Journal of Computational and Applied Mathematics 419 (2023) 114698

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

An L^p- primal–dual weak Galerkin method for convection–diffusion equations

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ARTICLE INFO

Article history:
Received 4 March 2022
Received in revised form 17 July 2022

MSC:
primary 65N30
65N15
65N12
74N20
secondary 35B45
35J50
35J35

Keywords:
Primal–dual weak Galerkin
Finite element methods
Second order elliptic problems
L^p error estimate
Polygonal or polyhedral meshes

ABSTRACT

In this article, the authors present a new L^p-primal–dual weak Galerkin method (L^p-PDWG) for convection–diffusion equations. Comparing with the standard L^2-PDWG method, the solution calculated from the L^p-PDWG may exhibit some important advantages and features (e.g., less jumps cross the element interface when $p \to 1$, or sparsity by using $p = 1$ and wavelet basis approximation). The existence and uniqueness of the numerical solution is discussed, and an optimal-order error estimate is derived in the L^q-norm for the primal variable, where $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Furthermore, error estimates are established for the numerical approximation of the dual variable in the standard $W^{m,p}$ norm, $0 \leq m \leq 2$. Numerical results are presented to demonstrate the efficiency and accuracy of the proposed L^p-PDWG method.

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1. Introduction

In this paper, the authors are concerned with the development of an L^p- primal–dual weak Galerkin (L^p-PDWG) finite element method for the second order elliptic boundary value problem that seeks u such that

$$\begin{aligned} -\Delta u + \nabla \cdot (\boldsymbol{\beta} u) &= f, \quad \text{in } \Omega, \\ u &= g, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

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1 The research of Waixiang Cao was partially supported by NSFC, China Grant No. 11871106, and Guangdong Basic and Applied Basic Research Foundation, China (No. 2019A1515111185).

2 The research of Chunmei Wang was partially supported by National Science Foundation, USA Award DMS-2136380.

3 The research of Junping Wang was supported in part by the NSF, USA IR/D program, while working at National Science Foundation. However, any opinion, finding, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

https://doi.org/10.1016/j.cam.2022.114698
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where $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is an open bounded and connected domain with piecewise smooth Lipschitz boundary $\partial\Omega$ and $f \in L^q(\Omega)$ is a given function with $q > 1$. We assume that the convection vector $\boldsymbol{\beta} \in [L^\infty(\Omega)]^d$ is bounded and piecewise continuous. By a weak solution of (1.1) we mean a function $u \in L^q(\Omega)$ with $q > 1$ such that

$$(u, \Delta\sigma + \boldsymbol{\beta} \cdot \nabla\sigma) = \langle g, \nabla\sigma \cdot \mathbf{n}\rangle_{\partial\Omega} - (f, \sigma) \quad \forall\sigma \in V, \tag{1.2}$$

where $V = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$.

The weak Galerkin (WG) method was first introduced by Wang and Ye in [1,2] for second order elliptic equations, where weak gradient and its discrete weak gradient were constructed to replace the standard gradient and its discrete gradient. Later, the authors in [3] developed a primal–dual weak Galerkin (PDWG) finite element method for the second order elliptic problem in non-divergence form, where a *discrete weak Hessian* operator in the weak formulation of the model PDEs was designed. This PDWG algorithm can be characterized as a constrained L^2 optimization problem with constraints given by the weak formulation of the model PDEs weakly defined on each element. In the past several years, many theoretical a priori error estimates for weak Galerkin methods have been established in L^2 and discrete H^m, $m = 1, 2$ norms. Readers are referred to [2,4–14] for an incomplete list of references.

The purpose of this paper is to present an L^p- primal–dual weak Galerkin method for the problem (1.1), and establish a general L^p theory for the numerical method. To our best knowledge, there is one existing result in the L^p $(1 < p < \infty)$ error estimate for the mixed finite element method developed by Duran [15] for second order elliptic problems in $\mathbb{R}^2$, but no results in L^p are known for the weak Galerkin finite element methods in the literature. Different from the method in [3], our numerical scheme is based on the weak formulation (1.2) together with a weak version of the dual operator applied to the test functions. The new PDWG method can be characterized as a constrained L^p optimization problem with constraints that satisfy the PDE weakly on each element, which extends the idea of L^2 minimization problem in [3] to a more general L^p setting.

It should be pointed out that our theory for the L^p- primal–dual weak Galerkin finite element method is based on the assumption that the solution to the following adjoint problem

$$\begin{aligned} -\Delta\varphi - \boldsymbol{\beta} \cdot \nabla\varphi &= \chi, \qquad \text{in } \Omega, \\ \varphi &= 0, \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

is $W^{2,p}$-regular in the sense that it has a unique solution in $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ and the solution satisfies

$$\|\varphi\|_{2,p} \le C\|\chi\|_{0,p}.$$

Here $W^{m,p}(D)$ denotes the standard Sobolev spaces on sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. When $D = \Omega$, we omit the index D; and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$, and $|\cdot|_{m,p,D} = |\cdot|_{m,D}$. Under this assumption, we shall derive an optimal order error estimate in the standard L^q norm for the primal variable u and the standard $W^{m,p}$, $0 \le m \le 1$ norms for the dual variable. Numerical experiments demonstrate that our error estimate for the primal variable is optimal; i.e., the error bound is sharp.

The rest of this paper is organized as follows. In Section 2, we briefly review the weak differential operators and their discrete versions. In Section 3, the primal–dual weak Galerkin scheme is introduced for the model problem (1.1) based on L^p theory. Section 4 is devoted to the establishment of the solution existence, uniqueness and stability. In Section 5, we derive an error equation for our numerical methods, which is of essential importance in our later error estimates. Sections 6 and 7 establish error estimates for the primal variable in L^q norms and for the dual variable in $W^{m,p}$, $0 \le m \le 1$ norms, respectively. Finally, a series of numerical examples are presented in Section 8 to verify the mathematical convergence theory.

2. Weak differential operators

The Laplacian and the gradient are the principle differential operators used in the weak formulation (1.2) for the second order elliptic model problem (1.1). This section gives a brief discussion of the weak Laplacian and gradient operators as well as their discrete analogies [16].

Let T be a polygonal or polyhedral domain with boundary ∂T. A weak function on T refers to a triplet $\sigma = \{\sigma_0, \sigma_b, \sigma_n\}$ such that $\sigma_0 \in L^p(T)$, $\sigma_b \in L^p(\partial T)$, and $\sigma_n \in L^p(\partial T)$. The first and second components σ_0 and σ_b can be identified as the value of σ in the interior and on the boundary of T. The third component σ_n is meant to represent the value of $\nabla\sigma \cdot \mathbf{n}$ on the boundary of the element T. Note that σ_b and σ_n might be totally independent of the trace of σ_0 and $\nabla\sigma_0 \cdot \mathbf{n}$ on ∂T, respectively. Denote by $\mathcal{W}(T)$ the space of all scalar-valued weak functions on T; i.e.,

$$\mathcal{W}(T) = \{\sigma = \{\sigma_0, \sigma_b, \sigma_n\} : \sigma_0 \in L^p(T), \sigma_b \in L^p(\partial T), \sigma_n \in L^p(\partial T)\}. \tag{2.1}$$

The weak Laplacian operator, denoted by Δ_w, is defined as a linear functional in $W^{2,p}(T)$ such that

$$(\Delta_w\sigma, w)_T = (\sigma_0, \Delta w)_T - \langle\sigma_b, \nabla w \cdot \mathbf{n}\rangle_{\partial T} + \langle\sigma_n, w\rangle_{\partial T},$$

for all $w \in W^{2,q}(T)$.

Denote by $P_r(T)$ the space of all polynomials on T with degree no more than r. A discrete analogy of $\Delta_w\sigma$ for $\sigma \in \mathcal{W}(T)$ is defined as the unique polynomial $\Delta_{w,r,T}\sigma \in P_r(T)$ satisfying

$$(\Delta_{w,r,T}\sigma, w)_T = (\sigma_0, \Delta w)_T - \langle \sigma_b, \nabla w \cdot \mathbf{n}\rangle_{\partial T} + \langle \sigma_n, w\rangle_{\partial T}, \quad \forall w \in P_r(T). \tag{2.2}$$

For smooth σ_0 such that $\sigma_0 \in W^{2,p}(T)$, we have from the integration by parts

$$(\Delta_{w,r,T}\sigma, w)_T = (\Delta\sigma_0, w)_T + \langle \sigma_0 - \sigma_b, \nabla w \cdot \mathbf{n}\rangle_{\partial T} - \langle \nabla\sigma_0 \cdot \mathbf{n} - \sigma_n, w\rangle_{\partial T}, \tag{2.3}$$

for all $w \in P_r(T)$. Similarly, the discrete weak gradient operator is defined as the unique polynomial $\nabla_{w,r,T}\sigma \in [P_r(T)]^d$ satisfying

$$(\nabla_{w,r,T}\sigma, \varphi)_T = -(\sigma_0, \nabla \cdot \varphi)_T + \langle \sigma_b, \varphi \cdot \mathbf{n}\rangle_{\partial T}, \quad \forall \varphi \in [P_r(T)]^d. \tag{2.4}$$

When $\sigma_0 \in W^{1,p}(T)$, the following identity holds true:

$$(\nabla_{w,r,T}\sigma, \varphi)_T = (\nabla\sigma_0, \varphi)_T + \langle \sigma_b - \sigma_0, \varphi \cdot \mathbf{n}\rangle_{\partial T}, \tag{2.5}$$

for all $\varphi \in [P_r(T)]^d$.

3. Numerical algorithm

Denote by $\mathcal{T}_h$ a partition of the domain Ω into polygons in 2D or polyhedra in 3D which is shape regular in the sense described in [2]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$ and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ the set of all interior edges or flat faces. Denote by h_T the meshsize of $T \in \mathcal{T}_h$ and $h = \max_{T\in\mathcal{T}_h} h_T$ the meshsize for the partition $\mathcal{T}_h$.

For any given integer $k \geq 1$, denote by $W_k(T)$ the local discrete space of the weak functions defined by

$$W_k(T) = \{\{\sigma_0, \sigma_b, \sigma_n\} : \sigma_0 \in P_k(T), \sigma_b \in P_k(e), \sigma_n \in P_{k-1}(e), e \subset \partial T\}.$$

Patching $W_k(T)$ over all the elements $T \in \mathcal{T}_h$ through a common value of σ_b and σ_n on the interior interface $\mathcal{E}_h^0$ yields a weak finite element space W_h:

$$W_h = \left\{\{\sigma_0, \sigma_b, \sigma_n\} : \{\sigma_0, \sigma_b, \sigma_n\}|_T \in W_k(T), \forall T \in \mathcal{T}_h\right\}. \tag{3.1}$$

Note that σ_n has two values σ_n^L and σ_n^R on each interior interface $e = \partial T_L \cap \partial T_R \in \mathcal{E}_h^0$ as seen from the two elements T_L and T_R, and they must satisfy $\sigma_n^L + \sigma_n^R = 0$. Denote by W_h^0 the subspace of W_h with homogeneous boundary condition; i.e.,

$$W_h^0 = \{\{\sigma_0, \sigma_b, \sigma_n\} : \{\sigma_0, \sigma_b, \sigma_n\}|_T \in W_h, \sigma_b|_{\partial\Omega} = 0,\ \forall e \in \partial T, T \in \mathcal{T}_h\}.$$

Denote by M_h the finite element space consisting of piecewise polynomials of degree s where $s = k - 1$; i.e.,

$$M_h = \{w : w|_T \in P_s(T), \forall T \in \mathcal{T}_h\}. \tag{3.2}$$

We emphasize that both the weak gradient and the weak Laplacian operators are defined by using piecewise polynomials of degree $s = k - 1$. For purely diffusive equations, one may assume the value of $s = k - 2$.

For simplicity of notation and without confusion, for any $\sigma \in W_h$, denote by $\Delta_w\sigma$ and $\nabla_w\sigma$ the discrete weak Laplacian $\Delta_{w,s,T}\sigma$ and discrete weak gradient $\nabla_{w,s,T}\sigma$ computed by (2.2) and (2.4) on each element T, respectively; i.e.,

$$(\Delta_w\sigma)|_T = \Delta_{w,s,T}(\sigma|_T), \quad (\nabla_w\sigma)|_T = \nabla_{w,s,T}(\sigma|_T),\ s = k - 1.$$

For any $\sigma, \lambda \in W_h$ and $u \in M_h$, we introduce the following forms

$$s(\lambda, \sigma) = \sum_{T\in\mathcal{T}_h} s_T(\lambda, \sigma), \tag{3.3}$$

$$b(u, \lambda) = \sum_{T\in\mathcal{T}_h} b_T(u, \lambda), \tag{3.4}$$

where

$$\begin{aligned} s_T(\lambda, \sigma) =& h_T^{1-2p}\int_{\partial T} |\lambda_0 - \lambda_b|^{p-1} sgn(\lambda_0 - \lambda_b)(\sigma_0 - \sigma_b)ds \\ &+ h_T^{1-p}\int_{\partial T} |\nabla\lambda_0 \cdot \mathbf{n} - \lambda_n|^{p-1} sgn(\nabla\lambda_0 \cdot \mathbf{n} - \lambda_n)(\nabla\sigma_0 \cdot \mathbf{n} - \sigma_n)ds \\ b_T(u, \lambda) =& (u, -\boldsymbol{\beta} \cdot \nabla_w\lambda - \Delta_w\lambda)_T. \end{aligned}$$

The numerical scheme for the second order elliptic model problem (1.1) based on the variational formulation (1.2) can be stated as follows:

Primal-Dual Weak Galerkin Algorithm 3.1. Find $(u_h; \lambda_h) \in M_h \times W_h^0$, such that

$$s(\lambda_h, \sigma) + b(u_h, \sigma) = (f, \sigma_0) - \langle g, \sigma_n \rangle_{\partial\Omega}, \qquad \forall \sigma \in W_h^0, \tag{3.5}$$

$$b(v, \lambda_h) = 0, \qquad \forall v \in M_h. \tag{3.6}$$

In the next section, we shall study the solution existence and uniqueness for the primal–dual weak Galerkin finite element algorithm (3.5)–(3.6). For simplicity of the analysis and presentation, we assume a constant value for the convection term $\boldsymbol{\beta}$ on each element $T \in \mathcal{T}_h$ in the rest of the paper. This assumption is nonessential since the analysis can be extended to variable and piecewise continuous functions $\boldsymbol{\beta}$.

4. Solution existence and uniqueness

Denote by Q_0 the L^2 projection operator onto $P_k(T)$ for each element T. For each edge or face $e \subset \partial T$, denote by Q_b and Q_n the L^2 projection operators onto $P_k(e)$ and $P_{k-1}(e)$, respectively. For any $w \in W^{2,p}(\Omega)$, define by $Q_h w$ the L^2 projection onto the weak finite element space W_h such that on each element T,

$$Q_h w = \{Q_0 w, Q_b w, Q_n(\nabla w \cdot \mathbf{n})\}.$$

Denote by $\mathcal{Q}_h$ the L^2 projection onto the finite element space M_h.

Lemma 4.1 ([16]). *The L^2 projection operators Q_h and $\mathcal{Q}_h$ satisfy the following commutative properties:*

$$\Delta_w(Q_h w) = \mathcal{Q}_h(\Delta w), \qquad w \in W^{2,p}(T), \tag{4.1}$$

$$\nabla_w(Q_h v) = \mathcal{Q}_h(\nabla v), \qquad v \in W^{1,p}(T). \tag{4.2}$$

To show the existence of solutions, we consider the functional

$$J(\sigma, v) := \frac{1}{p} s(\sigma, \sigma) + b(v, \sigma) - (F, \sigma),$$

where $(F, \sigma) = (f, \sigma_0) - \langle g, \sigma_n \rangle_{\partial\Omega}$ and $p > 1$. If $(u_h; \lambda_h) \in M_h \times W_h^0$ is the solution of (3.5)–(3.6), then we have from (3.6)

$$J(\lambda_h, v) = \frac{1}{p} s(\lambda_h, \lambda_h) - (F, \lambda_h) = J(\lambda_h, u_h), \quad \forall v \in M_h.$$

On the other hand, for $p > 1$, since the functional $J(\sigma, u_h)$ is convex in σ, Eq. (3.5) indicates that $\partial_\sigma J(\lambda_h, u_h)(\sigma) = 0$ for all $\sigma \in W_h^0$, where $\partial_\sigma J(\lambda_h, u_h)(\sigma)$ is the Gateaux partial derivative at λ_h in the direction of σ. It follows that λ_h is a global minimizer of the functional $\sigma \to J(\sigma, u_h)$; i.e.,

$$J(\lambda_h, u_h) \le J(\sigma, u_h), \qquad \forall \sigma \in W_h^0.$$

Combining the last two inequalities yields

$$J(\lambda_h, v) \le J(\lambda_h, u_h) \le J(\sigma, u_h), \qquad \forall v \in M_h, \sigma \in W_h^0. \tag{4.3}$$

The above inequality implies that the solution $(u_h; \lambda_h)$ is a saddle point of the functional $J(\cdot, \cdot)$. Thus, (3.5)–(3.6) can be formulated as the following min–max problem: Find $u_h \in M_h$ and $\lambda_h \in W_h^0$ such that

$$(\lambda_h, u_h) = \arg \min_{\sigma \in W_h^0} \max_{v \in M_h} J(\sigma, v).$$

Note that the exact solution u satisfies $b(u, \sigma) = (f, \sigma_0) - \langle g, \sigma_n \rangle_{\partial\Omega}$ for all σ. It follows from the second inequality of (4.3) that the exact solution (λ, u) satisfies

$$s(\lambda, \lambda) \le s(\sigma, \sigma), \quad \forall \sigma,$$

which yields $\lambda = 0$, i.e., the exact Lagrangian multiplier λ is equal to zero.

As a convex minimization problem, the above problem has a solution so that there must be a solution $(u_h; \lambda_h)$ satisfying (3.5)–(3.6).

The rest of this section is devoted to a discussion of the uniqueness of the numerical solution $(u_h; \lambda_h)$.

Theorem 4.2. *The numerical scheme (3.5)–(3.6) has one and only one solution $(u_h; \lambda_h)$ in the finite element space $M_h \times W_h^0$.*

Proof. Let $(u_h^{(1)}; \lambda_h^{(1)})$ and $(u_h^{(2)}; \lambda_h^{(2)})$ be two solutions of (3.5)–(3.6). Denote

$$\epsilon_h = \lambda_h^{(1)} - \lambda_h^{(2)} = \{\epsilon_0, \epsilon_b, \epsilon_n\}, \quad e_h = u_h^{(1)} - u_h^{(2)}.$$

For any constants θ_1, θ_2, we choose $\sigma = \theta_1 \lambda_h^{(1)} + \theta_2 \lambda_h^{(2)}$ in (3.5) and use (3.6) to obtain

$$s(\lambda_h^{(1)}, \theta_1 \lambda_h^{(1)} + \theta_2 \lambda_h^{(2)}) - s(\lambda_h^{(2)}, \theta_1 \lambda_h^{(1)} + \theta_2 \lambda_h^{(2)}) = 0.$$

In particular, by taking $(\theta_1, \theta_2) = (1, 0), (0, 1)$, we have

$$s(\lambda_h^{(1)}, \lambda_h^{(1)}) = s(\lambda_h^{(2)}, \lambda_h^{(1)}), \quad s(\lambda_h^{(2)}, \lambda_h^{(2)}) = s(\lambda_h^{(1)}, \lambda_h^{(2)}), \tag{4.4}$$

which yields, together with Young's inequality $|AB| \le \frac{|A|^p}{p} + \frac{|B|^q}{q}$, that

$$s(\lambda_h^{(1)}, \lambda_h^{(1)}) \le \frac{s(\lambda_h^{(2)}, \lambda_h^{(2)})}{q} + \frac{s(\lambda_h^{(1)}, \lambda_h^{(1)})}{p}, \quad s(\lambda_h^{(2)}, \lambda_h^{(2)}) \le \frac{s(\lambda_h^{(1)}, \lambda_h^{(1)})}{q} + \frac{s(\lambda_h^{(2)}, \lambda_h^{(2)})}{p},$$

which yields

$$s(\lambda_h^{(1)}, \lambda_h^{(1)}) = s(\lambda_h^{(2)}, \lambda_h^{(2)}). \tag{4.5}$$

On the other hand, for any two real numbers A, B, there holds

$$\left|\frac{A+B}{2}\right|^p \le (|A|^p + |B|^p)/2,$$

and the equality holds true if and only if $A = B$. It follows that

$$s(\frac{\lambda_h^{(1)} + \lambda_h^{(2)}}{2}, \frac{\lambda_h^{(1)} + \lambda_h^{(2)}}{2}) \le \frac{1}{2}\big(s(\lambda_h^{(1)}, \lambda_h^{(1)}) + s(\lambda_h^{(2)}, \lambda_h^{(2)})\big) = s(\lambda_h^{(1)}, \lambda_h^{(1)}). \tag{4.6}$$

By (4.4)–(4.5) and Young's inequality,

$$\begin{aligned} s(\lambda_h^{(1)}, \lambda_h^{(1)}) =\ & \frac{1}{2}\big(s(\lambda_h^{(1)}, \lambda_h^{(1)}) + s(\lambda_h^{(1)}, \lambda_h^{(2)})\big) = s(\lambda_h^{(1)}, \frac{\lambda_h^{(1)} + \lambda_h^{(2)}}{2}) \\ \le\ & \frac{1}{q} s(\lambda_h^{(1)}, \lambda_h^{(1)}) + \frac{1}{p} s(\frac{\lambda_h^{(1)} + \lambda_h^{(2)}}{2}, \frac{\lambda_h^{(1)} + \lambda_h^{(2)}}{2}), \end{aligned}$$

which indicates that

$$s(\lambda_h^{(1)}, \lambda_h^{(1)}) \le s(\frac{\lambda_h^{(1)} + \lambda_h^{(2)}}{2}, \frac{\lambda_h^{(1)} + \lambda_h^{(2)}}{2}).$$

In light of (4.6), we easily obtain that

$$s(\frac{\lambda_h^{(1)} + \lambda_h^{(2)}}{2}, \frac{\lambda_h^{(1)} + \lambda_h^{(2)}}{2}) = s(\lambda_h^{(1)}, \lambda_h^{(1)}) = s(\lambda_h^{(2)}, \lambda_h^{(2)}).$$

The above equality holds true if and only if

$$\begin{aligned} \lambda_0^{(1)} - \lambda_b^{(1)} &= \lambda_0^{(2)} - \lambda_b^{(2)}, \ \text{ on } \partial T, \\ \nabla \lambda_0^{(1)} \cdot \mathbf{n} - \lambda_n^{(1)} &= \nabla \lambda_0^{(2)} \cdot \mathbf{n} - \lambda_n^{(2)}, \ \text{ on } \partial T, \end{aligned}$$

or equivalently,

$$\epsilon_0 = \epsilon_b, \ \text{ on } \partial T, \tag{4.7}$$

$$\nabla \epsilon_0 \cdot \mathbf{n} = \epsilon_n, \ \text{ on } \partial T. \tag{4.8}$$

Let $(u_h^{(1)}; \lambda_h^{(1)})$ and $(u_h^{(2)}; \lambda_h^{(2)})$ be two solutions of (3.5)–(3.6). We have from (3.6) that $b(v, \epsilon_h) = 0$. Using (2.3) and (2.5), we have

$$\begin{aligned} 0 =& b(v, \epsilon_h) \\ =& \sum_{T \in \mathcal{T}_h} (v, -\boldsymbol{\beta} \cdot \nabla_w \epsilon_h - \Delta_w \epsilon_h)_T \\ =& \sum_{T \in \mathcal{T}_h} -(\nabla \epsilon_0, \boldsymbol{\beta} v)_T - \langle \epsilon_b - \epsilon_0, \boldsymbol{\beta} v \cdot \mathbf{n} \rangle_{\partial T} \\ & - (\Delta \epsilon_0, v) - \langle \epsilon_0 - \epsilon_b, \nabla v \cdot \mathbf{n} \rangle_{\partial T} + \langle \nabla \epsilon_0 \cdot \mathbf{n} - \epsilon_n, v \rangle_{\partial T} \\ =& \sum_{T \in \mathcal{T}_h} (-\boldsymbol{\beta} \cdot \nabla \epsilon_0 - \Delta \epsilon_0, v)_T, \end{aligned}$$

where we used (4.7)–(4.8), which gives $-\boldsymbol{\beta} \cdot \nabla \epsilon_0 - \Delta \epsilon_0 = 0$ on each $T \in \mathcal{T}_h$. Together with (4.7)–(4.8), we arrive at $-\boldsymbol{\beta} \cdot \nabla \epsilon_0 - \Delta \epsilon_0 = 0$ in Ω, with the boundary condition $\epsilon_0 = 0$ on $\partial \Omega$ due to the fact that (4.7) and $\epsilon_h \in W_h^0$. Therefore, we obtain $\epsilon_0 = 0$ in Ω. Furthermore, we have $\epsilon_b = 0$ and $\epsilon_n = 0$, which leads to $\lambda_h^{(1)} = \lambda_h^{(2)}$.

We next show $e_h = 0$. To this end, using $\lambda_h^{(1)} = \lambda_h^{(2)}$ and Eq. (3.5) we obtain

$$b(e_h, \sigma) = s(\lambda_h^{(1)}, \sigma) - s(\lambda_h^{(2)}, \sigma) + b(e_h, \sigma) = 0, \quad \forall \sigma \in W_h^0,$$

which, together with the definition of the weak Laplacian Δ_w and the weak gradient ∇_w, yields

$$
\begin{aligned}
0 = b(e_h, \sigma) &= \sum_{T\in\mathcal{T}_h} (e_h, -\boldsymbol{\beta}\cdot\nabla_w\sigma - \Delta_w\sigma)_T \\
&= \sum_{T\in\mathcal{T}_h} (\nabla\cdot(\boldsymbol{\beta}e_h) - \Delta e_h, \sigma_0)_T \\
&\quad + \sum_{T\in\mathcal{T}_h} \langle \sigma_b, \nabla e_h\cdot\mathbf{n}\rangle_{\partial T} - \langle \sigma_n, e_h\rangle_{\partial T} - \langle e_h, \boldsymbol{\beta}\cdot\mathbf{n}\sigma_b\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} (\nabla\cdot(\boldsymbol{\beta}e_h) - \Delta e_h, \sigma_0)_T \\
&\quad + \sum_{e\in\mathcal{E}_h} \int_e ([\![\nabla e_h - \boldsymbol{\beta}e_h]\!]\cdot\mathbf{n}_e\sigma_b - [\![e_h]\!]\sigma_n)\, ds
\end{aligned}
$$

for all $\sigma \in W_h^0$, where $\mathbf{n}_e$ is the assigned outward normal direction to $e \in \mathcal{E}_h$ and $[\![\cdot]\!]$ is the jump on the edge $e \in \mathcal{E}_h$. In particular, by taking $\sigma_0 = \nabla\cdot(\boldsymbol{\beta}e_h) - \Delta e_h$, $\sigma_n|_{\mathcal{E}_h} = -[\![e_h]\!]$, and $\sigma_b|_{\mathcal{E}_h^0} = [\![\nabla e_h - \boldsymbol{\beta}e_h]\!]\cdot\mathbf{n}_e$ we obtain on each $T \in \mathcal{T}_h$

$$
\begin{aligned}
&-\triangle e_h + \nabla\cdot(\boldsymbol{\beta}e_h) = 0, \text{ in } T, \\
&[\![e_h]\!] = 0, \quad [\![\nabla e_h - \boldsymbol{\beta}e_h]\!]\cdot\mathbf{n}_e = 0, \text{ on } \partial T.
\end{aligned}
$$

Consequently, from the solution uniqueness for (1.1) we have

$$
e_h \equiv 0, \quad \text{or equivalently} \quad u_h^{(1)} = u_h^{(2)}.
$$

This completes the proof. □

5. Error equation

Let u and $(u_h; \lambda_h) \in M_h \times W_h^0$ be the exact solution of (1.1) and its numerical solution arising from the PDWG scheme (3.5)–(3.6), respectively. Denote two error functions by

$$
e_h = \mathcal{Q}_h u - u_h, \tag{5.1}
$$

$$
\epsilon_h = Q_h\lambda - \lambda_h = -\lambda_h. \tag{5.2}
$$

Lemma 5.1. *Let u and $(u_h; \lambda_h) \in M_h \times W_h^0$ be the exact solution of (1.1) and its numerical solution arising from PDWG scheme (3.5)–(3.6). The error functions e_h and ϵ_h satisfy the following equations:*

$$
s(\epsilon_h, \sigma) + b(e_h, \sigma) = l_u(\sigma), \qquad \forall \sigma \in W_h^0, \tag{5.3}
$$

$$
b(v, \epsilon_h) = 0, \qquad \forall v \in M_h. \tag{5.4}
$$

Here

$$
\begin{aligned}
l_u(\sigma) = &\sum_{T\in\mathcal{T}_h} \langle u - \mathcal{Q}_h u, \sigma_n - \nabla\sigma_0\cdot\mathbf{n}\rangle_{\partial T} + \langle \nabla u\cdot\mathbf{n} - \nabla\mathcal{Q}_h u\cdot\mathbf{n}, \sigma_0 - \sigma_b\rangle_{\partial T} \\
&+ \sum_{T\in\mathcal{T}_h} \langle (u - \mathcal{Q}_h u)\boldsymbol{\beta}\cdot\mathbf{n}, \sigma_b - \sigma_0\rangle_{\partial T}.
\end{aligned} \tag{5.5}
$$

Proof. First, from (5.2) and (3.6) we may readily derive (5.4). Next, by using (3.4) for $b(\cdot,\cdot)$ and choosing $w = \mathcal{Q}_h u$ in (2.3) and (2.5), we have

$$
\begin{aligned}
b(\mathcal{Q}_h u, \sigma) &= \sum_{T\in\mathcal{T}_h} (\mathcal{Q}_h u, -\boldsymbol{\beta}\cdot\nabla_w\sigma - \Delta_w\sigma)_T \\
&= \sum_{T\in\mathcal{T}_h} (\mathcal{Q}_h u, -\boldsymbol{\beta}\cdot\nabla\sigma_0 - \Delta\sigma_0)_T - \langle \mathcal{Q}_h u\boldsymbol{\beta}\cdot\mathbf{n}, \sigma_b - \sigma_0\rangle_{\partial T} \\
&\quad + \langle \nabla\mathcal{Q}_h u\cdot\mathbf{n}, \sigma_b - \sigma_0\rangle_{\partial T} + \langle \mathcal{Q}_h u, \nabla\sigma_0\cdot\mathbf{n} - \sigma_n\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} (u, -\boldsymbol{\beta}\cdot\nabla\sigma_0 - \Delta\sigma_0)_T - \langle \mathcal{Q}_h u\boldsymbol{\beta}\cdot\mathbf{n}, \sigma_b - \sigma_0\rangle_{\partial T} \\
&\quad + \langle \nabla\mathcal{Q}_h u\cdot\mathbf{n}, \sigma_b - \sigma_0\rangle_{\partial T} + \langle \mathcal{Q}_h u, \nabla\sigma_0\cdot\mathbf{n} - \sigma_n\rangle_{\partial T}.
\end{aligned} \tag{5.6}
$$

Now applying the usual integration by parts to the integrals on T yields

$$\begin{aligned} & b(\mathcal{Q}_h u, \sigma) \\ =& \sum_{T\in\mathcal{T}_h} (-\Delta u + \nabla\cdot(\boldsymbol{\beta} u), \sigma_0)_T + \langle (u-\mathcal{Q}_h u)\boldsymbol{\beta}\cdot\mathbf{n}, \sigma_b - \sigma_0\rangle_{\partial T} \\ & + \langle \nabla(\mathcal{Q}_h u - u)\cdot\mathbf{n}, \sigma_b-\sigma_0\rangle_{\partial T} + \langle (\mathcal{Q}_h u - u), (\nabla\sigma_0\cdot\mathbf{n} - \sigma_n)\rangle_{\partial T} - \langle g, \sigma_n\rangle_{\partial\Omega} \\ =& (f,\sigma_0)_T - \langle g, \sigma_n\rangle_{\partial\Omega} \\ & + \sum_{T\in\mathcal{T}_h} \langle \nabla(\mathcal{Q}_h u - u)\cdot\mathbf{n}, \sigma_b-\sigma_0\rangle_{\partial T} + \langle (\mathcal{Q}_h u - u), (\nabla\sigma_0\cdot\mathbf{n} - \sigma_n)\rangle_{\partial T} \\ & + \sum_{T\in\mathcal{T}_h} \langle (u-\mathcal{Q}_h u)\boldsymbol{\beta}\cdot\mathbf{n}, \sigma_b-\sigma_0\rangle_{\partial T}, \end{aligned} \tag{5.7}$$

where we have used (1.1) and $\sigma_b = 0$ on $\partial\Omega$. From $\lambda = 0$, we have $s(Q_h\lambda, \sigma) = 0$. Subtracting (3.5) from (5.7) yields the error Eq. (5.3). This completes the proof of the lemma. □

Eqs. (5.3)–(5.4) are called *error equations* for the primal–dual WG finite element scheme (3.5)–(3.6).

6. L^q-Error estimate for the primal variable u_h

Recall that $\mathcal{T}_h$ is a shape-regular finite element partition of the domain Ω. For any $T \in \mathcal{T}_h$ and $\nabla w \in L^q(T)$ with $q > 1$, the following trace inequality holds true:

$$\|w\|^q_{L^q(\partial T)} \le Ch_T^{-1}(\|w\|^q_{L^q(T)} + h_T^q\|\nabla w\|^q_{L^q(T)}). \tag{6.1}$$

For simplicity, denote by ∇_h^k the kth order partial derivative operator taken on each element $T \in \mathcal{T}_h$.

Theorem 6.1. *Let $q > 1$ and $k \ge 1$. Let u and $(u_h; \lambda_h) \in M_h \times W_h^0$ be the exact solution of the second order elliptic model problem (1.1) and the numerical solution arising from PDWG scheme (3.5)–(3.6). The following error estimate holds true:*

$$s(\lambda_h, \lambda_h) \le Ch^{qk}\|\nabla_h^k u\|^q_{L^q(\Omega)} + C\delta_{k,1}h^{2q}\|\nabla_h^2 u\|^q_{L^q(\Omega)}, \tag{6.2}$$

where $\delta_{i,j}$ is the Kronecker delta.

Proof. Recall that $\epsilon_h = -\lambda_h$. By letting $\sigma = -\lambda_h$ in (5.3), we have from (5.4) and (5.5) that

$$\begin{aligned} s(\lambda_h, \lambda_h) =& \sum_{T\in\mathcal{T}_h} \langle u - \mathcal{Q}_h u, \nabla\lambda_0\cdot\mathbf{n} - \lambda_n\rangle_{\partial T} + \langle \nabla u\cdot\mathbf{n} - \nabla\mathcal{Q}_h u\cdot\mathbf{n}, \lambda_b - \lambda_0\rangle_{\partial T} \\ & + \sum_{T\in\mathcal{T}_h} \langle (u-\mathcal{Q}_h u)\boldsymbol{\beta}\cdot\mathbf{n}, \lambda_0 - \lambda_b\rangle_{\partial T}. \end{aligned} \tag{6.3}$$

For the first term on the right-hand side of (6.3), we use the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} & \Big|\sum_{T\in\mathcal{T}_h} \langle \mathcal{Q}_h u - u, \lambda_n - \nabla\lambda_0\cdot\mathbf{n}\rangle_{\partial T}\Big| \\ \le& \Big(\sum_{T\in\mathcal{T}_h} \|u-\mathcal{Q}_h u\|^q_{L^q(\partial T)}\Big)^{\frac{1}{q}} \Big(\sum_{T\in\mathcal{T}_h} \|\lambda_n - \nabla\lambda_0\cdot\mathbf{n}\|^p_{L^p(\partial T)}\Big)^{\frac{1}{p}}. \end{aligned} \tag{6.4}$$

For the term $\|u - \mathcal{Q}_h u\|^q_{L^q(\partial T)}$, we have from the trace inequality (6.1) that

$$\begin{aligned} \sum_{T\in\mathcal{T}_h} \|u-\mathcal{Q}_h u\|^q_{L^q(\partial T)} \le& \sum_{T\in\mathcal{T}_h} h_T^{-1}\Big(\|u-\mathcal{Q}_h u\|^q_{L^q(T)} + h_T^q\|\nabla(u-\mathcal{Q}_h u)\|^q_{L^q(T)}\Big) \\ \le& Ch^{kq-1}\|\nabla_h^k u\|^q_{L^q(\Omega)}. \end{aligned} \tag{6.5}$$

Substituting (6.5) into (6.4) gives

$$\begin{aligned} & \Big|\sum_{T\in\mathcal{T}_h} \langle \mathcal{Q}_h u - u, \lambda_n - \nabla\lambda_0\cdot\mathbf{n}\rangle_{\partial T}\Big| \\ \le& Ch^k\|\nabla^k u\|_{L^q(\Omega)}\Big(\sum_{T\in\mathcal{T}_h} h_T^{1-p}\|\lambda_n - \nabla\lambda_0\cdot\mathbf{n}\|^p_{L^p(\partial T)}\Big)^{\frac{1}{p}} \\ \le& C_1\sum_{T\in\mathcal{T}_h} h_T^{1-p}\|\epsilon_n - \nabla\epsilon_0\cdot\mathbf{n}\|^p_{L^p(\partial T)} + C_2h^{qk}\|\nabla_h^k u\|^q_{L^q(\Omega)}. \end{aligned} \tag{6.6}$$

As to the second term on the right-hand side in (6.3), using the Cauchy–Schwarz inequality we have

$$\begin{aligned}&\Big|\langle\nabla\mathcal{Q}_hu\cdot\mathbf{n}-\nabla u\cdot\mathbf{n},\lambda_b-\lambda_0\rangle_{\partial T}\Big|\\ \le&\Big(\sum_{T\in\mathcal{T}_h}\|\nabla u\cdot\mathbf{n}-\nabla\mathcal{Q}_hu\cdot\mathbf{n}\|^q_{L^q(\partial T)}\Big)^{\frac{1}{q}}\Big(\sum_{T\in\mathcal{T}_h}\|\lambda_b-\lambda_0\|^p_{L^p(\partial T)}\Big)^{\frac{1}{p}}.\end{aligned}\tag{6.7}$$

For the term $\|\nabla u\cdot\mathbf{n}-\nabla\mathcal{Q}_hu\cdot\mathbf{n}\|^q_{L^q(\partial T)}$, we have from the trace inequality (6.1) that

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h}\|\nabla u\cdot\mathbf{n}-\nabla\mathcal{Q}_hu\cdot\mathbf{n}\|^q_{L^q(\partial T)}\\ \le&\sum_{T\in\mathcal{T}_h}h_T^{-1}\Big(\|\nabla u-\nabla\mathcal{Q}_hu\|^q_{L^q(T)}+h_T^q\|\nabla(\nabla u-\nabla\mathcal{Q}_hu)\|^q_{L^q(T)}\Big)\\ \le&Ch^{(k-1)q-1}\|\nabla_h^ku\|^q_{L^q(\Omega)}+C\delta_{k,1}h^{q-1}\|\nabla_h^2u\|^q_{L^q(\Omega)}.\end{aligned}\tag{6.8}$$

Substituting (6.8) into (6.7) gives

$$\begin{aligned}&\Big|\sum_{T\in\mathcal{T}_h}\langle\nabla\mathcal{Q}_hu\cdot\mathbf{n}-\nabla u\cdot\mathbf{n},\lambda_b-\lambda_0\rangle_{\partial T}\Big|\\ \le&C\left(h^k\|\nabla_h^ku\|_{L^q(\Omega)}+\delta_{k,1}h^2\|\nabla_h^2u\|_{L^q(\Omega)}\right)\Big(\sum_{T\in\mathcal{T}_h}h_T^{1-2p}\|\lambda_b-\lambda_0\|^p_{L^p(\partial T)}\Big)^{\frac{1}{p}}\\ \le&C_3\sum_{T\in\mathcal{T}_h}h_T^{1-2p}\|\lambda_b-\lambda_0\|^p_{L^p(\partial T)}+C_4\left(h^{qk}\|\nabla_h^ku\|^q_{L^q(\Omega)}+\delta_{k,1}h^{2q}\|\nabla_h^2u\|^q_{L^q(\Omega)}\right).\end{aligned}\tag{6.9}$$

The third term can be analogously estimated by

$$\begin{aligned}&\Big|\sum_{T\in\mathcal{T}_h}\langle(u-\mathcal{Q}_hu)\boldsymbol{\beta}\cdot\mathbf{n},\lambda_0-\lambda_b\rangle_{\partial T}\Big|\\ \le&C_5\sum_{T\in\mathcal{T}_h}h_T^{1-2p}\|\lambda_b-\lambda_0\|^p_{L^p(\partial T)}+C_6h^{qk}\|\nabla_h^ku\|^q_{L^q(\Omega)}.\end{aligned}\tag{6.10}$$

Substituting (6.6), (6.9), and (6.10) into (6.3) gives

$$(1-C_1-C_3-C_5)s(\lambda_h,\lambda_h)\le Ch^{qk}\|\nabla_h^ku\|^q_{L^q(\Omega)}+C\delta_{k,1}h^{2q}\|\nabla_h^2u\|^q_{L^q(\Omega)},$$

which leads to

$$s(\lambda_h,\lambda_h)\le Ch^{qk}\|\nabla_h^ku\|^q_{L^q(\Omega)}+C\delta_{k,1}h^{2q}\|\nabla_h^2u\|^q_{L^q(\Omega)}$$

by choosing C_i such that $1-C_1-C_3-C_5\ge C_0>0$. This completes the proof of the theorem. □

Consider the auxiliary problem that seeks ϕ such that

$$\begin{aligned}-\Delta\phi-\boldsymbol{\beta}\cdot\nabla\phi&=\psi,\quad\text{in }\Omega,\\ \phi&=0,\quad\text{on }\partial\Omega,\end{aligned}\tag{6.11}$$

where $\psi\in L^p(\Omega)$ is a given function. The problem (6.11) is said to have the $W^{2,p}$-regularity if there exists a constant C independent of ψ satisfying

$$\|\phi\|_{2,p}\le C\|\psi\|_{0,p}.\tag{6.12}$$

The following is the main error estimate for the approximation u_h.

Theorem 6.2. *Let u be the strong solution of the second order elliptic problem (1.1) and $(u_h;\lambda_h)\in M_h\times W_h^0$ be its numerical solution arising from the PDWG scheme (3.5)–(3.6). Assume that $\nabla_h^ku\in L^q(\Omega),f\in W^{k-2,q}(\Omega)$, and the $W^{2,p}$-regularity estimate (6.12) holds true for the auxiliary problem (6.11). Then the following L^q-error estimate holds true:*

$$\|u-u_h\|_{L^q(\Omega)}\le Ch^k\left(\|\nabla_h^ku\|_{L^q(\Omega)}+\delta_{k,1}h^{2-k}\|\nabla_h^2u\|_{L^q(\Omega)}+\|f\|_{k-2,q}+\|g\|_{k-\frac{1}{q},q,\partial\Omega}\right).$$

Proof. For the solution ϕ of (6.11), by choosing $\sigma=Q_h\phi\in W_h^0$ in (3.5) we obtain

$$s(\lambda_h,Q_h\phi)+(u_h,-\boldsymbol{\beta}\cdot\nabla_wQ_h\phi-\Delta_wQ_h\phi)=(f,Q_0\phi)-\langle g,Q_n(\nabla\phi\cdot\mathbf{n})\rangle_{\partial\Omega}.$$

From (4.1) and (4.2) we have

$$s(\lambda_h,Q_h\phi)+(u_h,-\mathcal{Q}_h(\boldsymbol{\beta}\cdot\nabla\phi+\Delta\phi))=(f,Q_0\phi)-\langle g,Q_n(\nabla\phi\cdot\mathbf{n})\rangle_{\partial\Omega},$$

or equivalently,

$$s(\lambda_h, Q_h\phi) + (u_h, \psi) = (f, Q_0\phi) - \langle g, Q_n(\nabla\phi \cdot \mathbf{n})\rangle_{\partial\Omega}.$$

On the other hand, there holds, from (6.11) and the integration by parts, that

$$\begin{aligned}(u, \psi) = (u, -\Delta\phi - \boldsymbol{\beta} \cdot \nabla\phi) &= -(\nabla \cdot (\nabla u - \boldsymbol{\beta} u), \phi) - \langle u, \nabla\phi \cdot \mathbf{n}\rangle_{\partial\Omega}\\ &= (f, \phi) - \langle g, \nabla\phi \cdot \mathbf{n}\rangle_{\partial\Omega}.\end{aligned}$$

It follows from the last two inequalities that

$$(u - u_h, \psi) = s(\lambda_h, Q_h\phi) + (f - Q_0 f, \phi) - \langle g - Q_n g, \nabla\phi \cdot \mathbf{n}\rangle_{\partial\Omega}. \tag{6.13}$$

We next estimate the three terms appeared in the right hand side of the above equation. Using the approximation property of Q_0 and the Cauchy–Schwarz inequality yields

$$|(f - Q_0 f, \phi)| = |(f - Q_0 f, \phi - Q_0\phi)| \le Ch^k \|f\|_{k-2,q} \|\phi\|_{2,p}.$$

Similarly, there holds

$$\begin{aligned}|\langle g - Q_n g, \nabla\phi \cdot \mathbf{n}\rangle_{\partial\Omega}| &\le C\|g - Q_n g\|_{L^q(\partial\Omega)} \|\nabla\phi - Q_n\nabla\phi\|_{L^p(\partial\Omega)}\\ &\le Ch^k \|g\|_{k-\frac{1}{q},q,\partial\Omega} \|\phi\|_{2,p}.\end{aligned}$$

Here in the last step, we have used (6.1) and the trace inequality.

To deal with the term $s(\lambda_h, Q_h\phi)$, from (3.3) we have

$$\begin{aligned}s(\lambda_h, Q_h\phi) &= \sum_{T\in\mathcal{T}_h} h_T^{1-2p} \int_{\partial T} |\lambda_0 - \lambda_b|^{p-1} sgn(\lambda_0 - \lambda_b)(Q_0\phi - Q_b\phi) ds\\ &+ \sum_{T\in\mathcal{T}_h} h_T^{1-p} \int_{\partial T} |\nabla\lambda_0 \cdot \mathbf{n} - \lambda_n|^{p-1} sgn(\nabla\lambda_0 \cdot \mathbf{n} - \lambda_n)(\nabla Q_0\phi \cdot \mathbf{n} - Q_n\phi) ds\\ &= I_1 + I_2.\end{aligned} \tag{6.14}$$

For the term I_1, we use the Cauchy–Schwarz inequality and the trace inequality (6.1) to obtain

$$\begin{aligned}|I_1| &\le \sum_{T\in\mathcal{T}_h} h_T^{1-2p} \Big(\int_{\partial T} |\lambda_0 - \lambda_b|^p ds\Big)^{\frac{1}{q}} \Big(\int_{\partial T} (Q_0\phi - Q_b\phi)^p ds\Big)^{\frac{1}{p}}\\ &\le C \sum_{T\in\mathcal{T}_h} h_T^{1-2p} h_T^{(2p-1)/q} \Big(\int_{\partial T} h_T^{1-2p} |\lambda_0 - \lambda_b|^p ds\Big)^{\frac{1}{q}}\\ &\cdot \Big(h_T^{-1} \|Q_0\phi - \phi\|_{L^p(T)}^p + h_T^{p-1} \|\nabla(Q_0\phi - \phi)\|_{L^p(T)}^p\Big)^{\frac{1}{p}}\\ &\le Cs(\lambda_h, \lambda_h)^{\frac{1}{q}} \sum_{T\in\mathcal{T}_h} h_T^{\frac{1-2p}{p}} \Big(h_T^{2p-1} \|\nabla^2\phi\|_{L^p(T)}^p\Big)^{\frac{1}{p}}\\ &\le Cs(\lambda_h, \lambda_h)^{\frac{1}{q}} \|\nabla^2\phi\|_{L^p(\Omega)}.\end{aligned} \tag{6.15}$$

Similarly, for I_2 we again use the Cauchy–Schwarz inequality and the trace inequality (6.1) to obtain

$$\begin{aligned}I_2 &\le \sum_{T\in\mathcal{T}_h} h_T^{1-p} \Big(\int_{\partial T} |\nabla\lambda_0 \cdot \mathbf{n} - \lambda_n|^p ds\Big)^{\frac{1}{q}} \Big(\int_{\partial T} (\nabla Q_0\phi \cdot \mathbf{n} - Q_n\phi)^p ds\Big)^{\frac{1}{p}}\\ &\le Cs(\lambda_h, \lambda_h)^{\frac{1}{q}} \|\nabla^2\phi\|_{L^p(\Omega)}.\end{aligned} \tag{6.16}$$

Substituting (6.15) and (6.16) into (6.14) and using (6.2) and (6.12) yields

$$\begin{aligned}\Big|s(\lambda_h, Q_h\phi)\Big| &\le Cs(\lambda_h, \lambda_h)^{\frac{1}{q}} \|\nabla^2\phi\|_{L^p(\Omega)}\\ &\le C\left(h^k \|\nabla_h^k u\|_{L^q(\Omega)} + \delta_{k,1} h^2 \|\nabla_h^2 u\|_{L^q(\Omega)}\right) \|\psi\|_{L^p(\Omega)}.\end{aligned}$$

Combining the three estimates for the terms on the right hand side of (6.13), we easily get

$$\Big|(u - u_h, \psi)\Big| \le Ch^k \left(\|\nabla_h^k u\|_{L^q(\Omega)} + \delta_{k,1} h^{2-k} \|\nabla_h^2 u\|_{L^q(\Omega)} + \|f\|_{k-2,q} + \|g\|_{k-\frac{1}{q},q,\partial\Omega}\right) \|\psi\|_{L^p(\Omega)}.$$

Consequently,

$$\|u - u_h\|_{L^q(\Omega)} \le Ch^k \left(\|\nabla_h^k u\|_{L^q(\Omega)} + \delta_{k,1} h^{2-k} \|\nabla_h^2 u\|_{L^q(\Omega)} + \|f\|_{k-2,q} + \|g\|_{k-\frac{1}{q},q,\partial\Omega}\right).$$

This completes the proof of the theorem. □

7. Error estimates for the dual variable

In this section we shall establish some error estimates for the dual variable λ_h in $W^{1,p}$ and L^p. To this end, let φ be the solution of the following auxiliary problem

$$\begin{aligned} -\triangle\varphi+\boldsymbol{\beta}\cdot\nabla\varphi&=\theta, \quad \text{in } \Omega,\\ \varphi&=0, \quad \text{on } \partial\Omega, \end{aligned} \tag{7.1}$$

where θ is a given function in $L^q(\Omega)$. Assume the dual problem (7.1) has the $W^{2,q}$-regularity in the sense that there exists a constant C such that

$$\|\varphi\|_{2,q}\le C\|\theta\|_{0,q}. \tag{7.2}$$

From (2.3) and the usual integration by parts we have

$$\begin{aligned} &(\triangle_w v,\varphi)=(\triangle_w v,\mathcal{Q}_h\varphi)\\ =&\sum_{T\in\mathcal{T}_h}(\triangle v_0,\mathcal{Q}_h\varphi)_T+\langle v_0-v_b,\nabla(\mathcal{Q}_h\varphi)\cdot\mathbf{n}\rangle_{\partial T}+\langle v_n-\nabla v_0\cdot\mathbf{n},\mathcal{Q}_h\varphi\rangle_{\partial T}\\ =&\sum_{T\in\mathcal{T}_h}(\triangle v_0,\varphi)_T+\langle v_0-v_b,\nabla(\mathcal{Q}_h\varphi)\cdot\mathbf{n}\rangle_{\partial T}+\langle v_n-\nabla v_0\cdot\mathbf{n},\mathcal{Q}_h\varphi\rangle_{\partial T}\\ =&\sum_{T\in\mathcal{T}_h}(v_0,\triangle\varphi)_T+\langle v_0-v_b,\nabla(\mathcal{Q}_h\varphi-\varphi)\cdot\mathbf{n}\rangle_{\partial T}+\langle v_n-\nabla v_0\cdot\mathbf{n},\mathcal{Q}_h\varphi-\varphi\rangle_{\partial T} \end{aligned}$$

so that

$$\begin{aligned} (v_0,\triangle\varphi)=(\triangle_w v,\varphi)&+\sum_{T\in\mathcal{T}_h}\langle\varphi-\mathcal{Q}_h\varphi,v_n-\nabla v_0\cdot\mathbf{n}\rangle_{\partial T}\\ &+\sum_{T\in\mathcal{T}_h}\langle v_0-v_b,\nabla(\varphi-\mathcal{Q}_h\varphi)\cdot\mathbf{n}\rangle_{\partial T}. \end{aligned} \tag{7.3}$$

Analogously, from (2.5) we have (note that $s=k-1$ so that $\nabla v_0\in[P_s(T)]^d$)

$$\begin{aligned} (v_0,\boldsymbol{\beta}\cdot\nabla\varphi)_T&=-(\nabla v_0,\boldsymbol{\beta}\varphi)_T+\langle v_0,\varphi\boldsymbol{\beta}\cdot\mathbf{n}\rangle_{\partial T}\\ &=-(\nabla v_0,\boldsymbol{\beta}\mathcal{Q}_h\varphi)_T+\langle v_0,\varphi\boldsymbol{\beta}\cdot\mathbf{n}\rangle_{\partial T}\\ &=-(\boldsymbol{\beta}\cdot\nabla_w v,\varphi)_T+\langle v_b-v_0,\mathcal{Q}_h\varphi\boldsymbol{\beta}\cdot\mathbf{n}\rangle_{\partial T}+\langle v_0,\varphi\boldsymbol{\beta}\cdot\mathbf{n}\rangle_{\partial T}. \end{aligned} \tag{7.4}$$

Summing (7.4) over all $T\in\mathcal{T}_h$ yields

$$(v_0,\boldsymbol{\beta}\cdot\nabla\varphi)=-(\boldsymbol{\beta}\cdot\nabla_w v,\varphi)+\sum_{T\in\mathcal{T}_h}\langle v_0-v_b,(I-\mathcal{Q}_h)\varphi\boldsymbol{\beta}\cdot\mathbf{n}\rangle_{\partial T}. \tag{7.5}$$

We have the following error estimates for the variable λ_0.

Theorem 7.1. *Let u and $(u_h;\lambda_h)\in M_h\times W_h^0$ be the solutions of (1.1) and (3.5)–(3.6), respectively. Assume that the dual problem (7.1) has the $W^{2,q}(\Omega)$ regularity with the a priori estimate (7.2). Then the following estimate holds true:*

$$\|\lambda_0\|_{0,p}\le\begin{cases} Ch^q\|\nabla_h u\|_{0,q}^{q/p}+Ch^{2q-1}\|\nabla_h^2 u\|_{0,q}^{q/p}, & k=1,\\ Ch^{(q-1)k+2}\|\nabla_h^k u\|_{0,q}^{q/p}, & k>1. \end{cases} \tag{7.6}$$

Proof. For any given function $\theta\in L^q(\Omega)$, let φ be the solution of (7.1). From (7.3) and (7.5) we have

$$\begin{aligned} (\theta,\lambda_0)&=(\lambda_0,-\triangle\varphi+\boldsymbol{\beta}\cdot\nabla\varphi)\\ &=(-\triangle_w\lambda_h-\boldsymbol{\beta}\nabla_w\lambda_h,\varphi)\\ &\quad+\sum_{T\in\mathcal{T}_h}\langle\varphi-\mathcal{Q}_h\varphi,\nabla\lambda_0\cdot\mathbf{n}-\lambda_n\rangle_{\partial T}+\sum_{T\in\mathcal{T}_h}\langle\lambda_b-\lambda_0,\nabla(\varphi-\mathcal{Q}_h\varphi)\cdot\mathbf{n}\rangle_{\partial T}\\ &\quad+\sum_{T\in\mathcal{T}_h}\langle\lambda_0-\lambda_b,(I-\mathcal{Q}_h)\varphi\boldsymbol{\beta}\cdot\mathbf{n}\rangle_{\partial T}\\ &=I_1+I_2+I_3+I_4. \end{aligned}$$

We next estimate $I_i,\ 1\le i\le 4$, respectively. In light of (3.6), we have

$$\begin{aligned} |I_1|&=|(\triangle_w\lambda_h+\boldsymbol{\beta}\cdot\nabla_w\lambda_h,\varphi)|\\ &=|(\triangle_w\lambda_h+\boldsymbol{\beta}\cdot\nabla_w\lambda_h,\varphi-\mathcal{Q}_h\varphi)|\\ &=0. \end{aligned}$$

As to I_2, we have from the Cauchy–Schwarz inequality and the trace inequality (6.1),

$$\begin{aligned}
|I_2| &= \left|\sum_{T\in\mathcal{T}_h}\langle\varphi-\mathcal{Q}_h\varphi,\nabla\lambda_0\cdot\mathbf{n}-\lambda_n\rangle_{\partial T}\right|\\
&\le \left(\sum_{T\in\mathcal{T}_h}h_T^{1-p}\|\nabla\lambda_0\cdot\mathbf{n}-\lambda_n\|_{p,\partial T}^p\right)^{\frac{1}{p}}\left(\sum_{T\in\mathcal{T}_h}h_T\|\varphi-\mathcal{Q}_h\varphi\|_{q,\partial T}^q\right)^{\frac{1}{q}}\\
&\lesssim s(\lambda_h,\lambda_h)^{\frac{1}{p}}(\|\varphi-\mathcal{Q}_h\varphi\|_{0,q}+h\|\nabla(\varphi-\mathcal{Q}_h\varphi)\|_{0,q})\\
&\lesssim h^{m+1}s(\lambda_h,\lambda_h)^{\frac{1}{p}}\|\varphi\|_{m+1,q}.
\end{aligned}$$

Following the same argument, there holds

$$|I_3|\lesssim h^{m+1}s(\lambda_h,\lambda_h)^{\frac{1}{p}}\|\varphi\|_{m+1,q},\quad |I_4|\lesssim h^{m+2}s(\lambda_h,\lambda_h)^{\frac{1}{p}}\|\varphi\|_{m+1,q}.$$

Here $m=1$ for $k>1$ and $m=0$ for $k=1$. It is so because there holds $P_1(T)\subset M_h(T)$ for $k>1$ and $M_h(T)$ consists of only piecewise constants for $k=1$.

By combining all the estimates for I_i, $i\le 4$ and the estimate (6.2) in Theorem 6.1 we arrive at

$$|(\lambda_0,\theta)|\lesssim h^{m+1}\left(h^{k(q-1)}\|\nabla_h^k u\|_{0,q}^{q/p}+\delta_{k,1}h^{2(q-1)}\|\nabla_h^2 u\|_{0,q}^{q/p}\right)\|\varphi\|_{m+1,q}. \tag{7.7}$$

The estimate (7.6) then follows from the $W^{2,q}(\Omega)$-regularity (7.2) with $m=0$ and $m=1$, respectively. This completes the proof. □

Theorem 7.2. *Let u and $(u_h;\lambda_h)\in M_h\times W_h^0$ be the solutions of (1.1) and (3.5)–(3.6), respectively. Assume that the dual problem (7.1) has the $W^{2,q}(\Omega)$-regularity with the a priori estimate (7.2). Then we have*

$$\|\nabla\lambda_0\|_{0,p}\le Ch^{k(q-1)+1}\|\nabla_h^k u\|_{0,q}^{q/p}+C\delta_{k,1}h^{2q-1}\|\nabla_h^2 u\|_{0,q}^{q/p}. \tag{7.8}$$

Proof. For simplicity, consider the case of $k>1$. For any given function $\eta\in[C^1(\Omega)]^d$ with $\eta=0$ on $\mathcal{E}_h$, let φ be the solution of the dual problem (7.1) with $\theta=-\nabla\cdot\eta$. It is easy to see that

$$(\nabla\lambda_0,\eta)=-(\lambda_0,\nabla\cdot\eta)=(\lambda_0,\theta).$$

In light of (7.7), we get

$$|(\nabla\lambda_0,\eta)|\lesssim h^{k(q-1)+1}\|\nabla_h^k u\|_{0,q}^{q/p}\|\varphi\|_{1,q}\lesssim h^{k(q-1)+1}\|\nabla_h^k u\|_{0,q}^{q/p}\|\eta\|_{0,q}$$

with φ being the solution of (7.1). As the set of all such η is dense in $L^q(\Omega)$, we then obtain

$$\|\nabla\lambda_0\|_{0,p}\lesssim h^{k(q-1)+1}\|\nabla_h^k u\|_{0,q}^{q/p}.$$

This completes the proof. □

8. Numerical results

Our numerical experiments are based on the PDWG algorithm (3.5)–(3.6) with $k=1,2$ for the finite element spaces W_h and M_h defined in (3.1)–(3.2). The system of nonlinear Eqs. (3.5)–(3.6) is solved by using an iterative scheme similar to that for the L^1 minimization problem in [17]. Specifically, given an approximation (u_h^m,λ_h^m) at step m, the scheme shall compute a new approximate solution $(u_h^{m+1},\lambda_h^{m+1})\in M_h\times W_h^0$ such that

$$\begin{aligned}
\tilde{s}(\lambda_h^{m+1},\sigma)+b(u_h^{m+1},\sigma)&=(f,\sigma_0)-\langle g,\sigma_n\rangle_{\partial\Omega}, &&\forall\sigma\in W_h^0,\\
b(v,\lambda_h^{m+1})&=0, &&\forall v\in M_h,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{s}(\lambda_h^{m+1},\sigma)=&\sum_{T\in\mathcal{T}_h}h_T^{1-2p}\int_{\partial T}(|\lambda_0^m-\lambda_b^m|+\epsilon)^{p-2}(\lambda_0^{m+1}-\lambda_b^{m+1})(\sigma_0-\sigma_b)ds\\
&+\sum_{T\in\mathcal{T}_h}h_T^{1-p}\int_{\partial T}(|\nabla\lambda_0^m\cdot\mathbf{n}-\lambda_n^m|+\epsilon)^{p-2}(\nabla\lambda_0^{m+1}\cdot\mathbf{n}-\lambda_n^{m+1})(\nabla\sigma_0\cdot\mathbf{n}-\sigma_n)ds.
\end{aligned}$$

Here ϵ is a small, but positive constant. All the numerical results are obtained with $\epsilon=10^{-3}$ if not otherwise stated. Various approximation errors are computed for u_h and λ_h, including the L^q error for $e_h:=u-u_h$, and the $W^{2,p}$, $W^{1,p}$, and

Table 8.1
Numerical error and rate of convergence for the L^p-PDWG method with $k = 2, s = k - 1$ for Example 1.

	h	$\|\lambda_0\|_{2,p}$	rate	$\|\lambda_0\|_{1,p}$	rate	$\|\lambda_0\|_{0,p}$	rate	$\|e_h\|_{0,q}$	rate
	3.54e−01	2.05e−03	–	2.58e−04	–	1.50e−04	–	1.48e−02	–
	1.77e−01	9.95e−04	1.04	4.89e−05	2.40	3.60e−05	2.06	3.92e−03	1.92
$p = 1$	8.84e−02	4.92e−04	1.02	1.05e−05	2.22	8.85e−06	2.02	1.00e−03	1.97
	4.42e−02	2.45e−04	1.06	2.42e−06	2.11	2.19e−06	2.01	2.53e−04	1.98
	2.21e−02	1.22e−04	1.00	5.81e−07	2.06	5.46e−07	2.01	6.37e−05	1.99
	3.54e−01	1.05e−02	–	1.82e−03	–	4.75e−04	–	4.01e−03	–
	1.77e−01	2.54e−03	2.05	1.79e−04	3.35	2.95e−05	4.01	9.96e−04	2.01
$p = 2$	8.84e−02	6.26e−04	2.02	1.96e−05	3.18	1.84e−06	4.00	2.48e−04	2.00
	4.42e−02	1.55e−04	2.01	2.30e−06	3.09	1.15e−07	4.00	6.20e−05	2.00
	2.21e−02	3.87e−05	2.00	2.79e−07	3.05	7.20e−09	4.00	1.55e−05	2.00
	3.54e−01	7.36e−02	–	1.12e−02	–	1.35e−03	–	3.22e−03	–
	1.77e−01	9.05e−03	3.02	5.52e−04	4.34	2.65e−05	5.67	7.60e−04	2.08
$p = 3$	8.84e−02	1.03e−03	3.14	2.76e−05	4.32	5.77e−07	5.52	1.77e−04	2.11
	4.42e−02	1.12e−04	3.19	1.39e−06	4.31	1.42e−08	5.35	4.04e−05	2.13
	2.21e−02	1.21e−05	3.22	7.46e−08	4.22	3.85e−10	5.12	9.35e−06	2.11
	3.54e−01	4.27e−01	–	6.56e−02	–	3.00e−03	–	2.96e−03	–
	1.77e−01	2.73e−02	3.97	1.50e−03	5.45	4.83e−05	5.96	6.05e−04	2.29
$p = 4$	8.84e−02	1.20e−03	4.51	3.65e−05	5.36	7.17e−07	6.07	1.37e−04	2.14
	4.42e−02	6.39e−05	4.23	1.23e−06	4.90	1.17e−08	5.94	3.36e−05	2.03
	2.21e−02	3.83e−06	4.06	4.04e−08	4.93	1.86e−10	5.97	8.37e−06	2.00
	3.54e−01	2.73e−02	–	4.79e−02	–	5.48e−03	–	3.91e−03	–
$p = 5$	1.77e−01	6.50e−02	−1.25	4.14e−03	3.53	1.51e−04	5.19	5.41e−04	2.85
	8.84–02	1.84e−03.	5.14	8.16e−05	5.66	1.56e−06	6.60	1.31e−04	2.04
	4.42e−02	5.75e−05	5.00	1.32e−06	5.94	1.25e−08	6.97	3.28e−05	2.00

Table 8.2
Numerical error and rate of convergence for the L^p-PDWG method with $k = 2, s = k - 2$ for Example 1.

	h	$\|\lambda_0\|_{2,p}$	rate	$\|\lambda_0\|_{1,p}$	rate	$\|\lambda_0\|_{0,p}$	rate	$\|e_h\|_{0,q}$	rate
	3.54e−01.	4.27e−02	–	1.23e−01	–	3.23e−03	–	1.02e−01	–
	1.77e−01	4.16e−02	0.04	1.33e−01	−0.11	2.08e−03	0.64	5.46e−02	0.90
$p = 1$	8.84e−02	4.02e−02	0.05	1.40e−01	−0.06	1.60e−03	0.38	2.82e−02	0.95
	4.42e−02	3.90e−02	0.04	1.43e−01	−0.04	1.44e−03	0.14	1.45e−02	0.96
	2.21e−02	3.85e−02	0.02	1.45e−01	−0.02	1.40e−03	0.04	7.36e−03	0.97
	3.54e−01	8.54e−02	–	2.75e−01	–	1.32e−02	–	1.04e−01	–
	1.77e−01	3.70e−02	1.21	7.81e−02	1.82	2.37e−03	2.47	5.44e−02	0.93
$p = 2$	8.84e−02	1.60e−02	1.21	2.08e−02	1.91	4.95e−04	2.26	2.78e−02	0.97
	4.42e−02	7.12e−03	1.17	5.37e−03	1.95	1.16e−04	2.09	1.41e−02	0.98
	2.21e−02	3.28e−03	1.12	1.36e−03	1.98	2.88e−05	2.01	7.07e−03	0.99
	3.54e−01	3.12e−01	–	5.23e−01	–	3.21e−02	–	1.00e−01	–
	1.77e−01	1.38e−01	1.18	6.50e−02	3.01	2.50e−03	3.68	5.31e−02	0.92
$p = 3$	8.84e−02	2.89e−02	2.26	4.56e−03	3.83	1.36e−04	4.20	2.71e−02	0.97
	4.42e−02	5.42e−03	2.42	3.225e−04	3.82	1.70e−05	3.01	1.35e−02	1.01
	2.21e−02	1.01e−03	2.43	2.47e−05	3.71	2.29e−06	2.89	6.66e−03	1.02

(*continued on next page*)

L^p errors for λ_h. The finite element partition $\mathcal{T}_h$ is obtained through a successive refinement of a coarse triangulation of the domain, by dividing each coarse element into four congruent sub-elements by connecting the midpoints of the three edges of the triangle. The right-hand side function, the boundary condition is calculated from the exact solution.

Example 1. The domain in the model problem (1.1) is given by $\Omega = (0, 1)^2$. Vanishing convection $\boldsymbol{\beta} = 0$ is considered in this test problem. The functions f and g are chosen so that the exact solution is given by $u = \sin(x)\sin(y)$.

Tables 8.1–8.3 illustrate the approximation error and the rate of convergence for the primal variable u_h and the dual variable λ_0 with $k = 1, 2$ and $p = 1, \ldots, 5$. For $p > 1$, one observes a convergence rate of $\mathcal{O}(h^{s+1})$ for the error $\|e_h\|_{0,q}$ with $s = k{-}1, k{-}2$, which is consistent with the theory shown in Theorem 6.2. For $p = 1$ and $k = 2$, one observes a convergence rate of $\mathcal{O}(h^{s+1})$ for the error $\|e_h\|_{0,\infty}$ - an optimal order of convergence in the maximum norm. For the dual variable approximation λ_0, the tables suggest the following rates of convergence: $\mathcal{O}(h^p)$ for $\|\lambda_0\|_{2,p}$, $\mathcal{O}(h^{p+1})$ for $\|\lambda_0\|_{1,p}$, and $\mathcal{O}(h^{p+2})$ for $\|\lambda_0\|_{0,p}$ with $k = 2, s = k - 1$. For the case of $s = 0$ with $k = 1, 2$, one observes a convergence of $\mathcal{O}(h^{p-1})$ for $\|\lambda_0\|_{2,p}$ and $\mathcal{O}(h^p)$ for $\|\lambda_0\|_{m,p}$, $m = 0, 1$. In other words, this numerical experiment suggests $\|\lambda_0\|_{m,p} = \mathcal{O}(h^{p+2-m})$, $m \leq 2$, when the approximating space $W_h(T)$ contains linear functions on each element. The convergence is reduced to $\mathcal{O}(h^{p+1-\max(1,m)})$ if only piecewise constant functions are seen in $W_h(T)$. We emphasize that the numerical dual variable λ_0 outperforms

Table 8.2 (*continued*).

	h	$\|\lambda_0\|_{2,p}$	rate	$\|\lambda_0\|_{1,p}$	rate	$\|\lambda_0\|_{0,p}$	rate	$\|e_h\|_{0,q}$	rate
	3.54e−01	2.36e−01	–	1.49e−01	–	2.33e−02	–	1.12e−01	–
	1.77e−01	3.08e−01	−3.84	6.84e−02	1.12	3.75e−03	2.63	5.02e−02	1.16
$p=4$	8.84e−02	1.80e−02	4.10	2.06e−03	5.05	2.52e−04	3.90	2.48e−02	1.02
	4.412e−02	7.86e−04	4.51	7.88e−05	4.71	1.47e−05	4.10	1.28e−02	0.96
	2.21e−02	5.87e−05	3.74	4.13e−06	4.25	9.02e−07	4.03	6.46e−03	0.98
	3.54e−01	5.21e−03	–	1.23e−02	–	3.61e−03	–	4.38e−01	–
	1.77e−01	2.57e−01	−5.62	5.13e−02	−2.06	2.80e−03	0.37	5.10e−02	3.10
$p=5$	8.84e−02	1.32e−02	4.28	2.16e−03	4.57	4.07e−04	2.78	2.38e−02	1.10
	4.42e−02	6.97e−04	4.24	6.62e−05	5.03	1.41e−05	4.85	1.23e−02	0.95
	2.21e−02	3.87e−05	4.17	2.02e−06	5.03	4.41e−07	5.00	6.24e−03	0.98

Table 8.3
Numerical error and rate of convergence for the L^p-PDWG method with $k=1, s=k-1$ for Example 1.

	h	$\|\lambda_0\|_{1,p}$	rate	$\|\lambda_0\|_{0,p}$	rate	$\|e_h\|_{0,q}$	rate
	3.54e−01	1.19e−01	–	3.11e−03	–	1.12e−01	–
	1.77e−01	1.31e−01	0.14	1.77e−03	0.81	6.08e−02	0.89
$p=1$	8.84e−02	1.39e−01	−0.08	1.03e−03	0.78	3.60e−02	0.76
	4.42e−02	1.43e−01	−0.04	8.77e−04	0.23	2.22e−02	0.70
	2.21e−02	1.45e−01	−0.02	1.00e−03	−0.20	1.34e−02	0.72
	3.54e−01	2.50e−01	–	1.14e−02	–	3.45e−02	
	1.77e−01	7.44e−02	1.75	2.07e−03	2.46	1.75e−02	0.98
$p=2$	8.84e−02	2.04e−02	1.87	4.83e−04	2.10	8.46e−03	1.05
	4.42e−02	5.32e−03	1.94	1.22e−04	1.98	4.08e−03	1.05
	2.21e−02	1.36e−03	1.97	3.10e−05	1.98	2.01e−03	1.03
	3.54e−01	4.68e−01	–	2.89e−02	–	3.19e−02	–
	1.77e−01	5.87e−02	2.99	2.57e−03	3.49	1.54e−02	1.06
$p=3$	8.84e−02	4.18e−03	3.81	1.14e−04	4.50	7.48e−03	1.04
	4.42e−02	2.81e−04	3.90	8.05e−06	3.82	3.73e−03	1.00
	2.21e−02	1.96e−05	3.84	1.27e−06	2.66	1.86e−03	1.00
	3.54-01	1.41e−01	–	1.67e−02	–	3.38e−02	–
	1.77e−01	5.22e−02	1.44	2.96e−03	2.50	1.56e−02	1.12
$p=4$	8.84e−02	1.54e−03	5.08	1.90e−04	3.96	7.52e−03	1.05
	4.42e−02	6.88e−05	4.48	1.37e−05	3.79	3.75e−03	1.00
	2.21e−02	3.99e−06	4.11	8.87e−07	3.95	1.88e−03	1.00
	3.54e−01	2.62e−02	–	4.16e−03	–	6.20e−02	–
	1.77e−01	4.03e−02	−0.62	3.01e−03	0.47	1.70e−02	1.87
$p=5$	8.84e−02	1.88e−03	4.42	3.91e−04	2.94	7.76e−03	1.13
	4.42e−02	6.35e−05	4.89	1.40e−05	4.80	3.86e−03	1.01
	2.21e−02	1.98e−06	5.00	4.40e−07	4.99	1.93e−03	1.00

the theory predicted in Theorems 7.1–7.2 with rates depending on p. This p-dependence of the convergence remains mysterious to the authors.

Example 2. The domain in this test case is given by $\Omega = (0,1)^2$. The convection term has the following form $\boldsymbol{\beta} = (-y, x)$. The functions f and g are chosen such that the exact solution to the elliptic problem is given by $u = \frac{1}{2}\sin(x+y) + \cos(x-y) + \frac{3}{2}$.

Tables 8.4–8.5 show the approximation error and rates of convergence for the primal variable u_h and the dual variable λ_0 with $k = 1, 2$ and $p = 1, \ldots, 5$. As the model problem contains a non-trivial convection term, the space W_h is taken as (3.2) with $s = k-1$. From Tables 8.4–8.5 we observe the same convergence phenomenon as that for the purely diffusive equation in Example 1. More precisely, it is observed that the error $\|e_h\|_{0,q}$ converges to zero at the rate of $\mathcal{O}(h^{s+1})$ for $p > 1$, which is consistent with the theory developed in Theorem 6.2. For λ_0, it appears that the convergence is dependent upon the value of p: $\mathcal{O}(h^{p+2-m})$ for $k = 2$ and $\mathcal{O}(h^{p+2-\max(1,m)})$ for $k = 1$ in the metric $\|\lambda_0\|_{m,p}$, $m \le 2$. In other words, the construction of the finite element space M_h has effect on the convergence of the dual variable λ_0. Like in Example 1, the numerical convergence for the dual variable λ_0 is faster than the theory predicted in Theorems 7.1–7.2 with non-trivial convection terms in the model equation.

Tables 8.4–8.5 also show the approximation error and convergence rates for the L^1-PDWG method, i.e., $p = 1$. For the case of $k = 2$ and $s = 1$, we see an optimal order of convergence in the maximum norm for the primal variable. Like Example 1, the convergence for the dual variable λ_0 varies with respect to the choice of M_h. For $s = 1$, Table 8.4 shows a convergence rate of $\mathcal{O}(h^{p+2-\max(1,m)})$ for the error $\|\lambda_0\|_{m,p}$ with $m \le 2$. For $s = 0$, no convergence is seen from Table 8.5 for $\|\lambda_0\|_{m,p}$, $m \le 1$.

Table 8.4
Numerical error and rate of convergence for the L^p-PDWG method with $k = 2, s = k - 1$ for Example 2.

	h	$\|\lambda_0\|_{2,p}$	rate	$\|\lambda_0\|_{1,p}$	rate	$\|\lambda_0\|_{0,p}$	rate	$\|e_h\|_{0,q}$	rate
$p = 1$	3.54e−01	4.87e−03	–	8.18e−04	–	8.78e−04	–	2.47e−02	–
	1.77e−01	2.45e−03	0.99	2.18e−04	1.91	2.17e−04	2.01	6.16e−03	2.00
	8.84e−02	1.23e−03	1.00	5.58e−05	1.96	5.40e−05	2.01	1.58e−03	1.96
	4.42e−02	6.14e−04	1.00	1.41e−05	1.98	1.35e−05	2.00	3.98e−04	1.99
	2.21e−02	3.07e−04	1.00	3.54e−06	1.99	3.36e−06	2.00	1.00e−04	1.99
$p = 2$	3.54e−01	2.20e−02	–	2.26e−03	–	2.17e−03	–	5.73e−03	–
	1.77e−01	5.46e−03	2.01	2.37e−04	3.26	1.36e−04	4.00	1.41e−03	2.02
	8.84e−02	1.36e−03	2.00	2.74e−05	3.11	8.49e−06	4.00	3.52e−04	2.01
	4.42e−02	3.41e−04	2.00	3.33e−06	3.04	5.30e−07	4.00	8.80e−05	2.00
	2.21e−02	8.51e−05	2.00	4.11e−07	3.02	3.31e−08	4.00	2.20e−05	2.00
$p = 3$	3.54e−01	1.33e−01	–	1.40e−02	–	5.21e−03	–	4.70e−03	–
	1.77e−01	1.80e−02	2.89	8.77e−04	3.99	9.80e−05	5.73	1.16e−03	2.02
	8.84e−02	2.17e−03	3.05	5.22e−05	4.07	1.97e−06	5.64	2.87e−04	2.01
	4.42e−02	2.59e−04	3.07	3.14e−06	4.06	4.41e−08	5.48	7.14e−05	2.01
	2.21e−02	3.09e−05	3.07	1.92e−07	4.04	1.11e−09	5.32	1.79e−05	1.99
$p = 4$	3.54e−01	3.46e−01	–	4.97e−02	–	9.70e−03	–	4.43e−03	–
	1.77e−01	5.86e−02	2.56	3.23e−03	3.94	1.45e−04	6.06	1.11e−03	2.00
	8.84e−02	3.37e−03	4.12	9.82e−05	5.04	1.91e−06	6.25	2.83e−04	1.97
	4.42e−02	2.03e−04	4.06	3.06e−06	5.00	2.85e−08	6.07	7.15e−05	1.99
	2.21e−02	1.25e−05	4.01	9.61e−08	4.99	4.41e−10	6.01	1.79e−05	2.00
$p = 5$	3.54e−01	5.46e−01	–	9.69e−02	–	1.38e−02	–	4.83e−03	–
	1.77e−01	1.68e−01	1.70	1.03e−02	3.23	3.67e−04	5.24	1.13e−03	2.10
	8.84e−02	6.03e−03	4.80	1.97e−04	5.71	3.70e−06	6.63	2.84e−04	1.99
	4.42e−02	1.89e−04	5.00	3.10e−06	5.99	2.91e−08	6.99	7.08e−05	2.00

Table 8.5
Numerical error and rate of convergence for the L^p-PDWG method with $k = 1, s = k - 1$ for Example 2.

	h	$\|\lambda_0\|_{1,p}$	rate	$\|\lambda_0\|_{0,p}$	rate	$\|e_h\|_{0,q}$	rate
$p = 1$	3.54e−01	7.72e−02	–	1.04e−02	–	2.10e−01	–
	1.77e−01	8.44e−02	−0.13	9.49e−03	0.14	1.22e−01	0.78
	8.84e−02	8.90e−02	−0.08	9.27e−03	0.03	7.14e−02	0.78
	4.42e−02	9.16e−02	−0.04	9.31e−03	−0.00	4.49e−02	0.67
	2.21e−02	9.29e−02	−0.02	9.42e−03	−0.01	2.80e−02	0.68
$p = 2$	3.54e−01	1.60e−01	–	3.09e−02	–	4.96e−02	
	1.77e−01	4.76e−02	1.75	7.83e−03	1.98	2.47e−02	1.00
	8.84e−02	1.31e−02	1.86	2.02e−03	1.96	1.19e−02	1.06
	4.42e−02	3.47e−03	1.92	5.14e−04	1.97	5.64e−03	1.07
	2.21e−02	8.90e−04	1.96	1.30e−04	1.99	2.74e−03	1.04
$p = 3$	3.54e−01	3.16e−01	–	7.09e−02	–	4.43e−02	
	1.77e−01	4.17e−02	2.92	8.68e−03	3.03	2.07e−02	1.09
	8.84e−02	3.45e−03	3.60	7.93e−04	3.45	9.88e−03	1.07
	4.42e−02	3.04e−04	3.51	7.64e−05	3.38	4.86e−03	1.02
	2.21e−02	3.12e−05	3.28	8.05e−06	3.25	2.42e−03	1.01
$p = 4$	3.54e−01	9.35e−02	–	4.03e−02	–	5.01e−02	–
	1.77e−01	4.59e−02	1.03	1.02e−02	1.99	2.10e−02	1.25
	8.84e−02	3.84e−03	3.58	9.62e−04	3.40	9.70e−03	1.12
	4.42e−02	2.39e−04	4.01	5.76e−05	4.06	4.73e−03	1.03
	2.21e−02	1.49e−05	4.00	3.55e−06	4.02	2.35e−03	1.01
$p = 5$	3.54e−01	6.61e−03	–	8.87e−03	–	2.51e−01	–
	1.77e−01	2.82e−02	−2.09	4.51e−03	0.98	2.14e−02	3.55
	8.84e−02	7.21e−03	1.97	1.76e−03	1.36	9.70e−03	1.14
	4.42e−02	2.51e−04	4.85	5.98e−05	4.88	4.70e−03	1.04
	2.21e−02	7.85e−06	5.00	1.87e−06	5.00	2.34e−03	1.01

Example 3. Let $\Omega = (0, 1)^2$ and $\Omega_1 = (0.25, 0.75)^2$, $\Omega_2 = \Omega \backslash \bar{\Omega}_1$. Consider the following problem: Find an unknown function u satisfying

$$\begin{aligned} -\nabla \cdot (\alpha \nabla u) + \boldsymbol{\beta} u &= f, \quad \text{in } \Omega_1 \cup \Omega_2, \\ u &= g, \quad \text{on } \partial\Omega \\ [\![u]\!]_\Gamma = 0, \quad [\![(\alpha \nabla u - \boldsymbol{\beta} u) \cdot \mathbf{n}]\!]_\Gamma &= \psi, \quad \text{on } \Gamma = \partial\Omega_1 \cap \partial\Omega_2, \end{aligned} \tag{8.1}$$

where $\mathbf{n}$ is the unit outward normal of Γ with respect to Ω_1, and $[\![u]\!]_\Gamma$ denotes the jump of u across the interface Γ. We take $f = 0$, $\boldsymbol{\beta} = (0, 0)$ and a piecewise constant function α defined as $\alpha|_{\Omega_1} = 5$, $\alpha|_{\Omega_2} = 1$. The functions g and ψ are chosen so that the exact solution to this problem is given by $u = e^x \cos(y) + 10$.

Table 8.6
Numerical error and rate of convergence for the L^p-PDWG method with $k=2, s=k-1$ for Example 3.

	h	$\|\lambda_0\|_{2,p}$	rate	$\|\lambda_0\|_{1,p}$	rate	$\|\lambda_0\|_{0,p}$	rate	$\|e_h\|_{0,q}$	rate
	3.54e−01	1.19e−02	–	9.22e−04	–	8.57e−05	–	7.176e−02	–
	1.77e−01	4.90e−03	1.29	1.66e−04	2.47	1.11e−05	2.96	1.85e−02	1.96
$p=1$	8.84e−02	2.25e−03	1.12	3.34e−05	2.31	1.40e−06	2.99	4.58e−03	2.01
	4.42e−02	1.09e−03	1.04	7.57e−06	2.14	1.75e−07	2.99	1.18e−03	1.95
	2.21e−02	5.41e−04	1.01	1.82e−06	2.06	2.20e−08	3.00	3.16e−04	1.91
	3.54e−01	5.02e−02	–	7.14e−03	–	7.86e−04	–	1.21e−02	–
	1.77e−01	9.29e−03	2.43	6.07e−04	3.55	4.77e−05	4.04	3.06e−03	1.98
$p=2$	8.84e−02	1.97e−03	2.24	5.62e−05	3.43	2.91e−06	4.03	7.68e−04	2.00
	4.42e−02	4.57e−04	2.11	5.90e−06	3.25	1.79e−07	4.02	1.92e−04	2.00
	2.21e−02	1.11e−04	2.04	6.79e−07	3.12	1.11e−08	4.01	4.80e−05	2.00
	3.54e−01	6.98e−02	–	1.30e−02	–	1.37e−03	–	7.30e−03	
	1.77e−01	4.48e−03	3.96	4.41e−04	4.88	2.85e−05	5.59	1.66e−03	2.13
$p=3$	8.84e−02	3.59e−04	3.64	1.78e−05	4.63	5.95e−07	5.58	3.70e−04	2.17
	4.42e−02	3.91e−05	3.20	8.42e−07	4.40	1.29e−08	5.53	7.99e−05	2.21
	2.21e−02	4.94e−06	2.98	4.63e−08	4.19	2.98e−10	5.43	1.75e−05	2.19
	3.54e−01	9.83e−03	–	1.97e−03	–	1.94e−04	–	5.61e−03	–
	1.77e−01	3.93e−04	4.64	3.17e−05	5.95	1.54e−06	6.98	1.11e−03	2.33
$p=4$	8.84e−02	2.34e−05	4.07	8.58e−07	5.21	1.69e−08	6.50	2.71e−04	2.04
	4.42e−02	1.43e−06	4.03	2.81e−08	4.93	2.41e−10	6.14	6.76e−05	2.00
	2.21e−02	8.86e−08	4.02	8.98e−10	4.97	3.67e−12	6.03	1.67e−05	2.02
	3.54e−01	6.28e−03	–	1.17e−03	–	8.38e−05	–	2.16e−02	–
$p=5$	1.77e−01	1.70e−04	5.21	1.25e−05	6.54	4.63e−07	7.50	4.88e−03	2.15
	8.84e−02	4.92e−06	5.11	2.05e−07	5.94	3.49e−09	7.05	1.25e−03	1.96
	4.42e−02	1.49e−07	5.04	3.23e−09	5.98	2.71e−11	7.01	3.19e−04	1.98

Based on the variational formulation, we numerically solve the above problem by slightly modifying our algorithm (3.5)–(3.6) as follows: Find $(u_h; \lambda_h) \in M_h \times W_h^0$, such that

$$\begin{aligned} s(\lambda_h, \sigma) + b(u_h, \sigma) &= (f, \sigma_0) - \langle g, \sigma_n\rangle_{\partial\Omega} + \langle \psi, \sigma_b\rangle_{\Gamma}, \qquad && \forall \sigma \in W_h^0, \\ b(v, \lambda_h) &= 0, && \forall v \in M_h, \end{aligned}$$

where

$$\begin{aligned} s(\lambda, \sigma) = &\sum_{T\in\mathcal{T}_h} h_T^{1-2p} \int_{\partial T} |\lambda_0 - \lambda_b|^{p-1} sgn(\lambda_0 - \lambda_b)(\sigma_0 - \sigma_b) ds \\ &+ \sum_{T\in\mathcal{T}_h} h_T^{1-p} \int_{\partial T} |\alpha\nabla\lambda_0 \cdot \mathbf{n} - \lambda_n|^{p-1} sgn(\alpha\nabla\lambda_0\cdot\mathbf{n} - \lambda_n)(\alpha\nabla\sigma_0\cdot\mathbf{n} - \sigma_n) ds. \\ b(u, \lambda) = &\sum_{T\in\mathcal{T}_h} (u, -\boldsymbol{\beta}\cdot\nabla_w\lambda - \alpha\Delta_w\lambda)_T. \end{aligned}$$

Tables 8.6–8.8 show the numerical performance for the primal variable u_h and the dual variable λ_0 for $k = 1, 2$ and $p = 1, \ldots, 5$. It can be seen that, for both the linear (i.e., $k = 1$) and quadratic (i.e., $k = 2$) PDWG methods, the convergence rate for the error $\|e_h\|_{0,q}$ is of order $\mathcal{O}(h^{s+1})$. This numerical experiment suggests that the theoretical estimate in Theorem 6.2 should hold true for elliptic problems with piecewise constant diffusions. Analogously to Example 1, the error $\|\lambda_0\|_{m,p}$, $m = 0, 1, 2$, for the dual variable converges to zero at the rate of $\mathcal{O}(h^{p+2-m})$ when $k = 2, s = k-1$ and $p \geq 1$. As shown in Tables 8.7–8.8, the rate of convergence varies in ways that depend on the value of p for $k = 1, s = k-1$ and $k = 2, s = k-2$. Note that no convergence was seen from Tables 8.7–8.8 for the dual variable λ_0 for the case of $p = 1, s = 0$.

Example 4. In this test, we consider the model problem (8.1) in the domain $\Omega = (0, 1)^2 = \Omega_1 \cup \Omega_2$, where Ω_1 is the circular domain centered at $(0.5, 0.5)$ with radius 0.25 and $\Omega_2 = \Omega\backslash\Omega_1$. The diffusive and convective terms are given by

$$\alpha|_{\Omega} = 1, \quad \boldsymbol{\beta}|_{\Omega_1} = (0, 1), \quad \boldsymbol{\beta}|_{\Omega_2} = (1, 0).$$

The functions f, g and ψ are chosen so that the exact solution to this problem is $u = e^{x^2+y}$.

The same algorithm as that for Example 3 was employed for solving this problem. Tables 8.9–8.10 show the performance of the L^p-PDWG for u_h and λ_h for $k = 1, 2$ and $p = 1, \ldots, 4$. It can be seen that the convergence for the error $\|e_h\|_{0,q}$ is of order $\mathcal{O}(h^{s+1})$ for $k = 1, 2$ and $p > 1$, which suggests that the results of Theorem 6.2 may hold true for elliptic problems with piecewise constant or smooth convection. For $\|\lambda_0\|_{m,p}$, Tables 8.9–8.10 demonstrate a convergence

Table 8.7
Numerical error and rate of convergence for the L^p-PDWG method with $k = 2, s = k - 2$ for Example 3.

	h	$\|\lambda_0\|_{2,p}$	rate	$\|\lambda_0\|_{1,p}$	rate	$\|\lambda_0\|_{0,p}$	rate	$\|e_h\|_{0,q}$	rate
$p = 1$	3.54e−01	8.36e−00	–	3.79e−00	–	3.85e−01	–	6.45e−01	–
	1.77e−01	1.20e−00	2.79	9.03e−01	2.07	8.98e−02	2.10	3.62e−01	0.83
	8.84e−02	1.09e−00	0.15	8.55e−01	0.08	8.19e−02	0.13	3.12e−01	0.21
	4.42e−02	1.09e−00	0.00	8.76e−01	−0.03	8.30e−02	−0.02	2.69e−01	0.22
$p = 2$	3.54e−01	6.49e−01	–	5.77e−01	–	8.48e−02	–	1.39e−01	–
	1.77e−01	2.19e−01	1.57	1.72e−01	1.75	2.34e−02	1.86	8.11e−02	0.77
	8.84e−02	1.01e−01	1.11	5.10e−02	1.76	6.66e−03	1.82	4.59e−02	0.82
	4.42e−02	4.56e−02	1.14	1.46e−02	1.81	1.86e−03	1.84	2.41e−02	0.93
	2.21e−02	1.98e−02	1.20	3.96e−03	1.88	4.97e−04	1.91	1.21e−02	0.99
$p = 3$	3.54e−01	1.03e+00	–	8.11e−01	–	1.21e−01	–	1.21e−01	–
	1.77e−01	4.74e−01	1.12	1.64e−01	2.30	2.35e−02	2.36	6.25e−02	0.96
	8.84e−02	1.30e−01	1.87	1.37e−02	3.58	1.85e−03	3.67	2.93e−02	1.10
	4.42e−02	2.63e−02	2.30	9.76e−04	3.82	1.22e−04	3.92	1.43e−02	1.04
	2.21e−02	5.11e−03	2.37	6.87e−05	3.83	7.83e−06	3.97	7.07e−03	1.01
$p = 4$	3.54e−01	2.90e−02	–	7.04e−02	–	7.84e−03	–	1.19e−01	–
	1.77e−01	9.24e−01	−5.00	1.58e−01	−1.16	2.34e−02	−1.58	5.82e−02	1.03
	8.84e−02	7.81e−02	3.60	4.66e−03	5.08	5.39e−04	5.44	2.74e−02	1.09
	4.42e−02	3.95e−03	4.31	1.11e−04	5.39	1.32e−05	5.35	1.37e−02	1.00
	2.21e−02	2.83e−04	3.80	3.59e−06	4.96	5.06e−07	4.71	6.84e−03	1.00
$p = 5$	3.54e−01	1.66e−00	–	2.71e−01	–	3.93e−02	–	1.17e−01	–
	1.77e−01	5.55e−02	4.90	9.24e−03	4.88	1.15e−03	5.09	5.40e−02	1.20
	8.84e−02	1.83e−03	4.93	8.84e−05	6.71	8.65e−06	7.06	2.69e−02	1.00
	4.42e−02	8.89e−05	4.36	1.86e−06	5.58	1.50e−07	5.85	1.35e−02	1.00

Table 8.8
Numerical error and rate of convergence for the L^p-PDWG method with $k = 1, s = k - 1$ for Example 3.

	h	$\|\lambda_0\|_{1,p}$	rate	$\|\lambda_0\|_{0,p}$	rate	$\|e_h\|_{0,q}$	rate
$p = 1$	3.54e−01	4.35e−01	–	4.89e−02	–	5.31e−01	–
	1.77e−01	2.52e−01	0.79	2.66e−02	0.88	4.04e−01	0.39
	8.84e−02	2.46e−01	0.03	2.43e−02	0.13	3.41e−01	0.25
	4.42e−02	2.57e−01	−0.06	2.45e−02	−0.01	2.95e−01	0.21
	2.21e−02	2.71e−01	−0.07	2.55e−02	−0.05	2.56e−01	0.20
$p = 2$	3.54e−01	4.70e−01	–	7.58e−02	–	1.47e−01	–
	1.77e−01	1.48e−01	1.67	2.13e−02	1.83	8.87e−02	0.73
	8.84e−02	4.52e−02	1.71	6.04e−03	1.82	5.41e−02	0.71
	4.42e−02	1.33e−02	1.77	1.71e−03	1.82	3.11e−02	0.80
	2.21e−02	3.74e−03	1.83	4.72e−04	1.86	1.67e−02	0.90
$p = 3$	3.54e−01	6.49e−01	–	1.08e−01	–	1.26e−01	–
	1.77e−01	1.40e−01	2.22	2.08e−02	2.38	6.55e−02	0.94
	8.84e−02	1.22e−02	3.52	1.72e−03	3.60	3.10e−02	1.08
	4.42e−02	8.73e−04	3.80	1.20e−04	3.84	1.48e−02	1.07
	2.21e−02	5.78e−05	3.92	7.69e−06	3.96	7.17e−03	1.04
$p = 4$	3.54-01	8.70e−02	–	1.27e−02	–	1.20e−01	–
	1.77e−01	1.30e−01	−0.58	2.08e−02	−0.71	6.11e−02	0.98
	8.84e−02	3.35e−03	5.28	4.82e−04	5.43	2.75e−02	1.15
	4.42e−02	6.33e−05	5.73	9.16e−06	5.72	1.37e−02	1.01
	2.21e−02	1.87e−06	5.08	3.38e−07	4.76	6.83e−03	1.00
$p = 5$	3.54e−01	2.83e−01	–	4.90e−02	–	1.19e−01	–
	1.77e−01	6.13e−03	5.53	9.44e−04	5.70	5.40e−02	1.15
	8.84e−02	3.75e−05	7.35	6.05e−06	7.29	2.69e−02	1.00
	4.42e−02	7.05e−07	5.73	1.34e−07	5.49	1.35e−02	1.00
	2.21e−02	2.28e−08	4.95	4.35e−09	4.95	6.73e−03	1.00

Table 8.9
Numerical error and rate of convergence for the L^p-PDWG method with $k = 2, s = k - 1$ for Example 4.

	h	$\|\lambda_0\|_{2,p}$	rate	$\|\lambda_0\|_{1,p}$	rate	$\|\lambda_0\|_{0,p}$	rate	$\|e_h\|_{0,q}$	rate
$p = 1$	3.54e−01	4.52e−02	–	5.87e−03	–	3.69e−03	–	2.34e−01	–
	1.77e−01	1.62e−02	1.48	1.27e−03	2.21	8.91e−04	2.05	8.70e−02	1.43
	8.84e−02	6.81e−03	1.25	2.94e−04	2.11	2.21e−04	2.01	2.73e−02	1.67
	4.42e−02	3.13e−03	1.12	7.05e−05	2.06	5.52e−05	2.00	7.88e−03	1.79
	2.21e−02	1.51e−03	1.06	1.72e−05	2.03	1.38e−05	2.00	2.13e−03	1.89

(continued on next page)

Table 8.9 (*continued*).

	h	$\|\lambda_0\|_{2,p}$	rate	$\|\lambda_0\|_{1,p}$	rate	$\|\lambda_0\|_{0,p}$	rate	$\|e_h\|_{0,q}$	rate
	3.54e−01	2.10e−01	–	3.29e−02	–	1.24e−02	–	3.44e−02	
	1.77e−01	3.87e−02	2.44	2.86e−03	3.52	7.94e−04	3.97	8.92e−03	1.95
$p = 2$	8.84e−02	7.90e−03	2.29	2.62e−04	3.45	4.99e−05	3.99	2.25e−03	1.99
	4.42-02	1.75e−03	2.17	2.62e−05	3.32	3.12e−06	4.00	5.636e−04	2.00
	2.21e−02	4.14e−04	2.08	2.88e−06	3.19	1.95e−07	4.00	1.41e−04	2.00
	3.54e−01	4.75e−01	–	9.00e−02	–	2.47e−02	–	2.38e−02	
	1.77e−01	7.43e−02	2.68	7.34e−03	3.62	7.95e−04	4.96	5.18e−03	2.20
$p = 3$	8.84e−02	6.28e−03	3.56	3.22e−04	4.51	1.54e−05	5.69	1.19e−03	2.12
	4.42e−02	8.25e−04	2.93	1.64e−05	4.30	3.08e−07	5.65	2.76e−04	2.10
	2.21e−02	4.48e−05	2.75	9.48e−07	4.11	6.75e−09	5.51	6.55e−05	2.08
	3.54e−01	2.87e−01	–	6.20e−02	–	1.34e−02	–	1.97e−02	
	1.77e−01	1.21e−01	1.25	1.37e−02	2.18	8.25e−04	4.02	4.53e−03	2.12
$p = 4$	8.84e−02	1.91e−02	2.66	5.84e−04	4.55	1.18e−05	6.13	9.90e−04	2.19
	4.42e−02	1.39e−03	3.78	1.97e−05	4.89	1.64e−07	6.17	2.41e−04	2.04

Table 8.10
Numerical error and rate of convergence for the L^p-PDWG method with $k = 1, s = k - 1$ for Example 4.

	h	$\|\lambda_0\|_{1,p}$	rate	$\|\lambda_0\|_{0,p}$	rate	$\|e_h\|_{0,q}$	rate
	1.77e−01	1.07e−00	–	8.89e−02	–	1.09e−00	–
$p = 1$	8.84e−02	8.14e−01	0.39	4.59e−02	0.95	7.15e−01	0.60
	4.42e−02	8.01e−01	0.02	4.37e−02	0.07	5.42e−01	0.40
	2.21e−02	8.21e−01	−0.03	4.50e−02	−0.04	4.52e−01	0.26
	3.54e−01	1.16e−00	–	1.33e−01	–	3.01e−01	
	1.77e−01	3.46e−01	1.74	3.50e−02	1.93	1.57e−01	0.94
$p = 2$	8.84e−02	1.09e−01	1.67	1.07e−02	1.80	7.68e−02	1.03
	4.42e−02	3.23e−02	1.75	2.78e−03	1.85	3.54e−02	1.12
	2.21e−02	8.87e−03	1.86	7.31e−04	1.93	1.61e−02	1.13
	3.54e−01	9.50e−01	–	1.14e−01	–	2.29e−01	–
	1.77e−01	2.37e−01	2.00	2.91e−02	1.96	1.16e−01	0.98
$p = 3$	8.84e−02	2.69e−02	3.14	3.59e−03	3.02	5.40e−02	1.10
	4.42e−02	2.08e−03	3.69	3.33e−04	3.43	2.60e−02	1.05
	2.21e−02	1.61e−04	3.69	3.17e−05	3.40	1.29e−02	1.02
	3.54e−01	7.06e−00	–	1.24e−00	–	5.98e−01	–
	1.77e−01	1.67e−01	5.41	1.77e−02	6.13	1.35e−01	2.14
$p = 4$	8.84e−02	1.12e−02	3.90	2.44e−03	2.86	5.41e−02	1.32
	4.42e−02	1.03e−03	3.44	1.99e−04	3.61	2.61e−02	1.05
	2.21e−02	6.60e−05	3.96	1.21e−05	4.04	1.27e−02	1.04

of $\mathcal{O}(h^{p+2-m})$ when $k = 2$ and $\mathcal{O}(h^p)$ when $k = 1$. It was observed the numerical convergence for the dual variable λ_0 performs significantly better than the theory shown in Theorems 7.1–7.2. Tables 8.9–8.10 also show the convergence of the numerical approximations for $p = 1$ and $k = 1, 2$. The convergence for the dual variable λ_0 varies according to the choice of M_h. For $s = 1$, the convergence for $\|\lambda_0\|_{m,p}$ is at the rate of $\mathcal{O}(h^{p+2-\max(1,m)})$, but for $s = 0$, no convergence was observed.

Data availability

No data was used for the research described in the article.

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