ALGEBRAIC PROPERTIES OF HERMITIAN SUMS OF SQUARES, II

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Abstract. We study real bihomogeneous polynomials $r(z,\bar{z})$ in n complex variables for which $r(z,\bar{z})\,\|z\|^2$ is the squared norm of a holomorphic polynomial mapping. Such polynomials are the focus of the Sum of Squares Conjecture, which describes the possible ranks for the squared norm $r(z,\bar{z})\,\|z\|^2$ and has important implications for the study of proper holomorphic mappings between balls in complex Euclidean spaces of different dimension. Questions about the possible signatures for $r(z,\bar{z})$ and the rank of $r(z,\bar{z})\,\|z\|^2$ can be reformulated as questions about polynomial ideals. We take this approach and apply purely algebraic tools to obtain constraints on the signature of r.

1. Introduction

Let $r(z,\bar{z}) = \sum_{|\alpha|=|\beta|=m} c_{\alpha,\beta} z^\alpha \bar{z}^\beta$ be a real bihomogeneous polynomial on the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$ of bi-degree (m,m). Its *rank* and *signature* are the rank and signature of the corresponding Hermitian coefficient matrix $(c_{\alpha,\beta})$. Suppose the bihomogeneous polynomial $r(z,\bar{z})\,\|z\|^2 = r(z,\bar{z}) \sum_{j=1}^n |z_j|^2$ is a squared norm, so that

$$r(z,\bar{z})\,\|z\|^2 = \|h(z)\|^2 = \sum_{k=1}^{\rho} |h_k(z)|^2$$

for some holomorphic polynomial mapping $h = (h_1, \ldots, h_\rho)$ defined on $\mathbb{C}^n$. In this case, the coefficient matrix for $r(z,\bar{z})\,\|z\|^2$ is non-negative semi-definite.

Due to its connection to the Gap Conjecture Problem for proper holomorphic mappings between balls in complex Euclidean spaces of different dimensions, an important open problem in the theory of functions of several complex variables is to determine the possible ranks of $r(z,\bar{z})\,\|z\|^2$.

Conjecture 1 (Ebenfelt Sum of Squares (SOS) Conjecture, [Ebe17])**.** *Suppose $n \geq 2$. Let $r(z,\bar{z})$ be a real bihomogeneous polynomial, and suppose*

$$r(z,\bar{z})\,\|z\|^2 = \|h(z)\|^2$$

for some holomorphic polynomial mapping h. Let ρ be the rank of $\|h\|^2$ and let

$$k_0 = \max\left\{k \in \mathbb{N} : \frac{k(k+1)}{2} < n-1\right\}.$$

Then either

$$\rho \geq (k_0+1)n - \frac{k_0(k_0+1)}{2}, \tag{1}$$

or there exists an integer $0 \leq k \leq k_0 < n$ such that

$$nk - \frac{k(k-1)}{2} \leq \rho \leq nk. \tag{2}$$

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In [GK15], the first two authors prove the Sum of Squares Conjecture when r has signature $(P, 0)$, and [BG21] shows it holds when $n = 3$ and the coefficient matrix of r is diagonal. The gaps in possible rank described in (2) occur in the non-negative semi-definite case, and it appears that when the coefficient matrix of r has negative eigenvalues, the inequality (1) always holds.

Thus the task is to determine the possible ranks for $r(z, \bar{z}) \|z\|^2$ when r has signature (P, N) with $N > 0$ and its coefficient matrix is not diagonal. Even for $N = 1$, we encounter difficulties, leading to the following:

Question 1. *Suppose $r(z, \bar{z})$ has signature $(P, 1)$ and $r(z, \bar{z}) \|z\|^2$ is a squared norm. What is the minimum possible value for P?*

When the coefficient matrix for r is diagonal, [HL13] proves that $P \geq n$ and that $P = n$ is possible. So a natural question is whether the same bound holds for arbitrary r. Our main result answers this question:

Theorem 1. *Let $r(z, \bar{z})$ be a real bihomogeneous polynomial on the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$ of bi-degree (m, m) with signature $(P, 1)$. Suppose that $r(z, \bar{z}) \|z\|^2$ is a squared norm. If $n \leq 3$, then $P \geq n$. However, if $n \geq 4$, then $P < n$ is possible.*

1.1. **Algebraic Formulation.** In [GK15] and [BG21], the first two authors investigate $r(z, \bar{z}) \|z\|^2$ by translating the problem into one about homogeneous ideals in $R = \mathbb{C}[z_1, \ldots, z_n]$. The process begins with a *holomorphic decomposition* of r; we write $r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$ where $f \colon \mathbb{C}^n \to \mathbb{C}^P$ and $g \colon \mathbb{C}^n \to \mathbb{C}^N$ are holomorphic polynomial mappings, all of whose components are homogeneous polynomials on $\mathbb{C}^n$ of degree m. We refer to [D'A02] for further discussion of this technique and its applications.

A holomorphic decomposition of a bihomogeneous polynomial is not unique, but if we require the polynomials f_j, g_k to be linearly independent, then P and N are uniquely determined, (P, N) is the signature of the coefficient matrix of r, and $P + N$ is its rank.

For $r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$, consider two ideals: I^+ is the ideal generated by the P components of f and I^- is the ideal generated by the N components of g. The condition that $r(z, \bar{z}) \|z\|^2$ is a squared norm can be restated as a condition on I^+_{m+1} and I^-_{m+1} (the components of these ideals in degree $m + 1$), as described in the next proposition. This result first appeared in a somewhat more general form in [D'A05]; [BG20] reformulates it in algebraic terms as follows:

Proposition 1. *Let $r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$ be a real bihomogeneous polynomial of bi-degree (m, m), and define I^+ and I^- as above. If $r(z, \bar{z}) \|z\|^2$ is a squared norm, then $I^-_{m+1} \subseteq I^+_{m+1}$.*

In this language, Theorem 1 takes the form:

Theorem 2. *Let $I^+ = \langle f_1, \ldots, f_P \rangle$ and $I^- = \langle g \rangle$ for homogeneous polynomials f_j, g in R of degree m, and suppose $\{f_1, \ldots, f_P, g\}$ is a linearly independent set. If $I^-_{m+1} \subseteq I^+_{m+1}$, then $n \leq 3$ implies $P \geq n$, while if $n \geq 4$, then $P < n$ is possible.*

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2. Algebraic Preliminaries

We quickly review some algebraic background; see [Eis95] or [Sch03] for additional details.

Definition 1. Let I, J be ideals in a ring R. Then the ideal quotient (or colon ideal) is

$$I : J = \{f \in R \mid f \cdot j \in I \text{ for all } j \in J\}$$

Let $R = \mathbb{C}[z_1, \ldots, z_n]$, and let $\mathfrak{m} = \langle z_1, \ldots z_n \rangle$ denote the homogeneous maximal ideal. Then for homogeneous ideals I^+ and $\langle g \rangle$ generated in degree m, with $g \notin I^+$, the condition of §1:

$$\langle g \rangle_{m+1} \subseteq I^+_{m+1} \tag{3}$$

holds if and only if

$$I^+ : g = \mathfrak{m}. \tag{4}$$

Algebraically, this means that $\mathfrak{m}$ is an *associated prime* of I^+.

Definition 2. For an R-module M, a prime ideal P is an associated prime of M if $P = \operatorname{ann}(m)$ for some $m \in M$. Write $\operatorname{Ass}(M)$ for the set of associated primes.

Example 1. Let $R = \mathbb{C}[x, y]$ and $M = R/\langle x^2, xy \rangle$. Then the annihilator of $x \in M$ is the ideal $\langle x, y \rangle$, and the annihilator of $y \in M$ is the ideal $\langle x \rangle$. Recall that an ideal Q is primary if $xy \in Q$ implies $x \in Q$ or $y^n \in Q$ for some $n \in \mathbb{Z}$. Polynomial rings over a field are Noetherian, and in a Noetherian ring, every ideal I has a primary decomposition

$$I = \bigcap_{i=1}^{m} Q_i, \text{ where } Q_i \text{ is a primary ideal,}$$

and the associated primes of M are defined as the P_i such that $P_i = \sqrt{Q_i}$.

The relevance to our problem comes from Lemma 1.3.10 of [Sch03].

Lemma 1. *The associated primes satisfy* $\operatorname{Ass}(R/I) = \{\sqrt{I : f} \mid f \in R\}$.

We need one last definition before moving on to proofs of our results.

Definition 3. An ideal $I = \langle f_1, \ldots, f_k \rangle \subseteq R$ with k minimal generators is a *complete intersection* if each f_i is not a zero divisor on $R/\langle f_1, \ldots, f_{i-1} \rangle$. Equivalently, the map

$$R/\langle f_1, \ldots, f_{i-1} \rangle \xrightarrow{\cdot f_i} R/\langle f_1, \ldots, f_{i-1} \rangle \text{ is an inclusion.}$$

From a geometric standpoint, being a complete intersection means that the locus $V(f_1, \ldots, f_k)$ where the f_j simultaneously vanish has codimension equal to k.

3. Proof of the theorem

Proposition 2. *If* $n \leq 3$ *and* (3) *holds, then* $P \geq 3$.

Proof. First, if $P = 1$, then I^+ and $\langle g \rangle$ are principal ideals, and the only way two principal ideals generated in the same degree m can be equal in degree $m + 1$ is if the ideals are themselves equal, which contradicts our hypothesis. Note that this covers the case $n = 2$, since if $g \notin I^+$ and (3) holds, $P \geq 2$.

If $P = 2$ (regardless of the number of variables), then either

(1) Case 1: the generators f_1 and f_2 of I^+ share a common facto, say $f_1 = h_1 F$ and $f_2 = h_2 F$, with h_i having no common factor.

(2) Case 2: f_1 and f_2 have no common factor.

The ideals $H = \langle h_1, h_2 \rangle$ (in Case 1), and $F = \langle f_1, f_2 \rangle$ (in Case 2) are by definition complete intersections. By Corollary 18.14 of [Eis95], all associated primes of H and F are of codimension two. Again, this fact does not depend on the number of variables. Now suppose $n \geq 3$. Then (3) can only hold in Case 2 if $P \geq 3$. In Case 1, the associated primes of I^+ are the union of the associated primes of H (which are codimension two), along with the codimension one primes corresponding to irreducible factors of F. Thus in Case 1 we again have that all associated primes are of codimension at most two, so (3) can only hold if $P \geq 3$. $\square$

In order to show that this result is optimal, we need a bit more homological algebra. The result we need is a classical theorem of Bruns [Bru76] from 1976.

Theorem 3. *Given a free resolution $F_\bullet$ with differentials d_i, there exists a three-generated homogeneous ideal B such that the free resolution $F(B)_\bullet$ and differentials d_i^B satisfy*

$$F_i = F(B)_i \text{ and } d_i = d_i^B \text{ for all } i \geq 2.$$

We will apply Theorem 3 to the *Koszul complex*:

Definition 4. For $\mathbf{f} = \{f_1, \ldots, f_k\} \subseteq R$, the Koszul complex is

$$K(\mathbf{f}) : 0 \longrightarrow R^1 \xrightarrow{d_k} R^k \xrightarrow{d_{k-1}} \Lambda^2(R^k) \xrightarrow{d_{k-2}} \cdots$$

with differential

$$d_i(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{k=1}^{i} (-1)^k f_{j_k} e_{j_1} \wedge \cdots \widehat{e_{j_k}} \cdots \wedge e_{j_i}.$$

A computation shows that

$$d_i \circ d_{i+1} = 0,$$

so $K(\mathbf{f})$ is a complex; it is exact if and only if $\{f_1, \ldots, f_k\}$ is a complete intersection.

Proposition 3. *If $n \geq 4$, then* (3) *can hold with $P < n$.*

Proof. By Lemma 1, when (4) holds, then g satisfies $I^+ : g = \mathfrak{m}$ and $\mathfrak{m}$ is an associated prime of I^+, since it annihilates g. (Recall that g is a nonzero element of R/I^+.) By the Auslander-Buchsbaum Theorem (Exercise 19.8 of [Eis95]), the projective dimension of R/I^+ (i.e., the length of a minimal free resolution) is n. Our goal is to produce an ideal I^+ with this behavior–maximal projective dimension–but with a small number of generators.

To achieve this aim, we apply Theorem 3 to the Koszul complex of $\mathfrak{m}$. The result is a three-generated ideal I^+ with $\mathfrak{m}$ an associated prime of I^+. Lemma 1 then yields an element g such that

$$\sqrt{I^+ : g} = \mathfrak{m}.$$

For $K(\{z_1, \ldots, z_n\})$, the ideal I^+ has three homogeneous generators of degree $n-2$. By Lemma 1 there is a homogeneous element g satisfying $I^+ : g = \mathfrak{m}$, but g could be of degree greater than $n-2$. A computation using the software package `Macaulay2` [GS] shows that for $n = 4$, Theorem 3 applied to $K(\mathfrak{m})$ yields an ideal I^+ with three quadratic generators and that $I^+ : \mathfrak{m}$ contains a fourth quadric $q \notin I^+$. $\square$

Example 2. There are different choices for the ideal I^+. Perhaps the simplest is $I^+ = \langle z_4^2, z_2z_3 + z_1z_4, z_2^2 + z_2z_4 \rangle$, and a quick check shows $z_2^2 \in I^+ : \langle z_1, z_2, z_3, z_4 \rangle$. Thus, $g = z_2^2$ gives a minimal example where (3) holds with $P < n$.

3.1. **Directions for future work.** A complete answer to Question 1 will only allow us to prove the Sum of Squares Conjecture when $r(z, \bar{z})$ has signature (P, N) for N small. To extend this range requires answering the analogue of Question 1 when $N > 1$, which is the subject of our ongoing work.

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