Comparing Authenticity in Proof Activity in an In-Person and Online Setting

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The extent to which online course delivery allows students to engage in authentic mathematical activity has yet to be explored. In this preliminary report, we use the Authentic Mathematical Proof Activity (AMPA) framework to analyze data collected in a larger design-based research project. This data consisted of videos of the same lesson implemented online and in-person. Our results show that while it is possible to provide students with opportunities to engage in authentic mathematical activity in an online course, opportunities were limited compared to in-person courses. Researchers may want to use this framework to continue to explore how the dimensions of authenticity are similar or different across online and in-person course settings.

Keywords: online instruction, authentic proof activity

The transition to online courses necessitated by the COVID-19 pandemic has allowed for new exploration into online instruction (e.g., Jung & Brady, 2020). Many courses, such as those in advanced mathematics, are currently being offered online. The efficacy of online mathematics courses is generally poor (Trenholm, et al. 2019); although the majority of this literature focuses on asynchronous courses. Student perceptions of online courses range from quite positive to quite negative (Dobbs, Waid & del Carmen, 2009). For example, 40% of students surveyed by Jacqueline and Smita (2001) indicated higher participation in online courses than traditional courses. However, O'Malley and McCraw (1999) found that students found it difficult to contribute to discussions online whether synchronous or asynchronous. Recent work in the mathematics setting points to ways that "rich dialogic interactions" can be maintained by having students share strategies and engage with them using unique features of online settings such as shared Google Docs and breakout rooms for small group discussion (Jung & Brady, 2020). Further, Öner (2008) suggested the online setting may be particularly conducive to engaging students in authentic proof activity through collaboration and exploration using dynamic geometry software. Similarly, Yopp (2014) illustrated how asynchronous online discussions may serve as a productive space for authentic engagement with quantifiers and tasks via examples and example-generation. The literature further points to ways in which online collaboration may be different and need different support as students engage with features like text-based chat (e.g., Stahl, 2006) or Zoom (e.g., Jung & Brady, 2020).

We aim to contribute to this literature base by situating our study as a direct comparison between a lesson implemented in-person and implemented online. The lesson was developed through an iterative design-based research approach with an explicit focus on engaging students in authentic proof activity defined broadly as engagement in formal mathematics in ways that is consistent with the work of mathematicians. This includes not just creating formal proofs, but the informal activity and alternate goals such as comprehension and validation. The lesson is part of a standard introductory undergraduate abstract algebra course and focuses on comparing between two common proof approaches and analyzing proofs and statements (see Melhuish, et al., 2022 for an outline of the lesson goals.) As our overarching goal was to promote authentic proof activity, we share an analysis of these two lessons to explore how authenticity may have played out differently in the two contexts. We conclude with conjectures as to why the online setting may have led to different instructional choices and different opportunities for students.

Theoretical Framing

Underlying our work is the assumption that advanced mathematical courses provide an opportunity for students to apprentice into the mathematical work of research mathematicians. To this end, we have developed a literature-based framework to describe the activity of mathematicians that has potential for adoption to the undergraduate classroom (Melhuish, et al., 2021). We broadly use activity theory (Engeström, 2000) to frame our approach where activity can be decomposed into goal-directed actions consisting of tools (materials, concepts, procedures) used in service of an objective (including a motive and focal object). Activity occurs in systems that are historically (such as where tools originate) and socially situated (communities with rules and norms divide up labor). Our overarching framework includes three objects: proofs, propositional statements, concept/definitions and three motives: constructing, exploring/comprehending, testing/validating. In service of these goals we include tools: analyzing/refining, formalizing, deformalizing, warranting, analogizing/transferring, examples, diagrams, logic, structure/frameworks, and existent objects (definitions, proofs, statements). We then operationalize authenticity along a number of dimensions to capture multiple, often competing (e.g., Dawkins, et al., 2019; Herbst, 2002; Lampert, 1992) notions of authenticity guided by ideas of content, practice, discipline, and students. See Table 1 for the authenticity dimensions of the *Authentic Mathematical Proof Activity (AMPA)* framework.

Table 1. Dimensions of Authenticity Defined by Tool Use

Dimension	Description	Characteristic
Variety	The degree of variety of tools in use including formal, informal, generating, and translating tools	Disciplinary Tool Use
Complexity	The degree to which tools and outcomes are used in conjunction and succession versus in isolation	Disciplinary Tool Use
Accuracy	The degree to which tools and outcomes are accurate to discipline standards	Discipline Tools and Outcomes
Agency	The degree to which students are the ones generating and using various tools	Student Role in the Division of Labor
Authority	The degree to which students are the ones connecting tools and objectives to determine whether a goal is, or will be, met	Student Role in the Division of Labor
Alignment	The degree to which tools and outcomes reflect student contributions	Student Tools and Outcomes

Methods

This study is part of a larger design-based research project (Design-Based Research Collective, 2003). The focal lesson was developed through an interactive design process and included creating a hypothesized trajectory of student activity linked to particular tasks features and instructional moves. The lesson was developed, evaluated by a panel of experts, and tested and refined over two iterations with a small group of undergraduate students. The goal of this lesson is for students to *comprehend* two different approaches to proving the structural property:

if G is abelian and isomorphic to H, then H is abelian. By comparing approaches (one that begins with elements of the domain and concludes that the image of these elements commute; one that begins with elements of the codomain and concludes these elements commute), students are positioned to attend to structural features of the proof and engage in analysis of which assumptions from the statement are needed and why. This then leads to a discussion of modifying the proofs and the statements through a process of analysis and refinement.

The data for this report comes from two classroom implementations in consecutive semesters. Both implementations were facilitated by the same instructor at the same institution. The class consisted of a relatively even distribution of mathematics and mathematics education majors with less than 20 students in each class. The first implementation, in-person, was videotaped using Swivl. The second implementation was conducted over Zoom and was recorded using the Zoom interface. In both implementations, members of the research team observed small groups and took field notes. We conducted a retrospective video analysis using the AMPA framework to guide analysis. One researcher repeatedly viewed the videos and identified comparable episodes for the two lessons. This researcher and another member of the team independently viewed selected episodes and created analytic memos attending to the various features of the AMPA framework identifying tools at use, objectives (motives/objects), and describing authenticity across the six dimensions. The respective analyses were compared with discrepancies resolved through discussion.

Preliminary Results

The goal of this lesson was for students to analyze and comprehend two different proof approaches for the statement: *if G is abelian and isomorphic to H, then H is abelian*. For the scope of this report, we focus on two episodes. In the first episode, students spent time in their groups coming up with similarities and differences between each proof approach. These were then shared with the whole class to form a list of all the similarities and differences identified by the students. In the second episode, students were asked to discuss with their group whether they thought each assumption (abelian, homomorphism, one-to-one, and onto) was needed. After discussion in groups, the instructor had students share their ideas in a whole class discussion.

The similarity/differences episode consisted of students spending time in their groups attempting to identify similarities and differences between the two proof approaches. In the online section, this was done via breakout rooms on Zoom. After each group was allowed time to discuss, the instructor brought all of the students back to the main room. The instructor then asked the students to state some of the similarities and differences they observed between the two proof approaches. Most of the students' responses were typed in the chat. We observed evidence of students *deformalizing* a proof to explore its components as they used informal language to describe the similarities and differences between two formal proof approaches. We also saw one example of *warranting*: identifying that both proof approaches use isomorphism, and one example of using *structure* by identifying the difference: one proof starts in the domain, while the other starts in the co-domain. The instructor wrote each of the similarities and differences onto a shared document. Therefore, this segment had a high level of *alignment* with student contributions. Additionally, the instructor was not confirming or correcting any of them. Thus, in this segment, students had a high level of *agency* as during the comparison they were spontaneously warranting and identifying proof frameworks (tools), and maintained most of the *authority* as the instructor was not evaluating or linking the students' contributions to the larger motive (in service of exploring/understanding the proofs). At this point, *accuracy* was high as most of the students' noticings were valid, *complexity* was low (different tools/outcomes were not being used consecutively/together to achieve a goal), and *variety* was mid as a number of tools were in play, although they stayed largely formal.

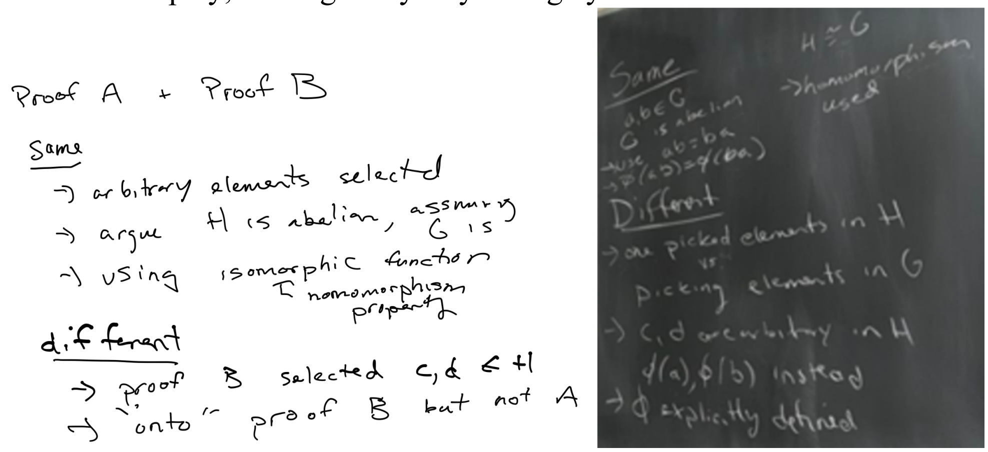

Figure 1. Public Record Documents of similarities and differences in online and in-class setting, respectively

The similarities/differences episode of the in-person section shared many of the same characteristics as the online counterpart. However, we note this occurred earlier in the lesson. Both episodes began with the instructor prompting students to share similarities and differences they observed across the two proofs. The instructor recorded each suggestion on the chalkboard, so this segment also had a high level of *alignment*. However, the instructor also prompted students to explain where each contribution was present in the proof. Thus, students were given slightly more opportunities to warrant in the in-person section than they were in the online section. The frequency of warranting appeared to be the only difference in authenticity across the two lessons. The authenticity dimensions in the in-person section were all comparable to those in the online section.

The assumptions episode consisted of students attempting to determine which assumptions were used in each proof. The online section spent time discussing the assumptions in breakout rooms based on a list of assumptions developed from a poll earlier in the class. When they returned as a whole class, the instructor used the poll function on Zoom to find out which assumptions the students thought were being used. The results of the polls were written down, and the instructor did not endorse any particular answer over another. Thus, students had some *agency* (although highly directed) to *analyze* the proofs, and the segment had a high level of *alignment.* However, due to the use of polls, the *variety* and *complexity* of contributions the students were able to make was limited. They were not given the opportunity to discuss where or why each assumption was being used (*warranting*). In general, students in the online section had limited opportunities to use *authority* -- connect their tools to the larger exploring proof motive -- in this episode *Alignment* remained high as the students' voting was recorded by the instructor in contrast to *accuracy* as many students did not respond in normative ways.

Students' opportunities to engage in authentic mathematical practices differed across the online and in-person sections in the assumptions episode significantly more than they did in the similarities/differences episode. In the in-person section, not only did the students state whether or not they thought each assumption was being used, but they were also asked to point out where each assumption was being used. For example, the students came to a consensus that the assumption that *G* is abelian is being used in the proof. The instructor then asked them to point out where it was being used in the proof. One student pointed at the proof being displayed on the projector and stated, "*a* operated with *b* equals *b* operated with *a*." Asking students to point out where each assumption was being used resulted in students having more *authority* as they took opportunities to warrant in service of exploring the proof . Additionally, allowing students to point to the proof (*warranting*), describe why each assumption is necessary (*analyzing*), and explain lines of a proof in their own words (*deformalizing*), increased both the *variety* and *complexity* of students' contributions. After some discussion, the class emerged with two conjectures about the assumptions needed for each proof: abelian and homomorphism, and abelian, homomorphism, one-to-one, and onto. The instructor wrote these down, *formalizing* them into conjecture form, and asked the class to discuss them in groups (increasing *complexity*). Therefore, the episode ended with mid *alignment* (as the instructor formalized the students' ideas), but low *accuracy* (as neither conjecture was valid.)

Discussion

Our results suggest that it is still possible for an instructor to provide students with opportunities to engage in authentic mathematical activity in an online setting. This was observed in both episodes we described above. The instructor was successful in giving students agency by providing them with opportunities to warrant and analyze. We also observed that online students had a high level of authority in both episodes, as it was their responsibility to determine what was valid and why. These results are important because they provide evidence that an online setting does not preclude authentic mathematical activity.

Although our results suggest that it is possible to provide students with opportunities to engage in authentic mathematical practices in an online setting, they also provide evidence that the extent to which this can be done may be limited compared to an in-person setting. The first major difference we saw across the sections was that in the online section the instructor tended to invite contributions by using polls and the chat window. This reduced the variety and complexity of students' contributions. Additionally, we saw that the use of the poll and chat in the second episode resulted in students having less opportunities to warrant and less overall authority. However, the use of these features meant that all students contributed, rather than just more vocal students in class. In some sense, this may serve the role of increasing student engagement in activity that parallels the in-class mechanism of a "turn and talk" which is not readily available online. Some of the differences may also be accounted for by pace. In the online version, the group work components of the lesson took more time. This may be partially due to the nature of going between a main room and a breakout room, as well as the time involved for the instructor to move from group to group. The instructor may have opted for polling rather than conversation with warranting due to time constraints. The slower pace accounts for the online version concluding with the assumptions task without further exploration of a conjecture that occurred after this episode in the in-person version. As this work is preliminary and situated in a particular lesson, we hesitate to make global claims about authenticity in activity online. We also acknowledge the difficulty of differentiating between constraints inherent to the course delivery mode and choices made by the instructor as a limitation of this report. However, the common setting, lesson, and instructor provided at least one case that points to similarities and differences across the contexts. Future researchers may want to use a similar approach to analyze authenticity to further understand the affordances and constraints of different class settings.

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