International Journal of Mathematics and
Computer Science, **18**(2023), no. 1, 11–22

Distance graphs on normed function spaces

Timothy Eller[1], **Peter Johnson**[2], **William Jones**[3], **Jakub Kraus**[4]

[1]Department of Mathematics
Georgia Southern University
Statesboro, GA 30460, U.S.A

[2]Department of Mathematics and Statistics
Auburn University
Auburn, AL 36849, U.S.A

[3]Department of Mathematics and Statistics
SUNY at Binghamton
Binghamton, NY 13902-6000, U.S.A

[4]Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1043, U.S.A

email: te02816@georgiasouthern.edu, johnspd@auburn.edu,
wjones4@binghamton.edu, jakraus@umich.edu

(Received July 13, 2022, Accepted August 23, 2022)

Abstract

It is easy to see that the unit distance graphs on the classical real normed sequence and function spaces — $L^p(\mathbb{N}), L^p(\mathbb{R}), 1 \leq p \leq \infty, c_0, C[0,1]$, and $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the continuous bounded functions from $\mathbb{R} = (-\infty, \infty)$ into itself — have infinite clique and chromatic numbers, because each graph contains a countably infinite clique. The question remains to determine exactly which infinite cardinals these numbers are. A related question is of interest as well: can the chromatic number be greater than the clique number?

Keywords and phrases: Distance graphs, chromatic number, clique number, independent sets of vertices, infinite cardinals.
AMS (MOS) Subject Classifications: 05C15, 05C63.
This paper was supported by NSF grant no. 1950563.
ISSN 1814-0432, 2023, http://ijmcs.future-in-tech.net

1 Introduction

$\mathbb{N}$ will stand for the set of non-negative integers and $\mathbb{R}$ will stand for the real numbers. If X and Y are sets, the notation X^Y will be used to denote the set of all functions from Y into X. As usual, $\{0,1\}^Y$ will be abbreviated 2^Y, and identified, in the usual way, with the set of all subsets of Y — also called the *power set* of Y.

Also as usual, $X^{\mathbb{N}}$ will be identified with the set of infinite sequences $(x_0, x_1, x_2, \dots)$ of elements of X; $f \in X^{\mathbb{N}}$ corresponds to the sequence $(f(0), f(1), \dots)$.

For any set X, the cardinality of X will be denoted $|X|$.

For any set $Y \neq \emptyset$, $\mathbb{R}^Y$ can be made into a vector space in a natural way via pointwise addition and scalar multiplication of functions.

Our questions concern single-distance graphs on infinite-dimensional normed subspaces of $\mathbb{R}^Y$ when $Y = \mathbb{N}$ or Y is a subinterval of $\mathbb{R}$. The spaces $L^p(I)$, where $1 \leq p \leq \infty$ and I is a real interval, do not exactly fall into this category, but bear with us.

If V is a real vector space with norm $\|\cdot\|$, and $d > 0$, the distance-d graph on V — let us denote it $G(V, \|\cdot\|, d)$, or just $G(V, d)$ if the norm is fixed in the discussion — is the graph with vertex set V in which two vectors $u, v \in V$ are adjacent if and only if they are a distance of d apart, i.e. $\|u-v\| = d$. Clearly scalar multiplication by d^{-1} induces a graph isomorphism $G(V, d) \to G(V, 1)$, so we restrict our discussion to $G(V, 1)$ in this paper.

A proper coloring of $G(V, \|\cdot\|, 1)$ with colors from a set C is a function $\phi : V \to C$ such that if $u, v \in V$ and $\|u - v\| = 1$, then $\phi(u) \neq \phi(v)$ — in other words, adjacent vertices cannot be the same color. The *chromatic number* of $G(V, \|\cdot\|, 1)$ is the least cardinal $|C|$ such that there is such a proper coloring; we shall denote this chromatic number by $\chi(V, \|\cdot\|, 1)$ or $\chi(V, 1)$, suppressing the superfluous "G."

A *clique* in a simple graph H is a complete subgraph of H. The *clique number* of H is

$$\omega(H) = \sup\{|V(K)| : K \text{ is a clique in } H\},$$

where $V(K)$ stands for the vertex set of K. We shall denote the clique number of $G(V, \|\cdot\|, 1)$ by $\omega(V, \|\cdot\|, 1)$ or $\omega(V, 1)$.

Certainly, if $\omega(H)$ is finite, the supremum in its definition is a maximum, but clearly this is not necessarily the case when $\omega(H)$ is infinite. It is easy to describe a graph H with no infinite clique, but containing cliques of all positive integer orders. For such an H, $\omega(H) = |\mathbb{N}|$ and there is no clique in H of order $\omega(H)$.

It is elementary that for any graph H, $\omega(H) \leq \chi(H)$. Each finite-dimensional vector space over $\mathbb{R}$ is isomorphic to $\mathbb{R}^n = \mathbb{R}^{\{1,\dots,n\}}$ for some $n \in \mathbb{N}$. It is easy to see that $\chi(\mathbb{R}^n, \|\cdot\|, 1)$ is finite for any norm $\|\cdot\|$ on $\mathbb{R}^n$: tile $\mathbb{R}^n$ with n-dimensional cubes of diameter < 1, and then color the tiles periodically with a copious, but finite, set of colors in such a way that all points in different cubes bearing the same color will be distances > 1 from each other. Therefore $\chi(V, \|\cdot\|, 1) < \infty$ for all finite-dimensional normed real vector spaces $(V, \|\cdot\|)$. Consequently, $\omega(V, \|\cdot\|, 1) < \infty$ for all such vector spaces $(V, \|\cdot\|)$. Therefore $\omega(V, \|\cdot\|, 1)$ is achieved.

Question 1. If V is an infinite-dimensional subspace of $\mathbb{R}^Y$ (so Y is infinite), with norm $\|\cdot\|$, is there necessarily a clique in $G(V, 1)$ whose vertex set has cardinality $\omega(V, 1)$?

The Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^n$ is defined by $\|(x_1, \dots, x_n)\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$. It is well known that $\omega(\mathbb{R}^n, \|\cdot\|_2, 1) = n+1$. Determining or even estimating $\chi(\mathbb{R}^n, \|\cdot\|_2, 1)$ is a famous problem; the journal *Geombinatorics* owes its existence to the posing of this problem in the case $n = 2$. One of the shocking early results in this area, due to Raiskii [2], was the discovery that for $n > 1$, $\chi(\mathbb{R}^n, \|\cdot\|_2, 1) \geq n + 2$. Thus, $\chi(\mathbb{R}^n, \|\cdot\|_2, 1) > \omega(\mathbb{R}^n, \|\cdot\|_2, 1)$ for all $n > 1$. It has been shown [1] that $\chi(\mathbb{R}^n, \|\cdot\|_2, 1)$ grows exponentially with n. Hence,

$$\frac{\chi(\mathbb{R}^n, \|\cdot\|_2, 1)}{\omega(\mathbb{R}^n, \|\cdot\|_2, 1)} \to \infty, \quad \text{as} \quad n \to \infty.$$

On the other hand, with the norm $\|\cdot\|_\infty$ defined on $\mathbb{R}^n$ by $\|(x_1, \dots, x_n)\|_\infty = \max_{1 \leq i \leq n} |x_i|$, it turns out that $\chi(\mathbb{R}^n, \|\cdot\|_\infty, 1) = \omega(\mathbb{R}^n, \|\cdot\|_\infty, 1) = 2^n$.

Question 2. For an infinite-dimensional normed real vector space V, is it possible that $\omega(V, 1) < \chi(V, 1)$?

We have questions auxiliary to Question 2 that are a bit embarrassing as they reveal our ignorance, but here they are, with discussion.

Question 3. For an infinite-dimensional normed real vector space V, is it possible that $\omega(V, 1) < \infty$?

Thanks to a famous result of P. Erdős about the existence of (finite) graphs with arbitrarily large girth and arbitrarily large chromatic number, it is easy to see that infinite graphs G exist such that $\omega(G) = 2$ and $\chi(G) = |\mathbb{N}|$.

Question 4. If G is a graph with $\omega(G) \geq |\mathbb{N}|$, can it be that $\omega(G) < \chi(G)$?

In the rest of this paper, we inspect the unit distance graphs on particularly well known normed sequence and function spaces.

2 $l^\infty(Y)$

Suppose that Y is an infinite set. Define

$$l^\infty(Y) = \left\{ f \in \mathbb{R}^Y : \text{ for some } M > 0, \ |f(y)| < M, \text{ for all } y \in Y \right\}.$$

In other words, $l^\infty(Y)$ is the subspace of $\mathbb{R}^Y$ consisting of all bounded functions. (Note: if Y is finite, then $\mathbb{R}^Y = l^\infty(Y) \simeq \mathbb{R}^{|Y|}$.) Clearly, $l^\infty(Y)$ is naturally equipped with the norm $\|\cdot\|_\infty$ defined by $\|f\|_\infty = \sup_{y \in Y} |f(y)|$.

Theorem 1. If Y is an infinite set, then
$\omega(l^\infty(Y), \|\cdot\|_\infty, 1) = \chi(l^\infty(Y), \|\cdot\|_\infty, 1) = |2^Y|$.

Proof. The proof uses some well-known facts about cardinal arithmetic, including $|2^{\mathbb{N}}| = |\mathbb{R}|$.

Since $\{0,1\}^Y \subseteq l^\infty(Y)$, and $\{0,1\}^Y$ is the vertex set of a clique in the graph

$$G(l^\infty(Y), \|\cdot\|_\infty, 1),$$

we have

$$\begin{aligned} |2^Y| = |\{0,1\}^Y| &\leq \omega(l^\infty(Y), \|\cdot\|_\infty, 1) \\ &\leq \chi(l^\infty(Y), \|\cdot\|_\infty, 1) \\ &\leq |l^\infty(Y)| \leq |\mathbb{R}^Y| \\ &= |(2^{\mathbb{N}})^Y| = |2^{\mathbb{N}\times Y}| = |2^Y|. \end{aligned}$$

□

Note that this result extends the result about $(\mathbb{R}^n, \|\cdot\|_\infty)$ mentioned earlier.

3 $l^p(Y)$, $1 \le p < \infty$

If Y is infinite and $1 \le p < \infty$, then $l^p(Y) = \left\{ f \in \mathbb{R}^Y : \sum_{y \in Y} |f(y)|^p < \infty \right\}$; clearly a subspace of $l^\infty(Y)$, $l^p(Y)$ is naturally normed by $\| \cdot \|_p$, which is defined by

$$\|f\|_p = \left(\sum_{y \in Y} |f(y)|^p \right)^{1/p}.$$

Theorem 2. If Y is an infinite set, and $1 \le p < \infty$, then

$$\omega(l^p(Y), \| \cdot \|_p, 1) = \chi(l^p(Y), \| \cdot \|_p, 1) = |Y|.$$

Proof. For $y \in Y$, let $e_y : Y \to \mathbb{R}$ be the characteristic function for the singleton set $\{y\}$, i.e.

$$e_y(z) = \begin{cases} 1 & \text{if } z = y \\ 0 & \text{if } z \ne y \end{cases}.$$

Then the functions $2^{-\frac{1}{p}} e_y$, $y \in Y$, are the vertices of a clique in $G(l^p(Y), \| \cdot \|_p, 1)$. Therefore $\omega(l^p(Y), \| \cdot \|_p, 1) \ge |Y|$.

With $\mathbb{Q}$ denoting the rational numbers, let

$$F(Y) = \left\{ f \in \mathbb{Q}^Y : \{y : f(y) \ne 0\} \text{ is finite} \right\}.$$

It is elementary that $F(Y)$ is dense in $(l^p(Y), \| \cdot \|_p)$ for all $p \in [1, +\infty)$ and, because Y is infinite and $\mathbb{Q}$ is countable, that $|F(Y)| = |Y|$. Therefore, $l^p(Y)$ can be covered with $|Y|$ metric balls of diameter 7/9. (Why 7/9? Feeble attempt at humor.) Each of these balls is an independent set in $G(l^p(Y), \| \cdot \|_p, 1)$. Therefore, we have $\chi(l^p(Y), \| \cdot \|_p, 1) \le |Y|$, which, with the previous inequality $\omega(l^p(Y), \| \cdot \|_p, 1) \ge |Y|$, establishes the claim. □

4 $C(I) \cap l^\infty(I)$

For a real interval I let

$$C(I) = \left\{ f \in \mathbb{R}^I : f \text{ is continuous on } I \right\},$$

where the continuity is with respect to the usual topologies on both spaces. Then, $C(I) \cap l^\infty(I)$ is the real vector space of continuous, bounded, real-valued functions on the interval I, normed by the restriction of $\| \cdot \|_\infty$ to $C(I) \cap l^\infty(I)$. The following are facts familiar to many, but not all.

1. If I is an open interval, then $(C(I) \cap l^\infty(I), \|\cdot\|_\infty)$ is linearly and isometrically isomorphic to $(C(\mathbb{R}) \cap l^\infty(\mathbb{R}), \|\cdot\|_\infty)$. To see this, one obtains a monotone, continuous surjection $\phi : I \to \mathbb{R}$ and then maps $C(\mathbb{R}) \cap l^\infty(\mathbb{R})$ one-to-one and onto $C(I) \cap l^\infty(I)$ by $f \mapsto f \circ \phi$, the composition of f and ϕ.

   Therefore, to study the unit distance graphs on $(C(I) \cap l^\infty(I), \|\cdot\|_\infty)$, it suffices to study the unit distance graph on $(C(\mathbb{R}) \cap l^\infty(\mathbb{R}), \|\cdot\|_\infty)$ when I is an open interval.

2. It is well-known that every continuous, real-valued function on an interval $I = [a, b] \subseteq \mathbb{R}$ achieves both a maximum and a minimum on I. Therefore, for $-\infty < a < b < \infty$,

$$C([a, b]) \cap l^\infty([a, b]) = C([a, b]),$$

   and for $f \in C([a, b])$,

$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|.$$

   Clearly, the different normed spaces $(C([a, b]), \|\cdot\|_\infty)$, $-\infty < a < b < \infty$ are isometrically isomorphic (see 1) to each other. So, their unit distance graphs are isomorphic as well. We will take $C([0, 1])$ as the representative of this tribe of spaces.

3. $(C([0, 1]), \|\cdot\|_\infty)$ is separable, i.e. $C([0, 1])$ has a countable subset which is dense with respect to the topology induced by $\|\cdot\|_\infty$. One way to see this arises from the well-known fact that each $f \in C([0, 1])$ is *uniformly continuous* on $[0, 1]$. This means that for all $\epsilon > 0$ there exists $\delta > 0$ such that $s, t \in [0, 1]$ and $|s - t| < \delta$ imply that $|f(s) - f(t)| < \epsilon$. It is then easy to see that f can be uniformly approximated by functions in $C([0, 1])$ devised as follows— choose a positive integer n, rational numbers

$$x_0 = 0 < x_1 < x_2 < \cdots < x_n = 1,$$

   more rational numbers $y_0, \ldots, y_n$, and use these choices to define a function g by the following:

   (a) $g(x_i) = y_i$, $i = 0, \ldots, n$;

   (b) g is linear on each interval $[x_{i-1}, x_i]$, $i = 1, \ldots, n$.

Such functions g, called *linear splines*, are continuous on $[0,1]$ and the collection of them is dense in $C([0,1])$. This collection is in one-to-one correspondence with a subset of the set of all finite sequences of ordered pairs of rational numbers; therefore, this collection of splines is countable.

4. On the other hand, $(C(\mathbb{R}) \cap l^\infty(\mathbb{R}), \|\cdot\|_\infty)$ is not separable. We can prove this by a "diagonal" argument. Let $\mathbb{Z}$ denote the set of integers, and let $\{g_k\}_{k\in\mathbb{Z}}$ be a subset of $C(\mathbb{R}) \cap l^\infty(\mathbb{R})$ indexed by $\mathbb{Z}$. Let f be the spline defined by

$$f(k) = \begin{cases} 1 & \text{if } g_k(k) < 0 \\ -1 & \text{if } g_k(k) \geq 0 \end{cases},$$

such that f is linear on $[k-1,k]$ for every $k \in \mathbb{Z}$. Then, $f \in C(\mathbb{R}) \cap l^\infty(\mathbb{R})$ (in fact, $\|f\|_\infty = 1$) and $\|f - g_k\|_\infty \geq 1$ for all $k \in \mathbb{Z}$.

Theorem 3. $\omega(C([0,1]), \|\cdot\|_\infty, 1) = \chi(C([0,1]), \|\cdot\|_\infty, 1) = |\mathbb{N}|$.

Proof. For each positive integer n, let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 0 & \text{if } x \in [0,1] \setminus \left(\frac{1}{n+1}, \frac{1}{n}\right) \\ 2n\left((n+1)x - 1\right) & \text{if } \frac{1}{n+1} < x < \frac{2n+1}{2n(n+1)} \\ -2(n+1)\left(nx - 1\right) & \text{if } \frac{2n+1}{2n(n+1)} \leq x < \frac{1}{n} \end{cases}.$$

The main thing is that f_n is continuous, takes values between 0 and 1, is zero outside the interval $(\frac{1}{n+1}, \frac{1}{n})$, and that $\|f_n\|_\infty = 1$. Clearly, $\|f_i - f_j\|_\infty = 1$ for all $1 \leq i < j$. Therefore $\omega(C([0,1]), \|\cdot\|_\infty, 1) \geq |\mathbb{N}|$.

On the other hand, the separability of $C([0,1])$ allows us to cover $C([0,1])$ with $|\mathbb{N}|$ sets that are independent in $G(C([0,1]), \|\cdot\|_\infty, 1)$, by an argument similar to that deployed in the proof of Theorem 2. Therefore

$$\begin{aligned} |\mathbb{N}| &\leq \omega(C([0,1]), \|\cdot\|_\infty, 1) \\ &\leq \chi(C([0,1]), \|\cdot\|_\infty, 1) \leq |\mathbb{N}|. \end{aligned}$$

□

Theorem 4. $\omega(C(\mathbb{R}) \cap l^\infty(\mathbb{R}), \|\cdot\|_\infty, 1) = \chi(C(\mathbb{R}) \cap l^\infty(\mathbb{R}), \|\cdot\|_\infty, 1) = |\mathbb{R}|$.

Proof. For each $f : \mathbb{Z} \to \{0,1\}$, extend f to a continuous function from $\mathbb{R}$ into $[0,1]$ by linear interpolation on each interval $[n-1,n], n \in \mathbb{Z}$. Clearly, if $f, g \in 2^{\mathbb{Z}}$, and $f \neq g$, then, letting $\tilde{f}, \tilde{g}$ denote the spline extensions of f and g to all of $\mathbb{R}$, $1 = \|\tilde{f} - \tilde{g}\|_\infty$. Thus,

$$\omega(C(\mathbb{R}) \cap l^\infty(\mathbb{R}), \|\cdot\|_\infty, 1) \geq |2^{\mathbb{Z}}| = |2^{\mathbb{N}}| = |\mathbb{R}|.$$

We are grateful for the prompt to complete this proof from posts on Mathematics Stack Exchange illustrating why $|C(\mathbb{R})| = |\mathbb{R}|$. Notice that $C(\mathbb{R}) \cap l^\infty(\mathbb{R})$ injects into $\mathbb{R}^{\mathbb{Q}}$ by mapping each function to its values on $\mathbb{Q}$ (this map is injective because the domain's functions are continuous). Therefore

$$\begin{aligned}\chi(C(\mathbb{R}) \cap l^\infty(\mathbb{R}), \|\cdot\|_\infty, 1) &\leq |C(\mathbb{R}) \cap l^\infty(\mathbb{R})| \\ &\leq |\mathbb{R}^{\mathbb{Q}}| = |\mathbb{R}^{\mathbb{N}}| = |2^{\mathbb{N}}| = |\mathbb{R}|.\end{aligned}$$

The penultimate equality comes from the proof of Theorem 1, where it is shown that $|\mathbb{R}^Y| = |2^Y|$ for any infinite set Y. □

What about half-open, half-closed intervals I? It is straightforward to see that for every such I, the metric space $(C(I) \cap l^\infty(I), \|\cdot\|_\infty)$ is isometrically isomorphic to the metric space $(C([0,\infty)) \cap l^\infty([0,\infty)), \|\cdot\|_\infty)$. By modifying previous arguments, it is easily seen that this space is not separable, and that the unit distance graph on this space has clique number no less than $|\mathbb{R}|$. (Come to think of it, the latter implies the former, since if the space were separable, then its chromatic number would not be greater than $|\mathbb{N}|$.) As this space has cardinality $\leq |C(\mathbb{R}) \cap l^\infty(\mathbb{R})| = |\mathbb{R}|$, the chromatic and clique numbers are both $|\mathbb{R}|$.

5 Step Functions

If $S \subseteq \mathbb{R}$, the characteristic function of S is the function $ch_S : \mathbb{R} \to \{0,1\}$ defined by

$$ch_S = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

A *step* function on an interval $I \subseteq \mathbb{R}$ is a finite linear combination $\sum_i \lambda_i ch_{S_i}$ in which each $\lambda_i \in \mathbb{R}$ and each $S_i \subseteq I$ is either a subinterval of I or a singleton. Clearly, each step function is bounded, so, with $Step(I)$ denoting the set of all step functions on the interval $I \subseteq \mathbb{R}$, we can equip $Step(I)$ with the norm $\|\cdot\|_\infty$.

Theorem 5. For each real interval I,

$$\omega(Step(I), \|\cdot\|_\infty, 1) = \chi(Step(I), \|\cdot\|_\infty, 1) = |Step(I)| = |\mathbb{R}|.$$

Proof. Since $Step(I)$ is the set of all real-valued functions on I representable as finite linear combinations, with coefficients from $\mathbb{R}$, of characteristic functions for singletons from I and subintervals of I, elementary cardinality arguments show that $|Step(I)| = |\mathbb{R}|$.

Clearly, $\chi(Step(I), \|\cdot\|_\infty, 1) \leq |Step(I)| = |\mathbb{R}|$. On the other hand, if J and K are different subintervals of I, then $\|ch_J - ch_K\|_\infty = 1$. Thus, the characteristic functions of the different intervals in I are the vertices of a clique in $G(Step(I), \|\cdot\|_\infty, 1)$, so $|\{\text{intervals in } I\}| = |\mathbb{R}| \leq \omega(Step(I), \|\cdot\|_\infty, 1)$.

□

6 Simple Functions

A *simple function* on a real interval I is a function representable as a finite linear combination $\sum_i \lambda_i ch_{S_i}$ in which each $\lambda_i \in \mathbb{R}$ and each S_i is a Lebesgue-measurable subset of I. We refer the reader to almost any graduate textbook on real analysis for the definition of "Lebesgue-measurable."

Let $Simple(I)$ denote the set of simple functions on I. Clearly, $Simple(I)$ is a vector subspace of $l^\infty(I)$. We will consider $Simple(I)$ with the norm $\|\cdot\|_\infty$.

Let $\mathcal{L}(I)$ denote the set of Lebesgue-measurable subsets of I. Clearly,

$$|\mathbb{R}| = |I| \leq |\mathcal{L}(I)| \leq |2^I| = |2^\mathbb{R}|.$$

By the proof of Theorem 5, we have the following.

Theorem 6. For any real interval I,

$$\begin{aligned}\omega(Simple(I), \|\cdot\|_\infty, 1) &= \chi(Simple(I), \|\cdot\|_\infty, 1)\\ &= |Simple(I)| = |\mathcal{L}(I)|.\end{aligned}$$

We have questions about $|\mathcal{L}(I)|$. We have been told by reliable sources that if the Zermelo-Fraenkel (ZF) axioms are consistent, then so are those axioms with the additional axiom LM: every subset of $\mathbb{R}$ is Lebesgue-measurable. In ZF+LM, clearly $|\mathcal{L}(I)| = |2^I| = |2^\mathbb{R}|$ for each real interval I. However, in

Zermelo-Fraenkel set theory with the axiom of choice (ZFC), the negation of the axiom LM is provable! We know from [3] and [4] that a change in axiom systems can change cardinalities, even the chromatic numbers of infinite graphs. If anyone can enlighten us on $|\mathcal{L}(I)|$ in ZFC, we will be most appreciative.

7 $L^p(I), 1 \leq p \leq \infty$

If I is a real interval and $p \in [1, \infty)$, $L^p(I)$ is usually thought of as the set of Lebesgue-measurable functions $f : I \to \mathbb{R}$ satisfying $\int_I |f|^p < \infty$, where $\int_I$ denotes the Lebesgue integral over I. (The Lebesgue integral agrees with the Riemann integral, but can be applied to a broader class of functions.) As all students of real analysis know, this definition is not quite right. Actually, the elements of $L^p(I)$ are *equivalence classes* of Lebesgue-measurable functions, with respect to the equivalence relation $\simeq$ defined by: $f \simeq g$ (for Lebesgue-measurable functions f and g) if and only if $\{x \in I : f(x) \neq g(x)\}$ has Lebesgue measure zero.

However, we shall, as is customary, treat the elements of $L^p(I)$ as functions. $L^p(I)$ is a normed vector space over $\mathbb{R}$ with the norm $\|f\|_p = \left(\int_I |f|^p\right)^{1/p}$. Note that we also used $\|\cdot\|_p$ to stand for the canonical norm on $l^p(Y)$. We hope that the distinction will be clear from context.

The elements of $L^\infty(I)$ are also equivalence classes of Lebesgue-measurable functions under the equivalence relation described above. The equivalence class of a Lebesgue-measurable function $f : I \to \mathbb{R}$ is in $L^\infty(I)$ if and only if, for some $M \geq 0$, $\mu(\{x \in I : |f(x)| > M\}) = 0$, where μ denotes the Lebesgue measure. Essentially, $L^\infty(I)$ is the set of $\simeq$ classes of Lebesgue-measurable functions whose members are bounded almost everywhere. We equip $L^\infty(I)$ with the norm $\|f\|_\infty = \inf\{M \geq 0 : \mu(\{x \in I : |f(x)| > M\}) = 0\}$, the essential supremum of $|f|$. As with $\|\cdot\|_p$, we underscore that this $\|\cdot\|_\infty$ is not the same as norms appearing previously in this paper with the same notation.

Theorem 7. For any real interval I and $1 \leq p < \infty$,

$$\omega(L^p(I), \|\cdot\|_p, 1) = \chi(L^p(I), \|\cdot\|_p, 1) = |\mathbb{N}|.$$

Proof. Let $J_n = (a_n, b_n)$ be a sequence of pairwise disjoint open subintervals of I, and let $f_n = (2(b_n - a_n))^{-\frac{1}{p}} ch_{J_n}$, where $n = 1, 2, \ldots$. Then, $\|f_s - f_t\|_p = 1$ whenever $1 \leq s < t$. Thus, $\omega(L^p(I), \|\cdot\|_p, 1) \geq |\mathbb{N}|$.

On the other hand, $(L^p(I), \|\cdot\|_p)$ is a separable normed space (this is a long story, which shall not be told here). Therefore, as in earlier proofs in this paper, $\chi(L^p(I), \|\cdot\|_p, 1) \leq |\mathbb{N}|$.

□

Theorem 8. For any real interval I,

$$\begin{aligned}|\mathbb{R}| &\leq \omega(L^\infty(I), \|\cdot\|_\infty, 1)\\ &\leq \chi(L^\infty(I), \|\cdot\|_\infty, 1) \leq |2^{\mathbb{R}}|.\end{aligned}$$

Proof. If J and K are two different open intervals in I, then $\|ch_J - ch_K\|_\infty = 1$. Thus, $\omega(L^\infty(I), \|\cdot\|_\infty, 1) \geq |\{(a,b) : a, b \in I \text{ and } a < b\}| = |\mathbb{R}|$.

On the other hand, clearly $\chi(L^\infty(I), \|\cdot\|_\infty, 1) \leq |L^\infty(I)| \leq |\mathbb{R}^{\mathbb{R}}| = |2^{\mathbb{R}}|$. (Again, recall from the proof of Theorem 1 that $|\mathbb{R}^Y| = |2^Y|$ for any infinite set Y.) □

Out of sheer curiosity, we hope that Theorem 8 can be improved so that both the clique number and chromatic number are completely determined.

8 Distance Graphs Defined By More Than One Distance

In this last section we shall consider distance graphs in a more general setting than we began this paper with. Suppose that (X, ρ) is a metric space and $D \subset (0, \infty)$. The distance graph $G(X, \rho, D)$ is the graph with vertex set X in which vertices $x, y \in X$ are adjacent if and only if $\rho(x, y) \in D$. When $D = \{d\}$ we write $G(X, \rho, d)$. If P is a graph parameter (e.g., χ) we write $P(X, \rho, D)$, suppressing the superfluous "G." When V is a real vector space with distance induced by a norm $\|\cdot\|$, we write $P(V, \|\cdot\|, D)$ as previously, except now D need not be a singleton.

Theorem 9. Suppose that (X, ρ) is a metric space, and $D \subseteq (0, \infty)$ is a finite, nonempty set. Then:

$$\max_{d \in D} \omega(X, \rho, d) \leq \omega(X, \rho, D) \leq \chi(X, \rho, D) \leq \prod_{d \in D} \chi(X, \rho, d)$$

Proof. The first two inequalities are straightforward to see. As for the last: for each $d \in D$, let C_d be a set of colors so that $|C_d| = \chi(X, \rho, d)$ and let $\varphi_d : X \to C_d$ be a proper coloring of $G(X, \rho, d)$. Then the function

$\varphi : X \to \prod_{d \in D} C_d$ defined by $\varphi(x) = (\varphi_d(x))_{d \in D}$ corresponds to a proper coloring of $G(X, \rho, D)$, since if $x, y \in X$ and $\rho(x, y) = d \in D$, then the dth coordinates of the $|D|$-tuples $\varphi(x), \varphi(y)$ will be different. Consequently:

$$\chi(X, \rho, D) \leq \left| \prod_{d \in D} C_d \right| = \prod_{d \in D} \chi(X, \rho, D)$$

□

Corollary. Suppose that $(V, \|\cdot\|)$ is a real normed space and $\chi(V, \|\cdot\|, 1)$ is infinite. Then for each finite, nonempty $D \subseteq (0, \infty)$:

$$\omega(V, \|\cdot\|, 1) \leq \omega(V, \|\cdot\|, D) \leq \chi(V, \|\cdot\|, D) = \chi(V, \|\cdot\|, 1)$$

Proof. The corollary follows from Theorem 9, the fact that $P(V, \|\cdot\|, d) = P(V, \|\cdot\|, 1)$ for each $d > 0$ when $P \in \{\omega, \chi\}$, and the fact that if c is an infinite cardinal and k is a positive integer, then $c^k = c$. □

The corollary implies generalizations of almost all the previous results of this paper from $P(V, \|\cdot\|, 1)$ to $P(V, \|\cdot\|, D)$ for $P \in \{\omega, \chi\}$ and finite $D \subseteq (0, \infty)$. Extensions of the proposition to the case of infinite $D \subseteq (0, \infty)$ are available, but we will leave these matters for another time.

References

[1] P. Frankl, R.M. Wilson, Intersection theorems with geometric consequences, Combinatorica, **1**, (1981), 357–358.

[2] D.E. Raiskii, Realizing all distances in a decomposition of the space $\mathbb{R}^n$ into $n + 1$ parts, Matematicheskie Zametki, **7,** (1970), 319–323, English translation in Math Notes, **7,** (1970), 194–196.

[3] Saharon Shelah, Alexander Soifer, Axiom of choice and the chromatic number of the plane, Journal of Combinatorial Theory, series A, **103**, (2003), 387–391.

[4] Saharon Shelah, Alexander Soifer, Chromatic number of the plane and its relatives, part III: its future, Geombinatorics, **13,** (2003), 41–46.