of its first vertex and by the orientation angle ϕ of its second one, and the average value of the binary function $M_k(\sigma, \phi)$ was determined, which takes the value 1 or 0 depending on whether an edge is mono-chromatic or not. Instead, we took an algebraic method, calculating the integral $p_k = \frac{1}{2\pi S} \int_S d\sigma \int_0^{2\pi} d\phi\, M_k(\sigma, \phi)$ in the `Mathematica` package, which allowed us to refine the optimal tiling parameters. However, we encountered some problems with the numerical integration function `NIntegrate` in the general case, and were forced to limit ourselves to relatively simple tilings.

As a result, we get the bounds: $e_5 \geq 99$, $v_5 \geq 28$, $e_6 \geq 182$, $v_6 \geq 42$.

We also tried Gwyn-Stavrianos' approach for the case $k = 7$, and obtained the following estimates for a tiling close to Pritikin's construction using pentagonal tiles: $e_7 \geq 232646$, $v_7 \geq 6456$. (For the construction using hexagonal tiles, we were able to get only $e_7 \geq 69451$ and $v_7 \geq 2608$.) This is better than [7], but falls short of [6]. However, there is still some hope of beating the latter one by using tiles with curved borders (the so-called "wavy edges" used in [6]).

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A Generalization of Szlam's Lemma

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Abstract

The best-known form of Szlam's Lemma asserts that if $\mathbb{R}^n$ is colored with red and blue so that no two blue points are at Euclidean distance 1 from each other, and there is a non-empty set $F \subseteq \mathbb{R}^n$ so that no translate of F is all red, then the chromatic number $\chi(\mathbb{R}^n, 1)$ of the Euclidean unit distance graph on $\mathbb{R}^n$ is no greater than $|F|$. Thus, the recent shocking discovery by Aubrey de Grey that $\chi(\mathbb{R}^2, 1) \geq 5$ implies an equally shocking improvement of a famous result of R. Juhász: if $\mathbb{R}^2$ is colored with red and blue so that Euclidean distance 1 is forbidden for the color blue, then the red set must contain translates of each 4-point subset of $\mathbb{R}^2$. Here we prove a theorem that contains Szlam's Lemma as a special case [1].

1 The Lemma and Its Generalization

Suppose (X, ρ) is a metric space and $D \subseteq (0, \infty)$. The distance graph $G = G((X, \rho), D)$ is defined by: $V(G) = X$ and $x, y \in X$ are adjacent in G if and only if $\rho(x, y) \in D$. The chromatic number of G, denoted $\chi((X, \rho), D)$ (note the suppression of the symbol G) is the minimum cardinality $|C|$ among sets C such that X can be colored by the elements of C so that all the distances in D are "forbidden" for each color in C.

This last bit, employing the jargon of the topic, requires explanation. Such a coloring is a function $\varphi : X \to C$ such that if $x, y \in X$ and $\rho(x, y) \in D$, then $\varphi(x) \neq \varphi(y)$. More generally, if $\varphi : X \to C$ is a coloring, $c \in C$ and $d > 0$, we say the coloring φ forbids the distance d for the color c if, whenever $x, y \in X$ and $\varphi(x) = \varphi(y) = c$, it must be that $\rho(x, y) \neq d$.

[1]This research was supported by NSF DMS grant no. 1950563

If ρ is fixed in the discussion, the notation for the chromatic number is simplified to $\chi(X, D)$, and if $D = \{d\}$, a singleton, to $\chi(X, d)$. In the common notation for the chromatic number of the plane, $\chi(\mathbb{R}^2, 1)$, it is understood that the metric involved is the usual Euclidean distance, $\rho((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Szlam's Lemma ([1],[4], [8]) now has several different forms, for application in different contexts, but it was originally discovered as a theorem that established a connection between the search for $\chi(\mathbb{R}^2, 1)$ and properties of certain 2-colorings of $\mathbb{R}^2$, in which the distance 1 is forbidden for one of the colors. We will give a version of the lemma that is a good deal more general than the original, yet far from the most abstruse of its derivatives.

Lemma 1 (Szlam's Lemma). *Suppose $(A, +, 0)$ is an abelian group, with a translation-invariant metric ρ, $\emptyset \neq B \subseteq A$ is closed under $+$, and $D \subseteq (0, \infty)$. Suppose that B can be colored with red and blue so that every $d \in D$ is forbidden for blue, and, for some set $F \subseteq B$, no translate $b + F, b \in B$ is all red. Then $\chi((B, \rho), D) \leq |F|$.*

The following theorem contains the preceding as a special case.

Theorem 1. *Suppose that $(A, +, 0)$ is an abelian group, with a translation-invariant metric ρ, $\emptyset \neq B \subseteq A$ is closed under $+$, and $D \subseteq (0, \infty)$. Suppose that B can be colored with red and k other colors, $c_1, ..., c_k$, so that each $d \in D$ is forbidden for each color $c_i, i = 1, ..., k$, and, for some set $F \subseteq B$, no translate $b + F$, $b \in B$, is all red. Then $\chi((B, \rho), D) \leq k|F|$*

Proof. We color B with ordered pairs $(c_i, f), i \in \{1, ..., k\}$, $f \in F$, as follows: $b \in B$ is colored by (c_i, f) satisfying the condition that $b + f$ is colored c_i. Such a pair can be found because $b + F$ is not all red.

It remains to be seen that no two elements b, b' of B with the same ordered pair color, say (c_i, f), are a distance $d \in D$ from each other. We have $\rho(b, b') = \rho(b + f, b' + f) \notin D$ because ρ is translation-invariant, and every $d \in D$ is forbidden for the color c_i. □

Applications and corollaries will be discussed in section 3.

2 Some History

Colorings of $\mathbb{R}^2$ with red and blue such that some single distance is forbidden for blue were introduced in [3] (for the first time, so far as we know) as phenomena of interest independent of the question of $\chi(\mathbb{R}^2, 1)$. It was ingeniously proven that for every such coloring, the red set must contain a translate of each 3-point

set in the plane. This result, quite spectacular in the 1970's, is an instantaneous corollary of Szlam's Lemma and the well-known (by 1970) fact that $\chi(\mathbb{R}^2, 1) \geq 4$.

Erdős, et al, went on in [3] to ask if $\mathbb{R}^2$ could be colored with red and blue so that distance 1 is forbidden for blue and no square of side length 1 has all red corners. Soon after, R. Juhász [6] answered the larger question represented by the no-red-unit-square problem, by showing that for every coloring of $\mathbb{R}^2$ with red and blue such that distance 1 is forbidden for blue, the red set must contain a congruent copy of each 4-point set in the plane. (Sets $F, F' \subseteq \mathbb{R}^2$ are congruent copies of each other if F can be mapped exactly onto F' by a translation followed by a rotation.)

Juhász's proof was a tour-de-force, and her result remained a sort of summit in the mountain range of Euclidean coloring results, until A. de Grey's proof [2] that $\chi(\mathbb{R}^2, 1) \geq 5$. As noted in [5], this result plus Szlam's Lemma immediately improves Juhász's theorem to: for every coloring of $\mathbb{R}^2$ with red and blue such that the distance 1 is forbidden for blue, the red set contains a translate of each 4-point set in the plane.

3 What's next?

We'll confine our attention to normed spaces $(\mathbb{R}^n, \|\|)$ and the single distance 1. The norms of most interest are the norms $\|\|_p$, $1 \leq p \leq \infty$, defined by

$$\|(x_1, ..., x_n)\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}, \qquad 1 \leq p < \infty, \text{ and}$$
$$\|(x_1, ..., x_n)\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

It is folkloric that

$$\chi((\mathbb{R}^2, \|\|_p), 1) = 4, \qquad p \in \{1, \infty\}, \text{ and that}$$
$$\chi((\mathbb{R}^2, \|\|_p), 1) \geq 4, \qquad 1 < p < \infty.$$

Thanks to A. de Grey [2], we now know that $\chi((\mathbb{R}^2, \|\|_2), 1) \geq 5$.

We define a RPkS (red-plus-k-special) coloring of $(\mathbb{R}^n, \|\|)$ to be a coloring of $\mathbb{R}^n$ with $k+1$ colors, red and k others, $c_1, ..., c_k$, such that the distance (defined by $\|\|$) 1 is forbidden for each color $c_1, ..., c_k$. Following [7], we define the Szlam k-number of $(\mathbb{R}^n, \|\|)$ by $Sz_k(\mathbb{R}^n, \|\|) = \min[|F|; F \subseteq \mathbb{R}^n$ and, for some RPkS coloring of $(\mathbb{R}^n, \|\|)$ no translate of F is all red].

Proposition 1. *If k and t are positive integers and $k < t$, then for all n and for all norms $\|\|$ on $\mathbb{R}^n$, $Sz_t(\mathbb{R}^n, \|\|) \le Sz_k(\mathbb{R}^n, \|\|)$.*

Proof. If $F \subseteq \mathbb{R}^n$ and there is a RPkS coloring of $(\mathbb{R}^n, \|\|)$ with colors red, $c_1, \dots, c_k$ which forbids red translates of F, then we can produce a RPtS coloring of $(\mathbb{R}^n, \|\|)$ which forbids red translates of F by changing the color of each of $t-k$ singletons to one of the new colors. □

Observe that a RPkS coloring of $(\mathbb{R}^n, \|\|)$ can forbid red translates of a singleton if and only if no point of $\mathbb{R}^n$ is colored red. This implies the following curious result.

Proposition 2. *Suppose that n and k are positive integers and $\|\|$ is a norm on $\mathbb{R}^n$. Then $Sz_k(\mathbb{R}^n, \|\|) = 1$ if and only if $k \ge \chi((\mathbb{R}^n, \|\|), 1)$.*

From the Theorem in Section 1 and propositions above, plus results in [7], we have the following.

Corollary 1. *A. For all positive integers n and k, and all norms $\|\|$ on $\mathbb{R}^n$, $\chi((\mathbb{R}^n, \|\|), 1) \le kSz_k(\mathbb{R}^n, \|\|)$.*

B. For $1 \le p \le \infty$,

- $4 \le Sz_1(\mathbb{R}^2, \|\|_p)$
- $5 \le Sz_1(\mathbb{R}^2, \|\|_2) \le 7$, *and*
- $4 = Sz_1(\mathbb{R}^2, \|\|_p), p \in \{1, \infty\}$

C. $3 \le Sz_2(\mathbb{R}^2, \|\|_2)$

Remarks

1. The inequality $Sz_1(\mathbb{R}^2, \|\|_2) \le 7$ is shown in [7].

2. Inequality C arises from the Theorem and $\chi((\mathbb{R}^2, \|\|_2), 1) \ge 5$. If $Sz_2(\mathbb{R}^2, \|\|_2) = 3$, then $\chi((\mathbb{R}^2, \|\|_2), 1) \le 6$, which would be a major advance. Further on in this paper we will describe an RP3S coloring of $(\mathbb{R}^2, \|\|_2)$ which forbids, for red, all translates of a 3-point set in $\mathbb{R}^2$. This does not tell us anything we do not already know about $\chi((\mathbb{R}^2, \|\|_2), 1)$, but it does show that $Sz_3(\mathbb{R}^2, \|\|_2) \in \{2, 3\}$.

3. To prove the equalities $4 = Sz_1(\mathbb{R}^2, \|\|_p), p \in \{1, \infty\}$, it suffices to prove the equality for $p = \infty$, because $(\mathbb{R}^2, \|\|_\infty)$ and $(\mathbb{R}^2, \|\|_1)$ are isometrically isomorphic. For $m, n \in \mathbb{Z}$, color the square $[m, m+1) \times [n, n+1)$ blue if both m and n are even, and red otherwise. This yields an RP1S coloring of $(\mathbb{R}^2, \|\|_\infty)$ which forbids red translates of $F = \{(0,0), (0,1), (1,0), (1,1)\}$. Thus $4 = \chi((\mathbb{R}^2, \|\|_\infty), 1) \le Sz_1(\mathbb{R}^2, \|\|_\infty) \le 4$. Incidentally, if we did not know that $\chi((\mathbb{R}^2, \|\|_\infty), 1) = 4$, the existence of the RP1S coloring of $(\mathbb{R}^2, \|\|_\infty)$ just given would prove it, in view of the fact that $(0,0), (0,1), (1,0), (1,1)$ induce a clique in $G((\mathbb{R}^2, \|\|_\infty), 1)$, whence $4 \le \chi((\mathbb{R}^2, \|\|_\infty), 1)$.

4 Application

Theorem 2. *There is a RP3S coloring of $(\mathbb{R}^2, \|\|_2)$ which forbids for red all translates of a 3-point set in $\mathbb{R}^2$.*

Proof. Let $\vec{v} = (0, \frac{2}{\sqrt{3}})$, $\vec{w} = (-1, -\frac{1}{\sqrt{3}})$. Let X be the set of lattice points generated by $\vec{v}, \vec{w}$, i.e. $X = \{a\vec{v} + b\vec{w} \mid a, b \in \mathbb{Z}\}$. Consider the coloring of $\mathbb{R}^2$ where for all points $\vec{O} \in X$, we have a non-red open disc of radius $\frac{1}{2}$ centered at $\vec{O}$. The non-red discs are colored blue, green, and purple in an alternating manner, as shown in the Figure 1. The rest is colored red. Note that any two discs of the same color are a distance of at least 1 apart; therefore this is a RP3S coloring.

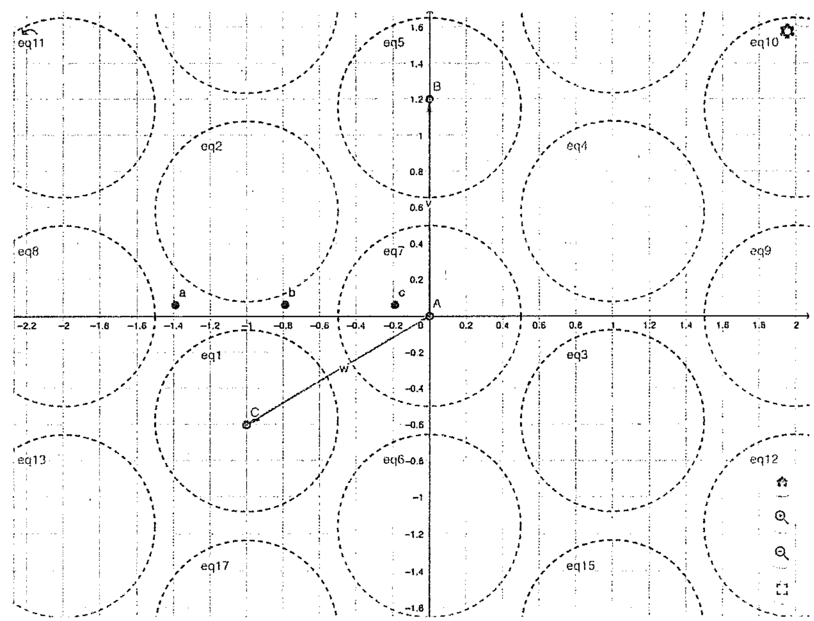

Figure 1: The RP3S coloring in Theorem 2, with the vectors $\vec{w}, \vec{v}$ and a translate of the 3-point set $\vec{a}, \vec{b}, \vec{c}$ shown.

Now we prove that translates of the 3-point configuration $\{\vec{a},\vec{b},\vec{c}\} = \{(-\frac{6}{10},0),(0,0),(\frac{6}{10},0)\}$ is forbidden for red. Assume for contradiction that there exists $\vec{x}$ such that $\{\vec{a}',\vec{b}',\vec{c}'\} = \{\vec{a},\vec{b},\vec{c}\} + \vec{x}$ is all red. In particular, $\vec{b}' = \vec{x}$, so the point $\vec{x}$ cannot be in any non-red discs. Note that for our proof here we do not distinguish between non-red colors, just between red and non-red. Clearly this 2-coloring is invariant with respect to translation by $\vec{v}$ or $\vec{w}$. Let $p_1 = (0,0), p_2 = \vec{v}, p_3 = \vec{w}, p_4 = \vec{v}+\vec{w}$. Then it suffices to consider $\vec{x}$ confined to the parallelogram $p_1p_2p_3p_4$. Note that as both the coloring and the 3-point configuration $\{\vec{a},\vec{b},\vec{c}\}$ is symmetric about the y-axis, it suffices to consider $\vec{x}$ confined to the triangle $p_1p_3p_4$, as triangle $p_1p_4p_2$ is a reflection and translation by $\vec{v}+\vec{w}$ of $p_1p_3p_4$. Let $p_5 = (-1,0)$. Moreover, note that as both the coloring and the 3-point configuration $\{\vec{a},\vec{b},\vec{c}\}$ is symmetric about the x-axis, it suffices to consider $\vec{x}$ confined to the triangle $p_1p_5p_4$. Note that the circular boundary of the disc centered at p_4 intersects the triangle $p_1p_5p_4$ at $p_6 = (-1,\frac{1}{\sqrt{3}}-\frac{1}{2})$ and $p_7 = (-1+\frac{\sqrt{3}}{4},\frac{1}{\sqrt{3}}-\frac{1}{4})$. Also note that the boundary of the disc centered at p_1 intersects triangle $p_1p_5p_4$ at $p_8 = (-\frac{1}{2},0)$ and $p_9 = (-\frac{\sqrt{3}}{4},\frac{1}{4})$. Therefore it suffices to consider $\vec{x}$ confined to the convex polygon $p_6p_5p_8p_9p_7$ (see Figure 2).

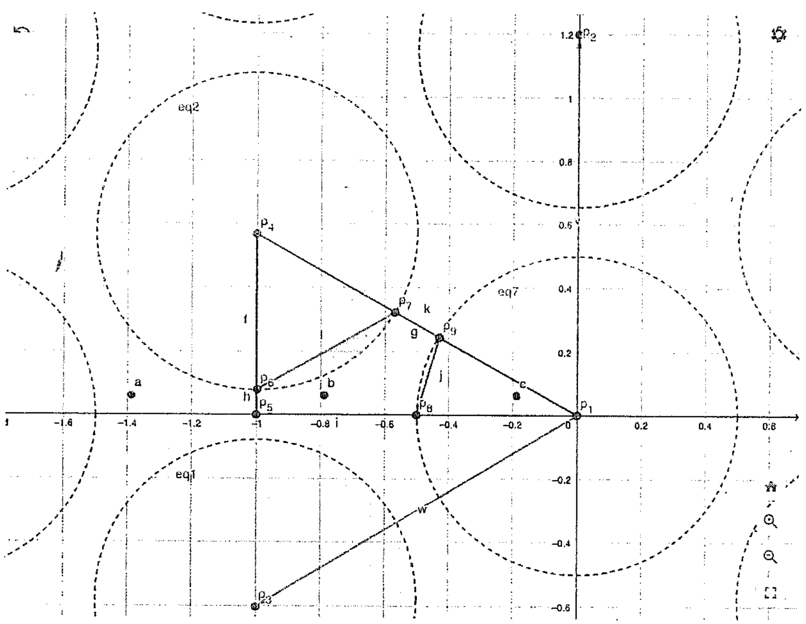

Figure 2: The RP3S coloring in Theorem 2, with points p_i shown for $i = 1,2,...,9$.

Let $p_i' = p_i + (\frac{6}{10},0)$ for $i = 5,6,7,8,9$. Then note that $\vec{c}'$ must be somewhere in the convex polygon $p_6'p_5'p_8'p_9'p_7'$. But note that $\|p_i'\|_2 < \frac{1}{2}$ for all $i = 5,6,7,8,9$, so p_i' is contained in the disc centered at p_1. Therefore the polygon $p_6'p_5'p_8'p_9'p_7'$ is contained in the disc centered at p_1, and so is $\vec{c}'$, which is a contradiction to the assumption that $\vec{c}'$ is red.

□

Corollary 2. $Sz_3(\mathbb{R}^2,\|\|_2) \leq 3$.

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