A SIMPLER KIND OF FROBENIUS THEOREM IN THE GAUSSIAN INTEGERS*

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Abstract

Suppose that $a, b, c, d \in \mathbb{N}$ and $\alpha = a + ib$, $\beta = c + id$ are linearly independent as planar vectors. Let $U(\alpha, \beta)$ be the intersection with $\mathbb{N}[i]$ of the smaller angular sector bounded by the half-lines through the origin and α, β, respectively, and let $SG(\alpha, \beta) = \{\lambda\alpha + \mu\beta \mid \lambda, \mu \in \mathbb{N}\}$. It is shown that there exists $w \in \mathbb{N}[i]$ such that $w + U(\alpha, \beta) \subseteq SG(\alpha, \beta)$ if and only if $|ad - bc| = 1$. Further, when this requirement is satisfied, $U(\alpha, \beta) = SG(\alpha, \beta) = \{w \in \mathbb{Z}[i] \mid w + U(\alpha, \beta) \subseteq SG(\alpha, \beta)\}$.

1 Introduction

Throughout, $\mathbb{Z}$ will denote the set of integers, $\mathbb{Z}^+$ the set of positive integers, $\mathbb{N}$ the set of non-negative integers, $\mathbb{Z}[i] = \{a + ib \mid a, b \in \mathbb{Z}\}$, the set of Gaussian integers, and $\mathbb{N}[i] = \{a + ib \mid a, b \in \mathbb{N}\}$, the set of Gaussian integers in the closed first quadrant of the complex plane.

What we will call the primordial Frobenius theorem is the following. It is from the 19^{th} century, and it is not due to Frobenius alone (Sylvester may have been the first to prove it), but somehow it has becoem associated with the great German mathematician Frobenius.

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Theorem F. *Suppose that* $a_1, \ldots, a_n \in \mathbb{Z}^+$ *and*

$$SG(a_1, \ldots, a_n) = \left\{ \sum_{j=1}^{n} \lambda_j a_j \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N} \right\}$$

Then $SG(a_1, \ldots, a_n)$ *contains a tail of* $\mathbb{N}$ *(i.e., all sufficiently large integers are in* $SG(a_1, \ldots, a_n)$ *if and only if* $a_1, \ldots, a_n$ *are relatively prime (meaning that the greatest common integer divisor of* $a_1, \ldots, a_n$ is 1).

Still of mathematical interest are the Frobenius problems associated with this theorem: Given $a_1, \ldots, a_n \in \mathbb{Z}^+$, relatively prime, find the largest integer not in $SG(a_1, \ldots, a_n)$; equivalently, find the successor of that integer, which will be the smallest element of the largest (with respect to inclusion) tail of $\mathbb{N}$ which is contained in $SG(a_1, \ldots, a_n)$. If we let w_0 denote that smallest element, then the tail of $\mathbb{N}$ of which w_0 is the smallest element vs $w_0 + \mathbb{N}$, the translate of $\mathbb{N}$ by w_0. Further, because $w_0 - 1 \notin SG(a_1, \ldots, a_n)$, $w_0 + \mathbb{N} = \{w \in \mathbb{Z} \mid w + \mathbb{N} \subseteq SG(a_1, \ldots, a_n)\}$. These comments are trivial to verify, but they are worth remembering, in view of what is to come shortly.

It should be mentioned that in the case $n = 2$, there is a formula solution to the Frobenius problem: If a, b are relatively prime positive integers, then the largest integer not in $SG(a, b)$ is $ab-(a+b)$. Therefore, in these case, the w_0 referred to above is $ab-(a+b)+1 = (a-1)(b-1)$. For $n > 2$, the Frobenius problems in $\mathbb{Z}$ are still being worked on, in some precincts.

Nicole Looper found a way to seek Frobenius-type theorems in arbitrary rings (usually commutative, with unit). We slightly modify the presentation in [4] : A Frobenius template in a ring R is a triple (A, C, U) in which each $A, C \subseteq R$ and U is a function which assigns to each finite sequence $(\alpha_1, \ldots, \alpha_n) \in A^n$, $n \in \mathbb{Z}^+$, a monoid (closed under $+$, containing 0) $U(\alpha_1, \ldots, \alpha_n) \subseteq R$ such that

$$SG(\alpha_1, \ldots, \alpha_n) = \left\{ \sum_{j=1}^{n} \lambda_j \alpha_j \mid \lambda_1, \ldots, \lambda_n \in C \right\} \subseteq U(\alpha_1, \ldots, \alpha_n)$$

For interesting answers to questions we shall pose about such a template, generally C (the coefficient set) is a monoid, and, often, so is

$A \cup \{0\}$, although this does not seem to be essential. What is essential is that for each finite sequence $\alpha_1, \ldots, \alpha_n \in A$, $U(\alpha_1, \ldots, \alpha_n)$ should be a monoid containing $SG(\alpha_1, \ldots, \alpha_n)$ with the vague property of having no "holes" – a property that $\mathbb{N}$ clearly posseses in $\mathbb{Z}$, as do the monoids $U(\alpha, \beta)$ described in the abstract, in $\mathbb{Z}[i]$. And $U(\alpha_1, \ldots, \alpha_n)$ should satisfy a strange condition vis-a-vis $SG(\alpha_1, \ldots, \alpha_n)$; besides containing $SG(\alpha_1, \ldots, \alpha_n)$, it should be the case that for any monoid $V \subseteq R$ property containing $U(\alpha_1, \ldots, \alpha_n)$, there is no $w \in R$ such that $w + V \subseteq SG(\alpha_1, \ldots, \alpha_n)$. If U does not satisfy these vague requirements, the answers to the questions we are about to pose will be trivial, or at least uninteresting.

Frobenius questions about this template are:

(1) For which sequences $(\alpha_1, \ldots, \alpha_n) \in A^n$ is $\mathrm{Frob}(\alpha_1, \ldots, \alpha_n) = \{w \in R \mid w + U(\alpha_1, \ldots, \alpha_n) \subseteq SG(\alpha_1, \ldots, \alpha_n)\}$ non-empty?

(2) When $\mathrm{Frob}(\alpha_1, \ldots, \alpha_n) \neq \varnothing$, can $\mathrm{Frob}(\alpha_1, \ldots, \alpha_n)$ be usefully described?

Clearly Theorem F answers question (1) in the classical case, in which the ring is $\mathbb{Z}$ and the template is $(\mathbb{N} \setminus \{0\}, \mathbb{N}, \mathbb{N})$. Note that U is a constant function in this template. Results mentioned earlier settle both questions for sequences $a, b \in \mathbb{N} \setminus \{0\}$ of length 2: From Theorem F, $\mathrm{Frob}(a, b) \neq \varnothing$ if and only if a and b are relatively prime, and if they are, $\mathrm{Frob}(a, b) = (a-1)(b-1) + \mathbb{N}$. Both questions for this template in the case of a single positive integer are pretty easy. For sequences $a_1, \ldots, a_n$, $n > 2$, there are algorithms and short answers for (2) in special cases, but there is still plenty of territory open for exploration.

Looper's clearing of the path to Frobenius theorems in contexts more general then the classical led to non-trivial results in [1], [3], [4], and [5]. In each of these the U component of the template was constant.

The very first Frobenius theorems in a non-classical context (that we know of) appeared in [2], before the publication of [4]. In terms not available at the time, the template in [2] was $(\mathbb{N}[i] \setminus \{0\}, \mathbb{N}[i], U)$ and the U was necessarily non-constant, due to the nature of complex multiplication and the (related) absence of a linear ordering on the ring of Gaussian integers $\mathbb{Z}[i]$ compatible with the ring operations. The Frobenius theorem that we will prove here is also in the ring $\mathbb{Z}[i]$, although

the ring structure is incidental, of no importance. But, again, U will be non-constant.

Before getting down to business, we want to acknowledge the contributions of two of the co-authors of [2], Chris Maier and Jordan Paschke, to the entire Frobenius-generalization project. Undergraduates at the time (summer, 2008, in a Research Experience for Undergraduates program), having been introduced to the 19^{th} century classical Frobenius results, they decided to search for analogous phenomena in $\mathbb{Z}[i]$. Without a definite idea of what they were looking for, they decided to take elements of $\mathbb{N}[i] \setminus \{0\}$ and form linear combinations of these elements with coefficients from $\mathbb{N}[i]$. Looking at two "easy" cases, they found gold. (And then the brilliant Ken Dutch appeared, and showed us how to find more gold, buried deeper.)

2 The problem, and the main results

For a non-zero complex number $z = x + iy$, $x, y \in \mathbb{R}$, let $\theta(z)$ be the unique number in $(-\pi, \pi]$ such that $z = |z|e^{i\theta(z)}$. Thus, if $x, y \geq 0$, $0 \leq \theta(z) \leq \pi/2$. If $x > 0$ and $y \geq 0$ then $\theta(z) = \arctan \frac{y}{x}$.

For $0 \leq \varphi \leq \psi \leq \pi/2$, let

$$A(\varphi, \psi) = \{\alpha \in \mathbb{N}[i] \setminus \{0\} \mid \varphi \leq \theta(\alpha) \leq \psi\} \cup \{0\}$$

Notice that we allow $\varphi = \psi$, in which case $A(\varphi, \psi)$ is the intersection with $\mathbb{Z}[i]$ of the single half-line starting at 0, making an angle φ with the positive x-axis in the plane. Otherwise, $A(\varphi, \psi)$ is the intersection with $\mathbb{Z}[i]$ of the angular sector with vertex 0, bounded by the half-lines with equations $\theta(z) = \varphi$ and $\theta(z) = \psi$. Observe that $A(\varphi, \psi)$ is a monoid – closed under addition and containing 0.

Strictly speaking, to have a Frobenius template we must have a ring. Ours shall be $\mathbb{Z}[i]$, and the template shall be $(\mathbb{N}[i] \setminus \{0\}, \mathbb{N}, U)$, in which U is defined by:

If $\alpha_1, \ldots, \alpha_n \in \mathbb{N}[i] \setminus \{0\}$, and $\theta(\alpha_1) \leq \ldots \leq \theta(\alpha_n)$, then $U(\alpha_1, \ldots, \alpha_n) = A(\theta(\alpha_1), \theta(\alpha_n))$. Observe that $U(\alpha_1, \ldots, \alpha_n)$ contains $SG(\alpha_1, \ldots, \alpha_n) = \{\sum_{i=1}^{n} \lambda_i \alpha_i \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$ and more: $A(\theta(\alpha_1), \theta(\alpha_n))$ is minimal, with respect to inclusion, among sets of the form $A(\varphi, \psi)$, $0 \leq \varphi \leq \psi \leq \pi/2$, which contain $SG(\alpha_1, \ldots, \alpha_n)$. Further, if $A(\theta(\alpha_1), \theta(\alpha_n)) \subsetneq$

$A(\varphi,\psi)$ (i.e., if $\varphi \le \theta(\alpha_1), \theta(\alpha_n) \le \psi$, and at least one of these inequalities is strict) then no translate of $A(\varphi,\psi)$ could possibly be contained in $SG(\alpha_1,\ldots,\alpha_n)$. Consequently, if we want $U(\alpha_1,\ldots,\alpha_n)$ to be a monoid containing $SG(\alpha_1,\ldots,\alpha_n)$, with the possibility that for some choices of $\alpha_1,\ldots,\alpha_n$, $\mathrm{Frob}(\alpha_1,\ldots,\alpha_n) = \{w \in \mathbb{Z}[i] \mid w + U(\alpha_1,\ldots,\alpha_n) \subseteq SG(\alpha_1,\ldots,\alpha_n)\}$ is non-empty, and with Mother Nature emphatically pointing at the sets $A(\varphi,\psi)$, monoids without "holes" in $\mathbb{Z}[i]$, our definition of U is virtually forced.

Our aim here is to completely determine the sets $\mathrm{Frob}(\alpha_1,\ldots,\alpha_n)$, $\alpha_1,\ldots,\alpha_n \in \mathbb{N}[i] \setminus \{0\}$, in the cases $n = 1, 2$.

Theorem 1. *Suppose that $\alpha = a + ib$, $a, b \in \mathbb{N}$, not both zero. With reference to the Frobenius template $(\mathbb{N}[i] \setminus \{0\}, \mathbb{N}, U)$ defined above, the following are equivalent:*

(a) $Frob(\alpha) \neq \varnothing$;

(b) $Frob(\alpha) = U(\alpha) = A(\theta(\alpha),\theta(\alpha)) = SG(\alpha) = \{k\alpha \mid k \in \mathbb{N}\}$;

(c) a and b are relatively prime in $\mathbb{Z}$.

Theorem 2. *Suppose that $\alpha = a + ib$, $\beta = c + id \in \mathbb{N}[i] \setminus \{0\}$, and $\theta(\alpha) = \theta(\beta)$, with reference to the Frobenius template $(\mathbb{N}[i] \setminus \{0\}, \mathbb{N}, U)$.*

(1) If $b = d = 0$ then $Frob(\alpha,\beta) \neq \varnothing$ if and only if a and c are relatively prime in $\mathbb{Z}$, in which case $Frob(\alpha,\beta) = (a-1)(c-1) + \mathbb{N}$.

(2) If $a = c = 0$ then $Frob(\alpha,\beta) \neq \varnothing$ if and only if b and d are relatively prime in $\mathbb{Z}$ in which case $Frob(\alpha,\beta) = ((b-1)(d-1) + \mathbb{N})i$.

(3) Suppose that $a, b, c, d \in \mathbb{N} \setminus \{0\}$, and let $\gamma = e + if$, $e, f \in \mathbb{N} \setminus \{0\}$, be the element of $\mathbb{N}[i] \setminus \{0\}$ closest to 0 on the line with the equation $y = \frac{b}{a}x$. For some positive integers q, r, $\alpha = r\gamma$ and $\beta = r\gamma$. Then $Frob(\alpha,\beta) \neq \varnothing$ if and only if q and r are relatively prime in $\mathbb{Z}$, in which case

$$Frob(\alpha,\beta) = (q-1)(r-1)\gamma + A(\theta(\alpha),\theta(\alpha)) = ((q-1)(r-1)+\mathbb{N})\gamma.$$

The astute reader will recognize that the conclusions in cases (1) and (2) in Theorem 2 are really just special cases of the conclusion in case

(3). It seemed to us to be wise to separate these cases, to make the conclusion in (3) easier to understand.

Theorem 3 *Suppose that* $a, b, c, d \in \mathbb{N}, \alpha = a + ib, \beta = c + id \in \mathbb{N}[i] \setminus \{0\}$*, and* $\theta(\alpha) \neq \theta(\beta)$*. With reference to the Frobenius template* $(\mathbb{N}[i] \setminus \{0\}, \mathbb{N}, U)$*, defined earlier, the following are equivalent:*

(a) $Frob(\alpha, \beta) \neq \varnothing$*;*

(b) $Frob(\alpha, \beta) = U(\alpha, \beta) = SG(\alpha, \beta)$

(c) $|ad - bc| = 1$*.*

3 Proofs

Lemma 1. *Suppose that* a *and* b *are positive integers, and* $\gamma = e + if$*,* $e, f \in \mathbb{N} \setminus \{0\}$*, is the point in* $\mathbb{N}[i] \setminus \{0\}$ *closest to* 0 *on the line with equation* $ay = bx$ *in the plane. Then* e *and* f *are relatively prime in* $\mathbb{Z}$ *and every point with integer coordinates on that line is an integer multiple of* γ*.*

Proof. If $r > 1$ is an integer which divides both e and f, then $\frac{e}{r} + i\left(\frac{f}{r}\right)$ is a point on the line with positive integer coordinates which is closer to 0 than γ. So e and f must be relatively prime, as integers.

To show that every integer point on the line L with equation $y = \frac{b}{a}x$ is an integer multiple of γ, it suffices to show that this holds for points $\delta = g + ih, g, h \in \mathbb{N} \setminus \{0\}$, on that line (so $h = \frac{b}{a}g$). For such a point δ, let m be the largest positive integer such that $me \leq g$. If $me = g$ then $\delta = m\gamma$, so suppose that $me < g$; we have $g < (m+1)e$, by the definition of m. Then $0 < g - me < e$, so $0 < h - mf = \frac{b}{a}(g - me) < \frac{b}{a}e = f$; thence $\delta - m\gamma$ is a point on that line with positive integer coordinates, and is closer to 0 than γ is. This is impossible, so $\delta = m\gamma$ after all. □

Proof of Theorem 1. $U(\alpha)$ consists of the integer points on the half-line starting at 0 and running through α, while $SG(\alpha)$ is $\{m\alpha \mid m \in \mathbb{N}\}$.

As in Lemma 1, let $\gamma = e + if$ be the non-zero integer point on that half-line closest to the origin. (If the half-line is horizontal or vertical, $e + if = 1$ or i.) By Lemma 1, $U(\alpha) = \gamma\mathbb{N}$.

Note that $w \in \text{Frob}(\alpha) \Rightarrow w = w + 0 \in w + U(\alpha) \subseteq SG(\alpha)$. Thus $\text{Frob}(\alpha) \subseteq SG(\alpha)$. Putting this together with the fact that $SG(\alpha)$ is

closed under addition, we see that $U(\alpha) = SG(\alpha)$ implies that $\mathrm{Frob}(\alpha) = U(\alpha) = SG(\alpha)$. Thus, (b) is equivalent to $U(\alpha) = SG(\alpha)$.

Since e and f are relatively prime integers by Lemma 1, and $\alpha = m\gamma$ for some positive integer m (again, by Lemma 1), we have that $\alpha = \gamma$ if and only if a and b are relatively prime positive integers. Thus $(c) \Rightarrow (b) \Rightarrow (a)$. On the other hand if a and b are not relatively prime positive integers, then $\alpha = m\gamma$ for some $m > 1$; then $\mathrm{SG}(\alpha) = \{k\alpha \mid k \in \mathbb{N}\} = \{q\gamma \mid q \in \mathbb{N}, q \equiv 0 \bmod m\}$. For any candidate w for $\mathrm{Frob}(\alpha)$, $w \in SG(\alpha) \Rightarrow w = t\alpha$ for some $t \in \mathbb{N}$. Take any $r \in \mathbb{N}$ such that $r \equiv 1 \bmod m$ and we have that $w + r\gamma = (tm + r)\gamma \notin SG(\alpha)$ and thus $w + U(\alpha) \subsetneq SG(\alpha)$. Thus, a, b, not relatively prime integers implies that $\mathrm{Frob}(\alpha) = \varnothing$. Therefore, $(a) \Rightarrow (c)$. □

Proof of Theorem 2. In all 3 cases, $U(\alpha, \beta)$ consists of the integer points on the half-line H containing α and β, anchored at 0. Let $\gamma = e + if \in U(\alpha, \beta)$ be the non-zero element of $U(\alpha, \beta)$ closest to 0. (In case (1), $\gamma = 1$; in case (2), $\gamma = i$.) By Lemma 1, $U(\alpha, \beta) = \{k\gamma \mid k \in \mathbb{N}\}$. Then $\alpha = q\gamma$ and $\beta = r\gamma$ for some $q, r \in \mathbb{N} \setminus \{0\}$, so $SG(\alpha, \beta) = \{(\lambda q + \mu r)\gamma \mid \lambda, \mu \in \mathbb{N}\}$. Thus the Frobenius questions about α and β, with respect to the template $(\mathbb{N}[i] \setminus \{0\}, \mathbb{N}, U)$, in the case when $\theta(\alpha) = \theta(\beta)$, are, in light disguise, just the same as the Frobenius questions about q and r with respect to the template $(\mathbb{N} \setminus \{0\}, \mathbb{N}, \mathbb{N})$ in the ring of integers. The answers are given by the 19^{th} century results described in the Introduction. □

Proof of Theorem 3. Without loss of generality, $\theta(\alpha) < \theta(\beta)$. Therefore, either $b = 0$ or $\frac{c}{a} < \frac{d}{b}$. In either case, $ad - bc > 0$. We interrupt this proof for a lemma.

Lemma 2. *Suppose that* $\gamma = e + if \in U(\alpha, \beta) \setminus \{0\}$, *then* $(i) \lceil \frac{cf}{d} \rceil \leq e$ *and* $(ii) \lceil \frac{be}{a} \rceil \leq f$.

Proof. $\gamma \in U(\alpha, \beta) \setminus \{0\} \Rightarrow 0 \leq \theta(\alpha) = \arctan \frac{b}{a} \leq \theta(\gamma) \leq \theta(\beta) \leq \pi/2$. Noting that $\tan = (\arctan)^{-1}$ is increasing on $[0, \pi/2)$, we have that if $\theta(\gamma) < \pi/2$ then $\theta(\alpha) = \arctan \frac{b}{a} \leq \theta(\gamma) = \arctan \frac{f}{e} \Rightarrow \frac{b}{a} \leq \frac{f}{e} \Rightarrow \frac{be}{a} \leq f \Rightarrow \lceil \frac{be}{a} \rceil \leq f$, because f is an integer. If $\theta(\gamma) = \pi/2$ then $e = 0$ and $f > 0$, so we still have $\lceil \frac{be}{a} \rceil = 0 \leq f$. Thus (ii) holds. If $\theta(\beta) < \pi/2$ then $c > 0$ and, from $\theta(\gamma) = \arctan \frac{f}{e} \leq \theta(\beta) = \arctan \frac{d}{c}$ we infer that $\lceil \frac{cf}{d} \rceil \leq e$. If $\theta(\beta) = \pi/2$ then $c = 0$, $d > 0$, so we still have $\lceil \frac{cf}{d} \rceil = 0 \leq e$. Thus (i) holds. □

We resume the proof of Theorem 3. Clearly, $(b) \Rightarrow (a)$.
$(c) \Rightarrow (b)$: We assume that $ad - bc = 1$ and prove that $U(\alpha, \beta) \subseteq SG(\alpha, \beta)$ which implies that $\text{Frob}(\alpha, \beta) = SG(\alpha, \beta) = U(\alpha, \beta)$. Suppose that $\gamma = e + if \in U(\alpha, \beta)$. We can assume that $\gamma \neq 0$. We will show the existence of $\lambda, \mu \in \mathbb{N}$ such that $\lambda\alpha + \mu\beta = \gamma$; λ and μ will be integer solutions of the linear system

$$\begin{aligned} &\lambda a + \mu c = e \\ &\lambda b + \mu d = f, \text{ which holds if and only if} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} &= \frac{1}{ad - bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} \\ &= \begin{bmatrix} de - cf \\ -be + af \end{bmatrix} \end{aligned}$$

By Lemma 2, because $\gamma \in U(\alpha, \beta)$, we have

$$\begin{aligned} &= \frac{cf}{d} \leq \left\lceil \frac{cf}{d} \right\rceil \leq e \Rightarrow \lambda = ed - cf \geq 0 \text{ and} \\ &= \frac{be}{a} \leq \left\lceil \frac{be}{a} \right\rceil \leq f \Rightarrow \mu = af - be \geq 0, \end{aligned}$$

so $\lambda, \mu \in \mathbb{N}$. This establishes $(c) \Rightarrow (b)$.

$(a) \Rightarrow (c)$: Suppose that $w \in \mathbb{Z}[i]$ and $w + U(\alpha, \beta) \subseteq (\alpha, \beta)$. As previously noted, $w = w + 0 \in w + U(\alpha, \beta) \subseteq SG(\alpha, \beta)$, so $w = \lambda'\alpha + \mu'\beta$ for some $\lambda', \mu' \in \mathbb{N}$.

For all $\gamma = e + if \in U(\alpha, \beta)$, $w + \gamma = (\lambda' a + \mu' c + e) + (\lambda' b + \mu' d + f)i \in SG(\alpha, \beta)$ implies that there exist $\lambda, \mu \in \mathbb{N}$ such that $\lambda\alpha + \mu\beta = w + \gamma$, which is equivalent to

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \lambda' a + \mu' c + e \\ \lambda' b + \mu' d + f \end{bmatrix},$$

and because $ad - bc > 0$ this implies

$$\begin{aligned} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} &= \frac{1}{ad - bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \begin{bmatrix} \lambda' a + \mu' c + e \\ \lambda' b + \mu' d + f \end{bmatrix} \\ &= \frac{1}{ad - bc} \begin{bmatrix} \lambda'(ad - bc) + de - cf \\ \mu'(ad - bc) + af - be \end{bmatrix}, \end{aligned}$$

from which we can conclude that $ad-bc \mid de-cf, af-be$ for all $e, f \in \mathbb{N}$ such that $e+if \in U(\alpha, \beta)$.

Because $\theta(\alpha) < \theta(\beta)$, $U(\alpha, \beta)$ "opens" without bound, so we can find $e, f \in \mathbb{N}$, quite large, such that $e+if$, $e+1+if$, $e+i(f+1) \in U(\alpha, \beta)$. From this we have that $ad-bc \mid de-cf, d(e+1)-cf, de-c(f+1), a(f+1)-be, af-b(e+1)$, and, from this, that $ad-bc \mid a, b, c, d$. From this it follows that $(ad-bc)^2 \mid ad-bc$. Since $ad-bc > 0$ and $ad-bc$ is an integer, we can conclude that $ad-bc = 1$. $\square$.

4 Remarks

This paper was inspired by [2], in which the ring was the same as here, $\mathbb{Z}[i]$, and the Frobenius template (not identified as such in [2], as the term did not exist at the time) was $(\mathbb{N}[i] \setminus \{0\}, \mathbb{N}[i], U)$ with U defined as follows: if $\alpha_1, \ldots, \alpha_n \in \mathbb{N}[i] \setminus \{0\}$, and $\theta(\alpha_1) \leq \ldots \leq \theta(\alpha_n)$, then $U(\alpha_1, \ldots, \alpha_n) = A(\theta(\alpha_1), \theta(\alpha_n) + \pi/2)$. Gaussian integers $\alpha_1, \ldots, \alpha_n$ are relatively prime if they have no non-unit common divisor in $\mathbb{Z}[i]$. One of the main results in [2] was the following.

Theorem G *If $\alpha_1, \ldots, \alpha_n \in \mathbb{N}[i] \setminus \{0\}$ then with reference to the Frobenius template of* [2] *(defined above), Frob$(\alpha_1, \ldots, \alpha_n) \neq \varnothing$ if and only if $\alpha_1, \ldots, \alpha_n$ are relatively prime in $\mathbb{Z}[i]$.*

In Theorems 1,2, and 3 there is no mention of Gaussian integers being relatively prime in $\mathbb{Z}[i]$; so you might ask: what is the relation between Frob$(\alpha_1, \ldots, \alpha_n)$ being non-empty, with reference to the Frobenius template $(\mathbb{N}[i] \setminus \{0\}, \mathbb{N}, U)$ of this paper, and the relative primeness of $\alpha_1, \ldots, \alpha_n$ in $\mathbb{Z}[i]$?

If $\theta(\alpha_1) = \cdots = \theta(\alpha_n) \in (0, \pi/2)$ then there is no relation, since, by Lemma 1, each α_j is divisible in $\mathbb{Z}[i]$ by the non-zero Gaussian integer $\gamma = e+if$ closest to 0 on the half-line on which the α_j lie, and γ is not a unit in $\mathbb{Z}[i]$ unless that half-line is horizontal or vertical. Yet the proof of Theorem 2 extended to cases $n > 2$ shows that Frob$(\alpha_1, \ldots, \alpha_n)$ can be empty or not.

However, in the cases $\theta(\alpha_1) = \cdots = \theta(\alpha_n) = 0$ or $\theta(\alpha_1) = \cdots = \theta(\alpha_n) = \pi/2$, the conditions for Frob$(\alpha_1, \ldots, \alpha_n) \neq \varnothing$ with reference to the template of this paper and the template of [2] are the same.

Lemma 3. *Suppose that* $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[i] \setminus \{0\}$ *and* $|\alpha_1|^2, \ldots, |\alpha_n|^2$ *are relatively prime in* $\mathbb{Z}$*. Then* $\alpha_1, \ldots, \alpha_n$ *are relatively prime in* $\mathbb{Z}[i]$*.*

Proof. If $\alpha_1, \ldots, \alpha_n$ have some non-unit common divisor in $\mathbb{Z}[i]$, say $\alpha_j = \gamma\beta_j$, $j = 1, \ldots, n$, and $\gamma, \beta_1, \ldots, \beta_n \in \mathbb{Z}[i]$, and γ is not a unit, then $|\gamma|^2 > 1$ divides each of $|\alpha_1|^2, \ldots, |\alpha_n|^2$, in $\mathbb{Z}$. □

Corollary. *If* $\alpha_1, \ldots, \alpha_n$ *are positive integers, then they are relatively prime in* $\mathbb{Z}$ *if and only if they are relatively prime in* $\mathbb{Z}[i]$*.*

Regarding the situation in Theorem 3: There are many, many choices of $a, b, c, d \in \mathbb{N}$ such that $\alpha = a + ib$ and $\beta = c + id$ are relatively prime in $\mathbb{Z}[i]$ yet $|ad - bc| > 1$. For instance, take $\alpha = 7 + i$, $\beta = 2 + 5i$. Then α and β are relatively prime in $\mathbb{Z}[i]$, by Lemma 3, since $|\alpha|^2 = 50$ and $|\beta|^2 = 29$, yet $ad - bc = 35 - 2 = 33$.

On the other hand, if $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$ then $\alpha = a + ib$ and $\beta = c + id$ are relatively prime in $\mathbb{Z}[i]$. To see this, suppose that $e, f, a', b', c', d' \in \mathbb{Z}$, $\gamma = e + if$, $\alpha' = a' + ib'$, $\beta' = c' + id'$, and $\alpha = \gamma\alpha'$, $\beta = \gamma\beta'$. Then after a short struggle one finds that $1 = ad - bc = (e^2 + f^2)(a'd' - b'c') = |\gamma|^2(a'd' - b'c')$. Thus $|\gamma|^2$ is a positive integer divisor of 1, from which it follows that γ is a unit in $\mathbb{Z}[i]$.

5 More Problems

Obviously the next big question concerning the template $(\mathbb{N}[i] \setminus \{0\}, \mathbb{N}, U)$ is: For which $\alpha_1, \ldots, \alpha_n \in \mathbb{N}[i] \setminus \{0\}, n > 2$, is $\mathrm{Frob}(\alpha_1, \ldots, \alpha_n) \neq \varnothing$? If $\theta(\alpha_1) \leq \ldots \leq \theta(\alpha_n)$, $\theta(\alpha_1) < \theta(\alpha_n)$, $\alpha_1 = a_1 + ib_1$ and $\alpha_n = a_n + ib_n$, then $a_1b_n - a_nb_1 = 1$ implies that $\mathrm{Frob}(\alpha_1, \ldots, \alpha_n) = SG(\alpha_1, \ldots, \alpha_n) = U(\alpha_1, \ldots, \alpha_n)$, by appeal to Theorem 3; but is $a_1b_n - a_nb_1 = 1$ necessary for $\mathrm{Frob}(\alpha_1, \ldots, \alpha_n) \neq \varnothing$ when $n > 2$?

Lemma 1 and the proofs of Theorems 1 and 2 show that when $\theta(\alpha_1) = \cdots = \theta(\alpha_n)$ the question of whether $\mathrm{Frob}(\alpha_1, \ldots, \alpha_n) = \varnothing$ or not can be dealt with by applying Theorem F to a sequence $r_1, \ldots, r_n$ of positive integers related to $\alpha_1, \ldots, \alpha_n$. Going in another direction: Some readers may have noticed that with our template $(\mathbb{N}[i] \setminus \{0\}, \mathbb{N}, U)$, multiplication in the ring $\mathbb{Z}[i]$ played no role in the pursuit of the associated Frobenius

questions. We imposed $\mathbb{Z}[i]$ as the context for this inquiry to make possible contrast with the work in [2], and because Looper's templates [4] normally require a ring. But in this case, we clearly could have replaced $\mathbb{Z}[i]$ by $\mathbb{Z}^2$ and $\mathbb{N}[i]$ by $\mathbb{N}^2$, and thus considered our template and our associated questions to be about a module ($\mathbb{Z}^2$) over a ring ($\mathbb{Z}$) with the coefficient part of the template being a monoid in the ring ($\mathbb{N} \subseteq \mathbb{Z}$, in this case).

Therefore, we can have an infinity of new problems: Take, in $\mathbb{Z}^n$, a Frobenius template $(\mathbb{N}^n \setminus \{(0,\ldots,0)\}, \mathbb{N}, U)$, and ask the Frobenius questions. As in the case $n = 2$, for $n > 2$, U will be a non-constant assignment of half cones to sequences $\alpha_1, \ldots, \alpha_n \in \mathbb{N}^n \setminus \{\underline{0}\}$. We leave exploration of this newly discovered island to the reader and to the future.

6 References

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