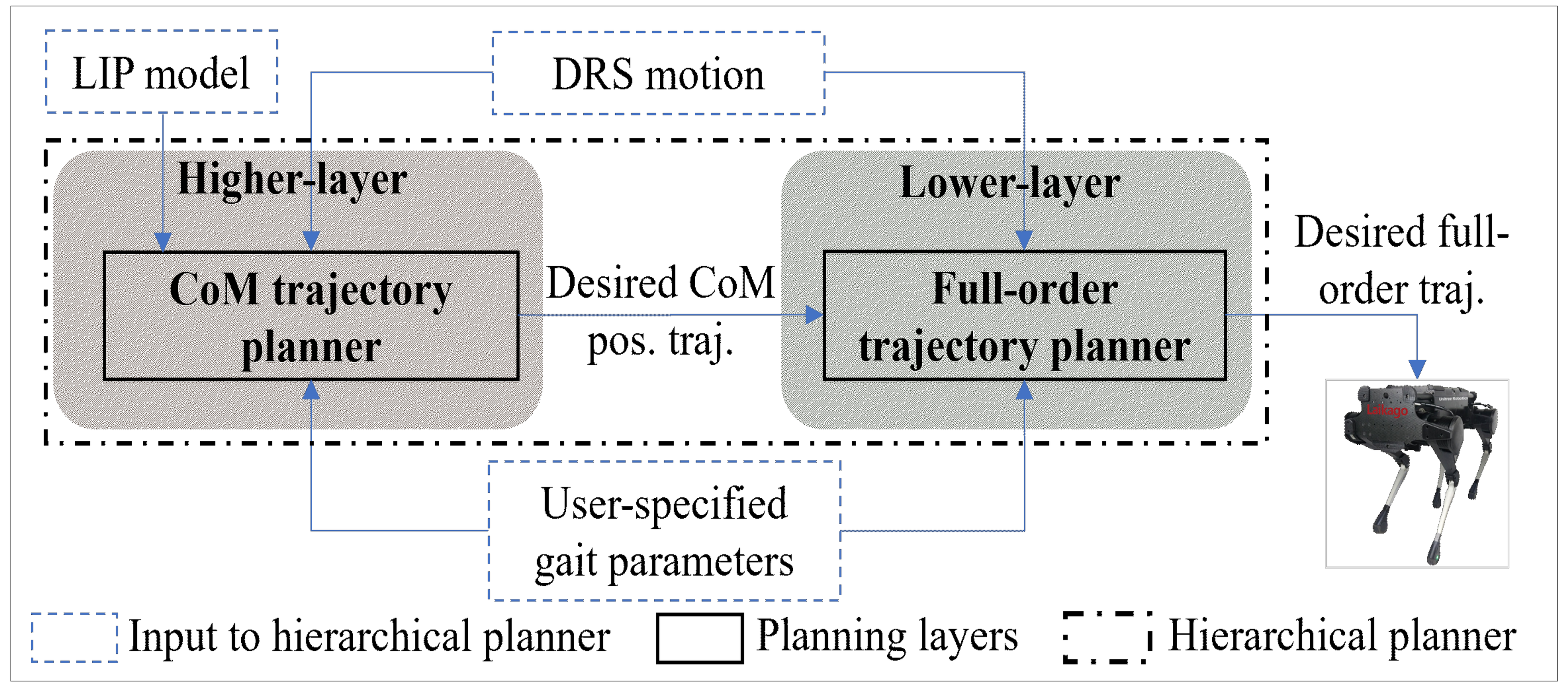
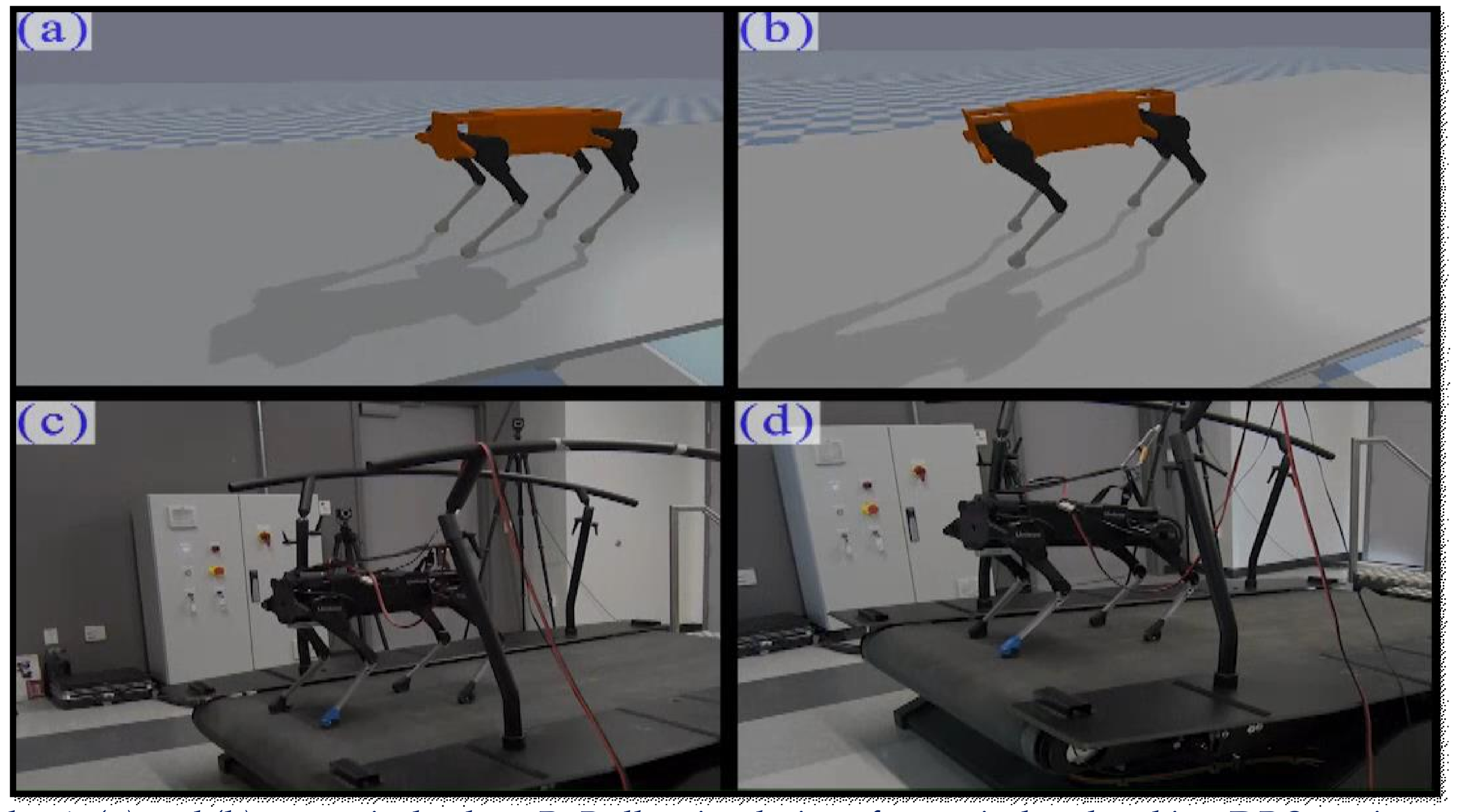


Fig. 2: Schematics of the proposed hierarchical planner.

SIMULATION AND EXPERIMENTAL RESULTS



Video 1: (a) and (b) respectively show PyBullet simulations for vertical and rocking DRS motions. (c) and (d) are the movies of experiments on a rocking treadmill.

CONCLUSION

This research derived the approximate analytical solution of a LIP model for legged locomotion on a nonstationary surface. We designed a hierarchical planner to highlight the advantage of the analytical solution in simultaneously ensuring the feasibility and efficiency of real-time planning. Both PyBullet simulations and hardware experiments validated the stability of the full-order desired walking trajectories produced by the hierarchical planner for various DRS motions. Our future work aims to address locomotion on unknown, general (periodic and aperiodic) DRS motions.

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*Department of Mechanical Engineering, University of Massachusetts Lowell (UML), Lowell, MA, 01854. Emails: amir_iqbal@student.uml.edu and yan_gu@uml.edu.
**NVIDIA Research, Santa Clara, CA 95051, U.S.A. This work was conducted while S. Veer was with Princeton University. Email: sveer@nvidia.com.

Terrain Robotics Advanced Control and Experimentation Lab