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REFERENCES

[1] A. D. N. J. de Grey. The chromatic number of the plane is at least 5. *ArXiv e-prints,* April 2018.
[2] P. Erdos. On sets of distances of n points. *The American Mathematical Monthly,* 53(5):248-250, 1946.
[3] Orrin Frink. Jordan measure and riemann integration. *Annals of Mathematics,* 34(3):518-526, 1933.
[4] Alejandra Garrido. An introduction to amenable groups. 2013.
[5] Kate Juschenko. *Amenability of discrete groups by examples.* 2015.
[6] Dan Pritikin. All unit-distance graphs of order 6197 are 6-colorable. *Journal of Combinatorial Theory, Series B,* 73(2):159-163, 1998.
[7] Polymath Project. Probabilistic formulation of hadwiger-nelson problem.
[8] Alexander Soifer. *The Mathematical Coloring Book.* Springer Science+Business Media, 2009.

On the Babai and upper chromatic numbers of $\mathbb{R}^n$ with non-Euclidean distance

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Abstract

With $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denoting the ℓ_1 and ℓ_∞ norms on $\mathbb{R}^n$, it is shown that the upper chromatic number of $(\mathbb{Z}^n, \|\cdot\|_1)$ is greater than $2n$, and the upper chromatic number of $(\mathbb{Z}^n, \|\cdot\|_\infty)$ is greater than $2n$.

1 Introduction

A **distance function** on a nonempty set X is a function $\rho : X \times X \to [0, \infty)$ satisfying, for all $x, y \in X$,

1. $\rho(x,y) = 0 \iff x = y$
2. $\rho(x,y) = \rho(y,x)$

If, in addition, for all $x, y, z \in X$:

3. $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$,

then ρ is said to be a *metric* on X, and we call the pair (X, ρ) a *metric space.* If ρ is a distance function on X, let us call the pair (X, ρ) a distance **space.**

A **coloring** of a set X is a function $\varphi : X \to C$ for some set C, a set of *colors.* A coloring φ does not need to be surjective.

If (X, ρ) is a distance space, $\varphi : X \to C$ is a coloring, $c \in C$, and $d \in (0, \infty)$, we say **distance d is forbidden for the color** c with respect to φ (omitting "with respect to φ" when context is clear), if and only if $x, y \in X$ and $\varphi(x) =$

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$\varphi(y) = c$ imply that $\rho(x, y) \neq d$. If d is forbidden for every $c \in C$, we say that φ *forbids the distance* d in (X, ρ).

If (X, ρ) is a distance space and $D \subseteq (0, \infty)$, the **chromatic number** of (X, ρ) with respect to D, written as $\chi((X, \rho), D)$, is the minimum cardinality $|C|$ of a set C such that there exists a coloring $\varphi : X \to C$ which forbids every $d \in D$. Since the identity function $\mathrm{id}_X : X \to X$ is such a coloring, it follows that $\chi((X,\rho),D)$ is well-defined and $\chi((X,\rho),D) \leq |X|$. If we define the **distance graph** $G((X, \rho), D)$ to have vertices $V(G) = X$ and edges $E(G) = \{xy : x, y \in X \text{ and } \rho(x, y) \in D\}$, then $\chi((X, \rho), D)$ is simply the chromatic number of the corresponding distance graph, i.e. the least cardinality of C such that there is a proper vertex coloring $\varphi : V(G) \to C$. When $D = \{d\}$, a singleton, we will omit the set brackets in notation involving D.

The kth **Babai number** of a distance space (X, ρ), denoted $B_k(X,\rho)$, is:

$$B_k(X,\rho) = \sup\{\chi((X, \rho), D) : D \subseteq (0, \infty) \text{ and } |D| = k\}.$$

For all of the distance spaces that we will encounter in this paper, $B_1(X, \rho)$ is finite; therefore, the "sup" in the definition is "max", as it was in the definition of the Babai numbers in [3], where they first appeared. This was a mistake; distance spaces (X, ρ) can be found (see [10]) such that:

$$\{\chi((X, \rho), d) : d > 0\} = \mathbb{Z}^+ = \{1, 2, 3, \dots\},$$

so $B_1(X, \rho) = \sup \mathbb{Z}^+ = \aleph_0$. A question in passing: is it the case that for every limit cardinal q, there exists a distance space (X, ρ) such that $\chi((X, \rho), d) < q$ for all $d > 0$, yet $B_1(X, \rho) = q$?

If S is a set and $k \in \mathbb{Z}^+$, then S^k will stand for the k-fold Cartesian product of S: $S^k = \times_{j=1}^k S = S \times \cdots \times S$. The kth **upper chromatic number** of a distance space (X, ρ), denoted $\chi^{(k)}(X, \rho)$, is the least cardinality of a set C satisfying:

For every function $d : C \to (0, \infty)^k$, there is a coloring $\varphi: X \to C$ such that for every $c \in C$, each distance d_j in the k-tuple $d(c)$ is forbidden for the color c, $j = 1, \dots, k$.

The class of such sets C is always nonempty, because X itself is such a set via the coloring $\varphi = \mathrm{id}_X$. Thus, by the fact that the cardinal numbers are well-ordered, we have that $\chi^{(k)}(X, \rho)$ is well-defined, and $\chi^{(k)}(X, \rho) \leq |X|$. When $k = 1$, we will omit k and use the notation $\chi(X, \rho)$ to stand for $\chi^{(1)}(X, \rho)$. If we use the term "upper chromatic number" without mentioning k, it is safe to assume that we are referencing the 1st upper chromatic number, $\chi(X, \rho)$.

Lemma 1. *For any distance space (X,ρ), $B_1(X, \rho) \leq \chi(X, \rho)$.*

Proof. Suppose that $|C| = \chi(X,\rho)$, so for every $d : C \to (0,\infty)$ there is a coloring $\varphi: X \to C$ such that for each $c \in C$ the distance $d(c)$ is forbidden for the color c.

For an arbitrary distance $d^* > 0$, let d stand for the constant function on C with constant value d^*. By the previous paragraph, there is a coloring $\varphi: X \to C$ that forbids the distance d^*, so $\chi((X, \rho), d^*) \leq |C|$. Therefore, $B_1(X,\rho) = \sup_{d^*>0}\chi((X,\rho),d^*) \leq |C| = \chi(X,\rho)$. □

The proof of Lemma 1 extends naturally to show $B_k(X,\rho) \leq \chi^{(k)}(X,\rho)$ for any $k \in \mathbb{Z}^+$.

Suppose that (X, ρ) is a distance space and $k, n \in \mathbb{Z}^+$. If D is a $k \times n$ matrix with entries from $(0, \infty)$, i.e. $D \in (0, \infty)^{k\times n}$, then let us say that D is **satisfiable** in (X, ρ) if there is a coloring of X with colors $(c_1, \dots, c_n)$ such that for $1 \leq i \leq n$, $1 \leq j \leq k$, the distance d_{ij} (row i, column j of D) is forbidden for the color c_i, i.e. no two points colored c_i are distance d_{ij} apart. If $D \in (0, \infty)^{1\times n}$, we might refer to D as a sequence $(d_1, \dots, d_n)$ rather than a matrix. Provided that $\chi^{(k)}(X,\rho) < \infty$, the definition of $\chi^{(k)}(X,\rho)$ may be restated as follows: $\chi^{(k)}(X, \rho)$ is the least positive integer n such that every matrix $D \in (0, \infty)^{k\times n}$ is satisfiable in (X, ρ). Notice that $B_k(X,\rho)$, when finite, is the least positive integer n such that every matrix $D \in (0, \infty)^{k\times n}$ with *identical* columns is satisfiable in (X, ρ); this framing makes it easy to see that $B_k(X, \rho) \leq \chi^{(k)}(X, \rho)$.

Lemma 2. *Suppose $n \in \mathbb{Z}^+$. If every matrix $D \in (0, \infty)^{k\times n}$ is satisfiable in (X,ρ), then for all $m \in \mathbb{Z}^+$, $m \geq n$, every matrix $D \in (0,\infty)^{k\times m}$ is satisfiable in (X,ρ).*

Proof. Given $D \in (0, \infty)^{k\times m}$, color X with $(c_1, \dots, c_n)$ so that for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k\}$, distance d_{ij} is forbidden for the color c_i. Do not use the colors $(c_{n+1}, \dots, c_m)$. □

Corollary 1. *For $k, n \in \mathbb{Z}^+$, $\chi^{(k)}(X, \rho) > n$ if and only if some matrix $D \in (0, \infty)^{k\times n}$ is not satisfiable in (X, ρ).*

We will let $(V, \|\cdot\|)$ denote a normed vector space over $\mathbb{R}$ or $\mathbb{C}$, where the norm is a function $\|\cdot\| : V \to [0, \infty)$ satisfying $\|x\| = 0 \Leftrightarrow x = 0$, $\|x + y\| \leq \|x\| + \|y\|$, and $\|kx\| = |k|\|x\|$ for all scalars k. For all subsets $U \subseteq V$, $(U, \|\cdot\|)$ will also stand for the metric space (U,ρ), with ρ defined by $\rho(x, y) = \|x - y\|$. In this paper we will see distance functions on $U \subseteq V$ defined by $\rho(x,y) = f(\|x-y\|)$, where $f: [0,\infty) \to [0,\infty)$ is a strictly increasing function satisfying $f(0) = 0$; in such cases, we shall let $(U, f(\|\cdot\|))$ stand for (U, ρ). The usual Euclidean norm on $\mathbb{R}^n$ will be denoted $|\cdot|$. Thus, on $\mathbb{R}$, $|\cdot|$ is the usual absolute value, and on $\mathbb{R}^2$, $|(x, y)| = (x^2 + y^2)^{1/2}$.

The norms $\|.\|_1$ and $\|.\|_\infty$ on $\mathbb{R}^n$ are defined by $\|(x_1,\dots,x_n)\|_1 = \sum_{i=1}^n |x_i|$ and $\|(x_1,\dots,x_n)\|_\infty = \max_{1\leqslant i\leqslant n}|x_i|$; the taxicab and maximum norms, respectively. Notice that distance spaces $(U,\|.\|)$ and $(U,f(\|.\|))$ always have translation-invariant distance functions ρ; that is, for all $x,y,z\in U$ such that $x+z, y+z\in U$, $\rho(x,y)=\rho(x+z,y+z)$.

When $(V,\|.\|)$ is a normed space (over $\mathbb{R}$ or $\mathbb{C}$), a simple change-of-scale argument shows that for any $d_1,d_2\in(0,\infty)$, $\chi((V,\|.\|),d_1)=\chi((V,\|.\|),d_2)=B_1(V,\|.\|)$. This fact does not necessarily hold for subsets of V equipped with the same metric. For instance:

$$\chi\left(\left(\mathbb{Q}^3,|.|\right),1\right)=2 \tag{[6]}$$

$$\chi\left(\left(\mathbb{Q}^3,|.|\right),\sqrt{2}\right)=4=B_1\left(\mathbb{Q}^3,|.|\right) \tag{[9]}$$

It is not known if $\hat{\chi}(\mathbb{R}^2,|.|)$ is finite. However, in a remarkable tour de force [5], Archer proves, among other things, that for all $n\in\mathbb{Z}^+$, $\hat{\chi}(\mathbb{R}^n,\|.\|_1)=\hat{\chi}(\mathbb{Z}^n,\|.\|_1)<\infty$ and $\hat{\chi}(\mathbb{R}^n,\|.\|_\infty)=\hat{\chi}(\mathbb{Z}^n,\|.\|_\infty)<\infty$. Archer also provides formulas for computing enormous upper bounds on these numbers; for example, $\hat{\chi}(\mathbb{R}^2,\|.\|_\infty)\leqslant 99$ and $\hat{\chi}(\mathbb{R}^3,\|.\|_\infty)\leqslant 106,623$. These are still the best known upper bounds to date.

Upper chromatic numbers were first defined in [7], where it was shown that $\hat{\chi}(\mathbb{R},|.|)=3$. Considerable advances were made in [1] and [5], then surveyed in [2], and then the topic lay fallow until [8], which concerned metric spaces naturally associated with connected graphs.

In the next section, we show that $\hat{\chi}(\mathbb{Z}^n,\|.\|_1)>2n$ and $\hat{\chi}(\mathbb{Z}^n,\|.\|_\infty)>2^n$ for all $n\in\mathbb{Z}^+$, which implies that the same lower bounds hold for $\mathbb{R}^n$. In the last section, we consider the Babai and upper chromatic numbers of distance spaces $(U,f(\|.\|))$.

2 Taxicab and maximum norm

Proposition 1. *For all $n\in\mathbb{Z}^+$, $\hat{\chi}(\mathbb{Z}^n,\|.\|_1)>2n$.*

Proof. Let the sequence of forbidden distances $(d_1,\dots,d_{2n-1},d_{2n})$ be $(2,\dots,2,1)$. Suppose this sequence is satisfiable in $(\mathbb{Z}^n,\|.\|_1)$, so there is a coloring of $\mathbb{Z}^n$ with colors $(c_1,\dots,c_{2n-1},c_{2n})$ such that each distance d_i is forbidden for the color c_i, $i=1,\dots,n$. Bestow the name "umber" upon color c_{2n}. Write $\pm\mathcal{E}$ for the set of all standard basis vectors $\mathbf{e}_i=(0,\dots,0,1,0,\dots,0)$ for $\mathbb{R}^n$ and their negatives $-\mathbf{e}_i$, $i=1,\dots,n$. Because all $2n$ points in $\pm\mathcal{E}$ are $\|.\|_1$ distance 2 apart, umber must appear at least once on every translate of $\pm\mathcal{E}$. Since recoloring by translation will produce a coloring satisfying our supposition, we may assume that $\mathbf{0}=(0,\dots,0)\in\mathbb{Z}^n$ is colored umber. Since $\mathbf{0}\notin\pm\mathcal{E}$, at least one point in $\pm\mathcal{E}$ is also colored umber, so we have two points $\|.\|_1$ distance 1 apart colored umber, contrary to the supposition. ☐

A weaker version of this result holds in many other metric spaces. If $p\geq 1$ is a real number, then the function $\|.\|_p:\mathbb{R}^n\longrightarrow[0,\infty)$ defined by $\|(x_1,\dots,x_n)\|_p=(\sum_{i=1}^n|x_i|^p)^{\frac{1}{p}}$ produces a normed space $(\mathbb{R}^n,\|.\|_p)$ with a translation-invariant metric. In $(\mathbb{Z}^n,\|.\|_p)$, the n standard basis vectors (not the $2n$ vectors in $\pm\mathcal{E}$) are all $\|.\|_p$ distance $\sqrt[p]{2}$ apart, so the above argument may be adapted, with distances $(d_1,\dots,d_{n-1},d_n)=(\sqrt[p]{2},\dots,\sqrt[p]{2},1)$, to prove that $\hat{\chi}(\mathbb{Z}^n,\|.\|_p)>n$. To gauge how weak this result may be, note that $\|.\|_2$ is the usual Euclidean norm.

Proposition 2. *For all $n\in\mathbb{Z}^+$, $\hat{\chi}(\mathbb{Z}^n,\|.\|_\infty)>2^n$. Generally, $\hat{\chi}^{(k)}(\mathbb{Z}^n,\|.\|_\infty)>(k+1)^n$ for all $k\in\mathbb{Z}^+$.*

Proof. First we will show $\hat{\chi}(\mathbb{Z}^n,\|.\|_\infty)>2^n$. Let the sequence of forbidden distances $(d_1,\dots,d_{2^n-1},d_{2^n})$ be $(1,\dots,1,2)$. Suppose this sequence is satisfiable in $(\mathbb{Z}^n,\|.\|_\infty)$, so there is a coloring of $\mathbb{Z}^n$ with colors $(c_1,\dots,c_{2^n-1},c_{2^n})$ such that each distance d_i is forbidden for the color c_i, $1\leqslant i\leqslant n$. Give the name "indigo" to color c_{2^n}. Since $\{0,1\}^n$, the vertices of a unit hypercube, has 2^n points that are all $\|.\|_\infty$ distance 1 apart, indigo must appear on every translate of $\{0,1\}^n$ in $\mathbb{Z}^n$. As in the proof of Proposition 1, we may assume that $\mathbf{0}=(0,\dots,0)\in\mathbb{Z}^n$ is colored indigo. Then no element of $\{2\}\times\{0,1\}^{n-1}$ is colored indigo. Therefore, because indigo must appear on the translate $(1,0,\dots,0)+\{0,1\}^n=(\{1\}\times\{0,1\}^{n-1})\cup(\{2\}\times\{0,1\}^{n-1})$, it follows that indigo must appear on $\{1\}\times\{0,1\}^{n-1}$. By a similar argument, indigo must appear on $\{-1\}\times\{0,1\}^{n-1}$. But this means that indigo appears on two elements in $\mathbb{Z}^n$ that are $\|.\|_\infty$ distance 2 apart.

To generalize this proof to $\hat{\chi}^{(k)}(\mathbb{Z}^n,\|.\|_\infty)$, construct the $k\times(k+1)^n$ distance matrix

$$D=\begin{bmatrix}1 & 1 & \dots & 1 & 2\\ 2 & 2 & \dots & 2 & 3\\ 3 & 3 & \dots & 3 & 4\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ k & k & \dots & k & k+1\end{bmatrix}$$

and let $\{0,1,\dots,k\}^n$ play the role on $\hat{\chi}$ that $\{0,1\}^n$ played above. ☐

Note that the lower bounds on $\hat{\chi}$ in propositions 2 and 3 are sharp for $n=1$ since $3=\hat{\chi}(\mathbb{Z},|.|)=\hat{\chi}(\mathbb{Z},\|.\|_1)=\hat{\chi}(\mathbb{Z},\|.\|_\infty)$ [7].

The existence of $\pm\mathcal{E}$ in Proposition 1 demonstrates that $B_1(\mathbb{Z}^n,\|.\|_1)\geqslant 2n$, and the existence of $\{0,1\}^n$ in Proposition 2 demonstrates that $B_1(\mathbb{Z}^n,\|.\|_\infty)\geqslant$

2^n. The latter result was already known, because a proof in [4] showed that if k and n are positive integers, then $B_k(\mathbb{Z}^n, \|.\|_\infty) = B_k(\mathbb{R}^n, \|.\|_\infty) = (k+1)^n$. Since $(\mathbb{R}^2, \|.\|_1)$ and $(\mathbb{R}^2, \|.\|_\infty)$ are isometrically isomorphic, it follows that $B_k(\mathbb{R}^2, \|.\|_1) = (k+1)^2$, so $B_1(\mathbb{R}^2, \|.\|_1) = 4$. Given that $B_1(\mathbb{Z}, \|.\|_1) = B_1(\mathbb{R}, \|.\|_1) = 2$, perhaps $B_1(\mathbb{Z}^n, \|.\|_1) = B_1(\mathbb{R}^n, \|.\|_1) = 2n$? The following result answers part of this question.

Proposition 3. $B_k(\mathbb{R}^n, \|.\|_1) = B_k(\mathbb{Z}^n, \|.\|_1)$ *for all* $k \in \mathbb{Z}^+$.

We owe the ideas behind this proof to Archer's [5] proof that $\hat{\chi}(\mathbb{R}^n, \|.\|_1) = \hat{\chi}(\mathbb{Z}^n, \|.\|_1)$.

Proof. Clearly $B_k(\mathbb{Z}^n, \|.\|_1) \leqslant B_k(\mathbb{R}^n, \|.\|_1)$. Now set $m = B_k(\mathbb{R}^n, \|.\|_1) - 1$, noting that $B_k(\mathbb{R}^n, \|.\|_1) < \infty$ by a proof in [3]. Since $m < B_k(\mathbb{R}^n, \|.\|_1)$, there exists a set $\hat{D} = \{\hat{d}_1, \ldots, \hat{d}_k\} \subseteq (0, \infty)$ such that the distance graph $G((\mathbb{R}^n, \|.\|_1), \hat{D})$ is not properly vertex colorable with m colors. By the De Bruijn-Erdős theorem, some finite subgraph of this infinite graph is not properly colorable with m colors, i.e. there exists a finite set $\hat{F} = \{\hat{\underline{x}}^{(1)}, \ldots, \hat{\underline{x}}^{(r)}\} \subseteq \mathbb{R}^n$ such that $\chi((\hat{F}, \|.\|_1), \hat{D}) > m$. Using $\hat{F}$ and $\hat{D}$, we will construct sets $F' \subseteq \mathbb{Q}^n$ and $D' = \{d'_1, \ldots, d'_k\} \subseteq (0, \infty)$ such that $G((F', \|.\|_1), D')$ is not properly colorable with m colors. Since we can scale such F' and D' to F'' and D'' such that $F'' \subseteq \mathbb{Z}^n$ and $\chi((F'', \|.\|_1), D'') > m$, this will complete the proof. To find F' and D', it is sufficient to construct a system of homogeneous linear equations involving the $nr + k$ unknowns $d_1, \ldots, d_k$ and $x_i^{(j)}$, $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, r\}$, such that

1. the coefficients in the equations are rational;
2. the vector $\left(\hat{x}_1^{(1)}, \ldots, \hat{x}_n^{(1)}, \ldots, \hat{x}_1^{(r)}, \ldots, \hat{x}_n^{(r)}, \hat{d}_1, \ldots, \hat{d}_k\right) \in \mathbb{R}^{nr+k}$ arising from $\left(\hat{F}, \hat{D}\right)$ is a solution; and
3. if a solution $\left(x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(r)}, \ldots, x_n^{(r)}, d_1, \ldots, d_k\right)$ of the system is sufficiently close, coordinatewise, to $\left(\hat{x}_1^{(1)}, \ldots, \hat{x}_n^{(1)}, \ldots, \hat{x}_1^{(r)}, \ldots, \hat{x}_n^{(r)}, \hat{d}_1, \ldots, \hat{d}_k\right)$ then, for $1 \leqslant p < q \leqslant r$ and $t = 1, \ldots, k$, we have $d_t > 0$, $\underline{x}^{(p)} = \left(x_1^{(p)}, \ldots, x_n^{(p)}\right)$ does not equal $\underline{x}^{(q)} = \left(x_1^{(q)}, \ldots, x_n^{(q)}\right)$, and
$$\left\|\hat{\underline{x}}^{(p)} - \hat{\underline{x}}^{(q)}\right\|_1 = \hat{d}_t \implies \left\|\underline{x}^{(p)} - \underline{x}^{(q)}\right\|_1 = d_t.$$

Property 1 ensures that the solution space of this system is the kernel of a matrix with rational entries, so the solution space has a basis of rational vectors. Combined with property 2, this means there exist rational solutions arbitrarily close to the vector associated with $(\hat{F}, \hat{D})$. Using the notation $F = \{\underline{x}^{(1)}, \ldots, \underline{x}^{(r)}\}$ and $D = \{d_1, \ldots, d_k\}$, property 3 guarantees that sufficiently close rational solutions will produce a distance graph $G((F, \|.\|_1), D)$ containing a subgraph isomorphic to $G((\hat{F}, \|.\|_1), \hat{D})$, so sufficiently close rational solutions will produce the F' and D' we want.

All that remains is to find such a system. For $i \in \{1, \ldots, n\}$ and $p, q \in \{1, \ldots, r\}$, let
$$\hat{s}^{p,q,i} = \begin{cases} 1 & \hat{x}_i^{(p)} > \hat{x}_i^{(q)} \\ 0 & \hat{x}_i^{(p)} = \hat{x}_i^{(q)} \\ -1 & \hat{x}_i^{(p)} < \hat{x}_i^{(q)} \end{cases}$$
and include the equation $\sum_i \hat{s}^{p,q,i}(x_i^{(p)} - x_i^{(q)}) - d_t = 0$ in the system whenever $\left\|\hat{\underline{x}}^{(p)} - \hat{\underline{x}}^{(q)}\right\|_1 = \hat{d}_t$. This is almost sufficient for Property 3, but we also need to ensure that for close solutions, the relationship between $x_i^{(p)}$ and $x_i^{(q)}$ matches with the relationship encoded in $\hat{s}^{p,q,i}$ between $\hat{x}_i^{(p)}$ and $\hat{x}_i^{(q)}$, so we also include $x_i^{(p)} - x_i^{(q)} = 0$ whenever $\hat{s}^{p,q,i} = 0$. This system satisfies properties 1, 2, and 3. □

Now we can revisit the earlier conjecture that $B_1(\mathbb{R}^n, \|.\|_1) = 2n$. Thanks to Proposition 3, it is sufficient to prove that $B_1(\mathbb{Z}^n, \|.\|_1) = 2n$. Note that $\chi((\mathbb{Z}^n, \|.\|_1), d) = 2$ when $d > 0$ is odd, since two points are an odd $\|.\|_1$ distance apart if and only if their $\|.\|_1$ norms have different parities; therefore, all efforts should concentrate on $\chi((\mathbb{Z}^n, \|.\|_1), d)$ when d is even.

One can obtain a crude lower bound on $B_k(\mathbb{Z}^2, \|.\|_1)$ as follows: set $D = \{1, 2, \ldots, k\}$, and consider the distance graph $G\left((\mathbb{Z}^2, \|.\|_1), D\right)$. If k is even so that $r = \frac{k}{2}$ is an integer, then the vertices in $\{\pm(r-i), \pm i) \in \mathbb{Z}^2 : 0 \leqslant i \leqslant r\}$ (such as $(r-1, 1), (r-1, -1), (-r+1, 1), (-r+1, -1)$) are all a $\|.\|_1$ distance in D apart. The same applies to all the points inside the rotated square boundary formed by these vertices, so counting the vertices in rows shows that there is a clique on $2(1 + 3 + \cdots + (k-1)) + k + 1 = \frac{1}{2}k^2 + k + 1$ vertices in the distance graph, so $B_k(\mathbb{Z}^2, \|.\|_1) \geqslant \frac{1}{2}k^2 + k + 1$. A similarly defined clique produces a lower bound for odd k. This technique highlights a general strategy for obtaining lower bounds on $B_k(X, \rho)$ for any distance space (X, ρ): find k distances such that there are large cliques in the distance graph. Abrams discusses several such bounds in [2]; for example, $B_3(\mathbb{R}^3, |.|) \geqslant 12$, since all the vertices of a regular icosahedron are one of three possible $|.|$ distances apart.

Here is one final result for this section, also inspired by Archer [5]. We will need the lemma below, which comes from [4].

Lemma 3. $B_k(\mathbb{R}^m, \|.\|_\infty) = (k+1)^m$ *for all* $k, m \in \mathbb{Z}^+$.

Proposition 4. $B_k(\mathbb{R}^n, \|.\|_1) \leqslant (k+1)^{(2^n-1)}$ *for all* $k, n \in \mathbb{Z}^+$.

Proof. Let $D = \{d_1, \ldots, d_k\}$ be an arbitrary sequence of distances in $(0, \infty)$, and let $r = (k+1)^{(2^n-1)}$. To complete the proof, we will construct a coloring of $(\mathbb{R}^n, \|.\|_1)$ with r colors that forbids all the distances in D.

Denote the power set of $\{1, \ldots, n\}$ by $2^{[n]}$. Construct a family of sets $\mathscr{F} \subseteq 2^{[n]}$ such that if $S, R \in 2^{[n]}$ and $S = \{1, \ldots, n\} \setminus R$, then exactly one of S or R is in $\mathscr{F}$, but not both. Use the 2^{n-1} sets in $\mathscr{F}$ to index the coordinates of $\mathbb{R}^{2^{n-1}}$, then define a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^{2^{n-1}}$ as follows: for each $S \in \mathscr{F}$, the Sth coordinate of $T((x_1, \ldots, x_n))$ is equal to the quantity $\left(\sum_{i \in S} x_i - \sum_{i \notin S} x_i\right)$. Notice that T is an isometry, i.e. $\|x\|_1 = \|T(x)\|_\infty$. By the above lemma, there is a coloring $\varphi : \mathbb{R}^{2^{n-1}} \longrightarrow \{c_1, \ldots, c_r\}$ such that for all $u, v \in \mathbb{R}^{2^{n-1}}$, $\varphi(u) = \varphi(v) \implies \|u - v\|_\infty \notin D$. To conclude, define a coloring $\psi : \mathbb{R}^n \longrightarrow \{c_1, \ldots, c_r\}$ by $\psi(x) = \varphi(T(x))$, and notice that

$$\begin{aligned}\psi(x) = \psi(y) &\implies \varphi(T(x)) = \varphi(T(y)) \implies \|T(x) - T(y)\|_\infty \notin D \\ &\implies \|x - y\|_1 \notin D.\end{aligned}$$

□

Although $2 \geqslant B_1(\mathbb{R}, \|.\|_1) = 2$ and $4 \geqslant B_1(\mathbb{R}^2, \|.\|_1) = 4$ are sharp and $16 \geqslant B_1(\mathbb{R}^3, \|.\|_1) \geqslant 6$ seems reasonable, $2^8 \geqslant B_1(\mathbb{R}^4, \|.\|_1) \geqslant 8$ suggests that this upper bound has significant room for improvement, not to mention $2^{16} \geqslant B_1(\mathbb{R}^5, \|.\|_1) \geqslant 10$.

3 Translation-invariant distance functions

As discussed in the first section, $(V, \|.\|)$ stands for a normed vector space over $\mathbb{R}$ or $\mathbb{C}$. If $f : [0, \infty) \longrightarrow [0, \infty)$ is a strictly increasing function that vanishes at 0, then $\rho(x, y) = f(\|x - y\|)$ is a distance function on any subset $U \subseteq V$, although it may or may not be a metric. Such distance spaces will be abbreviated as $(U, f(\|.\|))$.

Proposition 5. *If* $V \neq \{0\}$, *then* $\hat{\chi}(V, f(\|.\|)) > 2$.

Proof. Since V is nontrivial, there is a nonzero point $x \in V$. Consider the forbidden distance sequence

$$(d_1, d_2) = \big(f(\|x\|),\ f(\|2x\|)\big),$$

with respective colors c_1 and c_2. If $\mathbf{0} \in V$ is colored c_1, then x and $-x$ must both be colored c_2, but they are distance $f(\|2x\|)$ apart; therefore, $\mathbf{0}$ must be colored c_2. Then $2x$ cannot be colored c_2, so $2x$ must be colored c_1. And $f(\|2x - x\|) = f(\|x\|)$, so x must be colored c_2. A similar argument involving $-2x$ shows that $-x$ must be colored c_2, but then x and $-x$ are distance $f(\|2x\|)$ apart and both colored c_2. □

Proposition 6. $\hat{\chi}(\mathbb{R}, f(|.|)) = 3$.

Proof. We will show that for any $d_1, d_2, d_3 \in (0, \infty)$, the sequence (d_1, d_2, d_3) is satisfiable in $(\mathbb{R}, f(|.|))$ with colors c_1, c_2, and c_3. If some $d_i \notin f([0, \infty))$, i.e. if the distance d_i is never realized in $(\mathbb{R}, f(|.|))$, then we may color all of $\mathbb{R}$ with the color c_i. So we may as well assume that each of d_1, d_2, d_3 is in the range of f. And f is injective since f is strictly increasing, so there exist unique $g(d_1), g(d_2), g(d_3) \in (0, \infty)$ such that $f(|x - y|) = d_i \iff |x - y| = g(d_i)$, $i = 1, 2, 3$. By the main result in [7], there exists a coloring of $\mathbb{R}$ with c_1, c_2, c_3 such that ordinary $|.|$ distance $g(d_i)$ is forbidden for the color c_i, $i = 1, 2, 3$. □

The key part of this proof is that f is invertible when its codomain is restricted to its range, so points in the underlying set $\mathbb{R}$ are a given $f(|.|)$ distance apart if and only if they are some unique $|.|$ distance apart. The same correspondence exists between any normed space $(V, \|.\|)$ and $(V, f(\|.\|))$, where $f : [0, \infty) \longrightarrow [0, \infty)$ is strictly increasing and vanishing at 0. Thus, coloring $(V, \|.\|)$ is equivalent to coloring $(V, f(\|.\|))$.

For those who demand that the topology imposed by the distance function $\rho(x, y) = f(|x - y|)$ on $\mathbb{R}$ be the usual topology: all that is needed for that is continuity of f at 0. If f (strictly increasing, $f(0) = 0$) is not continuous at 0, then the topology imposed on $\mathbb{R}$ by $f(|.|)$ is the discrete topology, in which single points are open sets. The fact that $\hat{\chi}(\mathbb{R}, f(|.|)) = 3$ even in these cases sends the message that upper chromatic numbers do not have much to do with topology. This message will be further confirmed by a disturbing example at the end of this section.

Proposition 7. *For all* $k \in \mathbb{Z}^+$, $B_k(\mathbb{Z}, f(|.|)) = B_k(\mathbb{R}, f(|.|)) = k + 1$.

Proof. The k positive distances $f(1), \ldots, f(k)$ are distinct, and for any two points $x, y \in \{0, 1, \ldots, k\}$, $x \neq y$, we have $f(|x - y|) \in \{f(1), \ldots, f(k)\}$. Thus

$$\begin{aligned}B_k(\mathbb{R}, f(|.|)) &\geqslant B_k(\mathbb{Z}, f(|.|)) \\ &\geqslant \chi\big((\mathbb{Z}, f(|.|), \{f(1), \ldots, f(k)\}\big) \geqslant k + 1\end{aligned}$$

Suppose that $0 < d_1 < \cdots < d_k$. As before, we may as well assume that $d_1, \ldots, d_k \in f([0, \infty))$, so we can set $g(d_i)$ to be the unique point in $(0, \infty)$

such that $f(g(d_i)) = d_i$, $1 \leqslant i \leqslant k$. It is shown [4] and [11] that $B_k(\mathbb{R}, |.|) = k+1$, so $\mathbb{R}$ can be colored with $k+1$ colors to forbid the usual $|.|$ distances $g(d_1), \ldots, g(d_k)$, and the same coloring will forbid the $f(|.|)$ distances $d_1, \ldots, d_k$. $\square$

Proposition 7 is a special case of the following, which is a generalization of the result in [4] that $B_k(\mathbb{R}^n, \|.\|_\infty) = (k+1)^n$.

Proposition 8. *For all $n, k \in \mathbb{Z}^+$, $B_k(\mathbb{Z}^n, f(\|.\|_\infty)) = B_k(\mathbb{R}^n, f(\|.\|_\infty) = (k+1)^n$.*

Proof. The proof is very much like the proof of Proposition 7; $\{0, 1, \ldots, k\}^n$ plays the role that $\{0, 1, \ldots, k\}$ played there. First, notice that $\{0, 1, \ldots, k\}^n$ is a clique in the distance graph $G((\mathbb{Z}^n, \|.\|_\infty), \{f(1), \ldots, f(k)\})$.

Given distances $d_1, \ldots, d_k > 0$, color $\mathbb{R}$ with $k+1$ colors (possibly not using all of them) to forbid these distances in $(\mathbb{R}, f(|.|))$, and express this coloring as a coloring function $\varphi : \mathbb{R} \longrightarrow \{1, \ldots, k+1\}$. Then color each $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with $(\varphi(x_1), \ldots, \varphi(x_n)) \in \{1, \ldots, k+1\}^n$ to achieve a coloring of $\mathbb{R}^n$ with $(k+1)^n$ colors which forbids the $f(\|.\|_\infty)$ distances $d_1, \ldots, d_k$. $\square$

Here is a disturbing, but interesting, example. Let

$$f(x) = \begin{cases} x, & 0 \leqslant x < 1 \\ 1, & 1 \leqslant x \end{cases}$$

Then ρ, defined by $\rho(x, y) = f(|x-y|)$, is a translation-invariant metric on $\mathbb{R}$ that induces the usual topology on $\mathbb{R}$. Noting that $\rho(x, y) = 1$ if and only if $|x-y| \geqslant 1$, we can see that the integers compose a clique in the distance graph $G((\mathbb{R}, f(|.|)), 1)$, so $\aleph_0 \leqslant \chi((\mathbb{R}, f(|.|)), 1) \leqslant B_1(\mathbb{R}, f(|.|))$. In fact, the chromatic number is precisely $\aleph_0$, as we can color each interval $[n, n+1)$ with the color $n \in \mathbb{Z}$ to achieve a coloring of $(\mathbb{R}, f(|.|))$ that forbids the distance 1. Meanwhile, if $0 < d < 1$, then $\chi((\mathbb{R}, f(|.|), d) = \chi((\mathbb{R}, |.|), d) = 2$; and if $d > 1$, then clearly $\chi((\mathbb{R}, f(|.|)), d) = 1$. Thus $B_1(\mathbb{R}, f(|.|)) = \sup\{1, 2, \aleph_0\} = \aleph_0$ and $\hat{\chi}(\mathbb{R}, f(|.|)) \geqslant \aleph_0$. In fact, the same goes for any finite-dimensional normed space $(V, \|.\|)$: $\hat{\chi}(V, f(\|.\|)) \geqslant B_1(V, f(\|.\|)) = \aleph_0$. Proof omitted.

References

[1] Aaron Abrams. The kth upper chromatic number of the line. *Discrete Mathematics*, 169:157–162, 1997.

[2] Aaron Abrams. Upper chromatic numbers: an update. *Geombinatorics*, 10:4–11, July, 2000.

[3] Aaron Abrams and Peter Johnson. Yet another species of forbidden-distances chromatic number. *Geombinatorics*, 10:89–95, 2001.

[4] Loren Anderson and Peter Johnson. Constructively coloring the line. *Geombinatorics*, 23:52–64, October, 2013.

[5] Aaron F. Archer. On the upper chromatic number of the reals. *Discrete Mathematics*, 214:65–75, 2000.

[6] Miro Benda and Micha Perles. Colorings of metric spaces. *Geombinatorics*, 9:113–126, January, 2000.

[7] Don Greenwell and Peter Johnson. Forbidding prescribed distances for designated colors. *Geombinatorics*, 2(1):13–16, 1992.

[8] Peter Johnson and Alexis Krumpelman. On the Babai and upper chromatic numbers of graphs of diameter 2. (To appear in the *Tamkong Journal of Mathematics*).

[9] Peter Johnson, Andrew Schneider, and Michael Tiemeyer. $B_1(\mathbb{Q}^3) = 4$. *Geombinatorics*, 16:356–362, April, 2007.

[10] Peter Johnson and Michael Tiemeyer. Which pairs of distances can be forbidden by a 4-coloring of $\mathbb{Q}^3$? *Geombinatorics*, 18:161–169, April, 2009.

[11] Arnfried Kemnitz and Massimiliano Marangio. Coloring the line. *Ars Combinatoria*, 85:183–192, 2007.