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Secant varieties and the complexity of matrix multiplication

J. M. Landsberg

Dedicated to Giorgio Ottaviani on the occasion of his 60th birthday

Abstract. *This is a survey primarily about determining the border rank of tensors, especially those relevant for the study of the complexity of matrix multiplication. This is a subject that on the one hand is of great significance in theoretical computer science, and on the other hand touches on many beautiful topics in algebraic geometry such as classical and recent results on equations for secant varieties (e.g., via vector bundle and representation-theoretic methods) and the geometry and deformation theory of zero dimensional schemes.*

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1. Introduction

This is a survey of uses of secant varieties in the study of the complexity of matrix multiplication, one of many areas in which Giorgio Ottaviani has made significant contributions. I pay special attention to the use of deformation theory because at this writing, deformation theory provides the most promising path to overcoming lower bound barriers. For an introduction to more general uses of algebraic geometry in algebraic complexity theory see [35]. I begin by reviewing some classical results.

1.1. Symplectic bundles on the plane, secant varieties, and Lüroth quartics revisited [45]

In the 1860's, Darboux studied degree n curves in $\mathbb{P}^2$ that pass through all the $\binom{n+1}{2}$ vertices of a complete $(n+1)$-gon in $\mathbb{P}^2$ (i.e., the union of $n+1$ lines in $\mathbb{P}^2$ with no points of triple intersection). In 1869 Lüroth studied the $n=4$ case. A naïve dimension count indicates that all quartics should pass through the 10 vertices of some complete pentagon but Lüroth proved it is actually a

codimension one condition.

In 1902 Dixon [24] proved all degree n curves in $\mathbb{P}^2$ arise as a $n\times n$ symmetric determinant (also see [23] for the general determinantal case).

In 1977 Barth [6] studied the moduli space of stable (symplectic) vector bundles on $\mathbb{P}^2$. In particular he showed that the curve of jumping lines of a rank 2 stable bundle on $\mathbb{P}^2$ with Chern classes $(c_1, c_2) = (0, 4)$ is a Lüroth quartic. Barth also gave a new proof of Lüroth's theorem via vector bundles.

In [45] Giorgio Ottaviani explains these results via the defectivity of *secant varieties* of $Seg(\mathbb{P}^2 \times v_2(\mathbb{P}^{n-1}))$, where $Seg(\mathbb{P}^2 \times v_2(\mathbb{P}^{n-1})) \subset \mathbb{P}(\mathbb{C}^3\otimes S^2\mathbb{C}^n)$ is the set of points $[x\otimes z^2]$, where $[x] \in \mathbb{P}^2$ and $[z] \in \mathbb{P}^{n-1}$. The proof uses the bounded derived category version of Beilinson's monad Theorem [8], see [4] for an excellent introduction.

1.2. Secant varieties

Throughout this paper V, A, B, C denote finite dimensional complex vector spaces. Let $X \subset \mathbb{P}V$ be a projective variety, Define its *r-th secant variety*, or variety of secant $\mathbb{P}^{r-1}$'s, to be

$$\sigma_r(X) := \overline{\bigcup_{x_1,\ldots,x_r\in X} \langle x_1, \ldots, x_r\rangle}.$$

Here, for a set or subscheme $Z \subset \mathbb{P}V$, $\langle Z\rangle \subset \mathbb{P}V$ denotes its linear span, and the overline denotes Zariski closure.

In this article I will be particularly interested in the case $X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subset \mathbb{P}(A\otimes B\otimes C)$, the variety of rank one (3-way) tensors. Given $T \in A\otimes B\otimes C$, define the *border rank* of T, $\underline{\mathbf{R}}(T)$ to be the smallest r such that $[T] \in \sigma_r(X)$.

Secant varieties have a long history in algebraic geometry dating back to the 1800's. In the 20th century they were used by J. Alexander and A. Hirschowitz [1] to solve the polynomial interpolation problem, and by F. Zak [51, Chap II, §2] to solve a linearized version of R. Hartshorne's famous conjecture on complete intersections, called *Hartshorne's conjecture on linear normality*. L. Manivel and I used them to study the geometry of the exceptional groups and their homogeneous varieties, and even to obtain a new proof of the Killing-Cartan classification of complex simple Lie algebras and prove geometric consequences of conjectured categorical generalizations of Lie algebras by Deligne and Vogel, see [37] for a survey. In this article, I discuss their use in the context of *algebraic complexity theory*, more specifically, in proving lower and upper bounds on the *complexity of matrix multiplication*.

1.3. Matrix multiplication

In 1968, V. Strassen [47] discovered the usual way we multiply $\mathbf{n}\times\mathbf{n}$-matrices, which uses $\mathcal{O}(\mathbf{n^3})$ arithmetic operations, is not optimal. After much work, it was generally conjectured that one can in fact multiply matrices using $\mathcal{O}(\mathbf{n^{2+\epsilon}})$ arithmetic operations for any $\epsilon>0$. To fix ideas, define the *exponent of matrix multiplication* ω to be the infimum over all τ such that $\mathbf{n}\times\mathbf{n}$ matrices may be multiplied using $\mathcal{O}(\mathbf{n}^\tau)$ arithmetic operations, so the conjecture is that $\omega=2$. The *matrix multiplication tensor* $M_{\langle\mathbf{n}\rangle}:\mathbb{C}^{\mathbf{n^2}}\times\mathbb{C}^{\mathbf{n^2}}\to\mathbb{C}^{\mathbf{n^2}}$ executes the bilinear map of multiplying two matrices. Fortunately for algebraic geometry, Bini [9] showed $\underline{\mathbf{R}}(M_{\langle\mathbf{n}\rangle})=\mathcal{O}(\mathbf{n}^\omega)$ so we may study the exponent via secant varieties of Segre varieties.

Thus one way to prove complexity *lower* bounds for matrix multiplication would be to prove lower bounds on the border rank of $M_{\langle\mathbf{n}\rangle}$. I will give a history of such lower bounds. Perhaps more surprising, is that one way of showing *upper* bounds for the complexity of matrix multiplication would be to prove the border rank of certain auxiliary tensors is small, as I discuss in §4.

1.4. Dimensions of secant varieties

One expects $\dim\sigma_r(X)=\min\{r\dim X+r-1,\dim\mathbb{P}V\}$, because one can pick r points on X, and then a point on the $\mathbb{P}^{r-1}$ spanned by them. This always gives an upper bound on the dimension.

Strassen [48], motivated by the complexity of matrix multiplication, showed that this expectation fails for $X=Seg(\mathbb{P}^2\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1})\subset\mathbb{P}(\mathbb{C}^3\otimes\mathbb{C}^n\otimes\mathbb{C}^n)$, n odd, $r=\frac{3n-1}{2}$.

Previously, E. Toeplitz, in 1877 [50], had already shown it fails for $X=Seg(\mathbb{P}^2\times v_2(\mathbb{P}^3))\subset\mathbb{P}(\mathbb{C}^3\otimes S^2\mathbb{C}^4)$, $r=5$.

In 2007 Ottaviani [45] showed that more generally the expectation fails for $X=Seg(\mathbb{P}^2\times v_2(\mathbb{P}^{n-1}))\subset\mathbb{P}(\mathbb{C}^3\otimes S^2\mathbb{C}^n)$ with n even $r=\frac{3n}{2}-1$, and that this failure implies Lüroth's theorem. In the same paper he also partially recovers Barth's moduli results.

2. Koszul flattenings and variants

2.1. Idea of Proofs of results in §1.4

To prove the naïve dimension count for $\dim\sigma_r(X)$ is wrong (e.g., in the case of Lüroth's theorem that $\sigma_r(X)\neq\mathbb{P}V$), one can show that the ideal of $\sigma_r(X)$ is non-empty by exhibiting an explicit polynomial in the ideal.

Strassen did this and his result was revisited by Ottaviani: Consider $Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)\subset\mathbb{P}(A\otimes B\otimes C)$, $\dim A=3$, $\dim B=\dim C=m$. Let

$\{a_i\},\{b_j\},\{c_k\}$ be bases of A,B,C. Given $T=\sum T^{ijk}a_i\otimes b_j\otimes c_k\in A\otimes B\otimes C$, consider the linear map

$$\begin{aligned} T_A^{\wedge 1}: A\otimes B^* &\to \Lambda^2 A\otimes C\\ a\otimes\beta &\mapsto \sum_{i,j,k} T^{ijk}\beta(b_j)a\wedge a_i\otimes c_k \end{aligned}$$

Exercise: If $[T]\in Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$, then $\operatorname{rank}(T_A^{\wedge 1})=2$, and thus, by linearity, if $\operatorname{rank}(T_A^{\wedge 1})>2R$, then $[T]\notin\sigma_R(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C))$.

Ottaviani states in a remark that these minors are a reformulation of *Strassen's equations* (however see Remark 2.1 below), which, for tensors $T\in A\otimes B\otimes C=\mathbb{C}^{\mathbf{a}}\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ such that there exists $\alpha\in A^*$ with $\operatorname{rank}(T(\alpha))=m$, are naturally expressed as follows: consider $T(A^*)\subset B\otimes C$, and for $\alpha\in A^*$ with $\operatorname{rank}(T(\alpha))=m$, consider the linear isomorphism $T(\alpha):B^*\to C$. Then $T(A^*)T(\alpha)^{-1}\subset\operatorname{End}(C)$ is a space of endomorphisms. If $T=\sum_{j=1}^m e_j\otimes b_j\otimes c_j$ for some $e_j\in A$, then one obtains a space of diagonal matrices. In particular, the matrices commute. Since the property of commuting is closed, if $[T]\in\sigma_m(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C))$, then one still obtains a space of commuting endomorphisms. Moreover, the rank of the commutator (a measure of the failure of commutivity) may be computed from the rank of $T_A^{\wedge 1}$. Note that in both cases one restricts to a three dimensional subspace of A.

To see Strassen's equations as polynomials, for $X\in B\otimes C$, let $X^{\wedge m-1}\in\Lambda^{m-1}B\otimes\Lambda^{m-1}C\simeq B^*\otimes C^*$ denote the adjucate (cofactor matrix), and recall that X^{-1} is essentially the adjugate times the determinant. Then Strassen's equations for T to have border rank (at most) m [48] become, for all $X,Y,Z\in T(A^*)\subset B\otimes C$,

$$XY^{\wedge m-1}Z-ZY^{\wedge m-1}X=0. \tag{1}$$

These are equations of degree $m+1$.

Using a refinement of these equations that takes into account the rank of the commutator, Strassen proved $\underline{\mathbf{R}}(M_{\langle\mathbf{n}\rangle})\geq\frac{3}{2}\mathbf{n^2}$, the first non-classical lower bound on the border rank of the matrix multiplication tensor.

Call a tensor T which satisfies the genericity condition that there exists $\alpha\in A^*$ with $\operatorname{rank}(T(\alpha))=m$, 1_A-*generic*.

When T is 1_A-generic, taking Y of full rank and changing bases such that it is the identity element, the equations require the space to be abelian. If $T(A^*)$ is of bounded rank $m-1$, for each X,Y,Z, the set of m^2 equations reduces to a single equation. If $T(A^*)$ is of bounded rank $m-2$, then the equations become vacuous.

Lemma 3.1 of [45] states that if T is 1_A-generic, then the condition on $\operatorname{rank}(T_A^{\wedge 1})$ is indeed a reformulation of Strassen's equations.

REMARK 2.1. Recently, with my student Arpan Pal and Joachim Jelisiejew [32], we proved that if T is not 1_A-generic, then the condition on $\operatorname{rank}(T_A^{\wedge 1})$ is a stronger condition than the A-Strassen equations.

2.2. Generalizations: Young flattenings [43, 44]

2.2.1. Koszul flattenings

Consider $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subset \mathbb{P}(A\otimes B\otimes C)$, let $\dim A = 2p+1$, $\dim B = \dim C = m$. (If $\dim A > 2p+1$, restrict to a general $2p+1$ dimensional subspace.)

Given $T = \sum T^{ijk} a_i\otimes b_j\otimes c_k \in A\otimes B\otimes C$, consider the linear map

$$T_A^{\wedge p} : \Lambda^p A\otimes B^* \to \Lambda^{p+1} A\otimes C$$
$$a_{i_1}\wedge\cdots\wedge a_{i_p}\otimes\beta \mapsto \sum_{i,j,k} T^{ijk}\beta(b_j) a_{i_1}\wedge\cdots\wedge a_{i_p}\wedge a_i\otimes c_k$$

Exercise: If $[T] \in Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, then $\operatorname{rank}(T_A^{\wedge p}) = \binom{2p}{p-1}$. Thus if $\operatorname{rank}(T_A^{\wedge p}) > \binom{2p}{p-1}R$, then $[T] \notin \sigma_R(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C))$. Call these equations the *p-Koszul flattenings*.

When Ottaviani and I found the p-Koszul flattenings, we were sure we would get a new lower bound for matrix multiplication. Our first attempts were discouraging, we were attempting to take $2p+1 = \mathbf{n^2}$ or nearly so. It turns out that our initial attempts were too greedy, as such values do not give good lower bounds. Only months later, we finally tried taking $p = \mathbf{n-1}$ which enabled us to prove the first new lower bounds for border rank of matrix multiplication since 1983:

THEOREM 2.2 ([44]). $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle}) \geq 2\mathbf{n^2} - \mathbf{n}$.

It is worth noting that the absolute limit of Koszul flattenings, and any determinantal equations that we found, was $2\mathbf{n^2} - \mathbf{1}$, i.e., for tensors in $\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$, $2m-1$.

2.2.2. Young flattenings

We found the p-Koszul flattenings as part of a general program to systematically find equations for secant varieties, especially equations of secant varieties of homogeneous varieties, which we call *Young flattenings*. Giorgio likes to think of these in terms of degeracy loci of maps between vector bundles, and I prefer a more representation-theoretic perspective. The basic idea is for $X \subset \mathbb{P}V$, to find an inclusion of V into a space of matrices. Then if the matrices associated to points of X have rank at most q, the size $qr+1$ minors restricted to V give equations for $\sigma_r(X)$.

Vector bundle perspective

Let $E \to X$ be a vector bundle of rank e, write $L = \mathcal{O}_X(1)$ so $V = H^0(X, L)^* = H^0(L)^*$. Let $v \in V$ and consider the linear map $A_v^E : H^0(E) \to H^0(E^* \otimes L)^*$ induced by the multiplication map $H^0(E) \otimes H^0(L)^* \to H^0(E^* \otimes L)^*$. Then, assuming all spaces are sufficiently large, the size $(re+1)$ minors of A_v^E give equations for $\sigma_r(X)$. Here we have an inclusion $V = H^0(L)^* \subset H^0(E)^* \otimes H^0(E^* \otimes L)^*$.

Representation theory perspective

Let $X = G/P \subset \mathbb{P}V_\lambda$ where V_λ is an irreducible module for the reductive group G of highest weight λ and X is the orbit of a highest weight line, i.e., the minimal G-orbit in $\mathbb{P}V_\lambda$. Look for G-module inclusions $V_\lambda \subset V_\mu \otimes V_\nu$, so in coordinates one realizes V_λ as a space of matrices. Then for $x \in X$ if the associated matrix has rank k, the size $rk+1$ minors of $V_\mu \otimes V_\nu$ restricted to V_λ give equations in the ideal of $\sigma_r(X)$.

We spent some time trying to find more powerful Young flattenings. At first we just thought we were not being clever enough in our search for determinantal equations, but then we came to suspect that there was some limit to the method.

3. Beyond Koszul flattenings: steps forward and barriers to future progress

3.1. The cactus barrier

Around the same time, in both algebraic complexity theory [25] and algebraic geometry [7, 11] (see [34, Chap. 10] for an overview), it was proven that determinantal methods are subject to an absolute barrier that is at most $6m$ for tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$.

To explain the barrier from a geometric perspective, rewrite the definition of the secant variety as

$$\sigma_r(X) := \overline{\bigcup \{\langle R\rangle \mid \text{length}(R) = r,\ R \subset X,\ R : \text{smoothable}\}}.$$

Here $R \subset X$ denotes a zero dimensional scheme and the union is taken over all zero dimensional schemes satisfying the conditions in braces. Define the *cactus variety* [11]:

$$\kappa_r(X) := \overline{\bigcup \{\langle R\rangle \mid \text{length}(R) = r,\ R \subset X\}}.$$

It turns out that $\kappa_{6m-4}(Seg(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{m-1})) = \mathbb{P}(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m)$, compared with $\sigma_r(Seg(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}))$ which does not fill $\mathbb{P}(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m)$ until $r \sim \frac{m^2}{3}$.

The barrier results from the fact that *determinantal equations for $\sigma_r(X)$ are also equations for $\kappa_r(X)$*!

When I learned this, I became very discouraged.

3.2. A Phyrric victory

With M. Michałek, we were able to push things a little further for tensors with symmetry. Given $T \in A\otimes B\otimes C$, $\underline{\mathbf{R}}(T) \leq r$ if and only if there exists a curve $E_t \subset G(r, A\otimes B\otimes C)$, the Grassmannian of r planes through the origin in $A\otimes B\otimes C$, such that

- For $t \neq 0$, E_t is spanned by r rank one elements.
- $T \in E_0$.

Let $G_T := \{g \in GL(A)\times GL(B)\times GL(C) \mid g\cdot T = T\}$ denote the symmetry group of T. Then if we have such a curve E_t, then for all $g \in G_T$, gE_t also gives a border rank decomposition. Thus one can insist on normalized curves, those with E_0 Borel fixed for a Borel subgroup of G_T [40]. Then one can apply a border rank version of the classical substitution method (see, e.g., [2]) to reduce the problem to bounding the border rank of a smaller tensor. Applying this to the matrix multiplication tensor, we proved:

THEOREM 3.1 ([41]). $\underline{\mathbf{R}}(M_{\langle \mathbf{n}\rangle}) \geq 2\mathbf{n^2} - \log_{\mathbf{2}} \mathbf{n} - \mathbf{1}$.

Recall that the limit of lower bounds one can prove with Young flattening is $2m-1$. We wrote down an explicit sequence of tensors $T_m \in \mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ with a one-dimensional symmetry group and proved:

THEOREM 3.2 ([42]). $\underline{\mathbf{R}}(T_m) \geq (2.02)m$.

After that, I did not see any path to further lower bounds.

3.3. A vast generalization: border apolarity

W. Buczyńska and J. Buczyński [12] had the following idea: Consider not just a curve in the Grassmannian obtained by taking the spans of r moving points $\{T_{j,\epsilon}\}$, where $T = \lim_{\epsilon\to 0}\sum_{j=1}^r T_{j,\epsilon}$, but the curve of **ideals** $I_\epsilon \in Sym(A^*\oplus B^*\oplus C^*)$ that the points give rise to: let I_ϵ be the ideal of polynomials vanishing on $[T_{1,\epsilon}]\cup\cdots\cup[T_{r,\epsilon}] \subset \mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C$. Now consider the "limiting ideal". But how should one take limits? If one works in the usual Hilbert scheme the r points limit to some zero dimensional scheme and one could take the span of that scheme. But for secant varieties one is really taking the limit of the spans $\langle T_{1,\epsilon},\dots,T_{r,\epsilon}\rangle$ in the Grassmannian of r planes and the span of the limiting scheme can be strictly smaller than the limit of the spans, so this idea does not work.

The answer is to work in the Haiman-Sturmfels multigraded Hilbert scheme [29], which lives in a product of Grassmannians and keeps track of the entire Hilbert function rather than just the Hilbert polynomial. The price one pays is that now one must allow unsaturated ideals.

As with the border substitution method, one can insist that limiting ideal I_0 is Borel fixed, which for tensors with a large symmetry group reduces in small multi-degrees to a small search that has been exploited in practice.

Instead of single curve $E_\epsilon \subset G(r, A\otimes B\otimes C)$ limiting to a Borel fixed point, for each (i,j,k) one gets a curve in each $Gr(r, S^iA^*\otimes S^jB^*\otimes S^kC^*)$, each curve limiting to a Borel fixed point *and* satisfying compatibility conditions. Here $Gr(r,V)$ is the Grassmannian of *codimension* r subspaces in V. In this situation, $E_\epsilon = (I_\epsilon)^\perp_{(111)}$.

The upshot is an algorithm that either produces all normalized candidate I_0's or proves border rank $> r$. The caveat is that once one has a candidate I_0 one must determine if it actually came from a curve of ideals of r distinct points.

Using border apolarity, in [19] we proved numerous new matrix multiplication border rank lower bounds, as well as determining the border rank of the size three determinant polynomial considered as a tensor $\det_3 \in \mathbb{C}^9\otimes\mathbb{C}^9\otimes\mathbb{C}^9$, whose importance for complexity is explained below. In particular, our results include the first nontrivial lower bounds for "unbalanced matrix multiplication tensors", something that was untouchable using previous methods.

3.4. Border apolarity is subject to the cactus barrier

In practice, one attempts to construct an ideal by building it up from low multi-degrees. The main restrictions one obtains is when one has the ideal constructed up to multi-degrees $(s-1,t,u)$, $(s,t-1,u)$ and $(s,t,u-1)$, and one wants to construct the ideal in degree (s,t,u). In order that the construction may continue, one needs that the natural symmetrization and addition map

$$I_{s-1,t,u}\otimes A^* \oplus I_{s,t-1,u}\otimes B^* \oplus I_{s,t,u-1}\otimes C^* \to S^sA^*\otimes S^tB^*\otimes S^uC^* \qquad (2)$$

has image of codimension at least r. Here $I_{s-1,t,u} \subset S^{s-1}A^*\otimes S^tB^*\otimes S^uC^*$ denotes the component of the ideal in multi-degree $(s-1,t,u)$ etc. (Here and in what follows, the direct sum is the abstract direct sum of vector spaces, so there is no implied assertion that the spaces are disjoint when they live in the same ambient space.)

That is, the minors of the map of appropriate size must vanish. These are determinantal conditions and therefore subject to the cactus barrier.

Remark 3.3. Aside for the experts: J. Buczyński points out that not all components of the usual Hilbert scheme contain ideals with generic Hilbert func-

tion. Thus in those situations, border apolarity may give better lower bounds on border rank than on cactus border rank.

3.5. Deformation theory

Although border apolarity alone cannot overcome the cactus barrier, by placing calculations in the Haiman-Sturmfels multigraded Hilbert scheme, it provides a path to overcoming the cactus barrier. Namely one can apply the tools of *deformation theory* (see, e.g., [30] for an introduction) to determine if a candidate ideal is deformable to the ideal of a smooth scheme. Below, after motivating the problem, I describe a first implementation of this in the special case of tensors of minimal border rank.

4. Strassen's laser method for upper-bounding the exponent of matrix multiplication using tensors of minimal or near minimal border rank

The best way to prove an upper bound on matrix multiplication complexity would be to prove an upper bound for matrix multiplication directly. Fortunately for algebraic geometry, Bini [9] showed that this is measured by the border rank of the matrix multiplication tensor. However, there has been little progress in this direction since the early 1980's. Instead, border rank upper bounds for $M_{\langle \mathbf{n} \rangle}$ have been proven using indirect methods, the most important papers are those of Schönhage [46], Strassen [49] and Coppersmith-Winograd [22]. The resulting technique is called *Strassen's laser method*. Remarkably, it begins with a tensor of minimal (or near minimal) border rank, i.e., a concise tensor in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of border rank m or nearly m, and then builds a large tensor from it, using its Kronecker powers defined below, and then, using methods from probability and combinatorics, shows this large tensor admits a degeneration to a large matrix multiplication tensor.

For tensors $T \in A \otimes B \otimes C$ and $T' \in A' \otimes B' \otimes C'$, the *Kronecker product* of T and T' is the tensor $T \boxtimes T' := T \otimes T' \in (A \otimes A') \otimes (B \otimes B') \otimes (C \otimes C')$, regarded as 3-way tensor. Given $T \in A \otimes B \otimes C$, the *Kronecker powers* of T are $T^{\boxtimes N} \in A^{\otimes N} \otimes B^{\otimes N} \otimes C^{\otimes N}$, defined iteratively. Rank and border rank are submultiplicative under Kronecker product: $\mathbf{R}(T \boxtimes T') \leq \mathbf{R}(T)\mathbf{R}(T')$, $\underline{\mathbf{R}}(T \boxtimes T') \leq \underline{\mathbf{R}}(T)\underline{\mathbf{R}}(T')$, and both inequalities may be strict.

Given $T, T' \in A \otimes B \otimes C$, we say that T *degenerates* to T' if $T' \in \overline{GL(A) \times GL(B) \times GL(C) \cdot T}$, the closure of the orbit of T, the closures are the same in the Euclidean and Zariski topologies.

Strassen's *laser method* [49, 21] obtains upper bounds on ω by showing a certain explicit degeneration of a large Kronecker power of a tensor T satisfying certain combinatorial properties, admits a further degeneration to a large

matrix multiplication tensor. Since border rank is non-increasing under degenerations and one has an upper bound on $\underline{\mathbf{R}}(T^{\boxtimes N})$ inherited from the knowledge of $\underline{\mathbf{R}}(T)$, one obtains an upper bound on the border rank of the large matrix multilplication tensor.

An early success of the laser method was with a tensor of border rank $m+1$, now called the *small Coppersmith-Winograd tensor* $T_{cw,q} \in (\mathbb{C}^{q+1})^{\otimes 3}$. Coppersmith and Winograd showed that for all k and $q = m-1$, [22]

$$\omega \leq \log_q \left(\frac{4}{27} \left(\underline{\mathbf{R}} \left(T_{cw,q}^{\boxtimes k} \right) \right)^{\frac{3}{k}} \right). \tag{3}$$

They used this when $q = 8$ and the estimate $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes k}) \leq (q+2)^k$ to obtain $\omega \leq 2.41$. In particular, one could even potentially prove ω equaled two were $\lim_{k\to\infty}(\underline{\mathbf{R}}(T_{cw,2}^{\boxtimes k}))^{\frac{3}{k}}$ equal to 3 instead of 4. Using an enhancement of border apolarity, with A. Conner and H. Huang, in [20] we solved the longstanding problem of determining $\underline{\mathbf{R}}(T_{cw,2}^{\boxtimes 2})$. Unfortunately for matrix multiplication upper bounds, we proved that $\underline{\mathbf{R}}(T_{cw,2}^{\boxtimes 2}) = 4^2$. Previously, just using Koszul flattenings, analogous (and even higher Kronecker power) results for other small Coppersmith-Wingorad tensors were obtained with A. Conner, F. Gesmundo, and E. Ventura [18].

A more intriguing tensor is the "skew-cousin" of the small Coppersmith-Winograd tensor $T_{skewcw,q}$ occuring in odd dimensions, which similarly satisfies for all k and even q, [18]

$$\omega \leq \log_q \left(\frac{4}{27} \left(\underline{\mathbf{R}} \left(T_{skewcw,q}^{\boxtimes k} \right) \right)^{\frac{3}{k}} \right). \tag{4}$$

Again, the $q = 2$ case could potentially be used to prove the exponent is two. Here one begins with a handicap, as $\underline{\mathbf{R}}(T_{skewcw,2}) = 5 > 4$, but with A. Conner and A. Harper, using border apolarity for the lower bound and numerical search methods for the upper bound, in [19] we showed $\underline{\mathbf{R}}(T_{skewcw,2}^{\boxtimes 2}) = 17 < 25$. Unfortunately $17 > 16$. The problem of determining the border rank of the cube remains.

It is worth remarking that $T_{cw,2}^{\boxtimes 2}$ is isomorphic to the size three permanent polynomial considered as a tensor and $T_{skewcw,2}^{\boxtimes 2}$ is isomorphic to the size three determinant polynomial [18].

The best bounds on the exponent were obtained using the laser method applied to the *big Coppersmith-Winograd tensor* $T_{CW,q}$, which has minimal border rank. However, barriers to future progress using the laser method applied to this tensor have been discovered, first in [3], and then in numerous follow-up works. In particular, one cannot prove $\omega < 2.3$ using $T_{CW,q}$ in the laser method. A geometric interpretation of the barriers is given in [16].

Very recently, at an April 2022 workshop on geometry and complexity theory in Toulouse, France, J. Jelisiejew and M. Michałek announced a path to

improving the laser method. Their observation was that the border rank estimate for the "certain degeneration" of $T^{\boxtimes N}$ in the laser method mentioned above can be improved! The proof exploits properties of the algebra associated to $T^{\boxtimes N}$ (discussed in §7.1 below) that persist under the degeneration.

Even without that recent developement, other minimal border rank tensors could potentially prove $\omega < 2.3$ with the standard laser method. In fact in [31, Cor 4.3] and [17, Cor 7.5] it was observed that among tensors that are 1_A, 1_B and 1_C generic (such are called 1-*generic tensors*), $T_{CW,q}$ is the "worst" for the laser method in the sense that any bound one can prove using $T_{CW,q}$ can also be proved using any other minimal border rank 1-generic tensor. This provides strong motivation to understand tensors of minimal border rank. A second motivation is that it can serve as a case study for the implementation of deformation theory to overcome the cactus barrier.

5. Classical and neo-classical equations for tensors of minimal border rank

Before discussing recent developments for tensors of minimal border rank, I explain the previous state of the art. I already have discussed Strassen's equations and Koszul flattenings. What follows are additional conditions.

5.1. The equations of [36, 38]

Several modules of equations were found in [36, 38] using representation theory and variants of Strassen's equations. Many of these still lack a geometric interpretation.

5.2. The flag condition

If $\underline{\mathbf{R}}(T) = m$ there exists a flag $A_1 \subset \cdots \subset A_{m-1} \subset A$ such that for all j, $T(A_j^*) \subset \sigma_j(Seg(\mathbb{P}B \times \mathbb{P}C))$. This has been observed several times, dating back at least to [13, Exercise 15.14]. Note that to convert this condition to polynomial conditions, one would have to use elimination theory, even for the first step that there exists a line A_1^* such that $\mathbb{P}T(A_1^*) \in Seg(\mathbb{P}B \times \mathbb{P}C)$. The flag condition was essential to the results in [20].

5.3. The End-closed condition

Gerstenhaber [28] observed the following: Let $\langle x_1, \ldots, x_m\rangle \subset \mathrm{End}(\mathbb{C}^m)$ be a limit of spaces of simultaneously diagonalizable matrices. Then $\forall i, j$, $x_i x_j \in \langle x_1, \ldots, x_m\rangle$. Call this the "End-closed condition". To express the condition as

polynomials, let $\{\alpha_i\}$ be a basis of A^*, with α_1 chosen to maximize the rank of $T(\alpha_1)$, then for all $\alpha', \alpha'' \in A^*$, the End-closed condition is

$$(T(\alpha')T(\alpha_1)^{\wedge m-1}T(\alpha'')) \wedge T(\alpha_1) \wedge \cdots \wedge T(\alpha_m) = 0 \in \Lambda^{m+1}(\mathrm{End}(C)). \quad (5)$$

These are polynomials of degree $2m+1$. When T is 1_A-generic and one takes α_1 such that $\mathrm{rank}(T(\alpha_1)) = m$, these correspond to $T(A^*)T(\alpha_1)^{-1} \subset \mathrm{End}(C)$ being closed under composition of endomorphisms.

5.4. The symmetry Lie algebra condition

Let $\mathfrak{g} = \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$. Let $\hat{\mathfrak{g}}_T = \{X \in \mathfrak{g} \mid X.T = 0\}$. (This is the pullback of the symmetry Lie algebra of T to $\mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \oplus \mathfrak{gl}(C)$.) With T understood, write $\mathfrak{g}_{AB} = \{X \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \mid X.T = 0\}$ and similarly for $\mathfrak{g}_{BC}, \mathfrak{g}_{AC}$.

A concise tensor of rank m, $M_{\langle 1\rangle}^{\oplus m}$, has $\dim \hat{\mathfrak{g}}_{M_{\langle 1\rangle}^{\oplus m}} = 2m$ and $\dim \mathfrak{g}_{AB} = \dim \mathfrak{g}_{AC} = \dim \mathfrak{g}_{BC} = m$. The dimension of the symmetry Lie-algebra is semi-continuous under degenerations, thus if T is of minimal border rank $\dim \hat{\mathfrak{g}}_T \geq 2m$ and $\dim \hat{\mathfrak{g}}_{AB} \geq m$ and permuted statements.

Computing these dimensions amounts to determining the dimension of the kernel of a linear map. Precisely to check if $\dim \hat{\mathfrak{g}}_T \geq 2m$ are equations of degree $3m^2 - 2m + 1$ and $\dim \mathfrak{g}_{AB} \geq m$ are equations of degree $2m^2 - m + 1$.

6. The 111-equations and first consequences

6.1. The 111-equations

The *111-equations* are the rank conditions on the map (2) when $(s,t,u) = (1,1,1)$ and one is testing for border rank m. Note that in this case there are no choices for the ideal in degrees $(110), (101), (011)$, so they are really polynomial equations. These equations first appeared in [12, Thm 1.3].

The 111-equations for concise tensors of minimal border rank may be rephrased as the requirement that

$$\dim((T(A^*)\otimes A) \cap (T(B^*)\otimes B) \cap (T(C^*)\otimes C)) \geq m. \quad (6)$$

A special case of the 111-equations are the two-factor 111-equations, which have a natural geometric interpretation and are easier to implement because a pairwise intersection can be computed using inclusion-exclusion: Given subspaces X_1, X_2, X_3 of a vector space V, by inclusion-exclusion $\dim(X_i \cap X_j) = \dim(X_i) + \dim(X_j) - \dim\langle X_i, X_j\rangle$.

Thus the two-factor 111-test may be computed by checking if $\dim\langle T(A^*)\otimes A, T(B^*)\otimes B\rangle \geq 2m^2 - m + 1$ and permuted statements. These

are equations of degree $2m^2 - m + 1$ in the T^{ijk}. Notice that if $(X, Y) \in \mathfrak{g}_{AB}$, i.e., $X.T = -Y.T$, then $(X, -Y)$ gives rise to an element of $(T(A^*)\otimes A) \cap (T(B^*)\otimes B)$, i.e., the two factor 111-tests are equivalent to the dimension requirements on $\mathfrak{g}_{AB}, \mathfrak{g}_{AC}, \mathfrak{g}_{BC}$ for minimal border rank.

More generally, the full 111-equations may also be understood as a generalization of the lower bound on $\dim(\hat{\mathfrak{g}}_T)$, where one not just bounds dimension, but restricts the structure of $\hat{\mathfrak{g}}_T$ as well.

To compare the 111-equations with other previously known equations, we have:

PROPOSITION 6.1. *[32, Prop. 1.1, Prop. 1.2] The 111-equations imply both Strassen's equations and the End-closed equations. The 111-equations do not always imply the $p = 1$ Koszul flattening equations.*

Consider the situation of a concise tensor where each of the associated spaces of homomorphisms is of bounded rank $m - 1$. Strassen's equations do allow some assertions in this situation. A normal form for such tensors was proved by S. Friedland [26]. This normal form was generalized in [32, Prop. 3.3] by using the 111-equations instead of Strassen's equations. (In fact this generalized normal form allowed the proof that the 111-equations imply Strassen's equations and the End-closed equations.) These normal forms respectively allowed the characterization of concise tensors of minimal border rank when $m = 4$ and $m = 5$, in fact S. Friedland was even able to resolve the non-concise $m = 4$ case using additional equations he developed, solving the set-theoretic "salmon prize problem" posed by E. Allman.

Recall that Strassen's equations and the End-closed equations are trivial when a tensor gives rise to three linear spaces of bounded rank at most $m - 2$. The 111-equations do not share this defect. We are currently implementing them for such spaces of tensors. (The $p = 1$ Koszul flattenings are not trivial in this setting, we have yet to determine their utility for bounded rank $m - 2$ situations.)

7. Deformation theory for tensors of minimal border rank

For tensors $T \in \mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ satisfying genericity conditions, one has natural algebraic structures associated to them that can be utilized to help determine if they have minimal border rank.

7.1. Binding tensors and algebras

Say $T \in A\otimes B\otimes C$ is 1_A and 1_B generic with $T(\alpha_1) : B^* \to C$ and $T(\beta_1) : A^* \to C$ of full rank. (A tensor that is at least two of 1_A, 1_B or 1_C generic is called *binding*.) Use their inverses to obtain a tensor isomorphic to T, where

I abuse notation and also denote by T, $T \in C^*\otimes C^*\otimes C$, i.e., a bilinear map $T : C \times C \to C$, which gives C the structure of an associative algebra with left identity α_1 and right identity β_1. (The algebra is associative because matrix multiplication is associative.)

If T satisfies the A-Strassen equations then it is isomorphic to a partially symmetric tensor, see Proposition 8.1, and the associated algebra is abelian. Conversely, given such an algebra, one obtains its structure tensor.

Explicitly, let $\mathcal{I} \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal whose zero set in affine space is finite, more precisely so that $\mathcal{A}_{\mathcal{I}} := \mathbb{C}[x_1, \ldots, x_n]/\mathcal{I}$ is a finite dimensional algebra of dimension m. (This will be the case, e.g., if the zero set consists of m distinct points each counted with multiplicity one.) Let $\{p_I\}$ be a basis of $\mathcal{A}_{\mathcal{I}}$ with dual basis $\{p_I^*\}$ We can write the structure tensor of $\mathcal{A}_{\mathcal{I}}$ as

$$T_{\mathcal{A}_{\mathcal{I}}} = \sum_{p_I, p_J \in \mathcal{A}_{\mathcal{I}}} p_I^* \otimes p_J^* \otimes (p_I p_J \bmod \mathcal{I}).$$

Then [10] shows that a binding tensor T that is the structure tensor of a smoothable algebra is of minimal border rank, i.e., the tensor $M_{\langle 1\rangle}^{\oplus m}$ degenerates to T, where $M_{\langle 1\rangle}^{\oplus m}$ is the tensor whose associated algebra $\mathcal{A}_{M_{\langle 1\rangle}^{\oplus m}}$ comes from the ideal of m distinct points. The key step is showing that in this situation $T \in \overline{GL(A) \times GL(B) \times GL(C) M_{\langle 1\rangle}^{\oplus m}}$ if and only if (using the above identifications) $T \in \overline{GL(C) M_{\langle 1\rangle}^{\oplus m}}$.

Thus one may utilize deformation theory on the Hilbert scheme of points to determine if a binding tensor satisfying the A-Strassen equations has minimal border rank. In particular, such tensors automatically are of minimal border rank when $m \leq 7$ [14].

7.2. 1-Generic tensors: Gorenstein algebras

Now say T is 1_A, 1_B, and 1_C generic (such tensors are called 1-*generic*) and satisfies the A-Strassen equations. We have $\gamma_1 \in C^*$ such that $T(\gamma_1) \in \mathrm{End}(C)$ is invertible. What extra structure do we obtain?

Recall that an algebra $\mathcal{A}$ is *Gorenstein* if there exists $f \in \mathcal{A}^*$ such that any of the following equivalent conditions holds:

1) $T_{\mathcal{A}}(f) \in \mathcal{A}^* \otimes \mathcal{A}^*$ is of full rank,

2) the pairing $\mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$ given by $(a, b) \mapsto f(ab)$ is non-degenerate,

3) $\mathcal{A}f = \mathcal{A}^*$.

Thus $f = \gamma_1$ above tells us $\mathcal{A}_T$ is Gorenstein by (1). By the second assertion in Proposition 8.1, T is moreover symmetric.

In particular, such T is of minimal border rank when $m \leq 13$ [15]. For an algorithm that resolves the $m = 14$ case, see [27].

The additional algebraic structure of being Gorenstein makes the deformation theory easier to implement.

EXAMPLE 7.1. Consider $\mathcal{A} = \mathbb{C}[x]/(x^2)$, with basis $1, x$, so

$$T_{\mathcal{A}} = 1^*\otimes 1^*\otimes 1 + x^*\otimes 1^*\otimes x + 1^*\otimes x^*\otimes x.$$

Writing $e_0 = 1^*$, $e_1 = x^*$ in the first two factors and $e_0 = x$, $e_1 = 1$ in the third,

$$T_{\mathcal{A}} = e_0\otimes e_0\otimes e_1 + e_1\otimes e_0\otimes e_0 + e_0\otimes e_1\otimes e_0$$

That is, $T_{\mathcal{A}} = T_{WState}$ is a general tangent vector to $Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$.

EXAMPLE 7.2 (The big Coppersmith-Winograd tensor). Consider the algebra

$$\mathcal{A}_{CW,q} = \mathbb{C}[x_1,\dots,x_q]/(x_ix_j, x_i^2 - x_j^2, x_i^3,\ i\neq j)$$

Let $\{1, x_i, [x_1^2]\}$ be a basis of $\mathcal{A}$, where $[x_1^2] = [x_j^2]$ for all j. Then

$$\begin{aligned} T_{\mathcal{A}_{CW,q}} =& 1^*\otimes 1^*\otimes 1 + \sum_{i=1}^{q}(1^*\otimes x_i^*\otimes x_i + x_i^*\otimes 1^*\otimes x_i + x_i^*\otimes x_i^*\otimes [x_1^2]) \\ &+ 1^*\otimes [x_1^2]^*\otimes [x_1^2] + [x_1^2]^*\otimes 1^*\otimes [x_1^2]. \end{aligned}$$

Set $e_0 = 1^*$, $e_i = x_i^*$, $e_{q+1} = [x_1^2]^*$ in the first two factors and $e_0 = [x_1^2]$, $e_i = x_i$, $e_{q+1} = 1$ in the third to obtain

$$\begin{aligned} T_{\mathcal{A}_{CW,q}} =& T_{CW,q} = e_0\otimes e_0\otimes e_{q+1} + \sum_{i=1}^{q}(e_0\otimes e_i\otimes e_i + e_i\otimes e_0\otimes e_i + e_i\otimes e_i\otimes e_0) \\ &+ e_0\otimes e_{q+1}\otimes e_0 + e_{q+1}\otimes e_0\otimes e_0, \end{aligned}$$

which is the usual expression for the big Coppersmith-Winograd tensor.

7.3. 1_*-generic tensors: modules and the ADHM correspondence

When $\dim A = \dim B = \dim C = m$, one says T is 1_*-*generic* if it is 1_A or 1_B or 1_C generic.

Consider the case of a tensor that is 1_A-generic but not binding. What structure can we associate to it? Fixing α_1 as above we obtain $T\in A\otimes C^*\otimes C$, i.e., $T(A^*)T(\alpha_1)^{-1}\subset \mathrm{End}(C)$, and if Strassen's equations are satisfied, we have an abelian subspace, and if furthermore the End-closed condition holds, we may think of this space as defining an algebra action on $\mathrm{End}(C)$, which we may lift to an action of the polynomial ring $S := \mathbb{C}[y_2,\dots,y_m]$ by $y_s(c) := T(\alpha_s)T(\alpha_1)^{-1}c$. (The choice of indices is deliberate, as $T(\alpha_1)T(\alpha_1)^{-1} = \mathrm{Id}_C$ corresponds to $1\in S$.)

That is, the vector space C becomes a module over the polynomial ring. This association is called the *ADHM correspondence* in [33], after [5]. This leads one to deformation theory in the Quot scheme that parametrizes such modules.

This correspondence allowed Jelisiejew, Pal and myself [32] to characterize concise 1_*-generic tensors of border rank ≤ 6 as the zero set of Strassen's equations and the End-closed equations, and also as the zero set of the 111-equations. Strassen's equations, the 111-equations and the End-closed equations fail to characterize minimal border rank tensors when $m \geq 7$.

7.4. Concise tensors: the 111-algebra and its modules

Now say we just have a concise tensor. Previously there had not been any algebraic structure available for studying such tensors. Moreover, as remarked above, both Strassen's equations and the End-closed equations are trivially satisfied for such tensors when the three associated spaces of homomorphisms are of rank bounded above by $m-2$. Despite this, the 111-equations still give strong restrictions in these cases. I now explain that they also allow the implementation of deformation theory even in this situation.

For $X \in \mathrm{End}(A) = A^*\otimes A$, let $X \circ_A T$ denote the corresponding element of $T(A^*)\otimes A$. Explicitly, if $X = \alpha\otimes a$, then $X \circ_A T := T(\alpha)\otimes a$ and the map $(-)\circ_A T\colon \mathrm{End}(A) \to A\otimes B\otimes C$ is extended linearly.

Definition 7.3 ([32, Def. 1.9]). *Let T be a concise tensor. We say that a triple $(X,Y,Z) \in \mathrm{End}(A)\times\mathrm{End}(B)\times\mathrm{End}(C)$* is compatible with *$T$ if $X\circ_A T = Y\circ_B T = Z\circ_C T$. The* 111-algebra *of T is the set of triples compatible with T. We denote this set by $\mathcal{A}^T_{111}$.*

Thus a compatible triple gives a point in the triple intersection (6). The name 111-*algebra* is justified by the following theorem:

Theorem 7.4 ([32, Thm. 1.10]). *The 111-algebra of a concise tensor $T \in A\otimes B\otimes C$ is a commutative unital subalgebra of* $\mathrm{End}(A)\times\mathrm{End}(B)\times\mathrm{End}(C)$ *and its projection to any factor is injective.*

Using the 111-algebra, one obtains four consecutive obstructions for a concise tensor to be of minimal border rank [32]:

1. $\dim(\mathcal{A}^T_{111}) \geq m$. For the next three conditions, assume equality holds.
2. $\mathcal{A}^T_{111}$ must be smoothable.
3. Using the 111-algebra, A, B, C become modules for it and the polynomial ring S. These three S-modules, $\underline{A}, \underline{B}, \underline{C}$ (where the underline is there to emphasize their module structure) must lie in the principal component of the Quot scheme.

4. There exists a surjective module homomorphism $\underline{A}\otimes_{\mathcal{A}^T_{111}}\underline{B}\to\underline{C}$ associated to T and this homomorphism must be a limit of module homomorphisms $\underline{A}_\epsilon\otimes_{\mathcal{A}_\epsilon}\underline{B}_\epsilon\to\underline{C}_\epsilon$ for a choice of smooth algebras $\mathcal{A}_\epsilon$ and semisimple modules $\underline{A}_\epsilon$, $\underline{B}_\epsilon$, $\underline{C}_\epsilon$.

8. New proofs of existing results

In this section I present two significantly simpler proofs than the original that binding tensors satisfying Strassen's equations are partially symmetric and the original, more elementary proof that binding tensors satisfying Strassen's equations automatically satisfy the End-closed condition. These proofs were obtained in conversations with J. Jelisiejew and M. Michałek.

Let $A,B,C\simeq\mathbb{C}^m$ and let $T\in A\otimes B\otimes C$ be 1_A-generic. Say $\operatorname{rank}(T_A(\alpha_0))=m$.

Note the tautological identities: $T(\alpha,\beta)=T_A(\alpha)\beta=T_B(\beta)\alpha$.

The A-Strassen equations for minimal border rank say that for all $\alpha_1,\alpha_2\in A$,

$$T_A(\alpha_1)T_A(\alpha_0)^{-1}T_A(\alpha_2)=T_A(\alpha_2)T_A(\alpha_0)^{-1}T_A(\alpha_1).$$

PROPOSITION 8.1 ([39]). *Let T be 1_A and 1_B-generic and satisfy the A-Strassen equations. Then T is isomorphic to a tensor in $S^2C^*\otimes C$. If T is 1-generic then it is isomorphic to a symmetric tensor.*

Here are two proofs:

Proof. Assume $T(\alpha_0)\in B\otimes C$ and $T(\beta_0)\in A\otimes C$ are of full rank. Define $\tilde{T}\in C^*\otimes C^*\otimes C$ by $\tilde{T}(c_1,c_2):=T(T_B(\beta_0)^{-1}c_1,T_A(\alpha_0)^{-1}c_2)$.

Set $\alpha_1=T_B(\beta_0)^{-1}c_1$, $\alpha_2=T_B(\beta_0)^{-1}c_2$ so

$$\begin{aligned}
\tilde{T}(c_1,c_2)&=T(\alpha_1,T_A(\alpha_0)^{-1}T_B(\beta_0)\alpha_2) && \text{definition}\\
&=T(\alpha_1,T_A(\alpha_0)^{-1}T_A(\alpha_2)\beta_0) && \text{taut.id.}\\
&=T_A(\alpha_1)T_A(\alpha_0)^{-1}T_A(\alpha_2)\beta_0 && \text{taut.id.}\\
&=T_A(\alpha_2)T_A(\alpha_0)^{-1}T_A(\alpha_1)\beta_0 && \text{Strassen}\\
&=\tilde{T}(c_2,c_1) && \text{taut.id.}
\end{aligned}$$

The second assertion follows as $\mathfrak{S}_3$ is generated by the transpositions $(1,2)$ and $(1,3)$. □

Proof. Under the hypotheses $T_A^{\wedge 1}:A\otimes B^*\to\Lambda^2A\otimes C$ has rank m^2-m and the 1_B-genericity condition assures that the m-dimensional kernel contains an element of full rank, $\psi:A\to B$, which makes $(\psi\otimes\operatorname{Id}_{B\otimes C})(T)\in S^2B^*\otimes C$. The second assertion follows similarly. □

Note that Proposition 8.1 implies the B-Strassen equations are satisfied as well.

The following proposition appeared in [32] with a less elementary proof. Below is the original proof.

PROPOSITION 8.2. *If T is 1_A and 1_B generic and satisfies the A-Strassen equations, then $T(A^*)T(\alpha_0)^{-1} \subset \mathrm{End}(C)$ satisfies the End-closed condition.*

Proof. We need to show that for all α_1, α_2, that, there exists α' such that

$$T_A(\alpha_1)T_A(\alpha_0)^{-1}T_A(\alpha_2)T_A(\alpha_0)^{-1} = T_A(\alpha')T_A(\alpha_0)^{-1}.$$

It is sufficient to work with $\tilde{T} \in S^2C^* \otimes C$. Here, by symmetry $\tilde{T}_A(c) = \tilde{T}_B(c) =: \tilde{T}_{C^*}(c)$. We claim $\tilde{T}_{C^*}(c_1)\tilde{T}_{C^*}(c_2) = \tilde{T}_{C^*}(\tilde{T}(c_1, c_2))$. This will finish the proof as $c_1, c_2, \tilde{T}(c_1, c_2) \in C \cong A^*$ play the role of $\alpha_1, \alpha_2, \alpha'$ above.

To see this

$$\begin{aligned}
\tilde{T}_{C^*}(c_1)\tilde{T}_{C^*}(c_2)(c) &= \tilde{T}_{C^*}(c_1)\tilde{T}(c_2, c) && \text{taut.}\\
&= T_{C^*}(c_1)\tilde{T}(c, c_2)) && \text{sym.}\\
&= \tilde{T}_{C^*}(c_1)\tilde{T}_{C^*}(c)(c_2) && \text{taut.}\\
&= \tilde{T}_{C^*}(c)\tilde{T}_{C^*}(c_1)(c_2) && \text{Strassen}\\
&= \tilde{T}(c, \tilde{T}(c_1, c_2)) && \text{taut.}\\
&= \tilde{T}(\tilde{T}(c_1, c_2), c) && \text{sym.}\\
&= \tilde{T}_{C^*}(\tilde{T}(c_1, c_2))(c) && \text{taut.}
\end{aligned}$$

□

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Author's address:

J. M. Landsberg
Department of Mathematics
Texas A&M University
College Station, TX 77843-3368, USA
E-mail: jml@math.tamu.edu

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