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# Local cohomology bounds and the weak implies strong conjecture in dimension 4

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A B S T R A C T

We verify the weak implies strong conjecture for 4-dimensional finitely generated algebras over a field of prime characteristic $p > 5$ which has infinite transcendence degree over $\mathbb{F}_p$.



## 1. Introduction

Let $(R, \mathfrak{m}, k)$ be an excellent commutative Noetherian local ring of prime characteristic $p > 0$. Tight closure theory, introduced and developed by Hochster and Huneke in [11–16], is a subject central to prime characteristic commutative algebra, important to

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our understanding of characteristic 0 singularities, and a guiding force in the development of mixed characteristic singularities.

Let $N \subseteq M$ be $R$-modules and let $N^*_M$ denote the tight closure of $N$ inside of $M$, see [12] for a precise definition of tight closure. The ring $R$ is *weakly $F$-regular* if $N^*_M = N$ for all finitely generated modules $N \subseteq M$ and $R$ is *strongly $F$-regular* if $N^*_M = N$ for all $R$-modules $N \subseteq M$. The weak implies strong conjecture asserts that every weakly $F$-regular ring is strongly $F$-regular. The most notable cases that the weak implies strong conjecture has been proven for are for standard graded algebras over a field, [26], and all rings of Krull dimension at most 3, [36].

Outside of the graded scenario, developments around the weak implies strong conjecture depend upon the behavior of the anticanonical algebra of $R$. An unpublished result of Singh asserts that the weak implies strong conjecture holds for the class of rings whose anticanonical algebra is Noetherian,[2] see [6, Corollary 5.9]. Williams' proof of the weak implies strong conjecture in dimension 3 implicitly utilizes that the anticanonical algebra of a 3-dimensional weakly $F$-regular ring is Noetherian on the punctured spectrum, an assertion that relies on the algebraic-geometric methods of [25] and [31].

Our methods show that every 4-dimensional weakly $F$-regular ring is strongly $F$-regular, provided its anticanonical algebra is Noetherian when localized at a non-maximal prime ideal. Therefore a further relationship between tight closure and prime characteristic birational geometry is desired to progress the weak implies strong conjecture. This is indeed the scenario in 4-dimensional rings. Recent developments in the prime characteristic minimal model program, [8], allow us to conclude that all divisorial blowups rings, including the anticanonical algebra, are Noetherian away from maximal ideals for large classes of 4-dimensional weakly $F$-regular rings.

**Theorem A.** *Let $R$ be a 4-dimensional finitely generated algebra over a field of prime characteristic $p > 5$ which has infinite transcendence degree over $\mathbb{F}_p$. If $R$ is weakly $F$-regular then $R$ is strongly $F$-regular.*

The weak implies strong conjecture will be understood as a problem of understanding when an element of a non-injective direct limit system is mapped to 0 inside the direct limit. To this end, we introduce the notion of a *local cohomology bound* in Section 2. If $M$ is an $R$-module and $\underline{x} = x_1, \ldots, x_\ell$ a sequence of elements then we may identify the local cohomology module $H^i_{(\underline{x})}(M)$ as a direct limit of Koszul cohomologies

$$H^i_{(\underline{x})}(M) \cong \varinjlim_{t_1 \leq t_2} \left( H^i(x_1^{t_1}, \ldots, x_\ell^{t_1}; M) \xrightarrow{\alpha^i_{t_1,t_2}} H^i(x_1^{t_2}, \ldots, x_\ell^{t_2}; M) \right).$$

The $i$th local cohomology bound of $M$ with respect to the sequence $\underline{x}$ is bounded by an integer $k$ if for every $t$, if $\eta \in H^i(x_1^t, \ldots, x_\ell^t; M)$ represents the 0-element of $H^i_{(\underline{x})}(M)$

[2] It is conjectured that any divisorial blowup ring, including the anticanonical algebra, of any strongly $F$-regular ring is Noetherian.

then $\alpha^i_{t,t+k}(\eta) = 0$. Understanding when a module has bounded local cohomology bounds with respect to a sequence of elements is an interesting, challenging, and worth-while venture.

## 2. Local cohomology bounds

We do not present the basic theory of local cohomology bounds in full generality. We only present specific aspects needed in later sections. The interested reader should consult [2] for a thorough introduction to the theory of local cohomology bounds.

### *2.1. Definition of local cohomology bound*

Suppose $M$ is a module over a ring $R$ and $\underline{y} = y_2, y_3, y_4$[3] a sequence of elements. Then for each integer $t \in \mathbb{N}$ we let $\underline{y}^t = y_2^t, y_3^t, y_4^t$ and for each pair of integers $t_1 \leq t_2$ let $\tilde{\alpha}^{\bullet}_{M;\underline{y};t_1;t_2}$ denote the natural map of Koszul cocomplexes

$$K^{\bullet}(\underline{y}^{t_1}; M) \xrightarrow{\tilde{\alpha}^{\bullet}_{M;\underline{y};t_1;t_2}} K^{\bullet}(\underline{y}^{t_2}; M).$$

More specifically, $\tilde{\alpha}^{\bullet}_{M;\underline{y};t_1;t_2}$ is the following map of Koszul cocomplexes:

$$\begin{tikzcd}[column sep=large, row sep=huge]
K^{\bullet}(\underline{y}^{t_1}; M): \arrow[d, "\tilde{\alpha}^{\bullet}_{M;\underline{y};t_1;t_2}"] & 0 \arrow[r] & M \arrow[r, "\begin{bmatrix} y_4^{t_1} \\ y_3^{t_1} \\ y_2^{t_1} \end{bmatrix}"] \arrow[d, "\tilde{\alpha}^{0}_{M;\underline{y};t_1;t_2}"] & M^{\oplus 3} \arrow[r, "\begin{bmatrix} y_3^{t_1} & -y_4^{t_1} & 0 \\ -y_2^{t_1} & 0 & y_4^{t_1} \\ 0 & y_2^{t_1} & -y_3^{t_1} \end{bmatrix}"] \arrow[d, "\tilde{\alpha}^{1}_{M;\underline{y};t_1;t_2}"] & M^{\oplus 3} \arrow[r, "\begin{bmatrix} y_2^{t_1} \\ y_3^{t_1} \\ y_4^{t_1} \end{bmatrix}"] \arrow[d, "\tilde{\alpha}^{2}_{M;\underline{y};t_1;t_2}"] & M \arrow[r] \arrow[d, "\tilde{\alpha}^{3}_{M;\underline{y};t_1;t_2}"] & 0 \\
K^{\bullet}(\underline{y}^{t_2}; M): & 0 \arrow[r] & M \arrow[r, swap, "\begin{bmatrix} y_4^{t_2} \\ y_3^{t_2} \\ y_2^{t_2} \end{bmatrix}"] & M^{\oplus 3} \arrow[r, swap, "\begin{bmatrix} y_3^{t_2} & -y_4^{t_2} & 0 \\ -y_2^{t_2} & 0 & y_4^{t_2} \\ 0 & y_2^{t_2} & -y_3^{t_2} \end{bmatrix}"] & M^{\oplus 3} \arrow[r, swap, "\begin{bmatrix} y_2^{t_1} \\ y_3^{t_1} \\ y_4^{t_1} \end{bmatrix}"] & M \arrow[r] & 0
\end{tikzcd} \tag{2.1}$$

Where

- $\tilde{\alpha}^{0}_{M;\underline{y};t_1;t_2} = \mathrm{id}_M$;
- $\tilde{\alpha}^{1}_{M;\underline{y};t_1;t_2} = \begin{bmatrix} y_4^{t_2-t_1} & 0 & 0 \\ 0 & y_3^{t_2-t_1} & 0 \\ 0 & 0 & y_2^{t_2-t_1} \end{bmatrix}$
- $\tilde{\alpha}^{2}_{M;\underline{y};t_1;t_2} = \begin{bmatrix} (y_3y_4)^{t_2-t_1} & 0 & 0 \\ 0 & (y_2y_4)^{t_2-t_1} & 0 \\ 0 & 0 & (y_2y_3)^{t_2-t_1} \end{bmatrix}$
- $\tilde{\alpha}^{3}_{M;\underline{y};t_1;t_2} = \cdot(y_2y_3y_4)^{t_2-t_1}$

[3] We begin the sequence at $y_2$ instead of $y_1$ for ease of referencing this material in Section 3 and Section 4.

We let $\alpha^j_{M;\underline{y};t_1;t_2}$ denote the induced map of Koszul cohomologies

$$H^j(\underline{y}^{t_1};M) \xrightarrow{\alpha^j_{M;\underline{y};t_1;t_2}} H^j(\underline{y}^{t_2};M).$$

In particular,

$$\varinjlim_{t_1\leq t_2}\left(H^j(\underline{y}^{t_1};M) \xrightarrow{\alpha^j_{M;\underline{y};t_1;t_2}} H^j(\underline{y}^{t_2};M)\right) \cong H^j_{(\underline{y})}(M)$$

by [3, Theorem 3.5.6].

For each $0 \leq j \leq 3$ let $\alpha^j_{M;\underline{y};t;\infty}$ be the natural map

$$H^j(\underline{y}^{t};M) \xrightarrow{\alpha^j_{M;\underline{y};t;\infty}} H^j_{(\underline{y})}(M).$$

An element $\eta \in H^j(\underline{y}^t;M)$ belongs to $\mathrm{Ker}(\alpha^j_{M;\underline{y};t;\infty})$ if and only if there exists some $k \geq 0$ so that $\eta \in \mathrm{Ker}(\alpha^j_{M;\underline{y};t;t+k})$. If $\eta \in \mathrm{Ker}(\alpha^j_{M;\underline{y};t;\infty})$ we let

$$\epsilon^j_{\underline{y},t}(\eta) = \min\{k \mid \eta \in \mathrm{Ker}(\alpha^j_{M;\underline{y};t;t+k})\}.$$

**Definition 2.1.** Let $R$ be a ring, $\underline{y} = y_2, y_3, y_4$ a sequence of elements in $R$, and $M$ an $R$-module. The $j$th local cohomology bound of $M$ with respect to the sequence of elements $\underline{y}$ is

$$\mathrm{lcb}_j(\underline{y};M) = \sup\{\epsilon^j_{\underline{y},t}(\eta) \mid \eta \in \mathrm{Ker}(\alpha^j_{M;\underline{y};t;\infty}) \text{ for some } t\} \in \mathbb{N}\cup\{\infty\}.$$

Observe that if $M$ is an $R$-module and $\underline{y} = y_2, y_3, y_4$ a sequence of elements, then $\mathrm{lcb}_j(\underline{y};M) \leq N < \infty$ simply implies that if $\eta \in H^j(\underline{y}^t;M)$ represents the 0-element in the direct limit

$$\varinjlim_{t_1\leq t_2}\left(H^j(\underline{y}^{t_1};M) \xrightarrow{\alpha^j_{M;\underline{y};t_1;t_2}} H^j(\underline{y}^{t_2};M)\right) \cong H^j_{(\underline{y})}(M)$$

then $\alpha^j_{M;\underline{y};t;t+N}(\eta)$ is the 0-element of the Koszul cohomology group $H^j(\underline{y}^{t+N};M)$. Therefore finite local cohomology bounds correspond to a uniform bound of annihilation of zero elements in a choice of direct limit system defining a local cohomology module.

*2.2. Some basic properties of local cohomology bounds*

**Lemma 2.2.** *Let $R$ be a commutative Noetherian ring, $M$ an $R$-module, and $\underline{y} = y_2, y_3, y_4$ a sequence of elements, then $\mathrm{lcb}_j(\underline{y}^t;M) \leq \mathrm{lcb}_j(\underline{y};M)$. Furthermore, $\mathrm{lcb}_j(\underline{y};M) \leq tm$ for some integers $t, m$ if and only if $\mathrm{lcb}_j(\underline{y}^t;M) \leq m$.*

**Proof.** One only has to observe that $\alpha^j_{M;\underline{y}^t;k,k+m} = \alpha^j_{M;\underline{y};tk,tk+tm}$. □

**Proposition 2.3.** *Let $R$ be a commutative Noetherian ring and $M, N$ modules over $R$. Suppose $\underline{y} = y_2, y_3, y_4$ is a sequence of elements so that $(y_2, y_3, y_4)M = 0$ and $(y_2, y_3)N = 0$. Then*

*(1) $\alpha^j_{M;\underline{y};t,t+k} = 0$ for all $t$, $k \geq 1$, and $1 \leq j \leq 3$;*
*(2) $\alpha^j_{N;\underline{y};t,t+k} = 0$ for all $t$, $k \geq 1$, and $2 \leq j \leq 3$.*

*In particular, $\mathrm{lcb}_j(\underline{y}; M) \leq 1$ for $1 \leq j \leq 3$ and $\mathrm{lcb}_j(\underline{y}; N) \leq 1$ for $2 \leq j \leq 3$.*

**Proof.** Recall that $\alpha^j_{M;\underline{y};t,t+k}$ is the map of Koszul cohomologies induced from the map $\tilde{\alpha}^\bullet_{M;\underline{y};t,t+k}$ on Koszul cocomplexes. One can consult the diagram of (2.1) to observe that $\tilde{\alpha}^j_{M;\underline{y};t,t+k}$ is the 0-map for all $t$, $k \geq 1$, and $j \geq 1$. The second assertion follows by an identical argument. □

**Proposition 2.4.** *Let $(R, \mathfrak{m}, k)$ be a local ring and*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

*a short exact sequence of finitely generated $R$-modules. Let $\underline{y} = y_2, y_3, y_4$ a sequence of elements of $R$. If $(y_2, y_3)M_3 = 0$ then*

$$\mathrm{lcb}_3(y_2, y_3, y_4; M_1) \leq \mathrm{lcb}_3(y_2, y_3, y_4; M_2) + 1.$$

**Proof.** Consider the following commutative diagram, whose middle row is exact:

$$\begin{tikzcd}
 & H^3(\underline{y}^t; M_1) \arrow[r] \arrow[d, "\alpha^3_{M_1;\underline{y};t;t+k}"] & H^3(\underline{y}^t; M_2) \arrow[d, "\alpha^3_{M_2;\underline{y};t;t+k}"] \\
H^2(\underline{y}^{t+k}; M_3) \arrow[r] \arrow[d, "\alpha^2_{M_3;\underline{y};t+k;t+k+1}"] & H^3(\underline{y}^{t+k}; M_1) \arrow[r] \arrow[d, "\alpha^3_{M_1;\underline{y};t+k;t+k+1}"] & H^3(\underline{y}^{t+k}; M_2) \\
H^2(\underline{y}^{t+k+1}; M_3) \arrow[r] & H^3(\underline{y}^{t+k+1}; M_1) &
\end{tikzcd}$$

By Proposition 2.3 the map $\alpha^2_{M_3;\underline{y};t+k;t+k+1}$ is the 0-map. A straightforward diagram chase of the above diagram, which follows an element $\eta \in \mathrm{Ker}(\alpha^3_{M_1;\underline{y};t;\infty})$, shows that $\eta \in \mathrm{Ker}(\alpha^3_{M_1;\underline{y};t;t+k+1})$ whenever $k \geq \mathrm{lcb}_3(\underline{y}; M_2)$. In particular, $\mathrm{lcb}_3(\underline{y}, M_1) \leq \mathrm{lcb}_3(\underline{y}; M_2) + 1$. □

**Proposition 2.5.** *Let $R$ be a commutative Noetherian ring, $0 \to M_1 \to M_2 \to M_3 \to 0$ a short exact sequence of $R$-modules, and $\underline{y} = y_2, y_3, y_4$ a sequence of elements in $R$.*

*(1) If $\underline{y}$ is a regular sequence on $M_2$ then* $\mathrm{lcb}_j(\underline{y}; M_3) = \mathrm{lcb}_{j+1}(\underline{y}; M_1)$ *for all* $0 \leq j \leq 2$.
*(2) If $\underline{y}$ is a regular sequence on $M_3$ then* $\mathrm{lcb}_j(\underline{y}; M_1) = \mathrm{lcb}_j(\underline{y}; M_2)$ *for all* $0 \leq j \leq 3$.

**Proof.** The proofs of (1) and (2) are similar and for the sake of brevity we only provide the proof of (1). The sequence $\underline{y}$ is a regular sequence on $M_2$ and so $H^j(\underline{y}^t; M_2) = 0$ whenever $j \leq 2$. Therefore if $0 \leq j \leq 1$ there are commutative diagrams

$$\begin{tikzcd}
H^j(\underline{y}^t; M_3) \arrow[r, "\cong"] \arrow[d, "\alpha^{j+1}_{M_3;\underline{y};t;t+k}"] & H^{j+1}(\underline{y}^t; M_1) \arrow[d, "\alpha^{j}_{M_1;\underline{y};t;t+k}"] \\
H^j(\underline{y}^{t+k}; M_3) \arrow[r, "\cong"] & H^{j+1}(\underline{y}^{t+k}; M_1)
\end{tikzcd}$$

whose horizontal arrows are isomorphisms. It easily follows that $\mathrm{lcb}_j(\underline{y}; M_3) = \mathrm{lcb}_{j+1}(\underline{y}; M_1)$ whenever $0 \leq j \leq 1$. To verify that $\mathrm{lcb}_2(\underline{y}; M_3) = \mathrm{lcb}_3(\underline{y}; M_1)$ consider the following commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] & H^2(\underline{y}^t; M_3) \arrow[r, "\delta_t"] \arrow[d, "\alpha^{2}_{M_3;\underline{y};t;t+k}"] & H^3(\underline{y}^t; M_1) \arrow[r, "i_t"] \arrow[d, "\alpha^{3}_{M_1;\underline{y};t;t+k}"] & H^3(\underline{y}^t; M_2) \arrow[d, "\alpha^{3}_{M_2;\underline{y};t;t+k}"] \\
0 \arrow[r] & H^2(\underline{y}^{t+k}; M_3) \arrow[r, "\delta_{t+k}"] & H^3(\underline{y}^{t+k}; M_1) \arrow[r, "i_{t+k}"] & H^3(\underline{y}^{t+k}; M_2)
\end{tikzcd}$$

A simple diagram chase and utilizing the injectivity of the maps $\delta_t, \delta_{t+k}$, and $\alpha^3_{M_2;\underline{y};t,t+k}$ imply that $\mathrm{lcb}_2(\underline{y}; M_3) = \mathrm{lcb}_3(\underline{y}; M_1)$. □

## 3. Annihilation of Ext-modules and bounded local cohomology bounds

We are focused on the weak implies strong conjecture and every local weakly $F$-regular ring is a normal Cohen-Macaulay domain, [12, Theorem 4.9 and Lemma 5.9]. Therefore we assume throughout this section, and next, that $(R, \mathfrak{m}, k)$ is a normal Cohen-Macaulay domain. We further assume that $R$ is the homomorphic image of a regular local ring $S$. We write $R \cong S/I$ and $h$ will always denote the height of $I$, equivalently the codimension of $R$. In particular, the dimension of $S$ is $h + 4$.

### 3.1. Annihilation of Ext-modules

Our first proposition, Proposition 3.1, is fundamental toward providing linearly bounded local cohomology bounds needed to prove Theorem A.

**Proposition 3.1.** *Let $(R, \mathfrak{m}, k)$ be an excellent local normal Cohen-Macaulay domain of Krull dimension 4. Let $K \subseteq R$ be an ideal of pure height 1 of $R$ so that*

*(1) The inclusion of ideals $K^i \subseteq K^{(i)}$ is an equality on the punctured spectrum of $R$;*

*(2) The analytic spread of $KR_P$ is no more than $\mathrm{ht}(P) - 1$ for all non-maximal prime ideals $P$ of height at least 2;*

*(3) For each non-maximal prime ideal $P$ of $\mathrm{Spec}(R)$ the ideal $KR_P$ has reduction number 1 with respect to any reduction.*

*Then there exists an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ so that*

$$\mathfrak{a}^i \operatorname{Ext}_S^{h+3}(R/K^{(i)}, S) = 0$$

*for all $i \in \mathbb{N}$.*

**Proof.** Consider the short exact sequences

$$0 \to \frac{K^{(i)}}{K^i} \to \frac{R}{K^i} \to \frac{R}{K^{(i)}} \to 0.$$

The inclusion of ideals $K^i \subseteq K^{(i)}$ are an equality on the punctured spectrum of $R$ and hence

$$\operatorname{Ext}_S^{h+3}(R/K^{(i)}, S) \cong \operatorname{Ext}_S^{h+3}(R/K^i, S).$$

Choose an element $x_1 \in K$. The principal locus of $K$ is an open subset of $\mathrm{Spec}(R)$ and $K$ localizes to a principal ideal at all height 2 primes of $R$ by assumption. Indeed, an ideal of a normal domain has analytic spread 1 if and only if the ideal is principal. In particular, we can choose a parameter sequence $x_2, x_3$ of $R/x_1R$ so that $KR_{x_2}$ and $KR_{x_3}$ are principal ideals in their respective localizations. Even further, we replace $x_2$ and $x_3$ by $x_2^t$ and $x_3^t$ respectively and may assume that $x_2K \subseteq r_2R$ and $x_3K \subseteq r_3R$ for some elements $r_2, r_3 \in K$. Consider the short exact sequences

$$0 \to \frac{K^i}{(r_2^i)} \to \frac{R}{(r_2^i)} \to \frac{R}{K^i} \to 0$$

and

$$0 \to \frac{K^i}{(r_3^i)} \to \frac{R}{(r_3^i)} \to \frac{R}{K^i} \to 0.$$

Observe that $x_2^i$ annihilates $K^i/(r_2^i)$ and $x_3^i$ annihilates $K^i/(r_3^i)$. Therefore

$$\operatorname{Ext}_S^{h+3}(R/K^i, S) \cong \operatorname{Ext}_S^{h+2}(K^i/(r_2^i), S) \cong \operatorname{Ext}_S^{h+2}(K^i/(r_3^i), S)$$

and we find that the ideal $(x_1^i, x_2^i, x_3^i)$ annihilates $\operatorname{Ext}_S^{h+3}(R/K^i, S)$. To complete the proof of the theorem we aim to find a parameter element $x_4$ of $R/(x_1, x_2, x_3)$ so that $x_4^i$ annihilates $\operatorname{Ext}_S^{h+3}(R/K^i, S)$ for all $i$.

Let $\Lambda = \{P_1, \dots, P_m\}$ be the prime components of the height 3 parameter ideal $(x_1, x_2, x_3)$. If necessary, enlarge $\Lambda$ so that every component of $K$ is contained in some prime ideal of $\Lambda$ and let $W = R \setminus \bigcup_{P\in\Lambda} P$.

**Claim 3.2.** *There exist elements $a, b \in K$ such that*

*(1) $(a, b)R_W$ forms a reduction of $KR_W$;*
*(2) the element $a$ generates $K$ at its components;*
*(3) if $K'$ is the unique ideal of pure height 1 whose components are disjoint from $K$ and is such that $(a) = K \cap K'$ then $b$ avoids all components of $K'$.*

**Proof of Claim 3.2.** We are assuming the ideal $K$ has analytic spread at most 2 at each of the localizations $R_P$ as $P$ varies among the prime ideals in $\Lambda$. So for each $1 \le i \le m$ there exists $a_i, b_i \in K$ such that $(a_i, b_i)R_{P_i}$ forms a reduction of $KR_{P_i}$. For each $1 \le i \le m$ choose $r_i \in \bigcap_{P\in\Lambda-\{P_i\}} P - P_i$ and set $a' = \sum r_i a_i$ and $b' = \sum r_i b_i$. We claim $(a', b')R_{W_\ell}$ is a reduction of $KR_\ell$. By [18, Proposition 8.1.1] it is enough to check $(a', b')$ forms a reduction of $K$ at each of the localizations $R_{P_i}$ for $1 \le i \le m$. By [18, Proposition 8.2.4] it is enough to check that the fiber cone $R_P/PR_P \otimes R[Kt] \cong \bigoplus K^n R_{P_i}/P_i K^n R_{P_i}$ is finite over the subalgebra spanned by $((a', b')R_{P_i}, P_i K)/P_i K$. But $a' \equiv r_i a_i \mod P_i K$, $b' \equiv r_i b_i \mod P_i K$, $r_i$ is a unit of $R_{P_i}$, and therefore $(a', b')R_{W_\ell}$ does indeed form a reduction of $KR_{W_\ell}$ by a second application of [18, Proposition 8.2.4].

Now consider the set of primes $\Gamma = \{Q_1, \dots, Q_n\}$ which are the minimal components of $K$. The purpose of enlarging the set of height $\ell$ primes in the statement of the claim was to insure that each $Q_j \in \Gamma$ is a prime ideal of the localization $R_{W_\ell}$. In particular, $(a', b')R_{Q_i}$ forms a reduction of $KR_{Q_i}$ for each $1 \le i \le n$. But $R_{Q_i}$ is a discrete valuation ring and therefore for each $1 \le i \le \ell$ either $KR_{Q_i} = (a')R_{Q_i}$ or $KR_{Q_i} = (b')R_{Q_i}$. Without loss of generality we assume that $KR_{Q_i} = (a')R_{Q_i}$ for at least one value of $i$ and relabel the primes in $\Gamma$ so that $KR_{Q_i} = (a')R_{Q_i}$ for each $1 \le i \le j$ and $KR_{Q_i} \ne (a')R_{Q_i}$ for each $j + 1 \le i \le n$. Choose $r \in Q_1 \cap \cdots \cap Q_j - \bigcup_{i=j+1}^{n} Q_i$ and consider the element $a' + rb'$. We claim that $a' + rb'$ generates $KR_{Q_i}$ for each $1 \le i \le n$. First consider a localization at a prime $Q_i \in \Gamma$ with $1 \le i \le j$. Then $(a', b')R_{Q_i} = (a')R_{Q_i}$ by assumption and so $(b')R_{Q_i} \subseteq (a')R_{Q_i}$. Because $r \in Q_i$ there is a strict containment of principal ideal $(rb')R_{Q_i} \subsetneq (a')R_{Q_i}$ and it follows that $(a')R_{Q_i} = (a' + rb')R_{Q_i}$. Now consider a localization $R_{Q_i}$ with $j + 1 \le i \le n$. We are assuming that $a'$ does not generate $KR_{Q_i}$ and therefore $(a')R_{Q_i} \subsetneq (b')R_{Q_i} = KR_{Q_i}$. Moreover, $r$ is a unit of $R_{Q_i}$ and therefore $(b')R_{Q_i} = (a' + rb')R_{Q_i}$.

Let $a = a' + rb'$. Then $(a, b')R_{W_\ell} = (a', b')R_{W_\ell}$ forms a reduction of $KR_{W_\ell}$ and the element $a$ generates $K$ at each of its minimal components as desired. Suppose as an ideal of $R$ the principal ideal $(a)$ has decomposition $(a) = K \cap K' \cap K''$ so that

(1) $K, K', K''$ are pure height 1 ideals whose components are disjoint from one another;
(2) the components of $K'$ are height 1 prime ideals which do not contain $b$;

(3) the components of $K''$ are height 1 prime ideals which do contain $b$.

We take $K'$ or $K''$ to be $R$ if no such components of $(a)$ exist. If $K'' = R$ then we let $b = b'$ and the elements $a, b$ satisfy the conclusions of the claim. If $K'' \neq R$ then first observe that, because $(a, b')R_{W_\ell}$ forms a reduction of $KR_{W_\ell}$ and $a, b' \in K''$, we must have that $(a)R_{W_\ell} = (K \cap K')R_{W_\ell}$. Choose an element $r \in K \cap K'$ which avoids all components in $K''$ and consider the element $b = b' + r$. Then $(a, b)R_{W_\ell} = (a, b')R_{W_\ell}$ forms a reduction of $KR_{W_\ell}$. Moreover, the element $b$ avoids all minimal components of $K'$ and $K''$ by construction. □

We are assuming that $KR_W$ has reduction number 1 with respect to the reduction $(a, b)R_W$ provided above, i.e. $(a, b)KR_W = K^2 R_W$. The following $R$-modules localize to 0 over $R_W$:

(1) $K^2/(a, b)K$;
(2) $\mathrm{Ext}_S^{h+3}(R/(a, b)K, S)$;
(3) $\mathrm{Ext}_S^{h+3}(R/K, S)$.

Therefore there exists a parameter element $x_4$ of $R/(x_1, x_2, x_3)$ so that

(1) $x_4 K^2 \subseteq (a, b)K$;
(2) $x_4 \mathrm{Ext}_S^{h+3}(R/(a, b)K, S) = 0$;
(3) $x_4 \mathrm{Ext}_S^{h+3}(R/K, S) = 0$.

Even further, as $x_4 K^2 \subseteq (a, b)K$, observe that $x_4^{i-1} K^i \subseteq (a, b)^{i-1} K$ for all $i \in \mathbb{N}$. Consider the following short exact sequence:

$$0 \to \frac{K^i}{(a, b)^{i-1} K} \to \frac{R}{(a, b)^{i-1} K} \to \frac{R}{K^i} \to 0.$$

The left most term is annihilated by $x_4^{i-1}$. Thus, in order to show $x_4^i$ annihilates $\mathrm{Ext}_S^{h+3}(R/K^i, S)$ it suffices to show that $x_4 \in \bigcap_{i \in \mathbb{N}} \mathrm{Ann}_R(\mathrm{Ext}_S^{h+3}(R/(a, b)^{i-1} K, S))$. To this end we present a claim:

**Claim 3.3.** *For every integer $i$ there is a short exact sequence*

$$0 \to \frac{R}{ab^i R} \to \frac{R}{a(a, b)^{i-1} K} \oplus \frac{R}{b^i K} \to \frac{R}{(a, b)^i K} \to 0.$$

**Proof of Claim 3.3.** For any ideals $I, J$ there is a short exact sequence

$$0 \to \frac{R}{I \cap J} \to \frac{R}{I} \oplus \frac{R}{J} \to \frac{R}{I + J} \to 0.$$

Thus to prove the claim we need only to observe $a(a,b)^{i-1}K \cap b^iK = ab^iR$. Clearly $ab^i \in a(a,b)^{i-1}K \cap b^iKR$. On the other hand, an element of $a(a,b)^{i-1}K \cap b^iK$ is of the form $b^ir$ where $r \in K$ and $b^ir \in a(a,b)^{i-1}K$. To show that $b^ir \in ab^iR$ we must show that $r \in aR = K \cap K'$. The element $r \in K$ by assumption. Localizing at a component $P$ of $K'$, a component which does not contain $b$ by design, we find that $rb^i \in a(a,b)^{i-1}R_P = aR_P$ and thus $r \in aR$ as desired. □

Claim 3.3 provides to us isomorphisms

$$\operatorname{Ext}_S^{h+3}(R/(a,b)^iK, S) \cong \operatorname{Ext}_S^{h+3}(R/a(a,b)^{i-1}K, S) \oplus \operatorname{Ext}_S^{h+3}(R/b^iK, S). \quad (3.1)$$

There are short exact sequences

$$0 \to \frac{R}{(a,b)^{i-1}K} \xrightarrow{\cdot a} \frac{R}{a(a,b)^{i-1}K} \to \frac{R}{aR} \to 0$$

and

$$0 \to \frac{R}{K} \xrightarrow{\cdot b^i} \frac{R}{b^iK} \to \frac{R}{b^iR} \to 0.$$

Therefore the isomorphisms of (3.1) can be further rewritten as

$$\operatorname{Ext}_S^{h+3}(R/(a,b)^iK, S) \cong \operatorname{Ext}_S^{h+3}(R/(a,b)^{i-1}K, S) \oplus \operatorname{Ext}_S^{h+3}(R/K, S).$$

Inductively, we find that

$$\operatorname{Ext}_S^{h+3}(R/(a,b)^iK, S) \cong \bigoplus^{i} \operatorname{Ext}_S^{h+3}(R/K, S)$$

and we conclude that $x_4$ annihilates $\operatorname{Ext}_S^{h+3}(R/(a,b)^iK, S)$ as desired. □

Suppose that $I \subseteq R$ is an unmixed ideal. The Rees algebra of $I$ is the standard graded $R$-algebra $R[It] = \bigoplus_{N\geq 0} I^N$, the associated graded ring of $I$ is $\operatorname{Gr}_I(R) = R[It] \otimes_R R/I$, and the symbolic Rees algebra of $I$ is $\mathcal{R}_I = \bigoplus_{N\geq 0} I^{(N)}$. The inclusion of $\mathbb{N}$-graded $R$-algebras $R[It] \subseteq \mathcal{R}_I$ is an equality if and only if $\bigcup_{N\in\mathbb{N}} \operatorname{Ass}(R/I^N)$ agrees with the set of minimal prime ideals of $I$. The $R$-algebra $\mathcal{R}_I$ is Noetherian if and only if there exists an $m$ so that $\mathcal{R}_{I^{(m)}}$ is standard graded, i.e. $I^{(m)N} = I^{(mN)}$ for all $N$.

Suppose that $\mathcal{R}_I$ is Noetherian and $m$ is chosen so that $\mathcal{R}_{I^{(m)}}$ is standard graded. Let $\ell(I^{(m)}) = \dim \mathcal{R}_{I^{(m)}} \otimes_R R/\mathfrak{m}$, the analytic spread of $I^{(m)}$. If $m'$ is another integer so that $\mathcal{R}_{I^{(m')}}$ is standard graded then $I^{(m)m'} = I^{(m')m} = I^{(mm')}$ and so $\ell(I^{(m)}) = \ell(I^{(mm')}) = \ell(I^{(m)})$. We say that *the associated graded rings of $\mathcal{R}_I$ have negative a-invariants* if the graded local cohomology modules $H^{\ell}_{\mathcal{R}_I^+}(\operatorname{Gr}_{I^{(m)}}(R))$ are supported only in negative degrees where $\ell = \ell(I^{(m)})$ is the analytic spread of a choice of symbolic power of $I$ for which $R[I^{(m)}t] = \mathcal{R}_{I^{(m)}}$.

Suppose that $R$ is a normal domain, $J_1 \subset R$ is a choice of canonical ideal, and $x_1 \in J_1$ is a choice of generic generator. Then we can write $(x_1) = J_1 \cap K_1$ where $K_1$ is an ideal of pure height 1 whose components are disjoint from $J_1$. Then $K_1$ is an *anticanonical ideal* of $R$ and we refer to the symbolic Rees algebra $\mathcal{R}_{K_1}$ as the *anticanonical algebra of* $R$.

**Theorem 3.4.** *Let $(R, \mathfrak{m}, k)$ be an excellent local normal Cohen-Macaulay domain of Krull dimension 4. Suppose that the anticanonical algebra of $R$ is Noetherian on the punctured spectrum of $R$ so that its associated graded rings have negative $a$-invariant. Let $J_1 \subseteq R$ be a choice of canonical ideal of $R$. Then there exists an integer $m \in \mathbb{N}$ and $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq R$ so that $J_1^{(m)}$ is principal in codimension 2 and*

$$\mathfrak{a}^i \operatorname{Ext}_S^{h+2}(\operatorname{Ext}_S^{h+1}(R/J_1^{mi+1}, S), S) = 0$$

*for every integer $i \in \mathbb{N}$.*

**Proof.** Choose a generic generator $x_1 \in J_1$ and write $(x_1) = J_1 \cap K_1$ where $K_1$ is an anticanonical ideal of $R$ whose components are disjoint from the components of $J_1$. A theorem of Brodmann, [4], asserts that $\Gamma := \bigcup^\infty \operatorname{Ass}(R/K_1^n)$ is a finite set, cf. [19]. Let $P_1, \ldots, P_t \in \Gamma$ be the finitely many non-maximal primes of $\Gamma$ which are not of height 1. If $P$ is a non-maximal ideal of $R$ not belonging to $\{P_1, \ldots, P_t\}$ then $K_1^i R_P = K_1^{(i)} R_P$ for all $i$. If $P_j \in \{P_1, \ldots, P_t\}$ then our assumption that the anticanonical algebra is Noetherian on the punctured spectrum implies that there exists an integer $m_j$ so that $K_1^{(m_j)i} R_P = K_1^{(m_j i)} R_P$ and that the analytic spread of $K_1^{(m_j i)} R_P$ does not exceed $\operatorname{ht}(P) - 1$, see [24] and [7, Theorem 1.5].

Let $m$ be a common multiple of $m_1, \ldots, m_t$. Then the inclusion of ideals $K_1^{(m)i} \subseteq K_1^{(mi)}$ becomes an equality when localized at any non-maximal prime ideal of $R$. The ideal $K_1^{(m)}$ is principal in codimension 2 since its analytic spread in codimension 2 is 1. As $J_1^{(m)}$ is the inverse element of $K_1^{(m)}$ when viewed as elements of the divisor class group of $R$, $J_1^{(m)}$ is principal in codimension 2 as well. Even further, because we are assuming the $a$-invariant of the associated graded ring of the anticanonical algebra of $R_P$ is negative for each non-maximal prime ideal $R$, we can replace $m$ by a multiple of itself and assume that the ideal $K_1^{(m)} R_{P_j}$ has reduction number 1 with respect to any reduction, see [17, Theorem 2.1]. Let $K = K_1^{(m)}$ and $x = x_1^m$.

**Claim 3.5.** *For each integer $i$*

$$\operatorname{Ext}_S^{h+2}(\operatorname{Ext}_S^{h+1}(R/J_1^{mi+1}, S), S) \cong \operatorname{Ext}_S^{h+3}(R/K^{(i)}, S).$$

**Proof of Claim 3.5.** For each integer $i$ there is short exact sequence

$$0 \to \frac{J_1^{mi+1}}{x_1^{mi+1} J_1} \to \frac{R}{x_1^{mi+1} J_1} \to \frac{R}{J_1^{mi+1}} \to 0.$$

The ideal $x_1^{mi+1}J_1$ is a canonical ideal of $R$. Therefore

$$\operatorname{Ext}_S^{h+1}(R/x_1^{mi+1}, S) \cong R/x_1^{mi+1}J_1$$

and there are left exact sequences

$$0 \to \operatorname{Ext}_S^{h+1}(R/J_1^{mi+1}, S) \to \frac{R}{x_1^{mi+1}J_1} \to \operatorname{Ext}_S^{h+1}(J_1^{mi+1}/x_1^{mi+1}J_1, S).$$

Therefore $\operatorname{Ext}_S^{h+1}(R/J_1^{mi+1}, S) \cong L_i/x_1^{mi+1}J_1$ for some ideal $L_i \subseteq R$. Moreover, $R/L_i \subseteq \operatorname{Ext}_S^{h+1}(J_1^{mi+1}/x_1^{mi+1}J_1, S)$. Because $\operatorname{Ext}_S^{h+1}(J_1^{mi+1}/x_1^{mi+1}J_1, S)$ is an $(S_2)$-module over its support it follows that $R/L_i$ is an $(S_1)$-module over its support. Hence $L_i$, as an ideal of $R$, is unmixed of height 1. Moreover, every component of $L_i$ is a component of $x_1R$. Localizing at a component of $J_1$ we see that $L_i$ agrees with $x_1R$ and localizing at a component of $K_1$ we see that $L_i$ agrees with $x_1^{mi+1}$. Therefore $L_i$ agrees with the unmixed ideal $x_1K_1^{(mi)}$ and so

$$\operatorname{Ext}_S^{h+1}(R/J_1^{mi+1}, S) \cong x_1K_1^{(mi)}/x_1^{mi+1}J_1.$$

If we divide by $x_1$ we find that

$$x_1K_1^{(mi)}/x_1^{mi+1}J_1 \cong K_1^{(mi)}/x_1^{mi}J_1 = K^{(i)}/x^iJ_1.$$

Now we consider the short exact sequences

$$0 \to K^{(i)}/x^iJ_1 \to R/x^iJ_1 \to R/K^{(i)} \to 0.$$

The cyclic $R$-module $R/x^iJ_1$ is Cohen-Macaulay of dimension 3 and therefore

$$\operatorname{Ext}_S^{h+2}(K^{(i)}/x^iJ_1, S) \cong \operatorname{Ext}_S^{h+3}(R/K^{(i)}, S). \quad \square$$

To prove the theorem we aim to find an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ so that

$$\mathfrak{a}^i \operatorname{Ext}_S^{h+3}(R/K^{(i)}, S) = 0$$

for all $i$. Such an annihilation property is the content of Proposition 3.1. $\square$

*3.2. Existence of bounded local cohomology bounds*

Let $I \subseteq R$ be an ideal of pure height 1 and $(F_\bullet, \partial_\bullet)$ be an $S$-free resolution of $R/I$ and $G_\bullet$ an $S$-free resolution of $\omega_{R/I} \cong \operatorname{Ext}_S^{h+1}(R/I, S)$. Let $(-)^* = \operatorname{Hom}_S(-, S)$. Then $H^i(F_\bullet^*) = \operatorname{Ext}_S^i(R/I, S) = 0$ for all $i \leq h$ and if we let $\tilde{F}_\bullet^*$ be the truncation of $F_\bullet^*$ at the $h+1$st spot then $\tilde{F}_\bullet^*$ resolves $\operatorname{Coker}(\partial_{h+1}^*)$. There is a natural inclusion of $S$-modules

$\mathrm{Ext}_S^{h+1}(R/I, S) \subseteq \mathrm{Coker}(\partial_{h+1}^*)$. This inclusion produces a map of complex to cocomplex $G_\bullet \to \tilde{F}_\bullet^*$ and thus a natural map $R/I \to \mathrm{Ext}_S^{h+1}(\mathrm{Ext}_S^{h+1}(R/I, S), S)$ obtained from applying $\mathrm{Hom}_S(-, S)$ to $G_\bullet \to \tilde{F}_\bullet^*$. The map $R/I \to \mathrm{Ext}_S^{h+1}(\mathrm{Ext}_S^{h+1}(R/I, S), S)$ is injective and is an isomorphism in the Cohen-Macaulay locus of $R/I$.

**Theorem 3.6.** *Let $(R, \mathfrak{m}, k)$ be an excellent local normal Cohen-Macaulay domain of Krull dimension 4. Suppose that the anticanonical algebra of $R$ is Noetherian on the punctured spectrum of $R$ so that its associated graded rings have negative $a$-invariant. Let $J_1 \subseteq R$ be a choice of canonical ideal of $R$ and $\mathfrak{a}$ as in Theorem 3.4. There exists an integer $m$ and $x_1 \in J_1$, such that if $x_2, x_3, x_4 \in \mathfrak{a}$ are parameters on $R/x_1R$ chosen so that*

*(1) $x_2 J_1 \subseteq a_2 R$ for some $a_2 \in J_1$;*
*(2) $x_3 J_1^{(m)} \subseteq a_3 R$ for some $a_3 \in J_1^{(m)}$.*

*Then for each natural number $i$ there exists an integer $\ell$ such that*

$$\mathrm{lcb}_3(x_2^\ell, x_3^\ell, x_4; R/J_1^{(mi+1)}) \leq i + 1.$$

**Proof.** Let $m$ and $\mathfrak{a}$ be as in Theorem 3.4. Because $J_1$ is principal in codimension 1 and $J_1^{(m)}$ is principal in codimension 2 we can choose a parameter sequence $x_2, x_3, x_4 \in \mathfrak{a}$ on $R/x_1R$ so that

(1) $x_2 J_1 \subseteq a_2 R$ for some $a_2 \in J_1$;
(2) $x_3 J_1^{(m)} \subseteq a_3 R$ for some $a_3 \in J_1^{(m)}$.

For each integer $i$ there is a short exact sequence of the form

$$0 \to R/J_1^{(mi+1)} \to \mathrm{Ext}_S^{h+1}(\mathrm{Ext}_S^{h+1}(R/J_1^{(mi+1)}, S), S) \to C_i \to 0.$$

Inverting the element $x_2$ or $x_3$ has the effect of making the ideal $J_1$ principal. Therefore the first map in the above short exact sequence is an isomorphism whenever $x_2$ or $x_3$ is inverted and so there exists an integer $\ell$ so that $x_2^\ell, x_3^\ell$ annihilates $C_i$. By (3) of Proposition 2.4 we have that

$$\mathrm{lcb}_3(x_2^\ell, x_3^\ell, x_4; R/J_1^{(mi+1)}) \leq \mathrm{lcb}_3(x_2^\ell, x_3^\ell, x_4; \mathrm{Ext}_S^{h+1}(\mathrm{Ext}_S^{h+1}(R/J_1^{(mi+1)}, S), S)) + 1.$$

Our aim is to show $\mathrm{lcb}_3(x_2^\ell, x_3^\ell, x_4; \mathrm{Ext}_S^{h+1}(\mathrm{Ext}_S^{h+1}(R/J_1^{(mi+1)}, S), S)) \leq i$. By Lemma 2.2 it suffices to prove $\mathrm{lcb}_3(x_2^{\ell i}, x_3^{\ell i}, x_4^i; \mathrm{Ext}_S^{h+1}(\mathrm{Ext}_S^{h+1}(R/J_1^{(mi+1)}, S), S)) \leq 1$.

Let $(F_\bullet, \partial_\bullet)$ be the minimal free $S$-resolution of $\mathrm{Ext}_S^{h+1}(R/J^{(mi+1)}, S)$ and let $(-)^* = \mathrm{Hom}_R(-, S)$. The module $\mathrm{Ext}_S^{h+1}(R/J^{(mi+1)}, S)$ has depth at least 2 and so $F_{h+3} = F_{h+4} = 0$. It follows that there are short exact sequences

$$0 \to \mathrm{Im}(\partial_{h+2}^*) \to F_{h+2}^* \to \mathrm{Ext}_S^{h+2}(\mathrm{Ext}_S^{h+1}(R/J^{mi+1}, S), S) \to 0$$

and

$$0 \to \operatorname{Ext}_S^{h+1}(\operatorname{Ext}_S^{h+1}(R/J^{mi+1}, S), S) \to \operatorname{Coker}(\partial_{h+1}^*) \to \operatorname{Im}(\partial_{h+2}^*) \to 0.$$

The $S$-module $\operatorname{Coker}(\partial_{h+1}^*)$ has projective dimension $h+1$ and is annihilated by the height $h+1$ ideal $J^{mi+1}$. By a simple prime avoidance argument we may lift $x_2, x_3, x_4$ to elements of $S$ so that $x_2, x_3, x_4$ forms a regular sequence on $\operatorname{Coker}(\partial_{h+1}^*)$ and the free $S$-module $F_{h+2}^*$. By two applications of (1) of Proposition 2.5 applied to the two above short exact sequences

$$\begin{aligned}
&\operatorname{lcb}_3(x_2^{\ell i}, x_3^{\ell i}, x_4^i; \operatorname{Ext}_S^{h+1}(\operatorname{Ext}_S^{h+1}(R/J_1^{mi+1}, S), S)) \\
&= \operatorname{lcb}_2(x_2^{\ell i}, x_3^{\ell i}, x_4^i; \operatorname{Im}(\partial_{h+2}^*)) \\
&= \operatorname{lcb}_1(x_2^{\ell i}, x_3^{\ell i}, x_4^i; \operatorname{Ext}_S^{h+2}(\operatorname{Ext}_S^{h+1}(R/J_1^{mi+1}, S), S)).
\end{aligned}$$

By Theorem 3.4 we have that

$$(x_2^{\ell i}, x_3^{\ell i}, x_4^i) \operatorname{Ext}_S^{h+2}(\operatorname{Ext}_S^{h+1}(R/J_1^{mi+1}, S), S) = 0.$$

Therefore

$$\operatorname{lcb}_1(x_2^{\ell i}, x_3^{\ell i}, x_4^i; \operatorname{Ext}_S^{h+2}(\operatorname{Ext}_S^{h+1}(R/J_1^{mi+1}, S), S)) = 1$$

by (1) of Proposition 2.3. $\square$

## 4. Equality of test ideals

If $N \subseteq M$ are $R$-modules then the *finitistic tight closure of $N$ inside $M$* is the union of $(N \cap M')^*_{M'}$ where $M' \subseteq M$ runs through all finitely generated submodules of $M$. Let $E_R(k)$ be the injective hull of the residue field of $(R, \mathfrak{m}, k)$. Then $R$ is strongly $F$-regular if and only if $0 = 0^*_{E_R(k)}$, [12, Proposition 8.23] and $R$ is weakly $F$-regular if and only if $0^{*,fg}_{E_R(k)} = 0$, [30, Proposition 7.1.2].

With the exception of Lemma 4.3 and Corollary 4.5, we continue to assume that $(R, \mathfrak{m}, k)$ is an excellent normal Cohen-Macaulay domain of Krull dimension 4 and is the homomorphic image of a regular local ring $S$ so that the results of Section 3 are applicable. We fix the characteristic of $R$ to be of prime characteristic $p > 0$.

The following lemma is inspired by the methodology of Williams and MacCrimmon, [36,27]. The lemma is well-known by experts, can be pieced together by work of the first author in [1], and we refer the reader to [29, Lemma 6.7] for a more general statement.

**Lemma 4.1.** *Suppose that $(R, \mathfrak{m}, k)$ is a local normal Cohen-Macaulay domain of prime characteristic $p > 0$, of Krull dimension 4, and $J \subseteq R$ an ideal of pure height 1. Let $y_1 \in J$ and $y_2, y_3, y_4$ parameters on $R/y_1R$ and fix $e \in \mathbb{N}$.*

(1) *If $y_2 J \subseteq aR$ for some $a \in J$, then for any integers $N_2, N_3, N_4$ with $N_2 \geq 2$, we have that*

$$\begin{aligned}&((J^{(p^e)}, y_2^{N_2p^e}, y_3^{N_3p^e}, y_4^{N_4p^e}) : y_2^{(N_2-1)p^e})\\ &= ((J^{[p^e]}, y_2^{N_2p^e}, y_3^{N_3p^e}, y_4^{N_4p^e}) : y_2^{(N_2-1)p^e})\\ &= ((J^{[p^e]}, y_2^{2p^e}, y_3^{N_3p^e}, y_d^{N_4p^e}) : y_2^{p^e}).\end{aligned}$$

(2) *Suppose $y_3 J^{(m)} \subseteq bR$ for some $b \in J^{(m)}$, then for any non-negative integers $N_2, N_3, N_4$ with $N_3 \geq 2$, we have that*

$$\begin{aligned}&((J^{(p^e)}, y_2^{N_2p^e}, y_3^{N_3p^e}, y_4^{N_4p^e}) : y_3^{(N_3-1)p^e})\\ &\subseteq ((J^{(p^e)}, y_2^{N_2p^e}, y_3^{2p^e}, y_4^{N_dp^e}) : y_1^m y_3^{p^e}).\end{aligned}$$

**Theorem 4.2.** *Let $(R, \mathfrak{m}, k)$ be an excellent local normal Cohen-Macaulay domain of prime characteristic $p > 0$ and of Krull dimension 4. Suppose that the anticanonical algebra of $R$ is Noetherian on the punctured spectrum of $R$ so that its associated graded rings have negative $a$-invariant. Then*

$$0^{*,fg}_{E_R(k)} = 0^*_{E_R(k)}.$$

*In particular, if $R$ is weakly $F$-regular then $R$ is strongly $F$-regular.*

**Proof.** Let $J_1 \subset R$ be a canonical ideal of $R$ and let $x_1, x_2, x_3, x_4$ and $m$ be as in the statement of Theorem 3.6. Identify the injective hull $E_R(k)$ as

$$E_R(k) = \varinjlim \left( \frac{R}{(x_1^{t-1}J_1, x_2^t, x_3^t, x_4^t)} \xrightarrow{\cdot x_1x_2x_3x_4} \frac{R}{(x_1^tJ_1, x_2^{t+1}, x_3^{t+1}, x_4^{t+1})} \right).$$

Suppose that $\eta = [r + (x_1^{t-1}J_1, x_2^t, x_3^t, x_4^t)] \in 0^*_{E_R(k)}$. Equivalently, if $c \in R$ is a test element then for all $e \in \mathbb{N}$ there exists an integer $s$ so that

$$cr^{p^e}(x_1x_2x_3x_4)^{sp^e} \in (x_1^{t+s-1}J_1, x_2^{t+s}, x_3^{t+s}, x_4^{t+s})^{[p^e]}.$$

The element $x_1$ is regular on $R/(x_2, x_3, x_4)$ and therefore

$$cr^{p^e}(x_2x_3x_4)^{sp^e} \in (x_1^{t-1}J_1, x_2^{t+s}, x_3^{t+s}, x_4^{t+s})^{[p^e]} = (J, x_2^{t+s}, x_3^{t+s}, x_4^{t+s})^{[p^e]},$$

where $J = x_1^{t-1}J_1$. Multiplying by $(x_2x_3x_4)^{p^e(ts-(t+s))}$ we find that

$$cr^{p^e}(x_2^tx_3^tx_4^t)^{(s-1)p^e} \in (J, x_2^{ts}, x_3^{ts}, x_4^{ts})^{[p^e]}.$$

Let $y_2, y_3, y_4$ denote the parameter sequence $x_2^t, x_3^t, x_4^t$ so that

$$cr^{p^e}(y_2y_3y_4)^{(s-1)p^e} \in (J, y_2^s, y_3^s, y_4^s)^{[p^e]}.$$

Observe that $p^e \geq m(\lfloor \frac{p^e}{m} \rfloor - 1) + 1$ and hence $J^{[p^e]} \subseteq J^{(p^e)} \subseteq J^{(mi+1)}$ where we set $i = \lfloor \frac{p^e}{m} \rfloor - 1$. Therefore

$$cr^{p^e}(y_2y_3y_4)^{(s-1)p^e} \in (J^{(mi+1)}, y_2^{sp^e}, y_3^{sp^e}, y_4^{sp^e}). \tag{4.1}$$

Let $\ell$ be the integer depending on $i$ described in Theorem 3.6. Theorem 3.6 and Lemma 2.2 tell us that for each integer $i$ that there exists an integer $\ell$ so that

$$\mathrm{lcb}_3(y_2^\ell, y_3^\ell, y_4; R/J_1^{(mi+1)}) \leq i + 1.$$

Because $J = x_1^{t-1}J_1$ we have that for each integer $i$ there is a short exact sequence

$$0 \to \frac{R}{J_1^{(mi+1)}} \xrightarrow{\cdot x_1^{(t-1)(mi+1)}} \frac{R}{J^{(mi+1)}} \to \frac{R}{x_1^{(t-1)(mi+1)}R} \to 0.$$

The sequence $y_2^\ell, y_3^\ell, y_4$ is a regular sequence on $R/x_1^{(t-1)(mi+1)}R$. By (2) of Proposition 2.5 we have that

$$\mathrm{lcb}_3(y_2^\ell, y_3^\ell, y_4; R/J_1^{(mi+1)}) = \mathrm{lcb}_3(y_2^\ell, y_3^\ell, y_4; R/x_1^{t-1}J_1^{(mi+1)}) \leq i + 1. \tag{4.2}$$

We multiply the containment (4.1) by $(y_2y_3)^{(\ell-1)sp^e}$ and notice that

$$\begin{aligned} cr^{p^e}(y_2y_3y_4)^{(s-1)p^e}(y_2y_3)^{(\ell-1)sp^e} &= cr^{p^e}(y_2y_3)^{(\ell-1)p^e}(y_2^\ell y_3^\ell y_4)^{(s-1)p^e} \\ &\in (J^{(mi+1)}, y_2^{\ell sp^e}, y_3^{\ell sp^e}, y_4^{sp^e}). \end{aligned} \tag{4.3}$$

Consider the element

$$\zeta = [cr^{p^e}(y_2y_3)^{(\ell-1)p^e} + (y_2^{\ell p^e}, y_3^{\ell p^e}, y_4^{p^e})]$$

of the top Koszul cohomology group

$$H^3(y_2^{\ell p^e}, y_3^{\ell p^e}, y_4^{p^e}; R/J^{(mi+1)}).$$

Using the notation of Section 2, the containment of (4.3) is equivalent to the assertion that

$$\alpha_{R/J^{(mi+1)}; y_2^\ell, y_3^\ell, y_4; p^e; sp^e}(\zeta) = 0 \in H^3(y_2^{\ell sp^e}, y_2^{\ell sp^e}, y_4^{sp^e}; R/J_1^{(mi+1)}).$$

By (4.2) we have that

$$\mathrm{lcb}_3(y_2^\ell, y_3^\ell, y_4; R/J^{(mi+1)}) \leq i + 1 \leq p^e - 1 + 1 = p^e$$

and therefore

$$\alpha_{R/J^{(mi+1)};y_2^\ell,y_3^\ell,y_4;p^e;2p^e}(\zeta)=0\in H^3(y_2^{\ell 2p^e},y_2^{\ell 2p^e},y_4^{2p^e};R/J_1^{(mi+1)}).$$

Equivalently, the element

$$cr^{p^e}(y_2y_3)^{(\ell-1)p^e}(y_2^\ell y_3^\ell y_4)^{p^e}=c(ry_4)^{p^e}y_2^{(2\ell-1)}y_3^{(2\ell-1)p^e}\in(J^{(mi+1)},y_2^{2\ell p^e},y_3^{2\ell p^e},y_4^{2p^e}).$$

Recall that $i=\lfloor\frac{p^e}{m}\rfloor-1$ and so $m\lfloor\frac{p^e}{m}\rfloor\geq m(\frac{p^e}{m}-1)=p^e-m$. Hence $mi+1\geq p^e-(2m-1)$ and so

$$c(ry_4)^{p^e}y_2^{(2\ell-1)p^e}y_3^{(2\ell-1)p^e}\in(J^{(p^e-(2m-1))},y_2^{2\ell p^e},y_3^{2\ell p^e},y_4^{2p^e}).$$

Pick a nonzero element $z\in J^{(2m-1)}$. Then

$$zc(ry_4)^{p^e}y_2^{(2\ell-1)p^e}y_3^{(2\ell-1)p^e}\in(J^{(p^e)},y_2^{2\ell p^e},y_3^{2\ell p^e},y_4^{2p^e}).$$

We want to utilize Lemma 4.1 to simplify the above containment. Recall that $J=x_1^{t-1}J_1$, $y_2=x_2^t$, $y_3=x_3^t$, $x_2J_1\subseteq a_2R$ for some $a_2\in J_1$, and $x_3J_1^{(m)}\subseteq a_3R$ for some $a_3\in J_1^{(m)}$. Then $y_2J\subseteq x_1^{t-1}a_2R$, $x_1^{t-1}a_2\in J$, $y_3J^{(m)}\subseteq x_1^{(t-1)m}a_3R$, and $x_1^{(t-1)m}a_3\in J^{(m)}$. Moreover, $y_1=x_1^t\in J$ and $y_2,y_3,y_4$ is a parameter sequence on $R/y_1R$. Therefore we can apply (2) of Lemma 4.1 and conclude that

$$y_1^mzc(ry_3y_4)^{p^e}y_2^{(2\ell-1)p^e}\in(J^{(p^e)},y_2^{2\ell p^e},y_3^{2p^e},y_4^{2p^e}).$$

By (1) of Lemma 4.1 we are then able to assert that

$$y_1^mzc(ry_2y_3y_4)^{p^e}\in(J^{[p^e]},y_2^{2p^e},y_3^{2p^e},y_4^{2p^e}).$$

The element $y_1^mzc$ does not depend on $e$. Therefore

$$ry_2y_3y_4\in(J,y_2^2,y_3^2,y_4^2)^*$$

and hence, as an element of $E_R(k)$, $\eta=[ry_2y_3y_4+(J,y_2^2,y_3^2,y_4^2)]$ belongs to $0^{*,fg}_{E_R(k)}$. □

To utilize Theorem 4.2 and prove Theorem A we must observe that the $a$-invariant of the associated graded rings of the anticanonical algebra is negative whenever the ambient ring is strongly $F$-regular and the anticanonical algebra is Noetherian.

**Lemma 4.3.** *Let $(R,\mathfrak{m},k)$ be an excellent strongly $F$-regular ring of prime characteristic $p>0$ and Krull dimension $d\geq 2$. Suppose that $I\subseteq R$ is an ideal of pure height 1 such that $I^N=I^{(N)}$ for all $N$. Then the associated graded ring of $I$ has negative $a$-invariant.*

**Proof.** If $\ell(I) = 1$ then $I$ is principal and $H^1_{R[It]^+}(\mathrm{Gr}_I(R)) = 0$. So we may assume that $2 \leq \ell(I) \leq d$. We first observe that $a_i(R[It]) < 0$ for all $2 \leq i \leq d$. Because $I^N = I^{(N)}$ for all $N$ we have that $S := R[It]$ is a strongly $F$-regular graded $R$-algebra by [6, Lemma 3.1], see also [35, Theorem 0.1] and [28, Main Theorem]. The cohomology groups $H^i_{S_+}(S)$ are only supported in finitely many positive degrees. Indeed, let $X = \mathrm{Proj}(S)$ so that $H^i_{S_+}(S) \cong H^{i-1}(X, \mathcal{O}_X)$ for all $i \geq 2$, see [22, Theorem 12.41], and therefore $[H^i_{S_+}(S)]_N = H^{i-1}(X, \mathcal{O}_X(N)) = 0$ for all $N \gg 0$ by Serre vanishing, [9, Theorem 5.2]. It follows that there exists a homogeneous positive degree element $c \in S$ such that $c[H^i_{S_+}(S)]_{\geq 0} = 0$. Because $S$ is strongly $F$-regular the $S$-linear maps $S \xrightarrow{\cdot F^e_* c} F^e_* S$ are pure for all $e \gg 0$. Therefore the $e$th Frobenius action on $H^i_{S_+}(S)$ followed by multiplying by $c$, which is the map realized by tensoring the pure map $S \xrightarrow{\cdot F^e_* c} F^e_* S$ with $H^i_{S_+}(S)$, are injective. But the $e$th Frobenius action of $H^i_{S_+}(S)$ maps elements of degree $n$ to elements of degree $np^e$. Furthermore, $c$ was chosen to annihilate elements of non-negative degree and therefore $H^i_{S_+}(S)$ can only be supported in negative degree.

The ring $S = R[It]$ is Cohen-Macaulay and therefore $a_d(\mathrm{gr}_I(R)) < 0$ by [17, Theorem 3.1]. By [33, Theorem 3.1 (ii)] we have that $a_i(\mathrm{gr}_I(R)) = a_i(S)$ whenever $a_i(\mathrm{gr}_I(R)) \geq a_{i+1}(\mathrm{gr}_I(R))$. An easy descending induction argument now tells us that $a_i(\mathrm{gr}_I(R)) < 0$ for all $2 \leq i \leq d$ and this completes the proof of the theorem. □

**Corollary 4.4.** *Let $R$ be an excellent 4-dimensional weakly $F$-regular ring of prime characteristic $p > 0$. If the anticanonical algebra of $R$ is Noetherian on the punctured spectrum of $R$ then $R$ is strongly $F$-regular.*

**Proof.** It is well known that the properties of being weakly $F$-regular and strongly $F$-regular can be checked at localizations at the maximal ideals of $R$, see [12, Corollary 4.15]. Thus we may assume $R = (R, \mathfrak{m}, k)$ is local. The properties of weakly $F$-regular and strongly $F$-regular for a local ring can be checked after completion. In which case, the property of being weakly $F$-regular is equivalent to $0^{*,fg}_{E_R(k)}$ being 0 and the property of being strongly $F$-regular is equivalent is $0^*_{E_R(k)}$ being 0.

Using gamma constructions with respect to a choice of coefficient field, we may assume $R$ is $F$-finite, see [15, Section 6 and Theorem 7.24] and [10, Corollary 3.31]. Every complete local weakly $F$-regular ring is a normal Cohen-Macaulay domain by [12, Lemma 5.9 and Theorem 4.9]. Every weakly $F$-regular ring is a splinter, [16, Corollary 5.23]. The property of being a splinter localizes. Therefore $R$ is strongly $F$-regular on the punctured spectrum of $R$ by [5, Corollary 5.9]. Lemma 4.3 tells us the anticanonical algebra of $R$ is such that its associated graded ring has negative $a$-invariant on the punctured spectrum of $R$ and therefore $R$ is strongly $F$-regular by Theorem 4.2. □

**Corollary 4.5.** *Let $R$ be a 4-dimensional normal Cohen-Macaulay domain of prime characteristic $p$. Suppose that either*

*(1) $R$ is finitely generated over a field of prime characteristic $p > 5$ with infinite transcendence degree over $\mathbb{F}_p$ and is weakly $F$-regular;*
*(2) $R$ is essentially of finite type over a field of prime characteristic $p > 5$ and is $F$-regular.*

*Then $R$ is strongly $F$-regular.*

**Proof.** Every weakly $F$-regular ring that is finitely generated over a field of infinite transcendence degree over $\mathbb{F}_p$ is $F$-regular by [15, Theorem 8.1]. Thus it suffices to prove statement (2) only.

As in the proof of Corollary 4.4, we may assume $R = (R, \mathfrak{m}, k)$ is local and we can use gamma constructions to reduce to the scenario that $R$ is $F$-finite. The ring $R$ is strongly $F$-regular at non-maximal points by [36, Main Result]. By [32, Corollary 6.9], if $P$ is a nonmaximal prime ideal of $R$ then there exists an effective boundary divisor $\Delta$ such that $(\mathrm{Spec}(R_P), \Delta)$ is globally $F$-regular (or just $F$-regular since $\mathrm{Spec}(R_P)$ is affine) and therefore has KLT singularities by [21, Theorem 3.3]. By [8, Corollary 1.12] the anticanonical algebra of $R_P$ is Noetherian and therefore $R$ is strongly $F$-regular by Corollary 4.4. □

## 5. Remarks on annihilating Ext-modules, the weak implies strong conjecture, and the (LC)-conjecture

Let $(R, \mathfrak{m}, k)$ be an excellent local normal Cohen-Macaulay domain of Krull dimension 4 and of prime characteristic $p > 0$. Let $(S, \mathfrak{n}, k)$ be a regular local ring of dimension $h+4$ mapping onto $R$. Fundamental to the results of this article is Proposition 3.1 which shows that for an ideal $K \subseteq R$ of pure height 1, satisfying certain technical conditions, there exists an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ so that

$$\mathfrak{a}^i \operatorname{Ext}_S^{h+3}(R/K^{(i)}, S) = 0.$$

By Matlis duality,

$$\mathrm{Ann}(\operatorname{Ext}_S^{h+3}(R/K^{(i)}, S)) = \mathrm{Ann}(H^1_{\mathfrak{m}}(R/K^{(i)})).$$

Suppose that $a \in K$ generates $K$ at its components and write

$$aR = K \cap L$$

for some ideal $L$ of pure height 1 whose components are disjoint from $K$. By prime avoidance, we can choose an element $b \in L$ that generically generates $L$ but avoids all components of $K$. Then for every integer $i$ there is a short exact sequence

$$0 \to \frac{R}{K^{(i)}} \xrightarrow{\cdot b^i} \frac{R}{a^i R} \to \frac{R}{(a^i, b^i)} \to 0.$$

Therefore

$$H^0_{\mathfrak{m}}(R/(a^i, b^i)) \cong H^1_{\mathfrak{m}}(R/K^{(i)})$$

for every integer $i$. In particular, if $K$ satisfies the technical conditions of Proposition 3.1, then there exists an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ so that

$$\mathfrak{a}^{p^e} H^0_{\mathfrak{m}}(R/(a, b)^{[p^e]}) = 0$$

for every integer $e$. Such an annihilation property is the expectation of the Local Cohomology conjecture.

**Conjecture 5.1** *((LC)-conjecture). Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic $p > 0$ and $I \subseteq R$ an ideal. There exists an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ so that for every integer $e$*

$$\mathfrak{a}^{p^e} H^0_{\mathfrak{m}}(R/I^{[p^e]}) = 0.$$

If the (LC)-conjecture is true whenever $R$ is weakly $F$-regular and $I \subseteq R$ is an ideal of height $\dim(R) - 1$, then it would follow that the property of weak $F$-regularity localizes, cf. [12, Page 43], [20], [34, Conjecture 1], and [23, Conjecture 1].

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