Relationships Between Dimensions of Authenticity During an Inquiry- Oriented Abstract Algebra Activity

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RELATIONSHIPS BETWEEN DIMENSIONS OF AUTHENTICITY DURING AN INQUIRY-ORIENTED ABSTRACT ALGEBRA ACTIVITY

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One of the goals of advanced undergraduate mathematics courses is to engage students in activity that is authentic to the mathematics discipline. However, engaging students in such activity often involves managing tensions between authenticity-to-students and authenticity-to-the-discipline. In this paper, we use the Authentic Mathematical Proof Activity (AMPA) Framework to further explore potential relationships and tensions between different dimensions of authenticity. We analyzed classroom data from an inquiry-oriented abstract algebra course where instruction focused on unpacking the fundamental homomorphism theorem. Our results focus on the complexity dimension of authenticity and how this dimension relates to other dimensions of authenticity within instruction. We identify ways that instructor decisions shape authenticity even within the context of a carefully developed task.

Keywords: Undergraduate Education, Reasoning and Proof, Instructional Activities and Practices

Many mathematics educators value engaging students in "authentic" mathematical activity. The use of the term authentic often reflects ties to the work of the discipline where the goal is to engage students in activity reflective of research mathematicians (e.g., Watson, 2008). However, what constitutes authentic activity and the degree to which such activity should be the goal of school mathematics is an unsettled topic (Weber & Dawkins, 2020). Further, engendering students in authentic activity is often subject to intrinsic tensions between authenticity to students and authenticity to the discipline (e.g., Melhuish et al., 2021; Ball, 1993; Lampert, 1992; Herbst, 2002). A well-documented tension is that between authenticity-to-the-discipline (as in Weiss et al. 2009) in terms of accuracy of content and alignment with student contributions that often diverge from the norms of the discipline (e.g., Chazan & Ball, 1999). Herbst (2002) has further illustrated dilemmas in proof courses where there is a double bind on the teacher to progress a class in normative ways related to proof argumentation while also staying authentic to the contributions and activity of students in class. As noted in Dawkins et al.'s (2019) analysis of an inquiry-based instructor, this tension remains salient at the advanced undergraduate level.

In Melhuish and colleague's (2021) recent work, they have suggested a need to better operationalize authenticity for the context of proof-based courses. Drawing on diverse design-based research projects, they built on Weiss et al.'s (2009) initial decomposition of authenticity to suggest both student/discipline dimensions and practice/content dimensions of authenticity that are at play in the advanced proof-based settings. In this paper, we build directly on this work by adapting this framework (developed in the settings of task-based interviews) to the classroom setting to better explore how different dimensions of authenticity may align or diverge in order to better understand instruction in such classes. For the scope of this paper, we forefront elements of practice, that is the nature of activities, as they have been less explored than their parallels in relation to content.

Our study is situated in the context of abstract algebra, a course taken by mathematics majors and pre-service secondary teachers. The advanced undergraduate setting is often a place where students are apprenticed into the work of research mathematicians with a focus on formal proving. It is well-documented that students often struggle to grasp the abstract concepts that are central to the course (e.g., Dubinsky et al., 1994; Melhuish et. al, 2019). As a result, there has been a growing body of research on how to improve the teaching of abstract algebra and a continued development of inquiry-oriented abstract algebra curriculum (e.g., Larsen et al., 2013) with a focus on engaging students in authentic activity. We are building on such work by elaborating on the tensions and relationships amongst competing authenticity goals observable in instruction.

Theoretical Framework

Drawing from activity theory (Engeström, 2000), we frame both students and instructors as operating within activity systems. These activity systems relate goal-driven actions to how members of a community work together toward shared goals. The assumption underlying our work is that advanced mathematics courses provide students with an opportunity to engage in activities that align with the activities of research mathematicians. Advanced mathematics courses may provide students an opportunity to use tools (e.g., examples, warranting, deformalizing) to meet objectives that can be deconstructed into motives (e.g., explore, test, construct) in reference to an object (proof, statement, concept). Evidence of participatory learning can be seen through expansions in activity within an activity system. For example, students may introduce tools, thus adding more variety and increasing their role in the division of labor.

We draw from the Authentic Mathematical Proof Activity (AMPA) framework (Melhuish et al., 2021) which was developed to capture components of student activity that reflect the work of research mathematicians. The framework includes ten tools: analyzing/refining, formalizing, deformalizing, warranting, analogizing/transferring, examples, diagrams, logic, structure/frameworks, and existent objects. These tools are used alongside three motives: constructing, exploring, and testing, in reference to three objects: proofs, statements, and concepts. Authenticity is operationalized across six dimensions to account for different, sometimes competing, notions of authenticity. Notably, the three dimensions: variety, complexity, and accuracy reflect the discipline and stem from analysis of mathematician activity while agency, authority, and alignment reflect features of authenticity to student activity and contributions. See Table 1 below for the dimensions of the Authentic Mathematical Proof Activity (AMPA) framework along with our elaboration of four levels within each dimension.

Table 1: Dimensions of Authenticity

Dimension of Authenticity	Levels
Variety: Degree of variation within tools used: formal, informal, generative, translating	Low: Only one type of tool at play Low-Mid: Two of the four types of tools at play Mid-High: Three of the four types of tools at play High: Informal, formal, translating, and generating tools at play
Complexity: Degree in which tools are used in isolation or in conjunction	Low: Single tool Low-Mid: A variety of tools are used in isolation Mid-High: Many tools used in conjunction

(and shifts from outcomes to tools)	High: Objects shift to tools
Accuracy: Degree in which the tools used would be accurate within the mathematical community	Low: Inaccurate Low-Mid: Mixed - correct tools but incorrect outcome Mid-High: Similar but imprecise High: Tools and outcomes are accurate
Alignment: Degree in which tools aligned with student contributions (whose tools are endorsed)	Low: Teacher Contributions Low-Mid: Mostly Teacher/ Some Student Contributions Mid-High: Mostly Student Contributions/ Some Teacher or Refined Student Contribution High: Tools and Outcomes are Student Contributions
Agency: Degree in which students generate tools	Low: Teacher generates/uses Tools Low-Mid: Teachers generate tools and students Use Tools Mid-High: Students Generate Tools (prompted) High: Students Generate Tools (unprompted)
Authority: Degree in which students determine how tools/ outcomes connect (validity of tools/ outcomes at play)	Low: Instructor links & explains Low-Mid: Instructor links & students explain Mid-High: Students link & explain (sometimes) High: Students link & explain (mostly)

Background: Fundamental Homomorphism Theorem and Quotient Groups

The focal task in our study involves students exploring the proof of the Fundamental Homomorphism Theorem (FHT). Both the FHT and quotient groups are key topics in an abstract algebra curriculum, but they are also two of the most difficult topics for students to understand (Melhuish et al., 2021). The FHT (see figure 1 below) involves both a homomorphism and an isomorphism to show that the quotient group is isomorphic to the image of the homomorphism.

Literature suggests that students may struggle to coordinate the homomorphism and isomorphism in the FHT (Nardi, 2000). Nardi (2000) elaborated that mathematical abstraction is particularly challenging in such proof settings. For students to productively engage with this theorem and proof, they need to coordinate a number of abstract mathematical objects including functions and quotient groups. Yet, Hazzan (1999) suggests that students often try to create less abstract environments for themselves by relying on things like the coset algorithm which in turn hides the structure of quotient groups. With the inherent challenges of abstract functions (e.g., Melhuish et al., 2021, year) and quotient groups (e.g., Dubinsky et al., 1994), we anticipate substantial opportunities to study authenticity dimensions where student contributions may often be in tension with disciplinary norms.

Theorem 1 (The First Isomorphism Theorem or the Fundamental Homomorphism Theorem).
If $\phi : G \to H$ is a group homomorphism, then

$$\frac{G}{\ker\phi} \cong \phi(G).$$

Proof. First, we note that the kernel, $K = \ker\phi$ is normal in G.

Define $\beta : \frac{G}{K} \to \phi(G)$ by $\beta(gK) = \phi(g)$. We first show that β is a well-defined map. If $g_1K = g_2K$, then for some $k \in K, g_1k = g_2$; consequently,

$$\beta(g_1K) = \phi(g_1) = \phi(g_1)\phi(k) = \phi(g_1k) = \phi(g_2) = \beta(g_2K).$$

Thus, β does not depend on the choice of coset representatives, and the map $\beta : G/K \to \phi(G)$ is well-defined.

We must also show that β is a homomorphism:

$$\begin{aligned}\beta(g_1Kg_2K) &= \beta(g_1g_2K)\\ &= \phi(g_1g_2)\\ &= \phi(g_1)\phi(g_2)\\ &= \beta(g_1K)\beta(g_2K).\end{aligned}$$

Clearly, β is onto $\phi(G)$. To show that β is one-to-one, suppose that $\beta(g_1K) = \beta(g_2K)$. Then $\phi(g_1) = \phi(g_2)$. This implies that $\phi(g_1^{-1}g_2) = e$, or $g_1^{-1}g_2$ is in the kernel of ϕ; hence, $g_1^{-1}g_2K = K$; that is, $g_1K = g_2K$. □

Figure 1: Statement and Proof of the FHT

Methods

The data for this report was collected at a large public university in the United States as a part of a design-based research project focused on orchestrating discussion around proof. More specifically, the data comes from an activity in which students were tasked with engaging with the Fundamental Homomorphism Theorem. This activity spanned one and a half class periods. On the first day, students worked in small groups on two tasks. For the first task, each group was given a different homomorphism between groups and asked to draw a function diagram where the homomorphism, kernel, cosets, and isomorphism were all labeled. This task concluded with a whole class discussion.

The second task involved unpacking the proof of the FHT. Students were first tasked with partitioning the proof into sections based on the property being proved (well-defined, homomorphism property, one-to-one, and onto). Since there were four groups, each group was assigned a section and given a set of questions about their section of the proof to facilitate a small group discussion. After each small group spent some time discussing their section of the proof, a representative from each group went to the front of their class to present their answers to each question and discuss their overall understanding of their section of the proof. The presentations occurred during the following class period.

This activity was video recorded, audio recorded, and transcribed. Using the video data from the two class sessions, we segmented the class into segments based on activity. A new segment was created when shifts in conversation such as a new topic of conversation or more major changes such as whole class to small group discussion occurred. The coding was done in two stages by two of the authors. The first stage was a trial coding of two segments, the second stage consisted of coding the rest of the segments. Using the AMPA framework, 21 segments with an average length of 6 minutes and 40 seconds were coded along six dimensions and received a code of low, low-mid, mid-high, or high. Any discrepancies between the two coders were resolved through discussion.

After we coded our data, we looked for patterns and trends throughout the coded data to find relationships and conflicts. We identified any segments that conflicted with the general trends of the data set. We discuss the findings of our analysis below.

Results

For this report, we forefront the dimension of complexity in order to better understand the relationships between complexity and other dimensions of authenticity that emerged from our analysis. Complexity captures the degree to which a set of tools are used interrelatedly to make progress towards an outcome. We use quotes and descriptions of the video data to provide context for each relationship. Note that all students have been assigned a pseudonym in these segments.

Before discussing the relationship between complexity and the other dimensions of authenticity, we present a figure (Figure 2) that summarizes the authority dimension levels across our segments. The grey boxes correspond to small group segments (focusing on the small group reported later in the results) while the white boxes are whole class segments. We note that we transition between lessons at the 80-minute mark. We can observe that authenticity dimensions are infrequently all at the high-level, with both discipline (variety, complexity, accuracy) and student (agency, authority, alignment) varying in different segments.

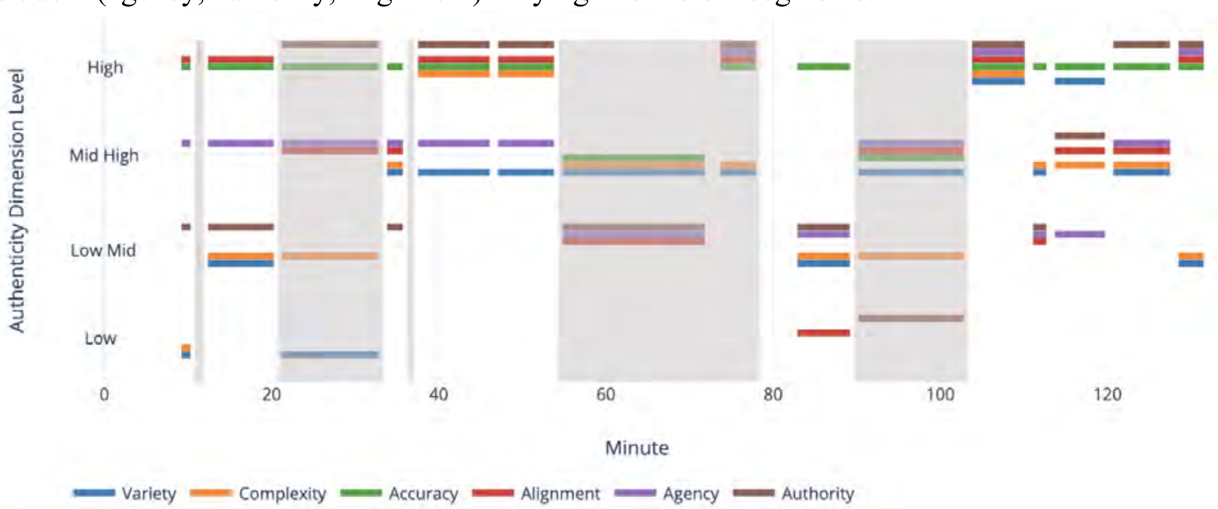

Figure 2: Coding Over Time By Whole Class And Small Group 1

Variety, a Necessary, but Not Sufficient Condition for Complexity

From a simplistic view, complexity and variety are interrelated. In order for multiple tools to be used in conjunction (high complexity), there must first be multiple tools available to use (high variety). For example, consider the following quote from the end of a segment where a student, Joe, is explaining a portion of the proof to his classmates:

> And then the last thing, which I hate. Oh, I hate this line so much is, what does it mean that beta does not depend on the choice of coset representatives, which is a really horrible way of saying that something's "well-defined". So, basically to translate what this means... I didn't know what it meant when I first saw it. A coset representative is like, if you think about our group over here, 1 and 4 would both be like coset representatives that are equal, right? Because they map to the same thing when paired with a kernel. And in our actual proof here, g_1 and g_2 are our coset representatives. And what we're trying to show in this proof is that it doesn't matter what you call them. Doesn't matter which one's being used, as long as they are equal, you will get the same result,

Joe generated several different types of tools in this segment. First, he is deformalizing when he translates the formal language in the proof to more informal language. Then, he is warranting by referring to an example, and formalizing by relating the example back to the formal proof. The variety of tools that were generated by Joe, as well as the tools being used in tandem, resulted in the segment having high levels of both variety and complexity. Across the lesson, students often generated tools with the intention of using them in conjunction with other tools. Similarly, with few tools at play (low variety), there are fewer opportunities for complex usage (low complexity.) Overall, levels of variety and complexity corresponded (both higher or both lower) in 20 of the 21 total segments.

A Deviation for the Trend of Variety and Complexity Co-Occurrence - and a Tension with Authority.

While most of the instances included variety and complexity playing out in tandem, one segment provided counter evidence to this trend. A small group of students was tasked with making sense of the onto portion of the FHT proof. Throughout this segment, students generated many different tools. For example, they referenced an example of a homomorphism between two groups, as well as deformalized the formal definition of onto by stating “everything in the codomain gets hit”. Despite generating a variety of tools, the students did not end up using any of the tools in tandem. This led to them having trouble making progress toward unpacking and understanding their portion of the proof. The instructor intervened and stated, “So, actually you have everything... So, let’s make sense of this”, and attempted to prompt students to begin using some of the tools they generated in tandem. This deviation from the trend we noticed provides evidence that students being able to successfully generate tools does not guarantee that they will use them together. The intervention by the instructor increased the level of complexity for the segment (tools began being used in tandem) but decreased authority (the instructor was primarily the one linking tools to outcomes – that is, evaluating that they had the right tools). Without the instructor’s intervention, the students appeared at an impasse in their activity. By decreasing authority (a student-related dimension), the instructor promoted increased complexity (a discipline-related dimension). Thus, this case suggests a tension between these dimensions, since linking between tools and outcomes often requires using a variety of tools in tandem.

Decreasing Agency to Increase Variety and Complexity

The final relationship we observed was that levels of agency related to levels of both variety and complexity. There were several instances where the students were not generating very many tools. If the instructor was not present, then this would necessarily lead to both low variety and complexity, because there would not be very many tools at play. However, it was often the case that if the instructor noticed that students were not generating many tools, they would step in to introduce a new tool, and sometimes even use the tool in tandem with another tool. This move by the instructor resulted in increasing both variety and complexity for the segment but decreasing agency –as the tools were no longer being generated strictly by the students. Consider the following exchange where one of the members of the group from the previous section (Nick) is presenting their explanation of the onto portion of the proof. The student started by writing out what was known: φ is onto. Then the student proceeded to attempt to explain why this necessarily meant that β was onto.

Nick: So, now it's time for us to address β. And we were given the definition for β, what it is. And I think if I remember it correctly, it has something to do with the kernel. Do you guys remember what the definition for β is? How is β defined?
Joe: Beta of G times the kernel equals □(□)?
Nick: Yeah. So, it's elements in G operated with the kernel, right?
Joe: Which are quotient group elements.
Nick: Yeah. So, would it be too much of a jump to say that □(□)is equal to □(□) operated with K?
Joe: You are allowed to do that
Nick: And why am I allowed to do that? I'm actually asking you.
Joe: That’s the definition of beta
Nick: Okay
Kevin: Because why is it □(□). Right so, it is an element in the big □(□), which we just said was equal to beta of little gK or in this case, little x big K. So, you've just replaced Y with that.

Nick then notes that his classmates’ contributions helped him “understand a little better” but still does not think he has the “best grasp on it.” After this comment, the instructor states they have a “general insight on what is going on with the onto,” but introduces a diagram (a new tool), to think through the argument. The introduction of a diagram of a homomorphism from $\mathbb{Z}_9$ to $\mathbb{Z}_6$ increased variety, since a new type of tool (informal) was being discussed. Additionally, the instructor used the tool in reference to existent objects (homomorphism, image) to warrant why the section of the proof was true. Thus, not only was variety increased, but complexity was increased as well. However, since the new tool being used was generated by the instructor, agency was decreased. This example shows that when a limited number of tools are at play and are being used in isolation, the instructor will often step in and introduce a new tool, as well as use that new tool in tandem with other tools. This results in increasing variety and complexity but decreasing agency.

Discussion

The AMPA framework was designed to document how students engage in authentic mathematical activity in a proof-based setting. Through our work, we have expanded the use of this framework to the classroom setting. We note that the course that we examined was an inquiry-oriented class which, in many ways, was naturally designed to foster an environment where students are engaging in many of these activities. Social norms within inquiry classes often include the expectation that the students are engaging actively in a disciplinary activity. Such a setting provides a robust opportunity to explore tensions in authenticity dimensions;

however, we caution that we cannot generalize these dimension relationships to a more traditional classroom.

Due to the inquiry nature of the classroom, the presence of the instructor varies from segment-to-segment. This influenced the levels of the dimensions of authenticity because when the instructor was not present in a small group, the dimensions of alignment, agency, and authority (presuming activity was occurring) all defaulted to high. The reason for this was because the instructor did not have an opportunity to prompt the students to introduce additional tools or give input on the validity of outcomes. In this paper, we shared three segments where the instructor was present throughout to better explore relationships between dimensions of authenticity.

If we turn back to the literature on authenticity, we can see evidence of the underlying tension between authenticity-to-students and authenticity-to-the-discipline found in the K-12 literature (e.g., Chazan & Ball, 1999; Lampert, 1992). The overarching goal for this abstract algebra lesson involved students engaging in authentic activities related to comprehending a theorem and a proof. In order to do this, the students needed to use a variety of tools in complex ways to make sense of a rather abstract theorem and proof. Variety was a necessity for complexity, and when variety was low, the instructor sometimes limited agency in order to introduce new tools. Further, variety did not assure complexity, and we observed the instructor lowering authority to assist students in connecting their tools to outcomes.

We also anticipate that the instructor's values and beliefs affected the authenticity profile of this class. For example, in the last exchange, the instructor introduced a diagram (which increased variety but decreased agency). We can reasonably conjecture this was for the purpose of supporting students' understanding of the onto portion of the proof. In other words, at this moment, the instructor appeared to value the students' engaging with a diagram more than ensuring that all of the tools at play were generated by the students. Further, this explanation means the increase in a disciplinary dimension was not necessarily motivated by authenticity intentions, but a consequence of other pedagogical intentions – attending to students' understanding.

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