EQUILIBRIUM STABILITY FOR MULTI-MODAL TRAFFIC IN AN URBAN ZONE

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ABSTRACT

This paper considers static models of traffic in urban regions that can produce multiple equilibria. Models with one mode of travel can have one equilibrium in the light congestion regime and multiple in hypercongestion. Models with two modes, which differ by occupancy, can have multiple in both light congestion and in hypercongestion due to mode-switching. We analyze whether the equilibria that arise are stable to local perturbations of the stock of people traveling at once. For one-mode models, unstable and stable equilibria can be distinguished by a simple rule-of-thumb. For two-mode models, the same rule-of-thumb can identify certain unstable equilibria, but to determine whether some equilibrium is stable requires performing a calculation for each equilibrium.

Keywords: Equilibrium, Stability, Multi-modal traffic

INTRODUCTION

Even the simplest economic models can have more than one equilibrium: the "stag hunt" game (*1*), wherein two people make one binary choice once, has two. It should probably not be surprising, then, that static[1] traffic models, wherein masses of people interact with the rules of traffic physics, also seem to yield multiple equilibria.

The canonical static model was first formulated in Walters (*4*). Consider a uniform highway with a variable demand, represented as traffic flow, that depends on the road's travel time. There are no entries or exits along the highway, so vehicles can only arrive from upstream. The model's logic can be understood by way of a diagram like Fig. 1, with vehicle flow on the horizontal axis and travel time (or some money cost index determined by travel time) on the vertical axis. In this coordinate system, a demand curve T^d (the blue line) gives the travel time which invites a certain flow. Another curve T^s, interpreted as a "supply curve," gives the locus of flow/travel time pairs consistent with the road's fundamental diagram (that is, with the microscopic behavior of drivers in stationary traffic). T^d is assumed to be non-increasing, because fewer people want to drive when driving takes longer. But T^s bends backwards when travel time is higher than a certain point. The lower, rising branch of T^s is sometimes called "light congestion" and the upper, declining branch "hypercongestion."

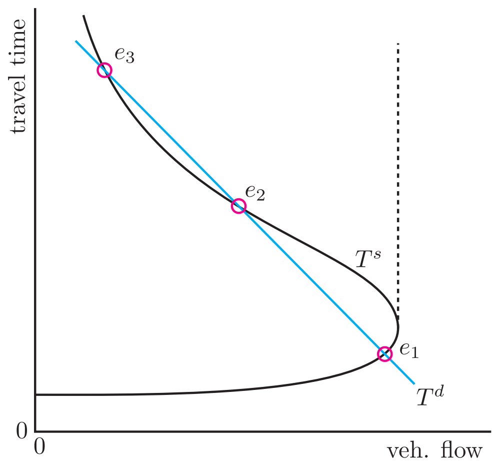


FIGURE 1: Static model example with equilibria in hypercongestion

In Walters (*4*), intersections of the T^d and T^s curves are said to be the model's equilibria, where traffic physics and demand coincide. Since T^d is non-increasing, there can only be one on the light congestion branch. But it is easy to see from the figure that there could be many equilibria on the hypercongestion branch. Figure 3 has three, with two in hypercongestion. The multiplicity of equilibria poses the question of which ones would or could actually obtain. A debate ensued, in particular, around the *stability* of hypercongested equilibria—that is, whether the system would return to a given equilibrium following a slight perturbation away from it. (See reviews of this debate in Small and Chu (*5*) and Arnott and Inci (*6*).) Some researchers favored the stability of hypercongested equilibria where T^d cuts T^s from above (such as e_2), and others where T^d cuts T^s from below (such as e_3).

[1]There are two broad classes of economic models of traffic: static and dynamic. (See Lindsey and Verhoef (*2*) for a review.) In dynamic models, such as the "bottleneck model" (*3*), travelers decide when to travel. In static models, they only decide whether or by what mode to travel.

In a series of papers (*7–9*), Verhoef cut the Gordian Knot by introducing the criterion of "dynamic stability" (or "dynamic consistency"). An equilibrium is dynamically stable if it can be reached from any other equilibrium by changing the arriving flow of traffic. In theory and in car-following microsimulations, equilibria that imply hypercongestion along the entirety of a route cannot be reached in this manner from equilibria in light congestion. Thus, whether or not hypercongested equilibria are stable to some proposed perturbation, they cannot transpire. In place of the declining branch of T^s, Verhoef argues for a vertical branch such as the dashed line in Figure 1, to capture the impact of a stationary queue that does not reduce capacity. Consequently, only one equilibrium can be obtained.

Granted that Verhoef is correct about the models he considers, the motivation for this paper is that there are *other* static traffic models with multiple equilibria. First, there are models in which the facility of interest has entries and exits dispersed throughout—such as a neighborhood of city streets. Vehicles can rush into city streets from parking spaces and the periphery at a rate higher than circulating traffic can possibly park or leave, and so hypercongestion is possible. These models are usually "bathtub models," meaning the rate trips finish is a well-defined function of vehicle density and speed. Most bathtub models are dynamic: e.g., Small and Chu (*5*), Geroliminis and Levinson (*10*), Arnott (*11*), Simoni et al. (*12*) and Lamotte and Geroliminis (*13*). But a few are static: e.g., Arnott and Inci (*6*), Lehe (*14*), Lehe (*15*) and Lehe and Pandey (*16*). Arnott and Inci (*6*) deals explicitly with the stability of various equilibria of a model in which drivers search for scarce parking in a downtown zone.

Second, Lehe and Pandey (*16*) shows that, under reasonable assumptions, a demand curve such as T^d can *rise* (that is, flow demanded can increase when speeds fall). The phenomenon is named *hyperdemand*. The mechanism has to do with the distinction between the rate of person-trips and vehicle flow: while the rate of *person-trips* reasonably falls with speed, the *vehicle flow* required by these trips may rise if falling speeds cause people to switch from high- to low-occupancy vehicles. For example, suppose people dislike spending time on the bus more than in cars, so that a fall in traffic speeds will persuade some people to switch from bus to car. When enough people switch as to outweigh the number of drivers who quit their trips altogether, the vehicle flow consistent with demands may rise with congestion—even though fewer people make trips. But whatever the particular justification for switching, any model with hyperdemand can support multiple equilibria even without hypercongestion. Imagine the T^d curve of Fig. 1 drawn with a rising portion; such could intersect the lower branch or dashed branch of T^s more than once.

This paper considers the stability of equilibria in models with hypercongestion and hyperdemand. There are two models: (i) a one-mode bathtub model with hypercongestion; (ii) a two-mode bathtub model with both hypercongestion and hyperdemand. In many places, the analysis resembles Arnott and Inci (*6*), but there are several differences: (i) there is no parking constraint; (ii) the two-mode system is capable of hyperdemand; (iii) stability is established by considering local perturbations at equilibria rather than via phase portraits of the global state space.

The paper is organized as follows. Section 2 describes a physical setting (a downtown zone) and its physics, lays out a static version of the one-mode model and characterizes the model's possible equilibria. Section 3 introduces clock time to the one-mode model and analyzes the stability of equilibria to perturbations of the stock of people traveling at once. Section 4 lays out a static version of the two-mode model for the same setting, shows how hyperdemand happens and characterizes its equilibria. Section 5 introduces clock time to the two-mode model and analyzes the stability of equilibria by considering the trace and determinant of its Jacobian. Section 6 gives a numerical simulation of the two-mode model with “nested logit” demands. Section 7 concludes.

ONE-MODE STATIC MODEL

This section describes a purely static one-mode model, including the physical setting, traffic relations and units—which will all persist throughout the rest of the paper. There is no mention of clock time here, because this section does not deal with out-of-equilibrium behavior whereby the system state could change over time.

Physics

The setting is a downtown zone. Vehicles are spread uniformly over a homogeneous street network, so that all vehicles move with the same speed. Distance units are arbitrary “du’s,” and time units are “tu’s.” All vehicles are, for physics purposes, the same; they do not take up different amounts of space. Hence, all traffic physics can be represented in units of “veh” (vehicles). Traffic physics revolves around the following three statistics:

— t (tu/du): *unit travel time*
— k (veh/lane-du): *vehicle density*
— q (veh/lane-tu): *vehicle flow*

The network is assumed to be sufficiently homogeneous such that all vehicles move at the same speed, so there exists a function $t = T(k)$ giving the unit travel time for a given vehicle density. Because traffic slows down as the network becomes crowded, $T(k)$ is always non-decreasing, and rising above a certain vehicle density, as in Fig. 2a.

By the Fundamental Identity of Traffic Flow $q = k/t$, there also exists a function $q = f(k) = k/T(k)$. This $f(k)$ is the relationship called the Network or Macroscopic Fundamental Diagram. Its existence in such downtown zones has been confirmed by some studies (e.g., Geroliminis and Daganzo (*17*) and Buisson and Ladier (*18*)), and its form can be derived theoretically from facts of the roadways (*19, 20*). $f(k)$ is unimodal, as in Fig. 2b. Its maximum occurs at the “critical density” k_c. For $k < k_c$, vehicle flow rises with density; this is the *light congestion* regime. The plots of Fig. 2 have that regime in pink. For $k > k_c$, where $f' < 0$, the regime is *hypercongestion*.

The same information can be represented as a flow/unit travel time curve via the parametric curve $\{f(k), T(k)\}$ of Fig. 2c, which is equivalent to the solid-line T^s in Fig. 1.

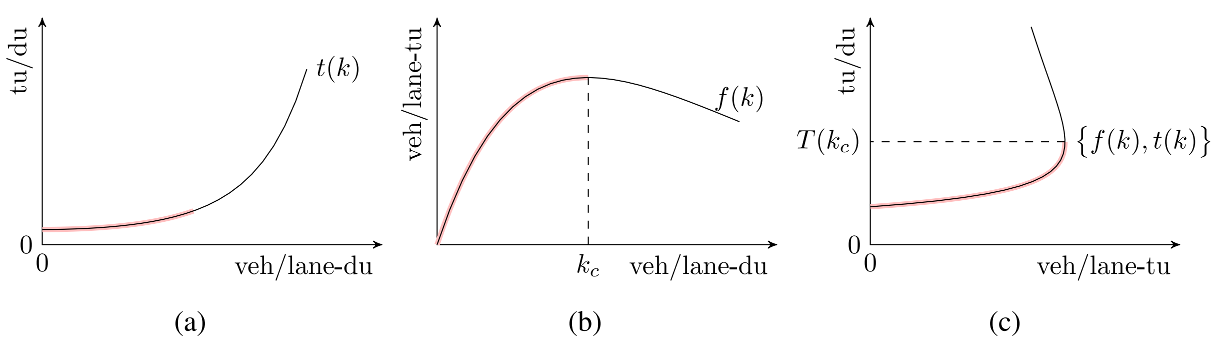


FIGURE 2: Traffic relations for the zone

Demand

At a constant rate, opportunities to travel appear and “passengers” decide whether to make a trip on the only mode available. Let the demand function $G(t)$ (pax/lane-du-tu) give the rate passengers begin trips, per lane-du of the zone, given a unit travel time t. $G(t)$ declines (that is, demand falls with t), as trips take longer. When passengers travel, they do so in a mode with an occupancy of ϕ (pax/veh). Passengers' mean trip length is l (du).

To tie traffic statistics to demands, let

$$Q(t) := G(t)l/\phi \quad \text{(veh/lane-tu)} \tag{1}$$

give the *vehicle flow demanded* for a given t. While passengers demand trips, rather than vehicle movement, $Q(t)$ gives the vehicle flow consistent with trip demands. Since G declines, and $Q(t)$ is its multiple, Q declines, too.

Since vehicle density maps to unit travel time via $T(k)$, we can also write vehicle flow demanded in terms of vehicle density. Define

$$D(k) := Q[T(k)] \quad \text{(veh/lane-tu).} \tag{2}$$

The derivative of D is $Q'T'$. Since $T' > 0$ and $Q' < 0$, D is declining. This function is useful in deriving equilibria below.

Equilibria

An equilibrium occurs when passengers' travel choices give rise to a unit travel time which, in turn, invites those same choices. At an equilibrium value of vehicle density $k = k^0$, we have

$$f\left(k^0\right) = D\left(k^0\right) \quad \text{(veh/lane-tu).} \tag{3}$$

Figure 3 plots $f(k)$ and $D(k)$ (the two sides of (3)) in (k,q) space. In this figure, $f(k)$ plays the role of a "supply curve" and $D(k)$ of a "demand curve." Figure 1 presents the same situation in the (q,t) space traditional to transportation economics, and has the same equilibria labeled.

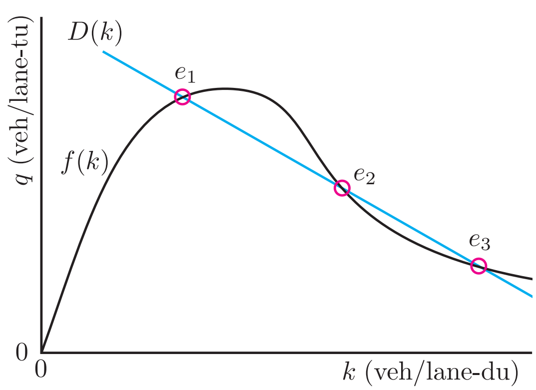


FIGURE 3: Equilibria in a one-mode system

There are three equilibria in the figures. Table 1 describes them according to two binary properties: (i) whether the equilibrium occurs in light or hyper congestion; (ii) the equilibrium's "cut." *Cut* describes whether, at equilibrium, D cuts f from *above* (i.e., $D' < f'$) or from *below* (i.e., $D' > f'$). The curves were drawn to illustrate all three feasible[2] combinations of cut and congestion regime.

[2]The combination of light congestion and a cut from below is infeasible, because f is rising in light congestion and D always declines.

equilibrium	congestion	cut
1	light	above
2	hyper	below
3	hyper	above

TABLE 1: Properties of feasible equilibria for one-mode model

Demand shifts

An equilibrium's cut implies lessons for public policy—though not necessarily good ones. Consider a continuous policy metric, indexed by ρ, which reduces the vehicle flow demanded: i.e., such that $\partial D / \partial \rho < 0$. ρ could be the level of a toll, a measure of transit service quality or any other initiative that would intuitively "get people out of their cars." At first blush, cutting the vehicle flow demanded ought to ameliorate congestion. But this is not necessarily so. Define

$$z(k) := D(k) - f(k), \tag{4}$$

and let $k = k_e$ be the density at an equilibrium, such that $z(k_e) = 0$. By applying the implicit function theorem at $k = k_e$, we have

$$\frac{\mathrm{d}k_e}{\mathrm{d}\rho} = -\frac{\partial z / \partial \rho}{\partial z / \partial k} - \frac{\partial D / \partial \rho}{D' - f'} \tag{5}$$

Since $\partial D / \partial \rho < 0$ by assumption (that is, the policy reduces the vehicle flow demanded), it follows that

$$\operatorname{sgn}\{ \mathrm{d}k_e / \mathrm{d}\rho \} = \operatorname{sgn}\{ D'(k_e) - f'(k_e) \}. \tag{6}$$

Thus, if the equilibrium exhibits a cut from below (i.e., if $D' > f'$), then a marginal increase in the policy will *raise* vehicle density and, by extension, the unit travel time.

This is an odd result. Why should reducing demand make traffic worse? The authors' suspicion is that it should not, and that this result is a clue that equilibria with cuts from above are of questionable relevance—mathematical artifacts of the model that could never be observed in real life.

ONE-MODE STABILITY ANALYSIS

This section develops a "dynamical" analysis of the above model and considers stability of equilibria. We mean "dynamical" in the sense that we consider perturbation in clock time away from equilibrium values, not in the sense that passengers schedule trips among different clock times.

Setup

We introduce a number of new variables and orient the discussion around instantaneous values of passenger stocks. Let u be the "clock time." Define the following instantaneous quantities at clock time u:

- $P(u)$ (pax/lane-du) *passenger density*: the density of passengers traveling at once (normalized per lane-du);
- $E(u)$ (pax/lane-du-tu) *exit rate*: the rate passengers complete trips per lane-du (and hence exit the network);

- $A(u)$ (pax/lane-du-tu) *arrival rate*: the rate passengers begin trips per lane-du (and hence arrive in the network);
- $t(u)$ (du/tu) instantaneous unit travel time;
- $k(u)$ (veh/lane-du) instantaneous vehicle density.

Let a dot over a quantity indicate its time derivative. Conservation of passengers implies

$$\dot{P}(u) = A(u) - E(u) \quad \text{(pax/lane-du-tu)}. \tag{7}$$

That is, the rate of change in passenger density on each mode is simply the flow of arriving passengers net of exiting passengers, normalized per lane-du of roadspace. Our goal is to turn (7) into an autonomous system where the *rates of change* of passenger stocks are written in terms of the instantaneous *levels* of passenger stocks. To start, we will make a pair of assumptions that let $A(u)$ and $E(u)$ be decomposed. The first concerns arrivals:

Assumption 1 (Myopia). *The arrival rate at clock time u depends only on* $t(u)$*:*

$$A(u) = G[t(u)] \quad \text{(pax/lane-du-tu)}. \tag{8}$$

We call this assumption "myopia" (nearsightedness) because passengers make decisions based only on current traffic; passengers do not forecast how traffic might change in the course of their trips.

The second assumption concerns exits.

Assumption 2 (Bathtub Physics). *The exit rate at clock time u is given by*

$$E(u) = \frac{P(u)}{lt(u)} \quad \text{(pax/lane-du-tu)}. \tag{9}$$

This is to say the model is assumed to have the "bathtub" physics coined by William Vickrey (in notes published as Vickrey (*21*)). Beyond the homogeneity and the function $T(k)$ we have given our setting, a requirement for Bathtub Physics to apply exactly and continuously is that the distribution of arriving traffic has a constant, negative exponential distribution of trip lengths (Jin (*22*) supplies a rigorous treatment of the question). Arnott and Inci (*6*) makes the same assumption.

Occupancy being ϕ (pax/veh), a passenger density of $P(u)$ means vehicle density is

$$k(u) = P(u)/\phi \quad \text{(veh/lane-du)}. \tag{10}$$

Swapping $T[k(u)]$ for $t(u)$ yields

$$E(u) = \frac{P(u)}{T[k(u)]l} \quad \text{(pax/lane-du-tu)} \tag{11}$$

$$A(u) = G\Big\{T[k(u)]\Big\} \quad \text{(pax/lane-du-tu)} \tag{12}$$

With (11) and (12), we can now write the autonomous system as:

$$\dot{P} = G[T(P/\phi)] - \frac{P}{T(P/\phi)l}. \tag{13}$$

Here the dependence on clock time has been dropped, because the system does not depend explicitly on clock time; only on the level of P.

Steady-state equilibria

Now consider steady-state equilibria. In a steady-state equilibrium, $\dot{P} = 0$: the stock of passengers is constant. Suppose that it is constant at a level $P = P^0$. By setting the LHS of (13) to 0, substituting P^0 for P and rearranging the RHS, P_0 can be written as the fixed point:

$$P_0 = G\Big[t(P_0/\phi)\Big] \cdot l \cdot T\Big(P_0/\phi\Big). \tag{14}$$

But economists traditionally look at traffic equilibria in terms of vehicle flows. So note that, via the Fundamental Identity $f = k/t$ and the fact $k = P/\phi$, we have

$$f[k] = f[P/\phi] = \frac{P}{\tau[P/\phi]\phi} \quad \text{(veh/lane-tu)}. \tag{15}$$

Next, rewrite (13) as

$$\dot{P} = G\Big[T(P/\phi)\Big] - \frac{\phi f[P/\phi]}{l}, \tag{16}$$

and using equation (2),

$$\dot{P} = \frac{\phi}{l}\Big[D(P/\phi) - f[P/\phi]\Big]. \tag{17}$$

It follows that, at $P = P_0$, we have

$$D\big(P^0/\phi\big) = f\big(P^0/\phi\big). \tag{18}$$

Stability

Not all equilibria that satisfy (18) are stable, which Strogatz (*23*) defines as: "An equilibrium is defined to be stable if all sufficiently small disturbances away from it damp out in time." More specifically, stability requires that

$$\mathrm{d}\dot{P}\big/\mathrm{d}P < 0. \tag{19}$$

If this is the case, following a small increase (decrease) in P at an equilibrium, P will tend to decline (rise) back towards its equilibrium level. Differentiating (17) yields

$$\mathrm{d}\dot{P}\big/\mathrm{d}P = \frac{1}{l}\big(D' - f'\big). \tag{20}$$

Hence, we have:

Proposition 1. *For the one-mode model, an equilibrium is stable if and only if it has a cut from above (i.e., if $D' < f'$).*

Equilibria with cuts from below ($D' > f'$), which have questionable policy implications, are unstable. Note that one advantage of analysis in the (k,q) plane is that e_1 and e_3 in Fig. 3 have the same cut. To state the same information using the (q,t) plane of Fig. 1, we would have to say that an equilibrium is stable if and only if it T^d cuts T^s from above in light congestion or from below in hypercongestion.

To summarise, equilibria e_1 and e_3 in Fig. 3 are stable because they have a cut from above ($D' < f'$). Equilibrium e_3 if Fig. 3 with questionable policy implications has a cut from below ($D' > f'$) and is unstable.

TWO-MODE STATIC MODEL

This section repeats the analysis of Sec. 2 but with two modes: L and H. A system with just one mode of travel has only one equilibrium in light congestion, but a system with two or more modes has the potentially for multiple equilibria in light congestion.

Physics

The setting is unchanged from the previous one-mode model: the same functions $T(k)$ and $f(k)$ give, respectively, the unit travel time and vehicle flow given a vehicle density k. In this case, however, our understanding of “vehicle density” must be broadened. Let the unit “veh,” previously an actual vehicle, now stand for a “vehicle unit”: a measure of space consumed by vehicles. A car, for instance, might take up one vehicle unit and a bus four. Mode $i = L,H$ has an occupancy ϕ_i (pax/veh) giving the number of passengers *per vehicle unit*. Without loss of generality, suppose that $\phi_L < \phi_H$, so that H carries more passengers per vehicle unit. A trip by H could be thought of as a bus trip (provided that the number of buses scales with their passengers) and a trip by L a trip driving oneself in a car. Thus, vehicle density k is the number of vehicle units per lane-du of network, and vehicle flow q gives the rate a roadside observer counts vehicle units pass (rather than actual vehicles).

Demand

In this model, in addition to choosing *whether* to travel, passengers also decide *how* to travel. Let $G_i(t)$ (pax/lane-du-tu) give the rate (per lane-du of network) that passengers demand trips on mode $i = L,H$, given a unit travel time t. When the unit travel time rises, a trip by either mode takes more time. Hence, total demand declines with t:

$$G'_L + G'_H < 0. \tag{21}$$

Since passengers can switch between modes, G'_L or G'_H could be positive, as long as total demand falls. This possibility leads to the *hyperdemand* phenomenon mentioned above.

Assume trips on mode $i = L,H$ have mean length l_i (du). In this case, the function

$$Q(t) = \frac{l_L}{\phi_L} G_L(t) + \frac{l_H}{\phi_H} G_H(t) \quad \text{(veh/lane-tu)} \tag{22}$$

gives *vehicle flow demanded* conditional on t. Next take the derivative

$$Q'(t) = \frac{l_L}{\phi_L} G'_L(t) + \frac{l_H}{\phi_H} G'_H(t) \quad \text{(veh/lane-tu}^2\text{)}. \tag{23}$$

Even though total demand for trips (the sum $G_L + G_H$) falls with t, it is possible that $Q'(t)$ is still positive, because $Q(t)$ is a *weighted sum* of G_L and G_H. For example, suppose $l_H/\phi_H < l_L/\phi_L$ and that, at some $t = t^*$, G_H falls and G_L rises such that

$$0 < -G'_H(t^*) \frac{l_H/\phi_H}{l_L/\phi_L} < G'_L(t^*) < -G'_H(t^*). \tag{24}$$

In this case, we have $Q'(t^*) > 0$ and $G'_L(t^*) + G'_H(t^*) < 0$: the vehicle flow demanded rises with the unit travel time, even though the total trip demand falls. For an explicit justification of why this might happen, see Lehe and Pandey (*16*), but for our purposes here what matters is the possibility.

As above, let

$$D(k) := Q[T(k)] = \frac{l_L}{\phi_L} G_L[T(k)] + \frac{l_H}{\phi_H} G_H[T(k)] \quad \text{(veh/lane-tu)} \tag{25}$$

give the vehicle flow demanded *conditional on vehicle density*. Like $q = Q(t)$, $q = D(k)$ has the potential to rise or fall over different intervals of its argument. When $D(k)$ rises over some interval of k or $Q(t)$ rises over an interval of t, we say that interval exhibits hyperdemand. When either falls, we say that interval exhibits *light demand*. The idea behind this choice of names is to match demand-side phenomena to light and hypercongestion.

Equilibria

The equilibrium condition for this model is the same as in the one-mode case: at an equilibrium value of vehicle density $k = k^0$, we have

$$f\left(k^0\right) = D\left(k^0\right) \quad \text{(veh/lane-tu)}. \tag{26}$$

A vehicle density is imply the sum of the density of vehicles engaged in each mode: $k = P_L/\phi_L + P_H/\phi_H$. So, the equilibrium can be expressed as a tuple $(P_L, P_H) = (P_L^0, P_H^0)$ such that

$$f\left(P_L^0/\phi_L^0 + P_H^0/\phi_H^0\right) = D\left(P_L^0/\phi_L^0 + P_H^0/\phi_H^0\right) \quad \text{(veh/lane-tu)}. \tag{27}$$

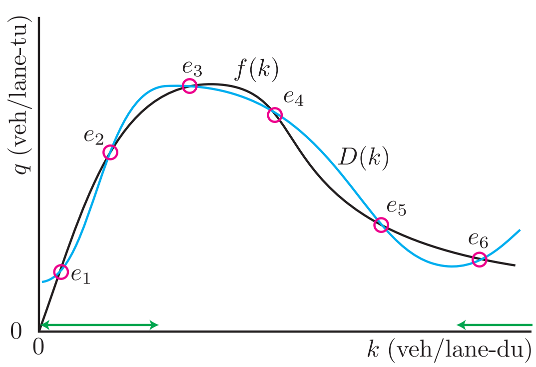


FIGURE 4: Equilibria for two-mode system

A key difference, relative to the one-mode situation, is that now more classes of qualitatively-distinct equilibria are possible. Figure 4 plots examples of $f(k)$ and $D(k)$, with $D(k)$ having two intervals of hyperdemand, the densities associated with which are highlighted in green. There are six equilibria. Table 1 describes the six equilibria by three binary properties: (i) whether the equilibrium occurs in light or hypercongestion; (ii) whether the equilibrium occurs in light or hyperdemand; (iii) the equilibrium's cut. The curves in Fig. 4 were drawn to illustrate all feasible[3] combinations of the three properties.

The same analysis as in Sec. 2.4 could be repeated here, and equilibria with cuts from below demonstrated to have absurd policy implications. The situation has not changed in that regard: a policy that reduced $D(k)$ in the vicinity of an equilibrium with an cut from below would make congestion worse. A difference is that, now, cuts from below can occur in light congestion.

[3] Two combinations are infeasible: (i) an equilibrium in light congestion and light demand with a "below" cut; (ii) an equilibrium in hypercongestion and hyperdemand with an "above" cut.

equilibrium	congestion	demand	cut
1	light	hyper	above
2	light	hyper	below
3	light	light	above
4	hyper	light	below
5	hyper	light	above
6	hyper	hyper	below

TABLE 2: Properties of feasible equilibria for two-mode model

TWO-MODE STABILITY ANALYSIS

We now derive a dynamic model and the stability of equilibria for the two-mode case. Differences between the one and two-mode case include:

- The dynamic model will now be a system of equations in two variables, forcing us to work with the Jacobian of our system instead of the derivative.
- Stability is impossible to establish from the cut in (k,q) space.

Setup

For mode $i = L,H$, let $P_i(u)$, $A_i(u)$ and $E_i(u)$ be, respectively, the passenger stock, arrival flow and exit rate of passengers at clock time u. $t(u)$ and $k(u)$ are still the instantaneous unit travel time and vehicle density, respectively. The conservation law for each mode's passengers is:

$$\dot{P}_i(u) = A_i(u) - E_i(u) \quad \text{(pax/lane-du-tu).} \tag{28}$$

We will now proceed to turn this into a two-equation autonomous system along the lines of (13).

Vehicle density evolves in clock time according to

$$k(u) = P_L(u)/\phi_L + P_H(u)/\phi_H \quad \text{(veh/lane-du),} \tag{29}$$

and so unit travel time evolves as

$$t(u) = T[k(u)] = T[P_L(u)/\phi_L + P_H(u)/\phi_H] \quad \text{(tu/du).} \tag{30}$$

Assumption 1 still holds, albeit in modified form: travelers decide whether and how to travel based only on the instantaneous unit travel time, so that, for mode $i = L,H$,

$$A_i(u) = G_i\Big\{T[P_L(u)/\phi_L + P_H(u)/\phi_H]\Big\} \quad \text{(pax/lane-du-tu).} \tag{31}$$

As for the exit rate, Assumption 2 still holds, but now applies to each mode. Hence,

$$E_i(u) = \frac{P_i(u)}{T[P_L(u)/\phi_L + P_H(u)/\phi_H]l_i} \quad \text{(pax/lane-du-tu).} \tag{32}$$

With (31) and (32), we can now write the autonomous system as:

$$\dot{P}_L = G_L\left[T\left(\frac{P_L}{\phi_L}+\frac{P_H}{\phi_H}\right)\right] - \frac{P_L}{T\left(\frac{P_L}{\phi_L}+\frac{P_H}{\phi_H}\right)l_L} \tag{33}$$

$$\dot{P}_H = G_H\left[T\left(\frac{P_L}{\phi_L}+\frac{P_H}{\phi_H}\right)\right] - \frac{P_H}{T\left(\frac{P_L}{\phi_L}+\frac{P_H}{\phi_H}\right)l_H}. \tag{34}$$

Or, equivalently, to write things in terms of vehicle flows, let

$$D_i(k) := \frac{l_i}{\phi_i}G_i[T(k)] \tag{35}$$

give the vehicle flow demanded on mode i, given k. The autonomous system becomes

$$\dot{P}_L = \frac{\phi_L}{l_L}\left\{D_L\left(\frac{P_L}{\phi_L}+\frac{P_H}{\phi_H}\right) - \frac{P_L}{T\left(\frac{P_L}{\phi_L}+\frac{P_H}{\phi_H}\right)l_L\phi_L}\right\} \tag{36}$$

$$\dot{P}_H = \frac{\phi_L}{l_H}\left\{D_H\left(\frac{P_L}{\phi_L}+\frac{P_H}{\phi_H}\right) - \frac{P_H}{T\left(\frac{P_L}{\phi_L}+\frac{P_H}{\phi_H}\right)l_H\phi_H}.\right\} \tag{37}$$

For each equation, the second term in curly brackets is the realized flow of vehicle units engaged in that mode.

Steady-state equilibria

In a steady-state equilibrium, $\dot{P}_L = \dot{P}_H = 0$; the density of passengers is unchanging for both the modes of travel. Let P_L^0 and P_H^0 be the densities of passengers at equilibria, and let $k^0 = P_L^0/\phi_L + P_H^0/\phi_H$ be the corresponding vehicle density. Thus, in equilibrium,

$$G_L\left[T\left(\frac{P_L^0}{\phi_L}+\frac{P_H^0}{\phi_H}\right)\right] = \frac{P_L^0}{T\left(\frac{P_L^0}{\phi_L}+\frac{P_H^0}{\phi_H}\right)l_L} \quad \text{(pax/lane-du-tu)} \tag{38}$$

$$G_H\left[T\left(\frac{P_L^0}{\phi_L}+\frac{P_H^0}{\phi_H}\right)\right] = \frac{P_H^0}{T\left(\frac{P_L^0}{\phi_L}+\frac{P_H^0}{\phi_H}\right)l_H} \quad \text{(pax/lane-du-tu).} \tag{39}$$

Alternatively, multiply both sides of (38) by l_L/ϕ_L, both sides of (39) by l_H/ϕ_H and then sum the results to obtain

$$D(k^0) = f\left(k^0\right) \quad \text{(veh/lane-tu).} \tag{40}$$

This is the equilibrium condition we gave for the two-mode static model earlier.

Stability

Having derived conditions for equilibria, we now consider their stability. To begin, note our two-dimensional dynamical system can be written in terms of the function

$$F\left(\begin{bmatrix}P_L\\P_H\end{bmatrix}\right) := \begin{bmatrix}\dot{P}_L\\\dot{P}_H\end{bmatrix} \tag{41}$$

$$= \begin{bmatrix} G_L\left[T\left(\frac{P_L}{\phi_L}+\frac{P_H}{\phi_H}\right)\right] - \dfrac{P_L}{T\left(\frac{P_L}{\phi_L}+\frac{P_H}{\phi_H}\right)l_L} \\ G_H\left[T\left(\frac{P_L}{\phi_L}+\frac{P_H}{\phi_H}\right)\right] - \dfrac{P_H}{T\left(\frac{P_L}{\phi_L}+\frac{P_H}{\phi_H}\right)l_L} \end{bmatrix} \tag{42}$$

F captures in a function the relationships given by (38)–(39).

The stability of a two-dimensional system depends on the Jacobian of the transformation, F, at equilibrium:

$$\nabla F = \begin{bmatrix} \mathrm{d}\dot{P}_L/\mathrm{d}P_L & \mathrm{d}\dot{P}_L/\mathrm{d}P_H \\ \mathrm{d}\dot{P}_H/\mathrm{d}P_L & \mathrm{d}\dot{P}_H/\mathrm{d}P_H\ . \end{bmatrix} \tag{43}$$

A two-dimensional dynamical system like ours is locally stable at an equilibrium when the real parts of both eigenvalues of the Jacobian, ∇F, are negative there Alligood1997. This occurs if and only if $\det(\nabla F) > 0$ and $\mathrm{tr}(\nabla F) < 0$; otherwise, the equilibrium is unstable.

Take derivatives to fill in ∇F. Below, for brevity, assume that all functions and derivatives are evaluated at an equilibrium vehicle density $k = k^0$. First,

$$\frac{\mathrm{d}\dot{P}_L}{\mathrm{d}P_L} = \frac{G_L' T'}{\phi_L} + \frac{T'}{T\phi_L}\frac{P_L^0}{l_L T} - \frac{1}{T l_L} \;\; = \frac{1}{l_L}\left\{D_L' + D_L\frac{T'}{T} - \frac{1}{T}\right\} \tag{44}$$

$$\frac{\mathrm{d}\dot{P}_H}{\mathrm{d}P_H} = \frac{G_H' T'}{\phi_H} + \frac{T'}{T\phi_H}\frac{P_H^0}{l_H T} - \frac{1}{T l_H} = \frac{1}{l_H}\left\{D_H' + D_H\frac{T'}{T} - \frac{1}{T}\right\}. \tag{45}$$

As for the cross partial derivatives, it can be shown that

$$\frac{\mathrm{d}\dot{P}_L}{\mathrm{d}P_H} = \frac{\phi_L}{\phi_H}\left\{\frac{\mathrm{d}\dot{P}_L}{\mathrm{d}P_L} + \frac{1}{T l_L}\right\} \tag{46}$$

$$\frac{\mathrm{d}\dot{P}_H}{\mathrm{d}P_L} = \frac{\phi_H}{\phi_L}\left\{\frac{\mathrm{d}\dot{P}_H}{\mathrm{d}P_H} + \frac{1}{T l_H}\right\}. \tag{47}$$

Thus,

$$\det[\nabla F] = \frac{\mathrm{d}\dot{P}_L}{\mathrm{d}P_L}\frac{\mathrm{d}\dot{P}_H}{\mathrm{d}P_H} - \frac{\mathrm{d}\dot{P}_L}{\mathrm{d}P_H}\frac{\mathrm{d}\dot{P}_L}{\mathrm{d}P_H}$$

simplifies to

$$\det[\nabla F] = \frac{1}{l_L l_H T}\left(f' - D'\right). \tag{48}$$

It follows that the determinant is positive if and only if the equilibrium exhibits a cut *from above* (i.e., if $D' < f'$). We have it that:

Proposition 2. *For the two-mode model, equilibria with cuts from below are unstable.*

Still, a cut from above does not imply stability. To be stable requires not only that the determinant of ∇F be positive but also that its trace be negative. With some work, the trace of ∇F in equilibrium can be simplified to

$$\mathrm{tr}(\nabla F) = \frac{\mathrm{d}\dot{P}_L}{\mathrm{d}P_L} + \frac{\mathrm{d}\dot{P}_H}{\mathrm{d}P_H} = \left(\frac{1}{l_H} + \frac{1}{l_L}\right)(D' - f') - \frac{1}{l_H}\left(D'_L + D_L\frac{T'}{T}\right) - \frac{1}{l_L}\left(D'_H + D_H\frac{T'}{T}\right). \tag{49}$$

While the sign of the first term depends on the equilibrium's cut, the signs and magnitudes of the second two terms do not. Hence, it is impossible to say whether the trace is positive from the cut in (k,q) space. Perhaps the most novel thing that can be said is that there is nothing keeping the trace from being negative, even if $D' > 0$. Hence, equilibria in hyperdemand can be stable as long as they have a cut from above.

NUMERICAL SIMULATION OF TWO-MODE SYSTEM

In this section we conduct an approximate numerical simulation of the two-mode system in which demand arises from by a nested logit[4] model. The simulation does not prove anything theoretically, but since the analysis so far has been very general it is illustrative to work through a concrete case. We say the simulation is "approximate" because, for simplicity, we assume that all travelers have the same trip length—in which case the bathtub physics granted by Assumption 2 are only approximate.

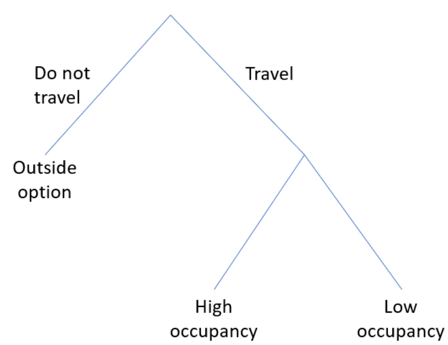


FIGURE 5: Nested Logit model for two mode system

Figure 5 shows the structure of tree diagram for mode choice. There is an "outside option" (subscript O) which stands for not traveling, and a "travel" nest that contains modes L and H. For a traveler n, the utility of mode $i = L,H,O$, given a unit travel time t, is

$$U_i(i) = V_i(t) + \varepsilon_{ni} \quad (\$), \tag{50}$$

where ε_{ni} is a random utility component and V_i is the "systematic utility" of mode i. The

[4]See Train (*24*) for a thorough exposition of the nested logit model and its assumption.

systematic utilities are

$$V_L(t) = a_L - \alpha l_L t \quad (\$) \tag{51}$$

$$V_H(t) = a_H - \alpha l_H t \quad (\$) \tag{52}$$

$$V_O(t) = 0 \quad (\$), \tag{53}$$

where a_L, a_H (\$) are mode-specific constants, α (\$/tu) is the (universal) value of time (which is the same on both modes) and l_L, l_H are the lengths of a trip taken on each mode. The second terms in V_L and V_H are the respective costs of travel time.

The cumulative distribution of the random components $\varepsilon_n = \{\varepsilon_{n0}, \varepsilon_{nL}, \varepsilon_{nH}\}$ is

$$\exp\left[-e^{-\varepsilon_{no}} - \left(e^{-\varepsilon_{nL}/\mu} + e^{-\varepsilon_{nL}/\mu}\right)^{\mu}\right], \tag{54}$$

where $\mu > 0$ is a metric of correlation between the two modes.

Given these assumptions, the probability of choosing to travel is

$$\mathscr{P}[\text{travel}](t) = \frac{[\exp(V_L(t)/\mu) + \exp(V_H(t)/\mu)]^{\mu}}{\exp(V_0) + [\exp(V_L(t)/\mu) + \exp(V_H(t)/\mu)]^{\mu}}, \tag{55}$$

and probability of choosing to travel by mode i, given that one chooses to travel, is

$$\mathscr{P}[i|\text{travel}](t) = \frac{\exp(V_i(t)/\mu)}{\exp(V_i(t)/\mu) + \exp(V_{-i}(t)/\mu)} \tag{56}$$

where $-i$ is the mode not chosen. Hence, the demand functions for the model are

$$G_L(t) = \gamma\, \mathscr{P}[L|\text{travel}](t)\, \mathscr{P}[\text{travel}](t) \quad \text{(pax/lane-du-tu)} \tag{57}$$

$$G_H(t) = \gamma\, \mathscr{P}[H|\text{travel}](t)\, \mathscr{P}[\text{travel}](t) \quad \text{(pax/lane-du-tu)}, \tag{58}$$

where γ (pax/lane-du-tu) is the maximum possible rate of demand.

The simulation is run with the following parameters: $\gamma = 45.0$ (pax/lane-du-tu), $a_L = 5.7$ (\$), $a_H = 8.0$ (\$), $l_L = 1$ and $l_H = 2$, $\alpha = 1.1$, $\phi_L = 1$, $\phi_H = 4$, $\mu = 0.4$. These are chosen arbitrarily by trial-and-error to produce hyperdemand. As for traffic physics, we use

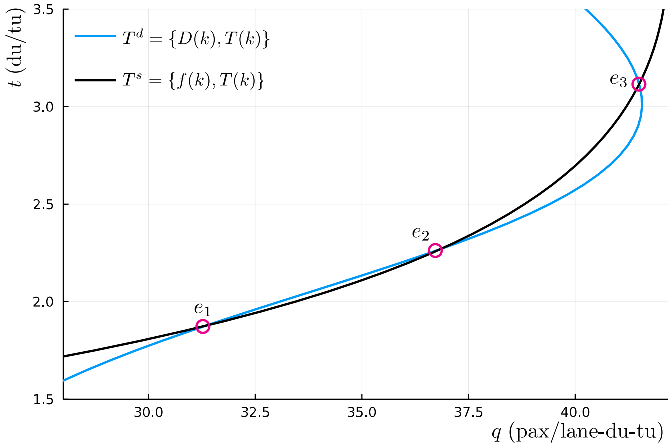
$$T(k) = \exp\left[\frac{(k/160)^{0.75}}{0.75}\right] \tag{59}$$

Figure 6 shows the vehicle demand in (q,t) and (k,q) space. e_1, e_2 and e_3 are the three equilibria, all in light congestion. e_1 and e_3 have cuts from above, e_2 from below. Table 3 gives the trace, determinant and eigenvalues λ_1 and λ_2 of ∇F evaluated at each equilibrium. For the values we have chosen, the trace happens to be negative for all three equilibria, but as expected the determinant is positive for e_2, which has a cut from below.

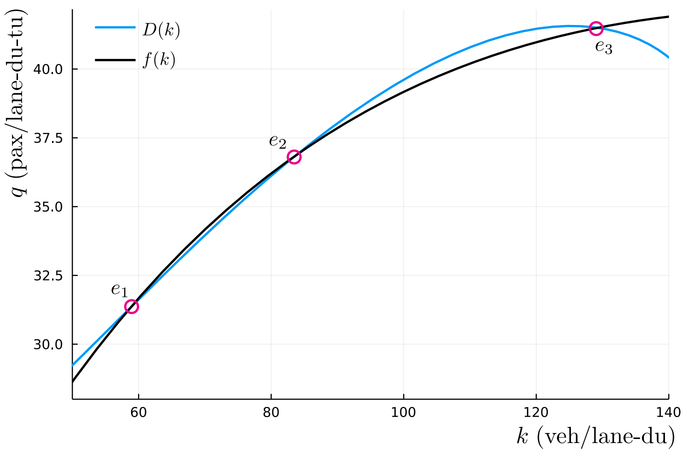
	tr(∇F)	det(∇F)	λ_1	λ_2
e_1	-0.30009	0.022329	-0.16361	-0.16361
e_2	-0.08016	-0.0100	-0.14793	0.06777
e_3	-0.20286	0.011606	-0.10143-0.036i	-0.10143+0.036i

TABLE 3: Trace, determinants and eigenvalues of ∇F at equilibria

Figure 7 shows the system's phase portrait. We start with selected values of our stock



(a)



(b)

FIGURE 6: Simulation results

variables P_L and P_H and let the system evolve over time according to F. As the figure shows, e_1 and e_3 are both sinks while e_2 is a saddle node.

CONCLUSION

This paper has the stability of equilibria for one- and two-mode static models of traffic in downtowns which can exhibit multiple equilibria. Via a pair of purely static models (Sections 2 and 4), we showed that a one-mode system can exhibit multiple equilibria in light congestion and multiple in hypercongested traffic, while a two-mode system can exhibit multiple equilibria even in light congestion. The reason for the latter is that the vehicle flow consistent with demands can rise with the unit travel time, due to switching between high- and low-occupancy modes.

Sections 3 and 5 conducted stability analysis on the static models of, respectively, the one- and two-mode models. Both stability analyses were conducted under a pair of assumptions: (i) that “passengers” (travelers or decision-makers) only care about traffic conditions when they begin their trips; (ii) that the rate of exits depends only on the stock of passengers and the unit travel time (the inverse of speed). For a one-mode model, the stability of an equilibrium depends only

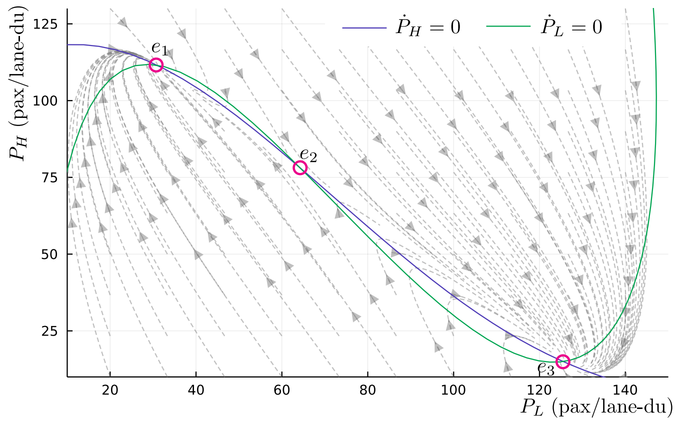


FIGURE 7: Phase portrait for nested logit simulation

on the relative slopes of aggregate "supply" and "demand" curves. For the two-mode model, some equilibria can be shown to be unstable by those relative slopes, but others cannot be shown to be stable. A numerical simulation in Sec. 6 confirms that such equilibria are stable and that the trace and determinant of the two-mode system's Jacobian have the requisite signs at stable equilibria.

Perhaps the broadest lesson of the paper is that traffic systems may be full of surprises. Particularly when mode-switching plays a prominent role, there may be multiple, stable equilibria. The last ten years have witnessed a proliferation of new modes for travel in downtowns, including ride-hailing of various kinds and electric-assist bicycles and scooters. New autonomous modes could be on the horizon. When there are multiple equilibria, policies such as tolls, transit subsidies and infrastructure changes not only affect traffic at the margin but can generate major, sustained changes by moving among these equilibria.

Further work could seek to verify how accurately the proposed theoretical models describe transportation systems. Since we can only observe the current state of a city's transportation system, we might only "observe" one equilibria, but demand and traffic models that have already been estimated might imply other equilibria which policymakers have not considered. Another avenue of research could be identifying optimal policies for jumping between equilibria —for example, in our numerical simulation, from e_3 to e_1. Such research would need to consider the day-to-day dynamics of adjustment and the role of credibility in setting travelers' expectations.

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