SINGULARITY MODELS IN THE THREE-DIMENSIONAL RICCI FLOW

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Abstract. The Ricci ow is a natural evolution equation for Riemann-ian metrics on a given manifold. The main goal is to understand singularity formation. In his spectacular 2002 breakthrough, Perelman achieved a qualitative understanding of singularity formation in dimension 3. More precisely, Perelman showed that every nite-time singular-ity to the Ricci ow in dimension 3 is modeled on an ancient -solution. Moreover, Perelman proved a structure theorem for ancient -solutions in dimension 3.

In this survey, we discuss recent developments which have led to a complete classication of all the singularity models in dimension 3. Moreover, we give an alternative proof of the classication of noncollapsed steady gradient Ricci solitons in dimension 3 (originally proved by the author in 2012).

1. Background on the Ricci flow

Geometric evolution equations play an important role in dierential geometry. The most important such evoluation equation is the Ricci ow for Riemannian metrics which was introduced by Hamilton [21]:

Denition 1.1 (R. Hamilton [21]). Let $g(t)$ be a one-parameter family of Riemannian metrics on a manifold M. We say that the metrics $g(t)$ evolve by the Ricci ow if

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)}. \tag{1}$$

In his paper [21], Hamilton established short time existence and uniqueness for the Ricci ow.

Theorem 1.2 (R. Hamilton [21]; D. DeTurck [18]). Let g_0 be a Riemannian metric on a compact manifold M. Then there exists a unique solution $g(t)$, $t \in [0,T)$, to the Ricci ow with initial metric $g(0) = g_0$. Here, T is a positive real number which depends on the initial data.

The main diculty in proving Theorem 1.2 is that the Ricci ow is weakly, but not strictly, parabolic. This is due to the fact that the Ricci ow is invari-ant under the dieomorphism group of M. This problem can be overcome

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using DeTurck's trick [18]. In the following, we sketch the argument (see [6] or [30] for details). Let us x a compact manifold M and a smooth one-parameter family of background metrics $h(t)$. The choice of the background metrics $h(t)$ is not important. In particular, we can choose the background metrics $h(t)$ to be independent of t. For each t, we denote by $_{g(t);h(t)}$ the Laplacian of a map from $(M; g(t))$ to $(M; h(t))$ (see [6], Denition 2.2). With this understood, we can dene Ricci-DeTurck ow as follows:

Denition 1.3. Let $\tilde{g}(t)$ be a one-parameter family of metrics on M. We say that the metrics $\tilde{g}(t)$ evolve by the Ricci-DeTurck ow if

$$\frac{\partial}{\partial t}\tilde{g}(t) = \quad 2\,\mathrm{Ric}_{\tilde{g}(t)} \quad L_{t}(\tilde{g}(t));$$

where the vector eld $_t$ is dened by $_t := {}_{\tilde{g}(t);h(t)}\mathrm{id}$.

The evolution of the metric under the Ricci-DeTurck ow can be written in the form

$$\frac{\partial}{\partial t}\tilde{g}_{ij} = \tilde{g}^{kl}\,\partial_k\partial_l\tilde{g}_{ij} + \text{lower order terms.}$$

Therefore, the Ricci-DeTurck ow is strictly parabolic, and admits a unique solution on a short time interval.

In the next step, we show that the Ricci ow is equivalent to the Ricci-DeTurck ow in the sense that whenever we have a solution to one equation, we can convert it into a solution of the other.

To explain this, suppose rst that we are given a solution $\tilde{g}(t)$ of the Ricci-DeTurck ow. Our goal is to produce a solution $g(t)$ of the Ricci ow. As above, we dene $_t := {}_{\tilde{g}(t);h(t)}\mathrm{id}$. We dene a one-parameter family of dieomorphisms φ_t by $\frac{\partial}{\partial t}\varphi_t(p) = {}_t|_{\varphi_t(p)}$ and $\varphi_0(p) = p$. Moreover, we dene a one-parameter family of metrics $g(t)$ by $g(t) = \varphi_t^*(\tilde{g}(t))$. Then $g(t)$ is a solution of the Ricci ow.

Conversely, suppose that we are given a solution $g(t)$ of the Ricci ow. Our goal is to produce a solution $\tilde{g}(t)$ of the Ricci-DeTurck ow. To that end, we solve the harmonic map heat ow $\frac{\partial}{\partial t}\varphi_t = {}_{g(t);h(t)}\varphi_t$ with initial condition $\varphi_0 = \mathrm{id}$. Moreover, we dene a one-parameter family of metrics $\tilde{g}(t)$ by $\varphi_t^*(\tilde{g}(t)) = g(t)$. Then $\tilde{g}(t)$ is a solution of the Ricci-DeTurck ow. This shows that the Ricci ow is equivalent to the Ricci-DeTurck ow.

A solution to the Ricci ow on a compact manifold can either be continued for all time, or else the curvature must blow up in nite time:

Theorem 1.4 (R. Hamilton [21]). Let g_0 be a Riemannian metric on a compact manifold M. Let $g(t)$, $t \in [0;T)$, denote the unique maximal solution to the Ricci ow with initial metric $g(0) = g_0$. If $T < \infty$, then the curvature of $g(t)$ is unbounded as $t \to T$.

A central problem is to understand the formation of singularities under the Ricci ow. To that end, it is often useful to consider a special class of solutions which move in a self-similar fashion. These are referred to as Ricci solitons:

Definition 1.5. Let (M, g) be a Riemannian manifold, and let f be a scalar function on M. We say that (M, g, f) is a steady gradient Ricci soliton if $\mathrm{Ric} = D^2 f$. We say that (M, g, f) is a shrinking gradient Ricci soliton if $\mathrm{Ric} = D^2 f + \lambda g$ for some constant $\lambda > 0$. We say that (M, g, f) is an expanding gradient Ricci soliton if $\mathrm{Ric} = D^2 f + \lambda g$ for some constant $\lambda < 0$.

We next discuss the global behavior of the Ricci flow in dimension 2. Hamilton [22] and Chow [15] showed that, for every initial metric on S^2, the Ricci flow shrinks to a point and becomes round after rescaling:

Theorem 1.6 (R. Hamilton [22]; B. Chow [15]). Let g_0 be a Riemannian metric on S^2. Let $g(t)$, $t \in [0, T)$, denote the unique maximal solution to the Ricci flow with initial metric $g(0) = g_0$. Then $T < \infty$. Moreover, as $t \to T$, the rescaled metrics $\frac{1}{2(T-t)}\, g(t)$ converge in C^∞ to a metric with constant Gaussian curvature 1.

Theorem 1.6 was first proved by Hamilton [22] under the additional assumption that the initial metric g_0 has positive scalar curvature. This condition was later removed by Chow [15]. In the following, we sketch the main ideas in Hamilton's proof. Full details can be found in [22] or [6], Section 4. Given a metric g on S^2 with positive scalar curvature, Hamilton defines the entropy $\mathcal{E}(g)$ by

$$(2) \qquad \mathcal{E}(g) = \int_{S^2} R \, \log\Big(\frac{A\,R}{8\pi}\Big) \, d\mu,$$

where A denotes the area of (S^2, g). The functional $\mathcal{E}(g)$ is invariant under scaling. By the Gauss-Bonnet theorem, $\int_{S^2} R \, d\mu = 8\pi$. Hence, it follows from Jensen's inequality that $\mathcal{E}(g)$ is nonnegative. Moreover, $\mathcal{E}(g)$ is strictly positive unless the scalar curvature of (S^2, g) is constant.

Hamilton's key insight is that the functional $\mathcal{E}(g)$ is monotone decreasing under the Ricci flow. From this, Hamilton deduced that the product $A\,R$ is uniformly bounded under the evolution. This implies that the flow converges to a shrinking gradient Ricci soliton, up to scaling. Finally, Hamilton showed that every shrinking gradient Ricci soliton on S^2 must have constant scalar curvature. This completes our discussion of Theorem 1.6.

In the three-dimensional case, Hamilton [21] showed that an initial metric with positive Ricci curvature shrinks to a point in finite time and becomes round after rescaling.

Theorem 1.7 (R. Hamilton [21]). Let g_0 be a Riemannian metric on a three-manifold M with positive Ricci curvature. Let $g(t)$, $t \in [0, T)$, denote the unique maximal solution to the Ricci flow with initial metric $g(0) = g_0$. Then $T < \infty$. Moreover, as $t \to T$, the rescaled metrics $\frac{1}{4(T-t)}\, g(t)$ converge in C^∞ to a metric with constant sectional curvature 1.

The proof of Theorem 1.7 is based on a pinching estimate for the eigenvalues of the Ricci tensor. To explain this, let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ denote the eigenvalues of the tensor $R\,g_{ij} \quad 2\,\mathrm{Ric}_{ij}$. With this understood, the scalar curvature is given by $_1 + \; _2 + \; _3$, and the eigenvalues of the Ricci tensor are given by $\frac{1}{2}(_2 + _3); \frac{1}{2}(_3 + _1); \frac{1}{2}(_1 + _2)$. In particular, the positivity of the Ricci tensor is equivalent to the inequality $_1 + \; _2 > 0$. Hamilton proved that

$$\frac{@}{@t}\,_1 \quad _1^2 + \; _1 + \; _2\,_3 \text{ and} \tag{3}$$

$$\frac{@}{@t}\,_3 \quad _3^2 + \; _3 + \; _1\,_2; \tag{4}$$

where both inequalities are understood in the barrier sense. In the special case when the initial metric has positive Ricci curvature, Hamilton proved a pinching estimate of the form $_3 \quad _1 \quad C\,(_1 + _2)^{1}$, where is a small positive constant depending on the initial data and C is a large constant depending on the initial data. The proof of this estimate relies on the maximum principle.

Theorem 1.7 has opened up two major lines of research. On the one hand, it is of interest to prove similar convergence theorems in higher dimensions, under suitable assumptions on the curvature. This direction led to the proof of the Dierentiable Sphere Theorem (see [5],[12]). On the other hand, it is important to understand the behavior of the Ricci ow in dimension 3 for arbitrary initial metrics. In this case, the ow will develop more complicated types of singularities, including so-called neck-pinch singularities. In a series of breakthroughs, Perelman [27],[28] achieved a qualitative understanding of singularity formation in dimension 3. This is sucient for topological conclusions, such as the Poincare conjecture.

In this survey, we will focus on issues related to singularity formation in dimension 3. In Section 2, we will review the concept of an ancient solution, and explain its relevance for the analysis of singularities. We next discuss Perelman's noncollapsing theorem. Moreover, we describe examples of ancient solutions to the Ricci ow in low dimensions. In Section 3, we discuss the classication of ancient solutions in dimension 2. In Section 4, we review results due to Perelman [27] concerning the structure of ancient -solutions in dimension 3. These are ancient solutions which have bounded and nonnegative curvature and satisfy a noncollapsing condition. In Section 5, we discuss the classication of ancient -solutions in dimension 3. In Section 6, we describe a quantitative version of the fact that the Ricci ow preserves symmetry. In Section 7, we discuss the Neck Improvement Theorem from [8]. This theorem asserts that a neck becomes more symmetric under the evolution. Finally, in Sections 8 and 9, we give an alternative proof of the classication of noncollapsed steady gradient Ricci solitons in dimension 3. This result was originally proved in [7]; the proof given here relies on the Neck Improvement Theorem from [8].

2. Ancient solutions and noncollapsing

The notion of an ancient solution plays a fundamental role in understanding the formation of singularities in the Ricci ow. This concept was introduced by Hamilton [23].

Denition 2.1. An ancient solution to the Ricci ow is a solution which is dened on the time interval $(\ 1; T]$ for some T.

The concept of an ancient solution to a parabolic PDE is analogous to the concept of an entire solution to an elliptic PDE.

Ancient solutions typically arise as blow-up limits at a singularity. In the Ricci ow, we are specically interested in ancient solutions which satisfy a noncollapsing condition.

Denition 2.2 (G. Perelman [27]). An ancient solution to the Ricci ow in dimension n is said to be -noncollapsed if $\mathrm{vol}_{g(t)}(B_{g(t)}(p; r)) \ \ r^n$ when-ever $\sup_{x 2 B_{g(t)}(p;r)} R(x; t) \ \ r^{\ 2}$.

Denition 2.2 is motivated by Perelman's noncollapsing theorem for the Ricci ow:

Theorem 2.3 (G. Perelman [27], Section 4). Let M be a compact manifold of dimension n, and let $g(t)$, $t \ 2 \ [0; T)$, be a solution to the Ricci ow, where $T <$ 1. Consider a sequence of times $t_j \ ! \ T$ and a bounded sequence of radii r_j. Finally, let p_j be a sequence of points in M such that

$$r_j^2 \sup_{x 2 B_{g(t_j)}(p_j; r_j)} R(x; t_j) < 1 :$$

Then

$$\liminf_{j \ ! \ 1} r_j^{\ n} \, \mathrm{vol}_{g(t_j)}(B_{g(t_j)}(p_j; r_j)) > 0:$$

Theorem 2.3 is a consequence of Perelman's monotonicity formula for the W-functional. In particular, Theorem 2.3 implies that every blow-up limit of the Ricci ow at a nite-time singularity must be -noncollapsed.

In dimension 3, the Hamilton-Ivey estimate gives a lower bound for the sectional curvature in terms of the scalar curvature:

Theorem 2.4 (R. Hamilton [23]; T. Ivey [25]). Let $g(t)$, $t \ 2 \ [0; T)$, be a solution to the Ricci ow on a compact three-manifold M. Let $_1$ denote the smallest eigenvalue of the tensor $R \, g_{ij} \ \ 2 \, \mathrm{Ric}_{ij}$. Then $_1$ satises a pointwise inequality of the form $_1 \ \ f(R)$, where the function f satises $\lim_{s ! 1} \frac{f(s)}{s} = 0$.

Theorem 2.4 implies that every blow-up limit of the Ricci ow in dimen-sion 3 must have nonnegative sectional curvature. The proof of Theorem 2.4 relies on the maximum principle together with the evolution equation for the Ricci tensor.

This motivates the following denition:

Definition 2.5. An ancient κ-solution to the Ricci flow in dimension $n \in \{2, 3\}$ is a complete, non-flat, κ-noncollapsed ancient solution with bounded and nonnegative curvature.

The notion of an ancient κ-solutions plays a key role in Perelman's the-ory. In particular, Perelman showed that, if a solution to the Ricci flow in dimension 3 forms a singularity in finite time, then the high curvature regions can be approximated by ancient κ-solutions (see [27], Section 12).

In the remainder of this section, we describe some of the known examples of ancient solutions to the Ricci flow in dimension 2 and 3.

Example 2.6. Let g_{S^2} denote the standard metric on S^2. Let us define a family of metrics $g(t)$ on S^2 by $g(t) = (-2t)\, g_{S^2}$ for $t \in (-\infty, 0)$. This is an ancient solution to the Ricci flow which shrinks homothetically. It is κ-noncollapsed.

Example 2.7. Let us define a one-parameter family of conformal metrics on $\mathbb{R}^2$ by

$$g_{ij}(t) = \frac{4}{e^t + |x|^2}\,\delta_{ij}$$

for $t \in (-\infty, \infty)$. This gives a rotationally symmetric solution to the Ricci flow on $\mathbb{R}^2$, which moves by diffeomorphisms. It is referred to as the cigar soliton. The cigar soliton has positive curvature and opens up like a cylinder near infinity. The cigar soliton fails to be κ-noncollapsed.

Example 2.8. Let us define a one-parameter family of conformal metrics on $\mathbb{R}^2$ by

$$g_{ij}(t) = \frac{8\sinh(-t)}{1 + 2\cosh(-t)\,|x|^2 + |x|^4}\,\delta_{ij}$$

for $t \in (-\infty, 0)$. For each $t \in (-\infty, 0)$, $g(t)$ extends to a smooth metric on S^2. This gives a rotationally symmetric solution to the Ricci flow on S^2. This is referred to as the King-Rosenau solution (cf. [26],[29]). The King-Rosenau solution is an ancient solution to the Ricci flow with positive curvature. The King-Rosenau solution fails to be κ-noncollapsed.

Example 2.9. Let g_{S^3} denote the standard metric on S^3. Let us define a family of metrics $g(t)$ on S^2 by $g(t) = (-4t)\, g_{S^3}$ for $t \in (-\infty, 0)$. This is an ancient solution to the Ricci flow which shrinks homothetically. It is κ-noncollapsed.

Example 2.10. Let again g_{S^2} denote the standard metric on S^2. Let us define a family of metrics $g(t)$ on $S^2 \times \mathbb{R}$ by $g(t) = (-2t)\, g_{S^2} + dz \otimes dz$ for $t \in (-\infty, 0)$. This is an ancient solution to the Ricci flow. It is κ-noncollapsed.

Example 2.11. Robert Bryant [13] has constructed a steady gradient Ricci soliton in dimension 3 which is rotationally symmetric. This can be viewed as the three-dimensional analogue of the cigar soliton. The Bryant soliton has positive sectional curvature and opens up like a paraboloid near innity. Unlike the cigar soliton, the Bryant soliton is -noncollapsed.

Example 2.12. Perelman has constructed an ancient solution to the Ricci ow on S^3 which is rotationally symmetric. This can be viewed as the three-dimensional analogue of the King-Rosenau solution. Perelman's ancient solution has positive sectional curvature. Unlike the King-Rosenau solution, Perelman's ancient solution is -noncollapsed.

The asymptotics of Perelman's ancient solution are by now well understood; see [2].

3. Classification of ancient solutions in dimension 2

In this section, we discuss the main classication results for ancient solution in dimension 2. In [27], Perelman gave a classication of ancient -solutions in dimension 2:

Theorem 3.1 (G. Perelman [27], Section 11). Let $(M; g(t))$ be an ancient -solution in dimension 2. Then $(M; g(t))$ is isometric to a family of shrinking spheres, or a Z_2-quotient thereof.

Let us sketch Perelman's proof of Theorem 3.1. Suppose that $(M; g(t))$ is an ancient -solution in dimension 2. After passing to a double cover if necessary, we may assume that M is orientable. By Proposition 11.2 in [27], we can nd a sequence of times t_j ! 1 and a sequence of points p_j 2 M with the following property: if we dilate the manifold $(M; g(t_j))$ around the point p_j by the factor $(\ t_j)^{\frac{1}{2}}$, then the rescaled manifolds converge in the Cheeger-Gromov sense to a non-at shrinking gradient Ricci soliton. Using Hamilton's classication of shrinking gradient Ricci solitons in dimension 2 (see [22]), we conclude that the limiting manifold must be a round sphere. Since the limiting manifold is dieomorphic to S^2, it follows that M is dieomorphic to S^2.

We next consider Hamilton's entropy functional dened in (2). Since the manifolds $(M; g(t_j))$ converge to a round sphere after rescaling, we know that $E(g(t_j))$! 0 as j ! 1. Moreover, it follows from Hamilton's work [22] that the function t ! $E(g(t))$ is monotone decreasing. Consequently, $E(g(t))$ 0 for each t. On the other hand, Jensen's inequality implies that $E(g(t))$ is strictly positive unless $(M; g(t))$ has constant scalar curva-ture. Putting these facts together, we conclude that the scalar curvature of $(M; g(t))$ is constant for each t. This completes our sketch of the proof of Theorem 3.1.

Daskalopoulos, Hamilton, and Sesum were able to classify compact ancient solutions in dimension 2 without noncollapsing assumptions:

Theorem 3.2 (P. Daskalopoulos, R. Hamilton, N. Sesum [17]). Let $(M; g(t))$ be a compact, non-at ancient solution to the Ricci ow in dimension 2. Then, up to parabolic rescaling, translation in time, and dieomorphisms,

$(M, g(t))$ coincides with the family of shrinking spheres, the King-Rosenau solution, or a $\mathbb{Z}_2$-quotient of these.

One of the main ideas behind Theorem 3.2 is to find a quantity to which the maximum principle can be applied and which vanishes on the King-Rosenau solution. To explain this, suppose that $(M, g(t))$ is a compact, non-flat ancient solution to the Ricci flow in dimension 2. After passing to a double cover if necessary, we may assume that M is orientable. Using the maximum principle, it is easy to see that $(M, g(t))$ has nonnegative scalar curvature for each t. Moreover, the strict maximum principle implies that the scalar curvature of $(M, g(t))$ is strictly positive. Since M is compact and orientable, it follows that $M = S^2$. By the uniformization theorem, we may assume that the metrics $g(t)$ are conformal to the standard metric on S^2.

Using the stereographic projection, we can identify $\mathbb{R}^2$ with the complement of the north pole in S^2. Thus, we obtain a family of conformal metrics on $\mathbb{R}^2$ which evolve by the Ricci flow. We may write the evolving metric in the form $v^{-1}\delta_{ij}$, where v satisfies the parabolic PDE

$$\frac{\partial}{\partial t} v = v\,\Delta v - |\nabla v|^2$$

on $\mathbb{R}^2$, where ∇v and Δv denote the gradient and Laplacian of v with respect to the Euclidean metric on $\mathbb{R}^2$. Daskalopoulos, Hamilton, and Sesum consider the quantity

$$Q = v\,\frac{\partial^3 v}{\partial z^3}\,\frac{\partial^3 v}{\partial \bar{z}^3},$$

where $\frac{\partial}{\partial z} = \frac{1}{2}\big(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\big)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\big(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\big)$ denote the usual Wirtinger derivatives. A straightforward calculation shows that the quantity Q is invariant under Möbius transformations. Moreover, Q satisfies the evolution equation

$$\begin{aligned}
\frac{\partial}{\partial t} Q &= v\,\Delta Q - 16\,v^{-1}\Big(v\,\frac{\partial^2 v}{\partial z\,\partial \bar{z}} - \frac{\partial v}{\partial z}\,\frac{\partial v}{\partial \bar{z}}\Big)\,Q \\
&\quad - 4\,\Big(v\,\frac{\partial^4 v}{\partial z^4} + 2\,\frac{\partial v}{\partial z}\,\frac{\partial^3 v}{\partial z^3}\Big)\Big(v\,\frac{\partial^4 v}{\partial \bar{z}^4} + 2\,\frac{\partial v}{\partial \bar{z}}\,\frac{\partial^3 v}{\partial \bar{z}^3}\Big) \\
&\quad - 4\,\Big(v\,\frac{\partial^4 v}{\partial z^3\,\partial \bar{z}} - \frac{\partial v}{\partial \bar{z}}\,\frac{\partial^3 v}{\partial z^3}\Big)\Big(v\,\frac{\partial^4 v}{\partial z\,\partial \bar{z}^3} - \frac{\partial v}{\partial z}\,\frac{\partial^3 v}{\partial \bar{z}^3}\Big).
\end{aligned}$$

The scalar curvature of the conformal metric $v^{-1}\delta_{ij}$ can be written in the form

$$R = v^{-1}(v\,\Delta v - |\nabla v|^2) = 4\,v^{-1}\Big(v\,\frac{\partial^2 v}{\partial z\,\partial \bar{z}} - \frac{\partial v}{\partial z}\,\frac{\partial v}{\partial \bar{z}}\Big).$$

This gives

$$\frac{\partial}{\partial t} Q \leq v\,\Delta Q - 4RQ$$

(compare [17], Section 5, or [16]). Here, ΔQ denotes the Laplacian of Q with respect to the Euclidean metric on $\mathbb{R}^2$. The term $v\,\Delta Q$ can be interpreted as the Laplacian of Q with respect to the evolving metric $v^{-1}\delta_{ij}$.

On the King-Rosenau solution, v is a quadratic polynomial in jxj^2, with coecients that depend on t. In particular, $\frac{@^3v}{@z^3}$ vanishes identically on the King-Rosenau solution. Therefore, Q vanishes identically on the King-Rosenau solution.

4. Structure of ancient -solutions in dimension 3

In [27], Perelman proved several fundamental results concerning the structure of ancient -solutions in dimension 3. One of the central results is the following pointwise estimate for the covariant derivatives of the curvature tensor.

Theorem 4.1 (G. Perelman [27], Section 11). Let $(M; g(t))$, $t\ 2\ (\ 1; 0]$, be an ancient -solution to the Ricci ow in dimension 3. Let m be a positive integer. Then the m-th order covariant derivatives of the curvature tensor satisfy the pointwise bound $jD^m Rmj\ \ C\ R^{\frac{m+2}{2}}$, where C is a positive constant that depends only on m and .

Another fundamental result in Perelman's work is the following longrange curvature estimate:

Theorem 4.2 (G. Perelman [27], Section 11). Let $(M; g(t))$, $t\ 2\ (\ 1; 0]$, be an ancient -solution to the Ricci ow in dimension 3. Then there exists a function $! : [0; 1)\ !\ \ [0; 1)$ (depending on) such that

$$R(y; t)\ \ R(x; t)\, !(R(x; t)\, d_{g(t)}(x; y)^2)$$

for all $x; y\ 2\ M$ and all $t\ \ 0$.

Perelman's longrange curvature estimate is extremely useful, in that it allows Perelman to take limits of sequences of ancient -solutions. One important consequence is that the space of ancient -solutions is compact in the following sense:

Theorem 4.3 (G. Perelman [27], Section 11). Let $(M^{(j)}; g^{(j)}(t))$, $t\ 2\ (\ 1; 0]$, be a sequence of ancient -solutions in dimension 3. Moreover, suppose that p_j $2\ M^{(j)}$ is a sequence of points such that $R(p_j; 0) = 1$ for each j. Then, after passing to a subsequence if necessary, the ows $(M^{(j)}; g^{(j)}(t); p\ )_j$ converge in the Cheeger-Gromov sense to a limit $(M^1; g^1(t))$, and this limit is again an ancient -solution.

Corollary 4.4 (G. Perelman [27], Section 11). Let $(M; g(t))$, $t\ 2\ (\ 1; 0]$, be a noncompact ancient -solution to the Ricci ow in dimension 3 with positive sectional curvature. Let us x a point p_0 in M. Let p_j be a sequence of points in M such that $d_{g(0)}(p_0; p_j)\ !\ \ 1$, and let $r_j\ ^2 := R(p_j; 0)$. Let us dilate the ow around the point $(p_j; 0)$ by the factor $r_j\ ^1$. Then, after passing to a subsequence if necessary, the rescaled ows converge in the Cheeger-Gromov sense to a family of shrinking cylinders.

Let us sketch how Corollary 4.4 follows from Theorem 4.3. The longrange curvature estimate gives

$$R(p_0; 0) \;\; R(p_j; 0) \,!(R(p_j; 0)\, d_{g(0)}(p_0; p_j)^2)$$

for each j. This implies

$$R(p_j; 0)\, d_{g(0)}(p_0; p_j)^2 \; ! \;\; 1$$

as j ! 1. In other words, $r_j^{\;1} d_{g(0)}(p_0; p_j)$! 1 as j ! 1. We next consider the rescaled metrics $g^{(j)}(t) := r_j^{\;2} g(r_j^2 t)$ for t 2 (1; 0]. By Theorem 4.3, the ows $(M; g^{(j)}(t); p_j)$ converge in the Cheeger-Gromov sense to an ancient -solution $(M^1; g^1(t))$. Since $r_j^{\;1} d_{g(0)}(p_0; p_j)$! 1, the limiting ow $(M^1; g^1(t))$ must split o a line. Using Perelman's classication of ancient -solutions in dimension 2 (see Theorem 3.1), it follows that the limiting ow $(M^1; g^1(t))$ must be a family of shrinking cylinders or a quo-tient thereof. On the other hand, M is dieomorphic to R^3 by the soul theorem. In particular, M does not contain an embedded RP^2. This implies that $(M^1; g^1(t))$ cannot be a non-trivial quotient of the cylinder. This completes the sketch of the proof of Corollary 4.4.

Denition 4.5. Let (M; g(t)) be a solution to the Ricci ow in dimension 3, and let (x; t) be a point in space-time with $R(x; t) = r^{\;2}$. We say that (x; t) lies at the center of an evolving "-neck if, after rescaling by the factor $r^{\;1}$, the parabolic neighborhood $B_{g(t)}(x; "^{\;1} r)$ [t "$^{1}r^2$; t] is "-close in $C^{[" \;\; 1]}$ to a family of shrinking cylinders.

The notion of a neck was introduced in Hamilton's work [24]. In particular, Hamilton showed that a neck admits a canonical foliation by constant mean curvature (CMC) spheres.

Corollary 4.4 implies the following structure theorem for noncompact ancient -solutions:

Corollary 4.6 (G. Perelman [27], Section 11). Let (M; g(t)), t 2 (1; 0], be a noncompact ancient -solution to the Ricci ow in dimension 3 with positive sectional curvature. Moreover, let " be a positive real number, and let $M_"$ denote the set of all points x 2 M with the property that (x; 0) does not lie at the center of an evolving "-neck. Then $M_"$ has nite diameter. Moreover, $\sup_{x 2 M_"} R(x; 0)$ C(; ") $\inf_{x 2 M_"} R(x; 0)$ and $\sup_{x 2 M_"} R(x; 0)$ C(; ") $\operatorname{diam}_{g(0)}(M_")$ 2.

5. Classification of ancient -solutions in dimension 3

We now turn to the classication of ancient -solutions in dimension 3. The rst major step was the classication of noncollapsed steady gradient Ricci solitons in dimension 3.

Theorem 5.1 (S. Brendle [7]). Let $(M; g)$ be a three-dimensional complete steady gradient Ricci soliton which is non-at and -noncollapsed. Then $(M; g)$ is rotationally symmetric, and is therefore isometric to the Bryant soliton up to scaling.

More recently, we classied all noncompact ancient -solutions in dimension 3:

Theorem 5.2 (S. Brendle [8]). Assume that $(M; g(t))$ is a noncompact ancient -solution of dimension 3. Then either $(M; g(t))$ is isometric to a family of shrinking cylinders (or a quotient thereof), or $(M; g(t))$ is isometric to the Bryant soliton up to scaling.

Theorem 5.2 conrms a conjecture of Perelman [27].

The proof of Theorem 5.2 consists of two main steps. In the rst step, we classify noncompact ancient -solutions with rotational symmetry. To do that, we need precise asymptotic estimates for such solutions. In the sec-ond step, we show that every noncompact ancient -solution is rotationally symmetric. This second step uses the classication of steady gradient Ricci solitons in Theorem 5.1, as well as the classication of ancient -solutions with rotational symmetry. Another crucial ingredient is the Neck Improve-ment Theorem which asserts that a neck tends to get more symmetric as it evolves under the Ricci ow. We will discuss the Neck Improvement Theo-rem in Section 7 below.

The following theorem is the counterpart of Theorem 5.2 in the compact case:

Theorem 5.3 (S. Brendle, P. Daskalopoulos, N. Sesum [10]). Assume that $(M; g(t))$ is a compact ancient -solution of dimension 3. Then, up to parabolic rescaling, translation in time, and dieomorphisms, $(M; g(t))$ is either a family of shrinking spheres or Perelman's ancient solution or a quotient of these.

The proof of Theorem 5.3 again requires two main steps. In the rst step, we show that every compact ancient -solution is rotationally sym-metric. This step uses the classication of noncompact ancient -solutions in Theorem 5.2, together with the Neck Improvement Theorem. In a second step, we classify compact ancient -solutions with rotational symmetry. To do that, we need to understand the asymptotic behavior of such solutions. These asymptotic estimates are established in [2].

Similar classication results exist for convex, noncollapsed ancient solutions to mean curvature ow in R^3. We refer to [9] for the classication in the noncompact case, and to [3] for the classication in the compact case.

6. Preservation of symmetry under the Ricci flow

Let $g(t)$, $t \in [0; T)$, be a solution to the Ricci ow on a compact manifold M. It follows from Hamilton's short time uniqueness theorem that the Ricci flow preserves symmetry. More precisely, every isometry of $(M, g(0))$ is also an isometry of $(M, g(t))$ for each $t \geq 0$. Consequently, every Killing vector field of $(M, g(0))$ is also a Killing vector field of $(M, g(t))$ for each $t \geq 0$.

In this section, we describe a quantitative version of this principle. We begin with a definition:

Definition 6.1. Let h be a symmetric $(0,2)$-tensor. The Lichnerowicz Laplacian of h is defined as

$$\Delta_L h_{ik} := \Delta h_{ik} + 2R_{ijkl}h^{jl} - \mathrm{Ric}_i^l h_{kl} - \mathrm{Ric}_k^l h_{il}. \tag{5}$$

The Lichnerowicz Laplacian arises naturally in the study of Einstein metrics (see [4], equation (1.180b)). It also comes up in connection with the evolution equation for the Ricci tensor under the Ricci flow. Indeed, if $(M, g(t))$ is a solution to the Ricci flow, then the Ricci tensor satisfies the evolution equation

$$\frac{\partial}{\partial t}\mathrm{Ric}_{g(t)} = \Delta_{L,g(t)}\mathrm{Ric}_{g(t)}$$

(see [21], Corollary 7.3).

The following results play a key role in our analysis:

Proposition 6.2. Let (M, g) be a Riemannian manifold. Let V be a smooth vector field on M, and let $h := \mathcal{L}_V(g)$. Then

$$\Delta V + \mathrm{Ric}(V) = \mathrm{div}\, h - \frac{1}{2}\nabla(\mathrm{tr}\, h).$$

Proposition 6.3 (S. Brendle [8], Section 5). Let $(M, g(t))$ be a solution of the Ricci flow. Let $V(t)$ be a time-dependent vector field such that

$$\frac{\partial}{\partial t}V(t) = \Delta_{g(t)}V(t) + \mathrm{Ric}_{g(t)}(V(t)), \tag{6}$$

and let $h(t) := \mathcal{L}_{V(t)}(g(t))$. Then the tensor $h(t)$ satisfies the evolution equation

$$\frac{\partial}{\partial t}h(t) = \Delta_{L,g(t)}h(t). \tag{7}$$

Proposition 6.3 has a natural geometric interpretation in terms of the linearized Ricci-DeTurck flow. To explain this, let us fix a solution $(M, g(t))$ of the Ricci flow. Suppose that φ_t is a one-parameter family of diffeomorphisms which solve the harmonic map heat flow with respect to the background metrics $g(t)$; that is,

$$\frac{\partial}{\partial t}\varphi_t = \Delta_{g(t),g(t)}\varphi_t. \tag{8}$$

Let us define a flow of metrics $\tilde{g}(t)$ by $\varphi_t^*(\tilde{g}(t)) = g(t)$. Then the metrics $\tilde{g}(t)$ solve the Ricci-DeTurck flow with respect to the background metrics $g(t)$. More precisely,

$$\frac{\partial}{\partial t}\tilde{g}(t) = -2\,\mathrm{Ric}_{\tilde{g}(t)} - \mathcal{L}_{\xi_t}(\tilde{g}(t)), \tag{9}$$

where $_t = {}_{g(t);g(t)}\mathrm{id}$. Clearly, $'_t :=$ id is a solution of (8), and $\tilde{g}(t) := g(t)$ is a solution of (9).

We now linearize the equations (8) and (9) around $'_t =$ id and $\tilde{g}(t) = g(t)$, respectively. Linearizing the harmonic map heat ow (8) around the identity, we obtain the equation

$$\frac{@}{@t}V(t) = {}_{g(t)}V(t) + \mathrm{Ric}_{g(t)}(V(t))$$

for a vector eld V (see [19], p. 11). Linearizing the Ricci-DeTurck ow (9) around $g(t)$ leads to the parabolic Lichnerowicz equation

$$\frac{@}{@t}h(t) = {}_{L;g(t)}h(t):$$

This completes our discussion of Proposition 6.3.

On a steady gradient Ricci soliton, Proposition 6.3 takes the following form:

Corollary 6.4 (S. Brendle [7]). Let $(M; g; f)$ be a steady gradient Ricci soliton, and let $X := rf$. Let V be a vector eld satisng

$$V + D_X V = 0; \tag{10}$$

and let $h := L_V(g)$. Then the tensor h satises the equation

$$_L h + L_X(h) = 0: \tag{11}$$

Let us sketch how Corollary 6.4 follows from Theorem 6.3. On a steady gradient Ricci soliton, the time derivative $\frac{@}{@t}$ reduces to a Lie derivative L_X. More precisely, let $(M; g; f)$ be a steady gradient Ricci soliton, let $X := rf$, and let $_t$ denote the ow generated by the vector eld X. Sup-pose that V satises $V + D_X V = 0$, and let $h := L_V(g)$. Using the identity $D_V X = \mathrm{Ric}(V)$, we obtain $V + L_X V + \mathrm{Ric}(V) = 0$. Consequently, the vec-tor elds (V) satisfy the parabolic PDE (6) on the evolving background $(M; (g))$. By Theorem 6.3, the tensors (h) satisfy the parabolic PDE (7) on the evolving background $(M; (g))$. This implies $_L h + L_X(h) = 0$. In order to apply Proposition 6.3 in practice, we need estimates for so-lutions of the parabolic Lichnerowicz equation. In dimension 3, this can be accomplished by applying the maximum principle to the quantity $\frac{jhj^2}{R^2}$:

Proposition 6.5 (G. Anderson, B. Chow [1]). Let $(M; g(t))$ be a solution to the Ricci ow in dimension 3 with positive scalar curvature. Let h be a solution of the parabolic Lichnerowicz equation $\frac{@}{@t}h(t) = {}_{L;g(t)}h(t)$. Then

$$\frac{@}{@t}\frac{jhj^2}{R^2} \quad \frac{jhj^2}{R^2} + \frac{2}{R}\Big\langle rR; r\frac{jhj^2}{R^2}\Big\rangle:$$

On a steady gradient Ricci soliton, Proposition 6.5 takes the following form:

Corollary 6.6 (G. Anderson, B. Chow [1]). Let (M, g, f) be a steady gradient Ricci soliton in dimension 3 with positive scalar curvature, and let $X := \nabla f$. Let h be a solution of the equation $\Delta_L h + \mathcal{L}_X(h) = 0$. Then

$$\Delta\Big(\frac{|h|^2}{R^2}\Big) + \Big\langle X, \nabla\Big(\frac{|h|^2}{R^2}\Big)\Big\rangle + \frac{2}{R}\Big\langle \nabla R, \nabla\Big(\frac{|h|^2}{R^2}\Big)\Big\rangle \leq 0.$$

7. Improvement of symmetry on a neck

In view of Proposition 6.3, it is important to understand the parabolic Lichnerowicz equation (7) on a Ricci flow background. As a starting point, we consider the special case when the background is given by a family of shrinking cylinders. To fix notation, we define a family of metrics $\bar{g}(t)$, $t \in (-\infty, 0)$ on $S^2 \times \mathbb{R}$ by

$$\bar{g}(t) = (-2t)\, g_{S^2} + dz \otimes dz, \qquad t \in (-\infty, 0).$$

Clearly, the metrics $\bar{g}(t)$, $t \in (-\infty, 0)$, evolve by the Ricci flow.

Proposition 7.1 (S. Brendle [8], Section 6). Let $(S^2 \times \mathbb{R}, \bar{g}(t))$ denote the family of shrinking cylinders. Let L be a large real number. Let $h(t)$ be a solution of the parabolic Lichnerowicz equation $\frac{\partial}{\partial t} h(t) = \Delta_{L,\bar{g}(t)} h(t)$ which is defined on $S^2 \times [-\frac{L}{2}, \frac{L}{2}]$ and for $t \in [-\frac{L}{2}, -1]$. Assume that $|h(t)|_{\bar{g}(t)} \leq 1$ for $t \in [-\frac{L}{2}, -\frac{L}{4}]$, and $|h(t)|_{\bar{g}(t)} \leq L^{10}$ for $t \in [-\frac{L}{2}, -1]$. Then we can find a rotationally invariant tensor of the form $\omega(z,t)\, g_{S^2} + \beta(z,t)\, dz \otimes dz$ and a scalar function $\psi : S^2 \to \mathbb{R}$ such that ψ lies in the span of the first spherical harmonics on S^2 and

$$|h(t) - \omega(z,t)\, g_{S^2} - \beta(z,t)\, dz \otimes dz - (-t)\,\psi\, g_{S^2}|_{\bar{g}(t)} \leq C\, L^{-\frac{1}{2}}$$

on $S^2 \times [-1000, 1000]$ and for $t \in [-1000, -1]$. Here, C is a constant which does not depend on L.

Proposition 7.1 asserts that, given sufficient time to evolve, a solution of the parabolic Lichnerowicz equation can be approximated by a sum of a rotationally invariant tensor and a tensor of the form $(-t)\,\psi\, g_{S^2}$, where $\psi : S^2 \to \mathbb{R}$ lies in the span of the first spherical harmonics on S^2. The tensor $(-t)\,\psi\, g_{S^2}$ can be written as a Lie derivative of the metric along a vector field. To see this, let us define a vector field ξ on S^2 by $g_{S^2}(\xi, \cdot) = -\frac{1}{4}\, d\psi$. Since ψ lies in the span of the first spherical harmonics on S^2, we obtain $\mathcal{L}_\xi(g_{S^2}) = \frac{1}{2}\,\psi\, g_{S^2}$, and consequently $\mathcal{L}_\xi(\bar{g}(t)) = (-t)\,\psi\, g_{S^2}$.

To prove Proposition 7.1, we decompose the tensor $h(t)$ into components, and perform a mode decomposition in spherical harmonics. This leads to a system of linear heat equations in one space dimension.

Using Proposition 7.1, we can show that a neck becomes more symmetric as it evolves under the Ricci flow. To state this result, we need a quantitative notion of ε-symmetry:

Definition 7.2 (S. Brendle [8], Section 8). Let $(M, g(t))$ be a solution to the Ricci flow in dimension 3, and let $(\bar{x}, \bar{t})$ be a point in space-time with

$R(\bar{x},\bar{t}) = r^{-2}$. We assume that $(\bar{x},\bar{t})$ lies at the center of an evolving ε_0-neck for some small positive number ε_0. We say that $(\bar{x},\bar{t})$ is ε-symmetric if there exist smooth, time-independent vector elds $U^{(1)}, U^{(2)}, U^{(3)}$ which are dened on an open set containing $B_{g(\bar{t})}(\bar{x}, 100r)$ and satisfy the following conditions:

- $\sup_{B_{g(\bar{t})}(\bar{x},100r)\times[\bar{t}-100r^2,\bar{t}]} \sum_{l=0}^{2} \sum_{a=1}^{3} r^{2l} |D^l(\mathcal{L}_{U^{(a)}}(g(t)))|^2 \leq \varepsilon^2$.
- If $t \in [\bar{t}-100r^2,\bar{t}]$ and $\Sigma \subset B_{g(\bar{t})}(\bar{x},100r)$ is a leaf of the CMC foliation of $(M, g(t))$, then $\sup_{\Sigma} \sum_{a=1}^{3} r^{-2} |\langle U^{(a)}, \nu\rangle|^2 \leq \varepsilon^2$, where ν denotes the unit normal vector to Σ in $(M, g(t))$.
- If $t \in [\bar{t}-100r^2,\bar{t}]$ and $\Sigma \subset B_{g(\bar{t})}(\bar{x},100r)$ is a leaf of the CMC foliation of $(M, g(t))$, then

$$\sum_{a,b=1}^{3} \Big|\delta_{ab} - \frac{\mathrm{area}_{g(t)}(\Sigma)^{-2}}{\dots} \int_{\Sigma} \langle U^{(a)}, U^{(b)}\rangle_{g(t)} \, d\mu_{g(t)}\Big|^2 \leq \varepsilon^2.$$

With this understood, we can now state the Neck Improvement Theorem from [8]:

Theorem 7.3 (S. Brendle [8], Section 8). We can nd a large constant L and small positive constant ε_1 such that the following holds. Let $(M, g(t))$ be a solution of the Ricci ow in dimension 3, and let (x_0, t_0) be a point in space-time which lies at the center of an evolving ε_1-neck and satises $R(x_0, t_0) = r^{-2}$. Moreover, we assume that every point in the parabolic neighborhood $B_{g(t_0)}(x_0, Lr) \times [t_0 - Lr^2, t_0]$ is ε-symmetric, where $\varepsilon \leq \varepsilon_1$. Then the point (x_0, t_0) is $\frac{\varepsilon}{2}$-symmetric.

8. Asymptotic behavior of noncollapsed steady gradient Ricci solitons in dimension 3

Let (M, g, f) be a non-at steady gradient Ricci soliton in dimension n, so that $\mathrm{Ric} = D^2 f$. For abbreviation, let $X := \nabla f$. Throughout this section, we x an arbitrary point $p \in M$.

Lemma 8.1. Given any point $\bar{x} \in M$, we can nd a smooth function ψ which is dened in an open neighborhood of $\bar{x}$ and satises the following conditions:

- $\psi(x) \geq d(p, x)^2$ in an open neighborhood of the point $\bar{x}$ $\psi(\bar{x}) = d(p, \bar{x})^2$.
- $|\nabla \psi|^2 = 4\psi$ at the point $\bar{x}$
- $\Delta\psi + \langle X, \nabla\psi\rangle \leq N_0 + N_1 \sqrt{\psi}$ at the point $\bar{x}$

Here, N_0 and N_1 are uniform constants which do not depend on $\bar{x}$

Proof. Let us x a positive real number r_0 such that r_0 is strictly smaller than the injectivity radius at p.

We rst consider the case $\bar{x} \in B(p, r_0)$. In this case, we dene a smooth function $\psi : B(p, r_0) \to \mathbb{R}$ by $\psi(x) := d(p, x)^2$. It is easy to see that $|\nabla\psi|^2 =$

4 at each point in $B(p; r_0)$. Moreover, at each point in $B(p; r_0)$, we have $+ \langle X, \nabla \rangle \leq C$ for some uniform constant C.

In the next step, we consider the case $x \in M \setminus B(p; r_0)$. Let $l := d(p; x) \geq r_0$. Moreover, let $: [0; l] \to M$ be a unit-speed geodesic with $(0) = p$ and $(l) = x$ Finally, let $: [0; 1) \to [0; 1)$ be a smooth cuto function such that $= 0$ on the interval $[0; \frac{r_0}{2}]$ and $= 1$ on the interval $[r_0; 1)$.

Let us x a positive real number r such that r is strictly smaller than the injectivity radius at x We dene a smooth function $: B(x; r) \to \mathbb{R}$ as follows. Given a point $x \in B(x; r)$, there exists a unique vector $w \in T_x M$ such that $|w| < r$ and $x = \exp_x(w)$. We denote by W the unique parallel vector eld along satisfying $W(l) = w$. We then dene $\overline{\sqrt{(x)}}$ to be the length of the curve

$$s \to \exp_{(s)}((s)\, W(s)); \qquad s \in [0; l]:$$

Clearly,

$$\overline{\sqrt{(x)}} \geq d\big(\exp_{(0)}((0)\, W(0)); \exp_{(l)}((l)\, W(l))\big) = d(p; \exp_x(w)) = d(p; x):$$

Moreover, in the special case when $x = x$ and $w = 0$, we obtain

$$\overline{\sqrt{(x)}} = l = d(p; x):$$

The formula for the rst variation of arclength implies that $\nabla \sqrt{} = {}^0(l)$ at the point x In particular, $|\nabla \sqrt{}|^2 = 1$ at the point x Consequently, $|\nabla|^2 = 4$ at the point x

Using the formula for the second variation of arclength, we obtain

$$(D^2 \sqrt{})_x(w; w) = \int_0^l {}^0(s)^2 \langle W(s); W(s)\rangle\, ds - \int_0^l {}^0(s)^2 \langle {}^0(s); W(s)\rangle^2\, ds - \int_0^l (s)^2 R({}^0(s); W(s); {}^0(s); W(s))\, ds$$

for every vector $w \in T_x M$, where W denotes the unique parallel vector eld along satisfying $W(l) = w$. Taking the trace over w gives

$$\sqrt{(x)} = (n - 1) \int_0^l {}^0(s)^2\, ds - \int_0^l (s)^2 \operatorname{Ric}({}^0(s); {}^0(s))\, ds:$$

Since $= 1$ on the interval $[r_0; l]$, we obtain

$$\sqrt{(x)} \leq - \int_0^l \operatorname{Ric}({}^0(s); {}^0(s))\, ds + C;$$

where C denotes a uniform constant that does not depend on x

On the other hand, using the identity $D^2 f = \operatorname{Ric}$, we obtain

$$\frac{d}{ds} \langle \nabla f((s)); {}^0(s)\rangle = \operatorname{Ric}({}^0(s); {}^0(s)):$$

Integrating this identity over $s \in [0; l]$ gives

$$\langle \nabla f((l)); {}^0(l)\rangle \leq \int_0^l \operatorname{Ric}({}^0(s); {}^0(s))\, ds + C;$$

where C is a uniform constant that does not depend on x Since (l) = x and ^{0}(l) = r (x), $\sqrt{\ }$ we obtain

$$h r f (x); r \sqrt{(x)} i \quad \int_0^l Ric(^0(s); ^0(s)) ds + C;$$

where C is a uniform constant that does not depend on x Putting these facts together, we conclude that

$$\sqrt{(x)} + h r f (x); r \sqrt{(x)} i \quad C;$$

where C is a uniform constant that does not depend on x This nally implies

$$(x) + h r f (x); r (x) i \quad 2 + C \sqrt{\ };$$

where C is a uniform constant that does not depend on x This completes the proof of Lemma 8.1.

Proposition 8.2 (B.L. Chen [14]). The manifold (M; g) has nonnegative scalar curvature.

Proof. As above, we x an arbitrary point p 2 M. Let N_0 and N_1 denote the constants in Lemma 8.1. Let us x a radius r > 0. We dene a continuous function u : B(p; r) ! R by

$$u(x) := R(x) + 2n(12 + N_0 + N_1 r) r^2 (r^2 \quad d(p; x)^2) \quad ^2$$

for x 2 B(p; r). We claim that u(x) 0 for all x 2 B(p; r). To prove this, we argue by contradiction. Let x be a point in B(p; r) where the function u attains its minimum, and suppose that u(x) < 0. The evolution equation for the scalar curvature implies

$$R + h X; r R i = \quad 2 jRicj^2:$$

By Lemma 8.1, we can nd a smooth function which is dened in an open neighborhood of x and satises the following conditions:

(x) d(p; x)^2 in an open neighborhood of the point x (x) = d(p; x)^2.
jrj^2 = 4 at the point x
+ h X; r i N_0 + N_1 $\sqrt{\ }$ at the point x

Then

$$R(x) + 2n(12 + N_0 + N_1 r) r^2 (r^2 \quad (x)) \quad ^2 \quad u(x) \quad u(x)$$

in an open neighborhood of x with equality at the point x Consequently, the function R + 2n(12 + N_0 + N_1 r)r^2(r^2) ^{2} attains a local minimum at the point x Thus, we conclude that

$$\begin{aligned}
0 \quad & R + \langle X, \nabla R\rangle \\
& + 2n(12 + N_0 + N_1 r)r^2 \;((r^2 \quad)^{2}) + \langle X, \nabla((r^2 \quad)^{2})\rangle \\
& = R + \langle X, \nabla R\rangle \\
& + 4n(12 + N_0 + N_1 r)r^2 (r^2 \quad)^{3} (+ \langle X, \nabla \rangle) + \\
& 12n(12 + N_0 + N_1 r)r^2 (r^2 \quad)^{4} |\nabla|^2 \\
& \quad 2\,|\mathrm{Ric}|^2 + 4n(12 + N_0 + N_1 r)(N_0 + N_1 \sqrt{\;})r^2 (r^2 \quad)^{3} \\
& + 48n(12 + N_0 + N_1 r)r^2 (r^2 \quad)^{4}
\end{aligned}$$

at the point x Since $u \quad 0$ at the point x we know that

$$R \quad 2n(12 + N_0 + N_1 r)r^2 (r^2 \quad)^{2}$$

at the point x This implies

$$n\,|\mathrm{Ric}|^2 \quad R^2 \quad 4n^2 (12 + N_0 + N_1 r)^2 r^4 (r^2 \quad)^{4}$$

at the point x Putting these facts together, we obtain

$$\begin{aligned}
0 \quad & 2\,|\mathrm{Ric}|^2 + 4n(12 + N_0 + N_1 r)(N_0 + N_1 \sqrt{\;})r^2 (r^2 \quad)^{3} \\
& + 48n(12 + N_0 + N_1 r)r^2 (r^2 \quad)^{4} \\
& 8n(12 + N_0 + N_1 r)^2 r^4 (r^2 \quad)^{4} \\
& + 4n(12 + N_0 + N_1 r)(N_0 + N_1 r)r^4 (r^2 \quad)^{4} \\
& + 48n(12 + N_0 + N_1 r)r^4 (r^2 \quad)^{4} \\
& = \quad 4n(12 + N_0 + N_1 r)^2 r^4 (r^2 \quad)^{4}
\end{aligned}$$

at the point x This is a contradiction.

Thus, we conclude that

$$R(x) + 2n(12 + N_0 + N_1 r)r^2 (r^2 \quad d(p,x)^2)^{2} \quad 0$$

for all $x \in B(p,r)$. Sending $r \to \infty$ gives $R(x) \quad 0$ for each point $x \in M$. This completes the proof of Proposition 8.2.

Corollary 8.3. There exists a large constant C such that $|\nabla f| \quad C$ at each point on M.

Proof. Since (M,g,f) is a a steady gradient soliton, the sum $R + |\nabla f|^2$ is constant. Since $R \quad 0$ by Proposition 8.2, we conclude that $|\nabla f|$ is uniformly bounded from above. This completes the proof of Corollary 8.3.

In the remainder of this section, we assume that M is three-dimensional. As in Section 1, we denote by $_1 \quad _2 \quad _3$ the eigenvalues of the tensor $R\,g_{ij} \quad 2\,\mathrm{Ric}_{ij}$. Then $R = {}_1 + {}_2 + {}_3$. Since $R \quad 0$, it follows that $_3 \quad 0$ at each point on M.

Proposition 8.4 (B.L. Chen [14]). Assume that $n = 3$. Then $k_1 + 2R \;\; 0$ for every nonnegative integer k.

Proof. The proof is by induction on k. For $k = 0$, the assertion follows from Proposition 8.2.

We now turn to the inductive step. Suppose that $k \;\; 1$ and $(k \quad 1)_1 \;\; + 2R \;\; 0$. As above, we x an arbitrary point p 2 M. Let N_0 and N_1 denote the constants in Lemma 8.1. Let us x a radius $r > 0$. We dene a continuous function $v : B(p; r)$! R by

$$v(x) := k_1(x) + 2R(x) + 4k(12 + N_0 + N_1 r)r^2(r^2 \quad d(p; x)^2) \;\; ^2$$

for x 2 $B(p; r)$. We claim that $v(x) \;\; 0$ for all x 2 $B(p; r)$. To prove this, we argue by contradiction. Let x be a point in $B(p; r)$ where the function v attains its minimum, and suppose that $v(x) < 0$. The evolution equation for the scalar curvature implies

$$R + hX; rRi = \quad 2\,jRicj^2:$$

The evolution equation for the Ricci tensor gives

$$_1 + hX; r_1 i \quad (_1 + {}_{23}); ^2$$

where the inequality is understood in the barrier sense. More precisely, we can nd a smooth function which is dened in an open neighborhood of x and satises the following conditions:

$(x) \;\; _1(x)$ in an open neighborhood of the point x (x) $= {}_1(x)$.

$+ hX; r \;\; i \quad (_1 + {}_{23})^2$ at the point x

For example, we may dene $:= Ric(;)$, where is a smooth unit vector eld satisfying $Ric(;) = {}_1$ at x $D = 0$ at x and $= 0$ at x

By Lemma 8.1, we can nd a smooth function which is dened in an open neighborhood of x and satises the following conditions:

$(x) \;\; d(p; x)^2$ in an open neighborhood of the point x $(x) = d(p; x)^2$.

$jrj^2 = 4$ at the point x

$+ hX; ri \;\; N_0 + N_1 \quad$ p at the point x

Then

$$k \;\; (x) + 2R(x) + 4k(12 + N_0 + N_1 r)r^2(r^2 \quad (x)) \;\; ^2 \;\; v(x) \;\; v(x)$$

in an open neighborhood of x with equality at the point x Consequently, the function $k \;\; + 2R + 4k(12 + N_0 + N_1 r)r^2(r^2 \quad) \;\; ^2$ attains a local minimum at the point x Thus, we conclude that

$$0 \;\; k(\;\; + hX; r \;\; i) + 2(R + hX; rRi)$$
$$+ 4k(12 + N_0 + N_1 r)r^2 \;\; ((r^2 \qquad) \;\; ^2) + hX; r((r^2 \qquad) \;\; ^2)i$$
$$= k(\;\; + hX; r \;\; i) + 2(R + hX; rRi)$$
$$+ 8k(12 + N_0 + N_1 r)r^2(r^2 \quad) \;\; ^3(+ hX; ri) +$$
$$24k(12 + N_0 + N_1 r)r^2(r^2 \quad) \;\; ^4 jrj^2$$
$$k(^2 + {}_1 {}_{23}) \qquad 4 jRicj^2$$
$$+ 8k(12 + N_0 + N_1 r)(N_0 + N_1 \sqrt{p}\;)r^2(r^2 \qquad) \;\; ^3$$
$$+ 96k(12 + N_0 + N_1 r)r^2(r^2 \qquad) \;\; ^4$$

at the point x Since v 0 at the point x we know that

$$(k_1 + 2R) \;\; 4k(12 + N_0 + N_1 r)r^2(r^2 \qquad) \;\; ^2$$

at the point x Note that R 0, (k 1)$_1$ + 2R 0, k$_1$ + 2R 0, $_1$ 0, and $_3$ 0 at the point x This implies

$$k^2(^2 {}_1 + {}_{23}) + 4k\, jRicj^2$$
$$= k^2(^2 {}_1 + {}_{23}) + 2k(^2 + {}^2 {}_1 + {}^2 + {}_{2\,12} + {}_{3\,23} + {}_{31}) = (k_1 + 2R)^2$$
$$2R(k_1 + 2R) + k((k \quad 1)_1 + 2R)_3$$
$$+ k^2(_2 \quad _1)_3 \qquad k_{13} + 2k^2 \; (k_{1\,2}$$
$$+ 2R)^2$$
$$16k^2(12 + N_0 + N_1 r)^2 r^4(r^2 \qquad) \;\; ^4$$

at the point x Putting these facts together, we obtain 0

$$k(^2 + {}_{23})\, {}_1 \qquad 4 jRicj^2$$
$$+ 8k(12 + N_0 + N_1 r)(N_0 + N_1 \sqrt{p}\;)r^2(r^2 \qquad) \;\; ^3$$
$$+ 96k(12 + N_0 + N_1 r)r^2(r^2 \qquad) \;\; ^4$$
$$16k(12 + N_0 + N_1 r)^2 r^4(r^2 \qquad) \;\; ^4$$
$$+ 8k(12 + N_0 + N_1 r)(N_0 + N_1 r)r^4(r^2 \quad) \;\; ^4$$
$$+ 96k(12 + N_0 + N_1 r)r^4(r^2 \quad) \;\; ^4$$
$$= \quad 8k(12 + N_0 + N_1 r)^2 r^4(r^2 \quad) \;\; ^4$$

at the point x This is a contradiction.

Thus, we conclude that

$$k_1(x) + 2R(x) + 4k(12 + N_0 + N_1 r)r^2(r^2 \qquad d(p;x)^2) \;\; ^2 \;\; 0$$

for all x 2 B(p; r). Sending r ! 1 gives k$_1$(x) + 2R(x) 0 for each point x 2 M. This completes the proof of Proposition 8.4.

Corollary 8.5 (B.L. Chen [14]). *Assume that $n = 3$. Then (M, g) has nonnegative sectional curvature.*

Proof. Sending $k \to \infty$ in Proposition 8.4 gives $\lambda_1 \geq 0$.

Corollary 8.6. *Assume that $n = 3$. Then (M, g) has bounded curvature.*

Proof. Since (M, g, f) is a a steady gradient soliton, the sum $R + |\nabla f|^2$ is constant. Consequently, the scalar curvature is uniformly bounded from above. Hence, the assertion follows from Corollary 8.5.

From now on, we will assume that (M, g) is κ-noncollapsed. Moreover, we will assume that (M, g, f) is normalized so that $R + |\nabla f|^2 = 1$ at each point on M. Let Φ_t denote the one-parameter group of diffeomorphisms generated by the vector field $-X$. It follows from Corollary 8.3 that Φ_t is defined for all $t \in (-\infty, \infty)$. In view of Corollary 8.5 and Corollary 8.6, the metrics $\Phi_t^*(g)$, $t \in (-\infty, 0]$, form an ancient κ-solution to the Ricci flow.

Proposition 8.7. *Assume that $n = 3$ and (M, g) is κ-noncollapsed. Then (M, g) has positive sectional curvature.*

Proof. By Corollary 8.5, (M, g) has nonnegative sectional curvature. We claim that (M, g) has strictly positive sectional curvature. Suppose this is false. By the strict maximum principle, the universal cover of (M, g) splits off a line. Using Perelman's classification of ancient κ-solutions in dimension 2 (see Theorem 3.1), we conclude that the universal cover of (M, g) is iso-metric to a cylinder $S^2 \times \mathbb{R}$, up to scaling. In particular, (M, g) has constant scalar curvature. This contradicts the fact that $R + \langle X, \nabla R\rangle = -2\,|\mathrm{Ric}|^2$ at each point on M. This completes the proof of Proposition 8.7.

Proposition 8.8. *Assume that $n = 3$ and (M, g) is κ-noncollapsed. Let p_j be a sequence of points going to infinity, and let $r_j^{-2} := R(p_j)$. Let us dilate the manifold (M, g) around the point p_j by the factor r_j^{-1}. Then, after passing to a subsequence if necessary, the rescaled manifolds converge in the Cheeger-Gromov sense to a cylinder of radius $\sqrt{2}$.*

Proof. Since (M, g) has positive sectional curvature, the assertion follows from Corollary 4.4.

Corollary 8.9. *Assume that $n = 3$ and (M, g) is κ-noncollapsed. Then $R \to 0$ at infinity.*

Proof. Proposition 8.8 implies that $R^{-2}\,\Delta R \to 0$ and $R^{-\frac{3}{2}}\,|\nabla R| \to 0$ at infinity. Since R is bounded from above, it follows that $R \to 0$ and $|\nabla R| \to 0$ at infinity. Since $|X|$ is bounded, we conclude that $R + \langle X, \nabla R\rangle \to 0$ at innity. On the other hand, the evolution equation for the scalar curvature gives $R + \langle X; rR \rangle = \ 2\, jRicj^2$ at each point in M. Putting these facts together, we conclude that $jRicj^2$! 0 at innity. This completes the proof of Corollary 8.9.

Corollary 8.10. Assume that $n = 3$ and $(M; g)$ is -noncollapsed. Let p_j be a sequence of points going to innity, and let $r_j^{\ 2} := R(p_j)$. Let us dilate the manifold $(M; g)$ around the point p_j by the factor $r_j^{\ 1}$. Then, after passing to a subsequence if necessary, the rescaled manifolds converge in the Cheeger-Gromov sense to a cylinder of radius $\text{p}\,\overline{2}$, and the rescaled vector elds $r_j X$ converge in C^1_{loc} to the axial vector eld on the cylinder.

Proof. The vector eld X satises the pointwise estimates jXj 1 and $jDXj = jRicj$ $C\,R$. Moreover, Perelman's pointwise derivative estimate (see Theorem 4.1) implies $jD^{m+1}Xj = jD^m Ricj$ $C\,R^{\frac{m+2}{}}$ for every positive integer m. Consequently, the rescaled vector elds $r_j X$ converge in C^1_{loc} to a limit vector eld on the cylinder. Since $jXj^2 = 1$ R ! 1 at innity, the limiting vector eld on the cylinder has unit length at each point. Since $jDXj$ $C\,R$, the limiting vector eld on the cylinder is parallel. This com-pletes the proof of Corollary 8.10.

Proposition 8.11. Assume that $n = 3$ and $(M; g)$ is -noncollapsed. Then the function f has a unique critical point p, and f attains its global mini-mum at the point p.

Proof. In view of Corollary 8.9, there exists a point p where the scalar curvature is maximal. In particular, $rR = 0$ at the point p. Since $R + jXj^2$ is constant, we know that $rR + 2\,Ric(X) = 0$ at each point on M. Con-sequently, $Ric(X) = 0$ at the point p. Since $(M; g)$ has positive Ricci curvature, it follows that $X = 0$ at the point p. In other words, p is a critical point of f. Since $(M; g)$ has positive Ricci curvature, the function f is strictly convex. Thus, p is the only critical point of f, and f attains its global minimum at the point p. This completes the proof of Proposition 8.11.

Corollary 8.12. Assume that $n = 3$ and $(M; g)$ is -noncollapsed. Then there exists a positive constant C such that

$$\frac{1}{C}\, d(p; x) \quad f(x) \quad C\, d(p; x)$$

outside some compact set.

Proof. The upper bound for f follows from Corollary 8.3. The lower bound follows from Proposition 8.11 together with the strict convexity of f.

Proposition 8.13 (H. Guo [20]). *Assume that $n = 3$ and (M, g) is κ-noncollapsed. Then $fR \to 1$ at infinity.*

Proof. Using the evolution equation for the scalar curvature, we obtain

$$\Delta R + \langle X, \nabla R\rangle = -2\,|\mathrm{Ric}|^2$$

at each point in M. This implies

$$\begin{aligned}\langle X, \nabla(R^{-1} - f)\rangle &= -R^{-2}\langle X, \nabla R\rangle - |\nabla f|^2 \\ &= R^{-2}\Delta R + 2R^{-2}|\mathrm{Ric}|^2 - |\nabla f|^2\end{aligned}$$

at each point in M. Using Proposition 8.8, we obtain $R^{-2}\Delta R \to 0$ and $R^{-2}|\mathrm{Ric}|^2 \to \frac{1}{2}$ at infinity. Moreover, Corollary 8.9 implies $|\nabla f|^2 = 1 - R \to 1$ at infinity. Putting these facts together, we conclude that

$$\langle X, \nabla(R^{-1} - f)\rangle \to 0$$

at infinity. Hence, if $\varepsilon > 0$ is given, then

$$\langle X, \nabla(R^{-1} - (1+\varepsilon) f)\rangle \leq 0$$

and

$$\langle X, \nabla(R^{-1} - (1-\varepsilon) f)\rangle \geq 0$$

outside a compact set. Integrating these inequalities along the integral curves of X, we obtain

$$\sup_M (R^{-1} - (1+\varepsilon) f) < \infty$$

and

$$\inf_M (R^{-1} - (1-\varepsilon) f) > -\infty.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $fR \to 1$ at infinity. This completes the proof of Proposition 8.13.

In particular, if s is sufficiently large and $f(x) = s$, then x lies at center of a neck, and the radius of the neck is $(1 + o(1))\sqrt{2s}$.

9. Rotational symmetry of noncollapsed steady gradient Ricci solitons in dimension 3 – the proof of Theorem 5.1

Throughout this section, we assume that (M, g, f) is a non-flat steady gradient Ricci soliton in dimension 3 which is κ-noncollapsed. Moreover, we assume that (M, g, f) is normalized so that $R + |\nabla f|^2 = 1$ at each point on M. It follows from Proposition 8.11 that f is bounded from below. After adding a constant to f if necessary, we may assume that f is positive at each point on M. For abbreviation, we put $X := \nabla f$. Let Φ_t denote the one-parameter group of diffeomorphisms generated by the vector field $-X$. By Corollary 8.3, Φ_t is defined for all $t \in (-\infty, \infty)$. Moreover, the metrics $\Phi_t^*(g)$, $t \in (-\infty, 0]$, form an ancient κ-solution to the Ricci flow.

Let us fix a large real number L and a small positive real number ε_1 so that the conclusion of the Neck Improvement Theorem holds. In view of

Proposition 8.13, we can nd a large constant with the following properties:

$f(x)\, R(x;0)^{\frac{1}{2}}\ 10^6 L$ for each point $x\ 2\ M$ with $f(x)\ \ _2$.
If $f(\bar{x})\ \ _2$, then $(\bar{x}, 0)$ lies at the center of an evolving $"_1$-neck.

By a repeated application of the Neck Improvement Theorem, we obtain the following result.

Proposition 9.1. Suppose that j is a nonnegative integer and x is a point in M with $f(x)\ \ 2^{\frac{j}{400}}$. Then the point $(x;0)$ is $2^{\ j}\ "_1$-symmetric.

Proof. The proof is by induction on j. For $j = 0$, the assertion is true by our choice of . Suppose next that $j\ \ 1$, and the assertion is true for $j\ \ \ 1$. Let us x a point $x 2\ M$ satisfying $f(\bar{x})\ \ 2^{\frac{j}{400}}$, and let $r^{\ 2} := R(\bar{x}, 0)$. By our choice of , $f(\bar{x})\, r^{\ 1} = f(\bar{x})\, R(\bar{x}, 0)^{\frac{1}{2}}\ \ 10^6 L$. Since $jrfj\ \ 1$ at each point on M, we obtain

$$f(x)\ \ f(\bar{x})\ \ \ \ L r\ \ (1\ \ \ 10^{\ 6})\, f(\bar{x})\ \ (1\ \ \ 10^{\ 6})\, 2^{\frac{j}{400}}\ \ 2^{\frac{j\ \ 1}{400}}$$

for each point $x\ 2\ B_g(\bar{x}, L r)$. This implies

$$f(\ _t(x))\ \ f(x)\ \ 2^{\frac{j\ \ 1}{400}}$$

for each point $x\ 2\ B_g(\bar{x}, L r)$ and each $t\ \ 0$. We now apply the induction hypothesis. Hence, if $x\ 2\ B_g(\bar{x}, L r)$ and $t\ \ 0$, then the point $(\ _t(x);0)$ is $2^{\ j+1}\ "_1$-symmetric. Since we are working on a self-similar solution, the point $(x;t)$ plays the same role as the point $(\ _t(x);0)$. Consequently, if $x\ 2\ B_g(\bar{x}, L r)$ and $t\ \ 0$, then the point $(x;t)$ is $2^{\ j+1}\ "_1$-symmetric. Using the Neck Improvement Theorem, we conclude that the point $(\bar{x}, 0)$ is $2^{\ j}\ "_1$-symmetric. This completes the proof of Proposition 9.1.

Corollary 9.2. If m is suciently large, then we can nd smooth vector elds $U^{(1;m)}; U^{(2;m)}; U^{(3;m)}$ on the domain $f m\ \ \ 80\ \ m\ \ \sqrt{m} + 80\ \ m g$ with $\sqrt{\ }$ the following properties:

$\sum_{l=0}^{2} \sum_{a=1}^{3} m^l\, |D^l (L_{U^{(a;m)}}(g))|^2\ \ C\ m^{\ 800}$.
If $f m\ \ \ 80\ \ m\ \sqrt{\ } m + 80\ \ m g$ is a $\sqrt{\ }$ leaf of the CMC foli-ation, then $\sup \sum_{a=1}^{3} m^{\ 1}\, |\langle U^{(a)}; \ \rangle|^2\ \ C\ m^{\ 800}$, where denotes the unit normal vector to .
If $f m\ \ \ 80\ \ m\ \sqrt{\ } m + 80\ \ m g$ is a leaf of the CMC foliation, then

$$\sum_{a,b=1}^{3} \Big(\ _{ab}\ \ \ \mathrm{area}()^{\ 2} \int \langle U^{(a;m)}; U^{(b;m)} \rangle\, d\Big)^2\ \ C\ m^{\ 800}.$$

As explained in Section 7 of [8], we can glue approximate Killing vector elds on overlapping necks. This allows us to draw the following conclusion:

Corollary 9.3. We can find smooth vector fields $U^{(1)}, U^{(2)}, U^{(3)}$ such that

$$|\mathcal{L}_{U^{(a)}}(g)| \leq C\,(f+100)^{-100},$$
$$|D(\mathcal{L}_{U^{(a)}}(g))| \leq C\,(f+100)^{-100},$$
$$|D^2(\mathcal{L}_{U^{(a)}}(g))| \leq C\,(f+100)^{-100}.$$

Finally, given any positive real number ε, we have

$$\sum_{a,b=1}^{3} \Big(\delta_{ab} - \mathrm{area}(\Sigma)^{-2} \int_\Sigma \langle U^{(a)}, U^{(b)} \rangle\Big)^2 \leq \varepsilon^2$$

whenever Σ is a leaf of the CMC foliation which is sufficiently far out near infinity (depending on ε).

Lemma 9.4. We have $|\langle [U^{(a)}, X], X\rangle| \leq C\,(f+100)^{-40}$.

Proof. The vector field X satisfies $|X|^2 = 1 - R$. Let us take the Lie derivative along $U^{(a)}$ on both sides. This gives

$$(\mathcal{L}_{U^{(a)}}(g))(X,X) + 2\,\langle \mathcal{L}_{U^{(a)}}(X), X\rangle = -\mathcal{L}_{U^{(a)}}(R).$$

Using the formula for the linearization of the scalar curvature (see [4], Theorem 1.174 (e)), we obtain

$$|\mathcal{L}_{U^{(a)}}(R)| \leq C\,|D^2(\mathcal{L}_{U^{(a)}}(g))| + C\,|\mathrm{Ric}|\,|\mathcal{L}_{U^{(a)}}(g)| \leq C\,(f+100)^{-40}.$$

Putting these facts together, we conclude that

$$|\langle \mathcal{L}_{U^{(a)}}(X), X\rangle| \leq C\,(f+100)^{-40}.$$

This completes the proof of Lemma 9.4.

Lemma 9.5. We have $|D([U^{(a)}, X])| \leq C\,(f+100)^{-40}$.

Proof. The vector field X satisfies $g_{jk}\, D_i X^j = \mathrm{Ric}_{ik}$. Let us take the Lie derivative along $U^{(a)}$ on both sides. Using the formula for the linearization of the Levi-Civita connection (see [4], Theorem 1.174 (a)), we obtain

$$\begin{aligned}
&(\mathcal{L}_{U^{(a)}}(g))_{jk}\, D_i X^j + g_{jk}\, D_i(\mathcal{L}_{U^{(a)}}(X))^j \\
&+ \frac{1}{2} D_i(\mathcal{L}_{U^{(a)}}(g))_{jk}\, X^j + \frac{1}{2} D_j(\mathcal{L}_{U^{(a)}}(g))_{ik}\, X^j - \frac{1}{2} D_k(\mathcal{L}_{U^{(a)}}(g))_{ij}\, X^j \\
&= (\mathcal{L}_{U^{(a)}}(\mathrm{Ric}))_{ik}.
\end{aligned}$$

The formula for the linearization of the Ricci tensor (see [4], Theorem 1.174 (d)) gives

$$|\mathcal{L}_{U^{(a)}}(\mathrm{Ric})| \leq C\,|D^2(\mathcal{L}_{U^{(a)}}(g))| + C\,|\mathrm{Rm}|\,|\mathcal{L}_{U^{(a)}}(g)| \leq C\,(f+100)^{-40}.$$

Putting these facts together, we conclude that

$$|D(\mathcal{L}_{U^{(a)}}(X))| \leq C\,(f+100)^{-40}.$$

This completes the proof of Lemma 9.5.

Lemma 9.6. We have $|[U^{(a)}, X]| \leq (f + 100)^{-20}$ outside a compact set.

Proof. Suppose that the assertion is false. Then there exists an index $a \in \{1, 2, 3\}$ and a sequence of points p_j going to infinity such that

$$|[U^{(a)}, X]| \geq (f + 100)^{-20}$$

at the point p_j. Let us define $s_j := f(p_j)$, and let A_j denote the norm of the vector field $[U^{(a)}, X]$ at the point p_j. By assumption, $A_j \geq (s_j + 100)^{-20}$ for each j. Since $|\nabla f| \leq 1$ at each point on M, we know that $\frac{s_j}{2} \leq f \leq \frac{3s_j}{2}$ at each point in $B_g(p_j, \frac{s_j}{2})$. Using Lemma 9.4 and Lemma 9.5, we obtain

$$\sup_{B_g(p_j, \frac{s_j}{2})} |\langle [U^{(a)}, X], X\rangle| \leq C\,(s_j + 100)^{-40} \leq C\,(s_j + 100)^{-20} A_j$$

and

$$\sup_{B_g(p_j, \frac{s_j}{2})} |D([U^{(a)}, X])| \leq C\,(s_j + 100)^{-40} \leq C\,(s_j + 100)^{-20} A_j.$$

In the next step, we integrate the bound for $D([U^{(a)}, X])$ along geodesics emanating from p_j. If j is sufficiently large, we obtain

$$\sup_{B_g(p_j, \frac{s_j}{2})} |[U^{(a)}, X]| \leq A_j + C\,(s_j + 100)^{-30} \leq 2A_j.$$

We now dilate the manifold (M, g) around the point p_j by the factor $s_j^{-\frac{1}{2}}$. By Corollary 8.10, the rescaled manifolds converge in the Cheeger-Gromov sense to a cylinder of radius $\sqrt{2}$, and the rescaled vector fields $s_j^{-\frac{1}{2}} X$ converge in C^∞_{loc} to the axial vector field on the cylinder. Moreover, the vector fields $s_j^{\frac{1}{2}} A_j^{-1} [U^{(a)}, X]$ converge in $C^{\frac{1}{2}}_{loc}$ to a non-trivial parallel vector field on the cylinder, and this limiting vector field is orthogonal to the axial vector field on the cylinder. This is a contradiction. This completes the proof of Lemma 9.6.

Lemma 9.7. We have $|U^{(a)} + D_X U^{(a)}| \leq C\,(f + 100)^{-20}$.

Proof. Using Proposition 6.2 and Corollary 9.3, we obtain

$$|U^{(a)} + \mathrm{Ric}(U^{(a)})| \leq C\,|D(\mathcal{L}_{U^{(a)}}(g))| \leq C\,(f + 100)^{-100}.$$

Moreover, Lemma 9.6 gives

$$|[U^{(a)}, X]| \leq C\,(f + 100)^{-20}.$$

Using the identity

$$\begin{aligned} U^{(a)} + D_X U^{(a)} &= U^{(a)} + D_{U^{(a)}} X - [U^{(a)}, X] \\ &= U^{(a)} + \mathrm{Ric}(U^{(a)}) - [U^{(a)}, X], \end{aligned}$$

we conclude that

$$|\Delta U^{(a)} + D_X U^{(a)}| \leq C\,(f+100)^{-20}.$$

This completes the proof of Lemma 9.7.

For abbreviation, we define smooth vector fields $Q^{(1)}, Q^{(2)}, Q^{(3)}$ by $Q^{(a)} := \Delta U^{(a)} + D_X U^{(a)}$.

Proposition 9.8. We can find smooth vector fields $W^{(1)}, W^{(2)}, W^{(3)}$ such that $|W^{(a)}| \leq C\,(f+100)^{-8}$ and $\Delta W^{(a)} + D_X W^{(a)} = Q^{(a)}$.

Proof. We consider a sequence of real numbers $s_j \to \infty$. For each j and each $a \in \{1,2,3\}$, we denote by $W^{(a,j)}$ the solution of the elliptic PDE

$$\Delta W^{(a,j)} + D_X W^{(a,j)} = Q^{(a)}$$

on the domain $\{f \leq s_j\}$ with Dirichlet boundary condition $W^{(a,j)} = 0$ on the boundary $\{f = s_j\}$. This Dirichlet problem has a solution by the Fredholm alternative. Moreover, since $Q^{(a)}$ is smooth, it follows that $W^{(a,j)}$ is smooth.

By Lemma 9.7, $Q^{(a)}$ satisfies a pointwise estimate of the form

$$|Q^{(a)}| \leq K\,(f+100)^{-20},$$

where K is a large constant that does not depend on j. Using Kato's inequality, we obtain

$$\Delta|W^{(a,j)}| + \langle X, \nabla|W^{(a,j)}|\rangle \geq -|Q^{(a)}| \geq -K\,(f+100)^{-20}$$

on the set $\{W^{(a,j)} \neq 0\}$. On the other hand, using the identity $\Delta f + \langle X, \nabla f\rangle = R + |\nabla f|^2 = 1$, we obtain

$$\begin{aligned}
&\Delta((f+100)^{-8}) + \langle X, \nabla((f+100)^{-8})\rangle \\
&= -8\,(f+100)^{-9}\,(\Delta f + \langle X, \nabla f\rangle) + 72\,(f+100)^{-10}\,|\nabla f|^2 \\
&\leq -8\,(f+100)^{-9} + 72\,(f+100)^{-10} \\
&\leq -(f+100)^{-9}.
\end{aligned}$$

Using the maximum principle, we conclude that

$$|W^{(a,j)}| \leq K\,(f+100)^{-8}$$

on the set $\{f \leq s_j\}$.

We now send $j \to \infty$. After passing to a subsequence, the vector fields $W^{(a,j)}$ converge in C^∞_{loc} to a smooth vector field $W^{(a)}$. The limiting vector field $W^{(a)}$ satisfies

$$|W^{(a)}| \leq K\,(f+100)^{-8}$$

and

$$\Delta W^{(a)} + D_X W^{(a)} = Q^{(a)}.$$

This completes the proof of Proposition 9.8.

Proposition 9.9. We have $|DW^{(a)}| \leq C\,(f+100)^{-8}$.

Proof. By Proposition 9.8, the vector field $W^{(a)}$ satisfies $\Delta W^{(a)} + D_X W^{(a)} = -Q^{(a)}$. This equation can be rewritten as $\Delta W^{(a)} + \mathcal{L}_X(W^{(a)}) + \text{Ric}(W^{(a)}) = -Q^{(a)}$. We next consider the vector fields $(\Phi_t^* W^{(a)})$ on the evolving background $(M, \Phi_t^*(g))$. These vector fields satisfy the parabolic PDE

$$\frac{\partial}{\partial t}(\Phi_t^* W^{(a)}) = \Delta_{\Phi_t^*(g)}(\Phi_t^* W^{(a)}) + \text{Ric}_{\Phi_t^*(g)}(\Phi_t^* W^{(a)}) - \Phi_t^*(Q^{(a)}).$$

The assertion follows now from interior estimates for parabolic PDE (see e.g. [11], Proposition C.2). This completes the proof of Proposition 9.9.

We next define smooth vector fields $V^{(1)}, V^{(2)}, V^{(3)}$ by $V^{(a)} := U^{(a)} - W^{(a)}$.

Proposition 9.10. *The vector field $V^{(a)}$ satisfies $\Delta V^{(a)} + D_X V^{(a)} = 0$.*

Proof. This follows immediately from Proposition 9.8.

Proposition 9.11. *The tensor $\mathcal{L}_{V^{(a)}}(g)$ vanishes identically.*

Proof. Recall that

$$\Delta V^{(a)} + D_X V^{(a)} = 0.$$

By Corollary 6.4, the tensor $h^{(a)} := \mathcal{L}_{V^{(a)}}(g)$ satisfies

$$\Delta_L h^{(a)} + \mathcal{L}_X(h^{(a)}) = 0.$$

Hence, Corollary 6.6 implies

$$\Delta\Big(\frac{|h^{(a)}|^2}{R^2}\Big) + \Big\langle X, \nabla\Big(\frac{|h^{(a)}|^2}{R^2}\Big)\Big\rangle + \frac{2}{R}\Big\langle \nabla R, \nabla\Big(\frac{|h^{(a)}|^2}{R^2}\Big)\Big\rangle \geq 0.$$

Using the maximum principle, we obtain

$$\sup_{\{f \leq s\}} \frac{|h^{(a)}|}{R} \leq \sup_{\{f = s\}} \frac{|h^{(a)}|}{R}$$

for each s. On the other hand, Corollary 9.3 and Proposition 9.9 imply

$$|h^{(a)}| \leq |\mathcal{L}_{U^{(a)}}(g)| + C\,|DW^{(a)}| \leq C\,(f + 100)^{-8}.$$

Using Proposition 8.13, we deduce that

$$\frac{|h^{(a)}|}{R} \leq C\,(f + 100)^{-7}.$$

In particular,

$$\sup_{\{f = s\}} \frac{|h^{(a)}|}{R} \to 0$$

as $s \to \infty$. Putting these facts together, we conclude that $h^{(a)}$ vanishes identically. This completes the proof of Proposition 9.11.

Corollary 9.12. *We have $[V^{(a)}, X] = 0$ and $\langle V^{(a)}, X\rangle = 0$.*

Proof. Note that
$$V^{(a)} + D_X V^{(a)} = 0$$
by Proposition 9.10. On the other hand, since $V^{(a)}$ is a Killing vector eld, we obtain
$$V^{(a)} + \mathrm{Ric}(V^{(a)}) = 0$$
by Proposition 6.2. This implies $D_X V^{(a)} = \mathrm{Ric}(V^{(a)})$. On the other hand, $D_{V^{(a)}} X = \mathrm{Ric}(V^{(a)})$. Consequently, $[V^{(a)}; X] = 0$. This proves the rst statement. We now turn to the proof of the second statement. Since $V^{(a)}$ is a Killing vector eld, we obtain
$$r(L_{V^{(a)}}(f)) = L_{V^{(a)}}(rf) = L_{V^{(a)}}(X) = 0:$$
Consequently, the function $L_{V^{(a)}}(f) = hV^{(a)}; Xi$ is constant. On the other hand, by Proposition 8.11, the vector eld X vanishes at some point p 2 M. Thus, we conclude that the function $hV^{(a)}; Xi$ vanishes identically. This completes the proof of Corollary 9.12.

Corollary 9.12 implies that the vector elds $V^{(1)}; V^{(2)}; V^{(3)}$ are tangential to the level sets of f.

Proposition 9.13. Given any positive real number ", we have
$$\sum_{a;b=1}^{3} \quad {}_{ab} \quad \mathrm{area}()^{2} \int hV^{(a)}; V^{(b)} i \, d \quad ^{2} \quad "^{2}$$
whenever is a leaf of the CMC foliation which is suciently far out near innity (depending on ").

Proof. This follows by combining Corollary 9.3 with Proposition 9.8.

Proposition 9.13 ensures that the vector elds $V^{(1)}; V^{(2)}; V^{(3)}$ are non-trivial near innity. From this, it is easy to see that $(M; g)$ is rotationally symmetric.

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