Matroid Partition Property and the Secretary Problem

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Abstract

A matroid $\mathcal{M}$ on a set E of elements has the α-partition property, for some $\alpha > 0$, if it is possible to (randomly) construct a partition matroid $\mathcal{P}$ on (a subset of) elements of $\mathcal{M}$ such that every independent set of $\mathcal{P}$ is independent in $\mathcal{M}$ and for any weight function $w : E \to \mathbb{R}_{\geq 0}$, the expected value of the optimum of the matroid secretary problem on $\mathcal{P}$ is at least an α-fraction of the optimum on $\mathcal{M}$. We show that the complete binary matroid, $\mathcal{B}_d$ on $\mathbb{F}_2^d$ does not satisfy the α-partition property for any constant $\alpha > 0$ (independent of d).

Furthermore, we refute a recent conjecture of [5] by showing the same matroid is $2^d/d$-colorable but cannot be reduced to an $\alpha 2^d/d$-colorable partition matroid for any α that is sublinear in d.

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1 Introduction

Since its formulation by Babaioff, Immorlica and Kleinberg in 2007 [3, 2], the matroid secretary conjecture has captured the imagination of many researchers [7, 1, 14, 12, 6, 13, 17, 9, 15, 11]. This beautiful conjecture states the following: Suppose that elements of a known matroid $\mathcal{M} = (E, \mathcal{I})$ with unknown weights $w : E \to \mathbb{R}_{\geq 0}$ arrive one at a time in a uniformly random order. When an element e arrives we learn its weight w_e and must make an irrevocable and immediate decision as to whether to "take it" or not, subject to the requirement that the set of elements taken must at all times remain an independent set in the matroid. The *matroid secretary conjecture* states that for any matroid, there is an (online) algorithm that guarantees that the expected weight of the set of elements taken is at least a constant fraction of the weight of the maximum weight base.

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More formally, we say the competitive ratio of a matroid secretary algorithm[1] A on a particular matroid M is

$$\inf_{\mathbf{w}} \frac{\mathbb{E}\left[A_{\mathcal{M}}(\mathbf{w})\right]}{\mathrm{opt}_{\mathcal{M}}(\mathbf{w})}$$

where $A_{\mathcal{M}}(\mathbf{w})$ is the weight of the set of elements selected by the online algorithm A, and $\mathrm{opt}_{\mathcal{M}}(\mathbf{w}) = \max_{I \in \mathcal{I}} \sum_{i \in I} w_i$. We drop the subscript $\mathcal{M}$ when the matroid is clear in the context. The expectation in the numerator is over the uniformly random arrival order of the elements and any randomization in the algorithm itself. The conjecture states that for any matroid, there is an algorithm with competitive ratio $O(1)$.

The matroid secretary conjecture is known to be true for a number of classes of matroids, including partition matroids, uniform matroids, graphic matroids and laminar matroids [3, 2, 7, 1, 14, 12, 13, 17]. In its general form, it remains open. At this time, the best known general matroid secretary algorithm has competitive ratio $O(1/\log\log r)$ where r is the rank of the matroid [15, 11]. Since our work focuses on binary matroids, it is worth mentioning that, to the best of our knowledge, there is no algorithm known for the binary matroids that has a better competitive ratio than $O(1/\log\log r)$.

A reasonably natural approach to proving the matroid secretary conjecture is by a *reduction to a partition matroid.*

▶ **Definition 1.** *A matroid $\mathcal{M}' = (E', \mathcal{I}')$ is a reduction of matroid $\mathcal{M} = (E, \mathcal{I})$, if $E' \subseteq E$ and $\mathcal{I}' \subseteq \mathcal{I}$.*

A matroid $\mathcal{M}$ is a partition matroid if its elements can be partitioned into disjoint sets $P_1, \ldots, P_d$ such that $S \subseteq E$ is independent iff $|S \cap P_i| \leq 1$ for all $1 \leq i \leq d$. Specifically, consider the following class of algorithms:

1. Wait until some number of elements have been seen without taking anything. We call this set of elements *the sample* and use S to denote this set.
2. Based on the elements in S and their weights, (randomly) reduce $\mathcal{M}$ to a partition matroid $\mathcal{P} = P_1 \cup P_2 \cup \ldots P_d$ on (a subset of) the non-sample $\overline{S}$.
3. In each part P_i, run a secretary algorithm which chooses at most one element; e.g. choose the first element in P_i whose weight is above a threshold τ_i (which may be based on S).

Some appealing applications of this approach which are constant competitive are for graphic matroids [14], laminar matroids, and transversal matroids [7, 14, 13]. The latter two algorithms rely crucially on first observing a random sample of elements and then constructing the partition matroid.

Consider the complete *binary matroid*, $\mathcal{B}_d$, which is the linear matroid defined on all vectors in $\mathbb{F}_2^d$ where a set $S \subseteq \mathbb{F}_2^d$ is independent if the vectors in S are linearly independent over the field $\mathbb{F}_2^d$. Our main result is that for complete binary matroids, no algorithm of the above type, that is, based on a reduction to a partition matroid, can yield a constant competitive ratio for the matroid secretary problem.

▶ **Theorem 2** (Informal)**.** *Any matroid secretary algorithm for complete binary matroids $\mathcal{B}_d$ that is based on a reduction to a partition matroid has competitive ratio $O(d^{-1/4})$.*

We say a matroid $\mathcal{M} = (E, \mathcal{I})$ has the α-partition property if it can be (randomly) reduced to a partition matroid $\mathcal{P}$ such that

$$\mathbb{E}\left[\mathrm{opt}_{\mathcal{P}}(w)\right] \geq \frac{1}{\alpha}\,\mathrm{opt}_{\mathcal{M}}(w).$$

[1] That is, an algorithm which decides as elements arrive whether to take them or not.

In a survey [8], Dinitz raised as an open problem whether every matroid $\mathcal{M}$ satisfies the *α-partition property* for some universal constant $\alpha > 0$. Dinitz observes that it is unlikely that the α-partition property holds for all matroids, but notes that there is no matroid known for which it is false. As a consequence of our main theorem, the complete binary matroid does not satisfy the α-partition property for $\alpha \leq O(d^{1/4})$. In fact, our negative result is stronger, since it allows for the partition matroid $\mathcal{P}$ to be constructed after seeing a sample and the weights of the sample. This shows that although this approach works for laminar and transversal matroids, it does not generalize to all matroids.

As a byproduct of our technique we also refute a conjecture of Bérczi, Schwarcz, and Yamaguchi [5]. The covering number of a matroid $\mathcal{M} = (E, \mathcal{I})$ is the minimum number of independent sets from $\mathcal{I}$ needed to cover the ground set E. A matroid is k-coverable if its covering number is at most k.

▶ **Conjecture 3** ([5]). *Every k-coverable matroid $\mathcal{M} = (E, \mathcal{I})$ can be reduced to a $2k$-coverable partition matroid on the same ground set E.*

Below we prove that $\mathcal{B}_d \setminus \{0\}$ refutes the above conjecture for $d \geq 17$. Note that we need to remove 0 from the complete binary matroid, since the covering number is not defined for matroids that have loops.

▶ **Theorem 4.** *For any $d \geq 17$ there exists a matroid $\mathcal{M}$ of rank d that is k-coverable for some $k \geq d$, but it cannot be reduced to a $2k$-coverable partition matroid with the same number of elements. In particular, such an $\mathcal{M}$ can only be reduced to $\Omega(kd)$ coverable partition matroids with the same number of elements.*

1.1 Related Work.

In recent independent work, Bahrani, Beyhaghi, Singla, and Weinberg [4] also studied barriers for simple algorithms for the matroid secretary problem. Most related to our result, they consider a restriction of the above class of algorithms in which a partition must be computed before seeing any weights. In this model, they show that no partition-based algorithm (which does not use the sample) can guarantee that each element of OPT is selected with constant probability. Note this does not imply such algorithms are not constant-competitive for the matroid secretary problem. In fact, their lower bound is constructed for the graphic matroid, for which we know a partition-based constant competitive algorithm exists [3, 14]. We refer the interested reader to [4] for further details of their contributions.

In an independent work, [16] proved that for any constant b, the complete binary matroid $\mathcal{B}_d$ is not $(b, o(\sqrt{d}))$-decomposable. For $b = 1$, their result implies that, for any positive integer d, there is a k-coverable matroid of rank d that can only be reduced to $\Omega(k\sqrt{d})$-coverable partition matroids with the same number of elements, thus refuting Conjecture 3. Subsequent to our results and the result of [16], [18] proved that for any $b, c \geq 1$, prime number q, and $p \in (0, 1]$, there is a sufficiently large d such that a random subset of the complete linear matroid on $\mathbb{F}_q^d$ that contains each element with probability p is not (b, c)-decomposable with a high probability. For $q = 2$ and $p = 1$, this refutes Conjecture 3. To the best of our knowledge, Theorem 2 does not follow from the results of [16] and [18]. Note that a proof of this theorem needs to account for the fact that in a reduction of matroid $\mathcal{M}$ to a partition matroid $\mathcal{P}$, the ground set of $\mathcal{P}$ can be any subset of the elements of $\mathcal{M}$.

2 Main Technical Theorem

For an integer $k \geq 1$, we write $[k] := \{1, \ldots, k\}$. The following is our main technical theorem:

▶ **Theorem 5** (Main Technical). *For any reduction of the complete binary matroid $\mathcal{B}_d = (\mathbb{F}_2^d, \mathcal{I})$ to a partition matroid $\mathcal{P} = P_1 \cup \cdots \cup P_d$, there is a subset $T \subseteq [d]$ such that $|T| \geq d - 8\sqrt{d}$ and $|\cup_{i \in T} P_i| \leq \frac{2^d}{\sqrt[4]{d}}$.*

Note that, throughout the paper, for any partition matroid specified by $P_1, \ldots, P_d$, we allow sets P_i to be empty. Therefore the partition matroid can effectively have less than d parts and the reduction does not have to be rank-preserving.

As a consequence of the above theorem, there are $O(\sqrt{d})$ parts in $[d] \setminus T$ that contain the vast majority of the elements of $\mathcal{B}_d$. For appropriately chosen weight vectors, this is bad, since only $O(\sqrt{d})$ elements can be taken from $\cup_{i \notin T} P_i$.

We use the following simple fact.

▶ **Fact 6.** *Let $\mathcal{P}$ be a partition matroid that is reduction of $\mathcal{B}_d$ with parts $P_1, \ldots, P_d$. Then if two elements x and y are in different parts (say P_i and P_j), then their sum $x + y$ is in P_i, P_j or $\mathbb{F}_2^d \setminus \cup_i P_i$.*

Proof. Let $z = x + y$ and suppose by way of contradiction that $x \in P_i, y \in P_j$, yet $z \in P_k$ for distinct indices i, j, k. Then, $\{x, y, z\}$ is independent in $\mathcal{P}$. But now by the definition of a matroid reduction, $\{x, z, y\}$ must be independent in $\mathcal{B}_d$. However, this set is not independent since we have the dependence $z = x + y$. ◀

▶ **Lemma 7.** *Let $\mathcal{P}$ be a partition matroid that is reduction of $\mathcal{B}_d$ with parts $P_1, \ldots, P_d$ and let $R := \mathbb{F}_2^d \setminus \mathcal{P}$. The number of pairs $a \in P_i, b \in P_j$ for $1 \leq i < j \leq d$ in which $a + b \in R$ is at most $\max_{1 \leq i \leq d} 2|P_i| \cdot |R|$.*

Proof. Create a hypergraph H whose vertices are elements in $\mathbb{F}_2^d$. Now, create a hyperedge $\{a, b, a + b\}$ for every $a \in P_i, b \in P_j$ for $1 \leq i < j \leq d$ in which $a + b \in R$.

Fix any $q \in R$. First, note that there are no two distinct hyperedges $\{a, b, q\}$, $\{a, b', q\}$, as this would imply $a + b = a + b'$ and therefore $b = b'$. Therefore, the sets $\{a, b\}$ such that $\{a, b, q\}$ is a hyperedge form a matching.

Now fix a hyperedge $\{a, b, q\}$ with $a \in P_i, b \in P_j$. If there is some other hyperedge $\{c, d, q\}$ such that $c, d \notin (P_i \cup P_j)$, then a, b, c, d are all in different partitions, which cannot occur as $a + b = c + d$, which is a linear dependence in the partition matroid. Therefore, every edge containing q must contain an element of $P_i \cup P_j$. Therefore the matching contains at most $|P_i| + |P_j| \leq 2 \max_{1 \leq k \leq d} |P_k|$ edges, from which the claim follows. ◀

▶ **Lemma 8.** *Let $\mathcal{P}$ be a reduction of $\mathcal{B}_d$ to a partition matroid with parts $P_1, \ldots, P_d$ with total size $\sum_{i=1}^d |P_i| = n = c \cdot 2^d$ for some $0 < c \leq 1$. Then, there exists an $1 \leq i \leq d$ such that $|P_i| > \frac{c}{8} n$.*

Proof. By way of contradiction, suppose $\max_{1 \leq i \leq d} |P_i| \leq \frac{cn}{8}$. Now, construct a (directed) graph whose vertices are the elements in P. First, create an edge (a, b) for all $a \in P_i, b \in P_j, i \neq j$ for which $a + b \notin R$.

For each such edge (a, b), by Fact 6 either $a + b \in P_i$ or $a + b \in P_j$. Direct the edge towards a if the former occurs, otherwise direct it towards b. Note that the in-degree of each element in a partition P_i is at most $|P_i| \leq \frac{cn}{8}$ (since if $a + b = a + d = a' \in P_i$ then $b = d$). Therefore, there are at most $\frac{cn^2}{8}$ such edges.

However, by the previous lemma, there are at least (using that $\max_i |P_i| \le \frac{cn}{8}$):

$$\binom{n}{2} - \sum_{i=1}^{d} \binom{|P_i|}{2} - 2\frac{cn}{8}|R| \ \ge\ \binom{n}{2} - \sum_{i=1}^{d} \binom{|P_i|}{2} - \frac{n^2}{4}$$

$$= \frac{n^2}{2} - \sum_{i=1}^{d} \frac{|P_i|^2}{2} - \frac{n^2}{4} > \frac{n^2}{8}$$

such edges, where in the first inequality we used that $|R| \le 2^d = \frac{1}{c}n$, in the equality we used $\sum_{i=1}^d |P_i| = n$ and in the last inequality we used that $\sum_{i=1}^d |P_i|^2$ is maximized when $|P_i| = \frac{cn}{8}$ on $8/c$ parts, and $c \le 1$, this is a contradiction with the above, which gives the lemma. ◀

Now, we finish the proof of Theorem 5.

Proof of Theorem 5. We start from $T = [d]$ and inductively repeat the following: if $|\cup_{i\in T} P_i| \ge \frac{2^d}{\sqrt[4]{d}}$, remove $j = \arg\max_{i\in T} |P_i|$ from T. Using Lemma 8, if $|\cup_{i\in T} P_i| \ge \frac{2^d}{\sqrt[4]{d}}$, the size of the partition that we remove is at least

$$\max_{i\in T} |P_i| \ge \frac{1}{8\sqrt[4]{d}} \cdot \frac{2^d}{\sqrt[4]{d}} = \frac{2^d}{8\sqrt{d}}.$$

Therefore, after at most $8\sqrt{d}$ steps, we get $|\cup_{i\in T} P_i| \le \frac{2^d}{\sqrt[4]{d}}$. This finishes the proof. ◀

3 Main Theorems

3.1 Matroid α-Partition Property (Proof of Theorem 4)

For a matroid $\mathcal{M}$, Edmonds defined:

$$\beta(M) := \max_{\emptyset \subset F \subseteq E} \frac{|F|}{\mathrm{rank}_M(F)}. \tag{1}$$

Note that the maximum in the RHS is attained at *flats* of $\mathcal{M}$, namely sets F that are the same as their closure.

▶ **Theorem 9** (Edmonds [10]). *For any matroid $\mathcal{M}$ on elements E and with no loops, the covering number of $\mathcal{M}$, namely the minimum number of independent sets whose union is E is equal to $\lceil \beta(M) \rceil$.*

Using this, we show that Theorem 4 is a corollary of Lemma 8.

▶ **Theorem 4.** *For any $d \ge 17$ there exists a matroid $\mathcal{M}$ of rank d that is k-coverable for some $k \ge d$, but it cannot be reduced to a $2k$-coverable partition matroid with the same number of elements. In particular, such an $\mathcal{M}$ can only be reduced to $\Omega(kd)$ coverable partition matroids with the same number of elements.*

Proof. It follows from Theorem 9 that the binary matroid $\mathcal{B}_d \setminus \{0\}$ on $\mathbb{F}_2^d$ satisfies $\beta(\mathcal{B}_d \setminus \{0\}) = (2^d - 1)/d$. This is because the flats of $\mathcal{B}_d \setminus \{0\}$ correspond to (linear) subspaces. A linear subspace of dimension k has exactly $2^k - 1$ many vectors. So, the maximum of (1) is attained at $F = \mathbb{F}_2^d \setminus \{0\}$ which has rank d.

Now, suppose $\mathcal{B}_d \setminus \{0\}$ is reduced to a partition matroid $\mathcal{P}$ with parts $P_1, \ldots, P_d$ such that $\cup_{i=1}^d P_i = \mathbb{F}_2^d \setminus \{0\}$. Observe that $\beta(\mathcal{P}) = \max_{1\le i\le d} |P_i|$. To refute Conjecture 3 and prove Theorem 4, it is enough to show that $\max_{1\le i\le d} |P_i| > \Omega(2^d - 1)$. However, by Lemma 8 (setting $c = 1 - 1/2^d$ to account for deleting the 0 element), this quantity is at least $\frac{2^d-1}{8}$, which gives the theorem. ◀

3.2 Matroid Secretary α-Partition Property (Proof of Theorem 2)

▶ **Definition 10.** *Let $\mathcal{P}(S, \mathbf{w}|_S)$ be any function that maps a sample $S \subset \mathbb{F}_2^d$ and weights $\mathbf{w}|_S$ of elements in the sample to a partition matroid that is a reduction of $\mathcal{B}_d$, where the elements of $\mathcal{P}(S, \mathbf{w}|_S)$ are a subset of $\overline{S} = \mathbb{F}_2^d \setminus S$. Let $\mathbf{P}$ be the collection of all such mappings.*

▶ **Definition 11** (Randomized Partition Reduction Algorithm)**.** *A (randomized)* partition reduction algorithm *$\mathcal{A}$ for a matroid $\mathcal{M}$ with n elements consists of two parts:*
- *$\mathcal{A}$ (randomly) chooses a sample size $0 \le |S| \le n$ before any elements have been seen; we denote this choice by $s_{\mathcal{A}}$.*
- *$\mathcal{A}$ (randomly) chooses a mapping $\mathcal{P}_{\mathcal{A}} \in \mathbf{P}$ and uses it to build $\mathcal{P}(S, \mathbf{w}|_S)$ after seeing the sample S.*

▶ **Theorem 12** (Main)**.** *For any randomized partition reduction algorithm $\mathcal{A}$ for $\mathcal{B}_d$, with $d \ge 2^{12}$ there is a weight function $\mathbf{w} : \mathbb{F}_2^d \to \mathbb{R}_{\ge 0}$ such that*

$$\mathbb{E}_{s_{\mathcal{A}}} \mathbb{E}_{S:|S|=s_{\mathcal{A}}} \mathbb{E}_{\mathcal{P}_{\mathcal{A}}}[\mathrm{opt}_{\mathcal{P}_{\mathcal{A}}}(\mathbf{w}|_{\overline{S}})] \le 4d^{-\frac{1}{4}}\, \mathrm{opt}_{\mathcal{B}_d}(\mathbf{w}).$$

For readability in the above, we have suppressed the fact that the partition matroid $\mathcal{P}_{\mathcal{A}}$ (whose elements are a subset of $\overline{S}$) depends on both S and $\mathbf{w}|_S$. Note that S is drawn from the uniform distribution over subsets of $\mathbb{F}_2^d$ of size $s_{\mathcal{A}}$.

Proof. Suppose that the weights of the elements in $\mathcal{B}_d$ are selected by setting

$$w_i = \mathbf{1}_{i \in X} \tag{2}$$

where $X = \{x_1, x_2, \ldots, x_d\}$ is a uniformly random sample of d elements from $\mathbb{F}_2^d$, selected with replacement. Then the optimal independent set has expected weight $\mathbb{E}_{\mathbf{w}}\, \mathrm{opt}_{\mathcal{B}_d}(\mathbf{w})$ equal to

$$\mathbb{E}_X(\mathrm{rank}(X)) = \sum_{i=1}^{d} \mathbf{1}_{x_i \notin \mathrm{span}\{x_1,\ldots,x_{i-1}\}} = \sum_{i=1}^{d} \frac{2^d - 2^{i-1}}{2^d - (i-1)} \ge \sum_{i=1}^{d} \frac{2^{d-1}}{2^d} = \frac{d}{2}. \tag{3}$$

Now let $\mathcal{A}$ be an arbitrary algorithm that chooses the sample size $s_{\mathcal{A}}$ and the mapping $\mathcal{P} \in \mathbf{P}$ deterministically. We claim that it suffices to show that for the weight vector given in Equation (2) when X is chosen uniformly at random, and for S a uniformly random sample of elements of any fixed size:

$$\mathbb{E}_{\mathbf{w}} \mathbb{E}_S\, \mathrm{opt}_{\mathcal{P}}(\mathbf{w}|_{\overline{S}}) \le 2d^{3/4}, \tag{4}$$

where $\mathcal{P} = \mathcal{P}(S, \mathbf{w}|_S)$ is any partition matroid on a subset of $\overline{S}$ constructed after seeing the elements in S and their weights. (Note that $\mathrm{opt}_{\mathcal{P}}$ is a upper bound on the performance of $\mathcal{A}$.)

To see why, observe that in the randomized case, by taking the expected value over the randomization in $\mathcal{A}$ and then interchanging the order of the expectations, we get

$$\mathbb{E}_{\mathbf{w}} \mathbb{E}_{\mathcal{A}} \mathbb{E}_S\, \mathrm{opt}_{\mathcal{P}_{\mathcal{A}}}(\mathbf{w}|_{\overline{S}}) = \mathbb{E}_{\mathcal{A}} \mathbb{E}_{\mathbf{w}} \mathbb{E}_S\, \mathrm{opt}_{\mathcal{P}_{\mathcal{A}}}(\mathbf{w}|_{\overline{S}}) \le 2d^{3/4}.$$

Therefore, for $\mathbf{w}$ chosen at random according to (2) and using (3),

$$\frac{\mathbb{E}_{\mathbf{w}} \mathbb{E}_{\mathcal{A}} \mathbb{E}_S\, \mathrm{opt}_{\mathcal{P}_{\mathcal{A}}}(\mathbf{w}|_{\overline{S}})}{\mathbb{E}_{\mathbf{w}}\, \mathrm{opt}_{\mathcal{B}_d}(\mathbf{w})} \le 4d^{-1/4}.$$

Applying the mediant inequality, we conclude that there is a set B of size d such that $\frac{\mathbb{E}_{\mathcal{A}} \mathbb{E}_S\, \mathrm{opt}_{\mathcal{P}_{\mathcal{A}}}(B|_{\overline{S}})}{\mathrm{rank}(B)}$ is at most $4d^{-1/4}$, completing the proof of the theorem.

It remains to prove (4). Observing that $\mathbf{w}$ (resp. $\mathbf{w}|_S$, $\mathbf{w}|_{\overline{S}}$) is fully determined by X (respectively $X \cap S$, $X \cap \overline{S}$) and letting $X_1 := X \cap S$, $X_2 := X \cap \overline{S}$ we write

$$\mathbb{E}_{\mathbf{w}}\mathbb{E}_S \operatorname{opt}_{\mathcal{P}}(\mathbf{w}|_{\overline{S}}) = \mathbb{E}_S\mathbb{E}_{X_1}\mathbb{E}_{X_2} \operatorname{opt}_{\mathcal{P}}(X_2). \tag{5}$$

For any choice of S and X_1, the partition matroid $\mathcal{P} = \mathcal{P}(S, X_1)$ on a subset of $\overline{S}$ consists of parts $P_1 \cup \cdots \cup P_d$ (some of these parts could be empty). By Theorem 5, there exists a set $T \subseteq [d]$ of size at least $d - 8\sqrt{d}$ such that $|\cup_{i\in T} P_i| \le \frac{2^d}{\sqrt[4]{d}}$. Therefore,

$$\operatorname{opt}_{\mathcal{P}}(X_2) \le |X_2 \cap \cup_{i\in T} P_i| + 8\sqrt{d}.$$

So, for a fixed S, we have

$$\begin{aligned}
\mathbb{E}_{X_1}\mathbb{E}_{X_2} \operatorname{opt}_{\mathcal{P}}(X_2) &\le \mathbb{E}_{X_1}\mathbb{E}_{X_2}\left(|X_2 \cap \cup_{i\in T} P_i| + 8\sqrt{d}\right)\\
&= \mathbb{E}_{X_1}\left((d - |X_1|)\frac{|\cup_{i\in T} P_i|}{|\overline{S}|} + 8\sqrt{d}\right).\\
&\le \mathbb{E}_{X_1}\left((d - |X_1|)\frac{2^d/\sqrt[4]{d}}{|\overline{S}|} + 8\sqrt{d}\right).\\
&= (d - \mathbb{E}(|X_1|))\frac{2^d}{\sqrt[4]{d}|\overline{S}|} + 8\sqrt{d},
\end{aligned}$$

where in the first equality, we used the fact that for any fixed X_1, the set X_2 is a uniformly random subset of size $d - |X_1|$ in $\overline{S}$, thus in expectation, $(d - |X_1|)\frac{|\cup_{i\in T} P_i|}{|\overline{S}|}$ many elements in X_2 are in $\cup_{i\in T} P_i \subseteq \overline{S}$. Finally, we observe that

$$\frac{(d - \mathbb{E}(|X_1|))}{|\overline{S}|} \cdot \frac{2^d}{\sqrt[4]{d}} = \frac{d - \frac{|S|\cdot d}{2^d}}{(1 - \frac{|S|}{2^d})2^d} \cdot \frac{2^d}{d^{1/4}} = d^{3/4}.$$

Thus, we get

$$\mathbb{E}_{X_1}\mathbb{E}_{X_2} \operatorname{opt}_{\mathcal{P}}(X_2) \le d^{3/4} + 8\sqrt{d} \le 2d^{3/4},$$

where in the last inequality we used our assumption that $d \ge 2^{12}$. Combining Equation (5) with this, Equation (4) follows. ◀

4 Conclusion

We note that for our bad example, the trivial algorithm for matroid secretary succeeds: one simply needs to take every improving element when it arrives.

On the positive side, probably the most interesting open problem is to solve the matroid secretary problem for the complete binary matroid. On the negative side: can we prove a lower bound against the following class of partition-based algorithms? After seeing a sample, partition the elements into sets $P_1, \ldots, P_k$ with corresponding matroids $M(P_1, \mathcal{I}_1), \ldots, M(P_k, \mathcal{I}_k)$ with the property that for any $I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k$, the set $\bigcup_{i=1}^k I_i$ is independent in the binary matroid. Then, run a simple greedy procedure on each $M(P_i, \mathcal{I}_i)$. In the special case that each M_i is a rank 1 matroid, our result gives a super-constant lower bound.

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