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Weak Galerkin finite element methods for quad-curl problems

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ABSTRACT

This article introduces a weak Galerkin (WG) finite element method for quad-curl problems in three dimensions. It is proved that the proposed WG method is stable and accurate in an optimal order of error estimates for the exact solution in discrete norms. In addition, an L^2 error estimate in an optimal order except the lowest order $k = 2$ is derived for the WG solution. Some numerical experiments are conducted to verify the efficiency and accuracy of our WG method and furthermore a superconvergence has been observed from the numerical results.

1. Introduction

In this paper we are concerned with the development of a weak Galerkin (WG) finite element method for the quad-curl problem in three dimensions which seeks $\mathbf{u}$ such that

$$\begin{aligned}
(\nabla\times)^4\mathbf{u} =&\mathbf{f}, \qquad \text{in} \quad \Omega,\\
\nabla\cdot\mathbf{u} =&0, \qquad \text{in} \quad \Omega,\\
\mathbf{u}\times\mathbf{n} =&0, \qquad \text{on} \quad \partial\Omega,\\
\nabla\times\mathbf{u}\times\mathbf{n} =&0, \qquad \text{on} \quad \partial\Omega,
\end{aligned} \tag{1.1}$$

for a given $\mathbf{f}$ defined on a bounded domain $\Omega\subset\mathbb{R}^3$.

The quad-curl problems arise in inverse electromagnetic scattering theory for nonhomogeneous media [1] and magneto-hydrodynamics equations [2]. Recently, some contributions have been made on the finite element methods for

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the quad-curl problems. The conforming finite element spaces for the quad-curl problem have been recently constructed in two dimensions (e.g. [3,4]) and in three dimensions (e.g. [5–7]). [2,8] proposed the nonconforming and low order finite element spaces for the quad-curl problems. [9–11] proposed the mixed methods for the quad-curl problems. [12] introduced a formulation using the Hodge decomposition for the quad-curl problems. [13] introduced a discontinuous Galerkin scheme. [14] proposed a novel weak Galerkin formulation using the conforming space for curl-curl problem as a nonconforming space for the quad-curl problem. [15] analyzed a posteriori error analysis for the quad-curl problems in two dimensions. [16] introduced a virtual element method for the quad-curl problems in two dimensions. [17] introduced a decoupled formulation for the quad-curl problems where the a priori and a posteriori error were analyzed.

In the literature, the existing WG methods for quad-curl problems proposed in [14] were curl-conforming and based on tetrahedral partitions. However, our WG method is not necessary to be curl-conforming and is based on any polyhedral partitions. Our WG numerical method (4.7)–(4.8) has provided an accurate and reliable numerical solution for the quad-curl system (1.1) in an optimal order of error estimates in discrete norms and in an optimal order of L^2 error estimates except the lowest order $k = 2$. In addition, we have observed some superconvergence phenomena from numerical experiments.

The paper is organized as follows. Section 2 is devoted to the derivation of a weak formulation for the quad-curl system (1.1). Section 3 briefly introduces the discrete weak gradient operator and the discrete weak curl-curl operator. Section 4 is dedicated to the presentation of the weak Galerkin algorithm for the quad-curl problem and a discussion of the solution existence and uniqueness for the WG scheme. In Section 5, the error equations are derived for the WG scheme. Section 6 establishes an optimal order of error estimates in discrete norms for the WG approximation. In Section 7, the L^2 error estimate for the WG solution is established in an optimal order except the lowest order $k = 2$ under some regularity assumptions. Section 8 demonstrates the numerical performance of the WG algorithm through some test examples.

We follow the standard notations for Sobolev spaces and norms defined on a given open and bounded domain $D \subset \mathbb{R}^3$ with Lipschitz continuous boundary. Denote by $\|\cdot\|_{s,D}$, $|\cdot|_{s,D}$ and $(\cdot,\cdot)_{s,D}$ the norm, seminorm and inner product in the Sobolev space $H^s(D)$ for any $s \geq 0$. The space $H^0(D)$ coincides with $L^2(D)$ (i.e., the space of square integrable functions), for which the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot,\cdot)_D$. When $D = \Omega$ or when the domain of integration is clear from the context, we shall drop the subscript D in the norm and the inner product notation. C denotes a generic constant independent of the meshsize and other physical or functional parameters.

2. A weak formulation

Let $s > 0$ be an integer. We first introduce

$$H(curl^s;\Omega) = \{\mathbf{u} \in [L^2(\Omega)]^3 : (\nabla\times)^j\mathbf{u} \in [L^2(\Omega)]^3, j = 1,\ldots,s\}$$

with the associated inner product $(\mathbf{u},\mathbf{v})_{H(curl^s;\Omega)} = (\mathbf{u},\mathbf{v}) + \sum_{j=1}^s((\nabla\times)^j\mathbf{u},(\nabla\times)^j\mathbf{v})$ and the norm $\|\mathbf{u}\|_{H(curl^s;\Omega)} = (\mathbf{u},\mathbf{u})^{\frac{1}{2}}_{H(curl^s;\Omega)}$. We further introduce

$$H_0(curl;\Omega) := \{\mathbf{u} \in H(curl;\Omega) : \mathbf{n}\times\mathbf{u} = 0 \text{ on } \partial\Omega\},$$

$$H_0(curl^2;\Omega) := \{\mathbf{u} \in H(curl^2;\Omega) : \mathbf{n}\times\mathbf{u} = 0 \text{ and } \nabla\times\mathbf{u}\times\mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

We introduce

$$H(div;\Omega) = \{\mathbf{u} \in [L^2(\Omega)]^3 : \nabla\cdot\mathbf{u} \in L^2(\Omega)\},$$

with the associated inner product $(\mathbf{u},\mathbf{v})_{H(div;\Omega)} = (\mathbf{u},\mathbf{v}) + (\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})$ and the norm $\|\mathbf{u}\|_{H(div;\Omega)} = (\mathbf{u},\mathbf{u})^{\frac{1}{2}}_{H(div;\Omega)}$. We further introduce

$$H(div^0;\Omega) = \{\mathbf{u} \in H(div;\Omega) : \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\}.$$

Using the usual integration by parts, we are ready to propose the weak formulation of the quad-curl problem (1.1) as follows: Given $\mathbf{f} \in H(div^0;\Omega)$, find $(\mathbf{u};p) \in H_0(curl^2;\Omega)\times H_0^1(\Omega)$ such that

$$\begin{aligned} ((\nabla\times)^2\mathbf{u},(\nabla\times)^2\mathbf{v}) + (\mathbf{v},\nabla p) &= (\mathbf{f},\mathbf{v}), & \mathbf{v} \in H_0(curl^2;\Omega),\\ -(\mathbf{u},\nabla q) &= 0, & \forall q \in H_0^1(\Omega). \end{aligned} \tag{2.1}$$

Theorem 2.1 (*[14]*). *Given $\mathbf{f} \in H(div^0;\Omega)$, the problem (2.1) has a unique solution $(\mathbf{u};p) \in H_0(curl^2;\Omega)\times H_0^1(\Omega)$. Furthermore, $p = 0$ and $\mathbf{u}$ satisfies*

$$\|\mathbf{u}\|_{H(curl^2;\Omega)} \leq C\|\mathbf{f}\|.$$

3. Weak differential operators

The principal differential operators in the weak formulation (2.1) for the quad-curl problem (1.1) are the gradient operator ∇ and the curl-curl operator $(\nabla\times)^2$. We shall briefly review the discrete weak gradient operator [18,19] and define the discrete weak curl-curl operator.

Let T be a polyhedral domain with boundary ∂T. A scalar-valued weak function on T refers to $\sigma = \{\sigma_0, \sigma_b\}$ with $\sigma_0 \in L^2(T)$ and $\sigma_b \in L^2(\partial T)$. Here σ_0 and σ_b are used to represent the value of σ in the interior and on the boundary of T. Note that σ_b may not necessarily be the trace of σ_0 on ∂T. Denote by $\mathcal{W}(T)$ the space of scalar-valued weak functions on T:

$$\mathcal{W}(T) = \{\sigma = \{\sigma_0, \sigma_b\} : \sigma_0 \in L^2(T), \sigma_b \in L^2(\partial T)\}. \tag{3.1}$$

A vector-valued weak function on T refers to a triplet $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b, \mathbf{v}_n\}$ where $\mathbf{v}_0$ and $\mathbf{v}_b$ are used to represent the values of $\mathbf{v}$ in the interior and on the boundary of T and $\mathbf{v}_n$ represents the value of $\nabla\times\mathbf{v}$ on ∂T. Note that $\mathbf{v}_b$ and $\mathbf{v}_n$ may not necessarily be the traces of $\mathbf{v}_0$ and $\nabla\times\mathbf{v}_0$ on ∂T respectively. Denote by $V(T)$ the space of vector-valued weak functions on T:

$$V(T) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b, \mathbf{v}_n\} : \mathbf{v}_0 \in [L^2(T)]^3, \mathbf{v}_b \in [L^2(\partial T)]^3, \mathbf{v}_n \in [L^2(\partial T)]^3\}. \tag{3.2}$$

The weak gradient of $\sigma \in \mathcal{W}(T)$, denoted by $\nabla_w\sigma$, is defined as a linear functional on $[H^1(T)]^3$ such that

$$(\nabla_w\sigma, \boldsymbol{\psi})_T = -(\sigma_0, \nabla\cdot\boldsymbol{\psi})_T + \langle\sigma_b, \boldsymbol{\psi}\cdot\mathbf{n}\rangle_{\partial T},$$

for all $\boldsymbol{\psi} \in [H^1(T)]^3$.

The weak curl-curl operator of any $\mathbf{v} \in V(T)$, denoted by $(\nabla\times)_w^2\mathbf{v}$ is defined in the dual space of $H(curl^2; T)$, whose action on $\mathbf{q} \in H(curl^2; T)$ is given by

$$((\nabla\times)_w^2\mathbf{v}, \mathbf{q})_T = (\mathbf{v}_0, (\nabla\times)^2\mathbf{q})_T - \langle\mathbf{v}_b\times\mathbf{n}, \nabla\times\mathbf{q}\rangle_{\partial T} - \langle\mathbf{v}_n\times\mathbf{n}, \mathbf{q}\rangle_{\partial T}.$$

Denote by $P_r(T)$ the space of polynomials on T with degree no more than r.

A discrete version of $\nabla_w\sigma$ for $\sigma \in \mathcal{W}(T)$, denoted by $\nabla_{w,r,T}\sigma$, is defined as a unique polynomial vector in $[P_r(T)]^3$ satisfying

$$(\nabla_{w,r,T}\sigma, \boldsymbol{\psi})_T = -(\sigma_0, \nabla\cdot\boldsymbol{\psi})_T + \langle\sigma_b, \boldsymbol{\psi}\cdot\mathbf{n}\rangle_{\partial T}, \quad \forall\boldsymbol{\psi} \in [P_r(T)]^3, \tag{3.3}$$

which, from the usual integration by parts, gives

$$(\nabla_{w,r,T}\sigma, \boldsymbol{\psi})_T = (\nabla\sigma_0, \boldsymbol{\psi})_T - \langle\sigma_0 - \sigma_b, \boldsymbol{\psi}\cdot\mathbf{n}\rangle_{\partial T}, \quad \forall\boldsymbol{\psi} \in [P_r(T)]^3, \tag{3.4}$$

provided that $\sigma_0 \in H^1(T)$.

A discrete version of $(\nabla\times)_w^2\mathbf{v}$ for $\mathbf{v} \in V(T)$, denoted by $(\nabla\times)_{w,r,T}^2\mathbf{v}$, is defined as a unique polynomial vector in $[P_r(T)]^3$ satisfying

$$((\nabla\times)_{w,r,T}^2\mathbf{v}, \mathbf{q})_T = (\mathbf{v}_0, (\nabla\times)^2\mathbf{q})_T - \langle\mathbf{v}_b\times\mathbf{n}, \nabla\times\mathbf{q}\rangle_{\partial T} - \langle\mathbf{v}_n\times\mathbf{n}, \mathbf{q}\rangle_{\partial T}, \tag{3.5}$$

for any $\mathbf{q} \in [P_r(T)]^3$.

4. Weak Galerkin algorithm

Let $\mathcal{T}_h$ be a finite element partition of the domain $\Omega \subset \mathbb{R}^3$ consisting of polyhedra that are shape-regular [19]. Denote by $\mathcal{E}_h$ the set of all faces in $\mathcal{T}_h$ and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ the set of all interior faces. Denote by h_T the meshsize of $T \in \mathcal{T}_h$ and $h = \max_{T\in\mathcal{T}_h} h_T$ the meshsize for the partition $\mathcal{T}_h$.

For any given integer $k \geq 2$, denote by $W_k(T)$ the local discrete space of the scalar-valued weak functions given by

$$W_k(T) = \{\{\sigma_0, \sigma_b\} : \sigma_0 \in P_k(T), \sigma_b \in P_k(e), e \subset \partial T\}.$$

Furthermore, denote by $V_k(T)$ the local discrete space of the vector-valued weak functions given by

$$V_k(T) = \{\{\mathbf{v}_0, \mathbf{v}_b, \mathbf{v}_n\} : \mathbf{v}_0 \in [P_k(T)]^3, \mathbf{v}_b \in [P_k(e)]^3, \mathbf{v}_n \in [P_{k-1}(e)]^3, e \subset \partial T\}.$$

Patching $W_k(T)$ over all the elements $T \in \mathcal{T}_h$ through a common value σ_b on the interior interface $\mathcal{E}_h^0$, we arrive at the following scalar-valued weak finite element space, denoted by W_h; i.e.,

$$W_h = \Big\{\{\sigma_0, \sigma_b\} : \{\sigma_0, \sigma_b\}|_T \in W_k(T), \forall T \in \mathcal{T}_h\Big\},$$

and the subspace of W_h with vanishing boundary values on $\partial\Omega$, denoted by W_h^0; i.e.,

$$W_h^0 = \{\{\sigma_0, \sigma_b\} \in W_h : \sigma_b = 0 \text{ on } \partial\Omega\}. \tag{4.1}$$

Similarly, patching $V_k(T)$ over all the elements $T \in \mathcal{T}_h$ through a common value $\mathbf{v}_b$ on the interior interface $\mathcal{E}_h^0$, we arrive at the following vector-valued weak finite element space, denoted by V_h; i.e.,

$$V_h = \left\{\{\mathbf{v}_0, \mathbf{v}_b, \mathbf{v}_n\} : \{\mathbf{v}_0, \mathbf{v}_b, \mathbf{v}_n\}|_T \in V_k(T), \forall T \in \mathcal{T}_h\right\},$$

and the subspace of V_h with vanishing boundary values on $\partial\Omega$, denoted by V_h^0; i.e.,

$$V_h^0 = \left\{\{\mathbf{v}_0, \mathbf{v}_b, \mathbf{v}_n\} \in V_h : \mathbf{v}_b \times \mathbf{n} = 0 \text{ and } \mathbf{v}_n \times \mathbf{n} = 0 \ \text{ on } \partial\Omega\right\}. \tag{4.2}$$

For simplicity of notation and without confusion, for any $\sigma \in W_h$ and $\mathbf{v} \in V_h$, denote by $\nabla_w\sigma$ and $(\nabla\times)_w^2\mathbf{v}$ the discrete weak actions $\nabla_{w,k,T}\sigma$ and $(\nabla\times)_{w,k-2,T}^2\mathbf{v}$ computed by using (3.3) and (3.5) on each element T; i.e.,

$$(\nabla_w\sigma)|_T = \nabla_{w,k,T}(\sigma|_T), \qquad \sigma \in W_h,$$

$$(\nabla\times^2)_w\mathbf{v}|_T = (\nabla\times^2)_{w,k-2,T}(\mathbf{v}|_T), \qquad \mathbf{v} \in V_h.$$

For any $\sigma, \lambda \in W_h$ and $\mathbf{u}, \mathbf{v} \in V_h$, we introduce the following bilinear forms

$$a(\mathbf{u}, \mathbf{v}) = \sum_{T\in\mathcal{T}_h} a(\mathbf{u}, \mathbf{v}), \tag{4.3}$$

$$b(\mathbf{u}, \lambda) = \sum_{T\in\mathcal{T}_h} b_T(\mathbf{u}, \lambda), \tag{4.4}$$

$$s_1(\mathbf{u}, \mathbf{v}) = \sum_{T\in\mathcal{T}_h} s_{1,T}(\mathbf{u}, \mathbf{v}), \tag{4.5}$$

$$s_2(\sigma, \lambda) = \sum_{T\in\mathcal{T}_h} s_{2,T}(\sigma, \lambda), \tag{4.6}$$

where

$$\begin{aligned}
a_T(\mathbf{u}, \mathbf{v}) =&\ ((\nabla\times)_w^2\mathbf{u}, (\nabla\times)_w^2\mathbf{v})_T,\\
b_T(\mathbf{u}, \lambda) =&\ (\mathbf{u}_0, \nabla_w\lambda)_T,\\
s_{1,T}(\mathbf{u}, \mathbf{v}) =&\ h_T^{-3}\langle \mathbf{u}_0 \times \mathbf{n} - \mathbf{u}_b \times \mathbf{n}, \mathbf{v}_0 \times \mathbf{n} - \mathbf{v}_b \times \mathbf{n}\rangle_{\partial T}\\
&+ h_T^{-1}\langle \nabla\times\mathbf{u}_0 \times \mathbf{n} - \mathbf{u}_n \times \mathbf{n}, \nabla\times\mathbf{v}_0 \times \mathbf{n} - \mathbf{v}_n \times \mathbf{n}\rangle_{\partial T},\\
s_{2,T}(\sigma, \lambda) =&\ h_T^3\langle \sigma_0 - \sigma_b, \lambda_0 - \lambda_b\rangle_{\partial T}.
\end{aligned}$$

The following is the weak Galerkin scheme for the quad-curl problem (1.1) based on the variational formulation (2.1).

Weak Galerkin Algorithm 4.1. Given $\mathbf{f} \in H(div^0; \Omega)$, find $(\mathbf{u}_h; p_h) \in V_h^0 \times W_h^0$, such that

$$s_1(\mathbf{u}_h, \mathbf{v}_h) + a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_0), \qquad \forall \mathbf{v}_h \in V_h^0, \tag{4.7}$$

$$s_2(p_h, q_h) - b(\mathbf{u}_h, q_h) = 0, \qquad \forall q_h \in W_h^0. \tag{4.8}$$

Theorem 4.1. *The weak Galerkin finite element scheme (4.7)–(4.8) has a unique solution.*

Proof. It suffices to prove that $\mathbf{f} = 0$ implies that $\mathbf{u}_h = 0$ and $p_h = 0$ in Ω. To this end, taking $\mathbf{v}_h = \mathbf{u}_h$ in (4.7) and $q_h = p_h$ in (4.8) gives

$$((\nabla\times)_w^2\mathbf{u}_h, (\nabla\times)_w^2\mathbf{u}_h) + s_1(\mathbf{u}_h, \mathbf{u}_h) + s_2(p_h, p_h) = 0.$$

This yields

$$(\nabla\times)_w^2\mathbf{u}_h = 0, \quad \text{in each } T, \tag{4.9}$$

$$\nabla\times\mathbf{u}_0 \times \mathbf{n} = \mathbf{u}_n \times \mathbf{n}, \quad \text{on each } \partial T, \tag{4.10}$$

$$\mathbf{u}_0 \times \mathbf{n} = \mathbf{u}_b \times \mathbf{n}, \quad \text{on each } \partial T, \tag{4.11}$$

$$p_0 = p_b, \quad \text{on each } \partial T. \tag{4.12}$$

Using (4.9), (3.5), (4.10)–(4.11), and the integration by parts, we obtain

$$\begin{aligned}
0 =&\ ((\nabla\times)_w^2\mathbf{u}_h, \mathbf{w})_T\\
=&\ ((\nabla\times)^2\mathbf{u}_0, \mathbf{w})_T - \langle \mathbf{w}, (\mathbf{u}_n - \nabla\times\mathbf{u}_0) \times \mathbf{n}\rangle_{\partial T} + \langle \nabla\times\mathbf{w}, (\mathbf{u}_0 - \mathbf{u}_b) \times \mathbf{n}\rangle_{\partial T}\\
=&\ ((\nabla\times)^2\mathbf{u}_0, \mathbf{w})_T,
\end{aligned}$$

for any $\mathbf{w} \in [P_{k-2}(T)]^3$. This gives $(\nabla\times)^2\mathbf{u}_0 = 0$ in each $T \in \mathcal{T}_h$. It follows from (4.10)–(4.11) that $\mathbf{u}_0 \times \mathbf{n}$ and $\nabla \times \mathbf{u}_0 \times \mathbf{n}$ are continuous across the interior interface $\mathcal{E}_h^0$. Thus, $\mathbf{u}_0 \in H(curl^2; \Omega)$ and $(\nabla\times)^2\mathbf{u}_0 = 0$ in Ω. Therefore, there exists a potential function ϕ such that $\nabla \times \mathbf{u}_0 = \nabla\phi$ in Ω. This gives

$$
\begin{aligned}
(\nabla\phi, \nabla\phi) &= \sum_{T\in\mathcal{T}_h} (\nabla \times \mathbf{u}_0, \nabla\phi)_T \\
&= \sum_{T\in\mathcal{T}_h} (\mathbf{u}_0, \nabla \times \nabla\phi)_T + \langle \nabla\phi, \mathbf{n} \times \mathbf{u}_0 \rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} \langle \nabla\phi, \mathbf{n} \times \mathbf{u}_b \rangle_{\partial T} \\
&= \langle \nabla\phi, \mathbf{n} \times \mathbf{u}_b \rangle_{\partial\Omega} \\
&= 0,
\end{aligned} \tag{4.13}
$$

where we used the usual integration by parts, (4.11) and $\mathbf{n}\times\mathbf{u}_b = 0$ on $\partial\Omega$. This leads to $\phi = C$ in Ω, and thus $\nabla\times\mathbf{u}_0 = 0$ in Ω. Furthermore, there exists a potential function ψ such that $\mathbf{u}_0 = \nabla\psi$ in Ω.

From (4.12), (3.3) and (4.8), we have

$$
\begin{aligned}
0 &= \sum_{T\in\mathcal{T}_h} (\nabla_w q_h, \mathbf{u}_0)_T \\
&= \sum_{T\in\mathcal{T}_h} -(q_0, \nabla \cdot \mathbf{u}_0)_T + \langle q_b, \mathbf{u}_0 \cdot \mathbf{n} \rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} -(q_0, \nabla \cdot \mathbf{u}_0)_T + \sum_{e\in\mathcal{E}_h^0} \langle q_b, [\![\mathbf{u}_0 \cdot \mathbf{n}]\!] \rangle_e,
\end{aligned} \tag{4.14}
$$

where $[\![\mathbf{u}_0 \cdot \mathbf{n}]\!]$ is the jump of $\mathbf{u}_0 \cdot \mathbf{n}$ on edge $e \in \mathcal{E}_h^0$ and we used $q_b = 0$ on $\partial\Omega$. Letting $q_0 = 0$ and $q_b = [\![\mathbf{u}_0 \cdot \mathbf{n}]\!]$ in (4.14) yields that $[\![\mathbf{u}_0 \cdot \mathbf{n}]\!] = 0$ on $e \in \mathcal{E}_h^0$ which means $\mathbf{u}_0 \cdot \mathbf{n}$ is continuous along the interior interface $e \in \mathcal{E}_h^0$. This follows that $\mathbf{u}_0 \in H(div; \Omega)$. Taking $q_0 = \nabla \cdot \mathbf{u}_0$ and $q_b = 0$ in (4.14) gives $\nabla \cdot \mathbf{u}_0 = 0$ on each T and further $\nabla \cdot \mathbf{u}_0 = 0$ in Ω due to $\mathbf{u}_0 \in H(div; \Omega)$. Recall that there exists a potential function ψ such that $\mathbf{u}_0 = \nabla\psi$ in Ω. Hence, $\nabla \cdot \mathbf{u}_0 = \Delta\psi = 0$ strongly holds true in Ω with the boundary condition $\nabla\psi \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n} = 0$ on $\partial\Omega$. This implies that $\psi = C$ in Ω. Thus, $\mathbf{u}_0 = \nabla\psi = 0$ in Ω. Using (4.10)–(4.11) gives $\mathbf{u}_b = 0$ and $\mathbf{u}_n = 0$ in Ω. Therefore, we obtain $\mathbf{u}_h = 0$ in Ω.

Using $\mathbf{u}_h = 0$ gives $s_1(\mathbf{u}_h, \mathbf{v}_h) + a(\mathbf{u}_h, \mathbf{v}_h) = 0$ for any $\mathbf{v}_h \in V_h^0$. It follows from the assumption $\mathbf{f} = 0$ and (4.7) that $b(\mathbf{v}_h, p_h) = 0$, which, together with (3.3) and (4.12) and the usual integration by parts, gives

$$
0 = b(\mathbf{v}_h, p_h) = -\sum_{T\in\mathcal{T}_h} (p_0, \nabla \cdot \mathbf{v}_0)_T + \langle p_b, \mathbf{v}_0 \cdot \mathbf{n} \rangle_{\partial T} = \sum_{T\in\mathcal{T}_h} (\nabla p_0, \mathbf{v}_0)_T.
$$

Letting $\mathbf{v}_0 = \nabla p_0$ gives rise to $\nabla p_0 = 0$ on each $T \in \mathcal{T}_h$; i.e., $p_0 = C$ on each $T \in \mathcal{T}_h$. The facts that $p_0 = p_b$ on each ∂T and $p_b = 0$ on $\partial\Omega$ give $p_0 = p_b = 0$ in Ω and further $p_h = 0$ in Ω.

This completes the proof of the theorem. □

Let $k \geq 2$. Let $\mathbf{Q}_0$ be the L^2 projection operator onto $[P_k(T)]^3$. Analogously, for $e \subset \partial T$, denote by $\mathbf{Q}_b$ and $\mathbf{Q}_n$ the L^2 projection operators onto $[P_k(e)]^3$ and $[P_{k-1}(e)]^3$, respectively. For $\mathbf{w} \in [H(curl; \Omega)]^3$, define the L^2 projection $\mathbf{Q}_h\mathbf{w} \in V_h$ as follows

$$
\mathbf{Q}_h\mathbf{w}|_T = \{\mathbf{Q}_0\mathbf{w}, \mathbf{Q}_b\mathbf{w}, \mathbf{Q}_n(\nabla \times \mathbf{w})\}.
$$

For $\sigma \in H^1(\Omega)$, the L^2 projection $Q_h\sigma \in W_h$ is defined by

$$
Q_h\sigma|_T = \{Q_0\sigma, Q_b\sigma\},
$$

where Q_0 and Q_b are the L^2 projection operators onto $P_k(T)$ and $P_k(e)$ respectively. Denote by $\mathcal{Q}_h^{k-2}$ and $\mathcal{Q}_h^k$ the L^2 projection operators onto $P_{k-2}(T)$ and $P_k(T)$, respectively.

Lemma 4.2. *The operators $\mathbf{Q}_h$, Q_h, $\mathcal{Q}_h^k$ and $\mathcal{Q}_h^{k-2}$ satisfy the following commutative properties:*

$$
(\nabla\times)_w^2(\mathbf{Q}_h\mathbf{w}) = \mathcal{Q}_h^{k-2}((\nabla\times)^2\mathbf{w}), \qquad \forall \mathbf{w} \in H(curl^2; T), \tag{4.15}
$$

$$
\nabla_w(Q_h\sigma) = \mathcal{Q}_h^k(\nabla\sigma), \qquad \forall \sigma \in H^1(T). \tag{4.16}
$$

Proof. For any $\mathbf{q} \in [P_{k-2}(T)]^3$, using (3.5) and the usual integration by parts gives

$$\begin{aligned} ((\nabla\times)_w^2 \mathbf{Q}_h\mathbf{w}, \mathbf{q})_T &= (\mathbf{Q}_0\mathbf{w}, (\nabla\times)^2\mathbf{q})_T - \langle \mathbf{Q}_b\mathbf{w}\times\mathbf{n}, \nabla\times\mathbf{q}\rangle_{\partial T} - \langle \mathbf{Q}_n(\nabla\times\mathbf{w})\times\mathbf{n}, \mathbf{q}\rangle_{\partial T} \\ &= (\mathbf{w}, (\nabla\times)^2\mathbf{q})_T - \langle \mathbf{w}\times\mathbf{n}, \nabla\times\mathbf{q}\rangle_{\partial T} - \langle \nabla\times\mathbf{w}\times\mathbf{n}, \mathbf{q}\rangle_{\partial T} \\ &= ((\nabla\times)^2\mathbf{w}, \mathbf{q})_T \\ &= (\mathcal{Q}_h^{k-2}((\nabla\times)^2\mathbf{w}), \mathbf{q})_T. \end{aligned}$$

This completes the proof of (4.15).

The proof of (4.16) can be found in [18,19]. □

5. Error equations

The goal of this section is to derive the error equations for the weak Galerkin method (4.7)–(4.8) for solving the quad-curl problem (1.1), which play a critical role in the forthcoming convergence analysis.

Let $(\mathbf{u}, p)$ be the solution of (2.1) and assume that $\mathbf{u} \in H(curl^4; \Omega)$. Then $(\mathbf{u}, p)$ satisfies

$$((\nabla\times)^4\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \nabla p) = (\mathbf{f}, \mathbf{v}), \tag{5.1}$$

$$(\nabla\cdot\mathbf{u}, q) = 0, \tag{5.2}$$

for $\mathbf{v} \in [L^2(\Omega)]^3$ and $q \in L^2(\Omega)$. Let $(\mathbf{u}_h, p_h)$ be the WG solutions of (4.7)–(4.8). Define the error functions $\mathbf{e}_h$ and ϵ_h by

$$\mathbf{e}_h = \{\mathbf{e}_0, \mathbf{e}_b, \mathbf{e}_n\} = \{\mathbf{Q}_0\mathbf{u} - \mathbf{u}_0, \mathbf{Q}_b\mathbf{u} - \mathbf{u}_b, \mathbf{Q}_n(\nabla\times\mathbf{u}) - \mathbf{u}_n\}, \tag{5.3}$$

$$\epsilon_h = \{\epsilon_0, \epsilon_b\} = \{Q_0 p - p_0, Q_b p - p_b\}. \tag{5.4}$$

Lemma 5.1. *Let $\mathbf{u} \in H(curl^4; \Omega)$ and $(\mathbf{u}_h; p_h) \in V_h^0 \times W_h^0$ be the exact solution of quad-curl model problem (1.1) and the numerical solution arising from the WG scheme (4.7)–(4.8) respectively. The error functions $\mathbf{e}_h$ and ϵ_h defined in (5.3)–(5.4) satisfy the following error equations; i.e.,*

$$s_1(\mathbf{e}_h, \mathbf{v}_h) + a(\mathbf{e}_h, \mathbf{v}_h) + b(\mathbf{v}_h, \epsilon_h) = s_1(\mathbf{Q}_h\mathbf{u}, \mathbf{v}_h) + \ell_1(\mathbf{u}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h^0, \tag{5.5}$$

$$-b(\mathbf{e}_h, q_h) + s_2(\epsilon_h, q_h) = s_2(Q_h p, q_h) - \ell_2(\mathbf{u}, q_h), \quad \forall q_h \in W_h^0. \tag{5.6}$$

Here

$$\begin{aligned} \ell_1(\mathbf{u}, \mathbf{v}_h) &= \sum_{T\in\mathcal{T}_h} \langle (\mathbf{v}_0 - \mathbf{v}_b)\times\mathbf{n}, \nabla\times(\mathcal{Q}_h^{k-2} - I)((\nabla\times)^2\mathbf{u})\rangle_{\partial T} \\ &\quad + \langle (\nabla\times\mathbf{v}_0 - \mathbf{v}_n)\times\mathbf{n}, (\mathcal{Q}_h^{k-2} - I)((\nabla\times)^2\mathbf{u})\rangle_{\partial T}, \\ \ell_2(\mathbf{u}, q_h) &= \sum_{T\in\mathcal{T}_h} \langle q_0 - q_b, (I - \mathbf{Q}_0)\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T}. \end{aligned}$$

Proof. Note that $\mathbf{v}_h = \{\mathbf{v}_0, \mathbf{v}_b, \mathbf{v}_n\}$. Using (4.15), (3.5) and the usual integration by parts, we have

$$\begin{aligned} &((\nabla\times)_w^2\mathbf{Q}_h\mathbf{u}, (\nabla\times)_w^2\mathbf{v}_h)_T \\ =\ & (\mathcal{Q}_h^{k-2}((\nabla\times)^2\mathbf{u}), (\nabla\times)_w^2\mathbf{v}_h)_T \\ =\ & (\mathbf{v}_0, (\nabla\times)^2\mathcal{Q}_h^{k-2}((\nabla\times)^2\mathbf{u}))_T - \langle \mathbf{v}_b\times\mathbf{n}, \nabla\times\mathcal{Q}_h^{k-2}((\nabla\times)^2\mathbf{u})\rangle_{\partial T} \\ & - \langle \mathbf{v}_n\times\mathbf{n}, \mathcal{Q}_h^{k-2}((\nabla\times)^2\mathbf{u})\rangle_{\partial T} \\ =\ & ((\nabla\times)^2\mathbf{v}_0, \mathcal{Q}_h^{k-2}((\nabla\times)^2\mathbf{u}))_T + \langle (\mathbf{v}_0 - \mathbf{v}_b)\times\mathbf{n}, \nabla\times\mathcal{Q}_h^{k-2}((\nabla\times)^2\mathbf{u})\rangle_{\partial T} \\ & + \langle (\nabla\times\mathbf{v}_0 - \mathbf{v}_n)\times\mathbf{n}, \mathcal{Q}_h^{k-2}((\nabla\times)^2\mathbf{u})\rangle_{\partial T} \\ =\ & ((\nabla\times)^2\mathbf{v}_0, ((\nabla\times)^2\mathbf{u}))_T + \langle (\mathbf{v}_0 - \mathbf{v}_b)\times\mathbf{n}, \nabla\times\mathcal{Q}_h^{k-2}((\nabla\times)^2\mathbf{u})\rangle_{\partial T} \\ & + \langle (\nabla\times\mathbf{v}_0 - \mathbf{v}_n)\times\mathbf{n}, \mathcal{Q}_h^{k-2}((\nabla\times)^2\mathbf{u})\rangle_{\partial T}. \end{aligned} \tag{5.7}$$

Taking $\mathbf{v} = \mathbf{v}_0$ in (5.1) where $\mathbf{v}_h = \{\mathbf{v}_0, \mathbf{v}_b, \mathbf{v}_n\} \in V_h^0$ and using the usual integration by parts, we get

$$\begin{aligned} &\sum_{T\in\mathcal{T}_h} ((\nabla\times)^2\mathbf{u}, (\nabla\times)^2\mathbf{v}_0)_T + \langle (\nabla\times)^3\mathbf{u}, (\mathbf{v}_0 - \mathbf{v}_b)\times\mathbf{n}\rangle_{\partial T} \\ &+ \langle (\nabla\times)^2\mathbf{u}, \nabla\times\mathbf{v}_0\times\mathbf{n} - \mathbf{v}_n\times\mathbf{n}\rangle_{\partial T} + (\nabla p, \mathbf{v}_0)_T = \sum_{T\in\mathcal{T}_h} (\mathbf{f}, \mathbf{v}_0)_T, \end{aligned} \tag{5.8}$$

where we used the facts that

$$\sum_{T\in\mathcal{T}_h}\langle(\nabla\times)^2\mathbf{u},\mathbf{v}_n\times\mathbf{n}\rangle_{\partial T}=\langle(\nabla\times)^2\mathbf{u},\mathbf{v}_n\times\mathbf{n}\rangle_{\partial\Omega}=0,$$

$$\sum_{T\in\mathcal{T}_h}\langle(\nabla\times)^3\mathbf{u},\mathbf{v}_b\times\mathbf{n}\rangle_{\partial T}=\langle(\nabla\times)^3\mathbf{u},\mathbf{v}_b\times\mathbf{n}\rangle_{\partial\Omega}=0.$$

Substituting (5.8) into (5.7) gives

$$\begin{aligned}&((\nabla\times)^2_w\mathbf{Q}_h\mathbf{u},(\nabla\times)^2_w\mathbf{v}_h)\\ =&(\mathbf{f}-\nabla p,\mathbf{v}_0)+\langle(\mathbf{v}_0-\mathbf{v}_b)\times\mathbf{n},\nabla\times(\mathcal{Q}_h^{k-2}-I)((\nabla\times)^2\mathbf{u})\rangle_{\partial T}\\ &+\langle(\nabla\times\mathbf{v}_0-\mathbf{v}_n)\times\mathbf{n},(\mathcal{Q}_h^{k-2}-I)((\nabla\times)^2\mathbf{u})\rangle_{\partial T}.\end{aligned}\tag{5.9}$$

It follows from (4.16) that

$$b(\mathbf{v}_h,Q_hp)=(\nabla_w(Q_hp),\mathbf{v}_0)=(\mathcal{Q}_h^k(\nabla p),\mathbf{v}_0)=(\nabla p,\mathbf{v}_0).\tag{5.10}$$

Combining (5.9)–(5.10) gives

$$\begin{aligned}&s_1(\mathbf{Q}_h\mathbf{u},\mathbf{v}_h)+a(\mathbf{Q}_h\mathbf{u},\mathbf{v}_h)+b(\mathbf{v}_h,Q_hp)\\ =&\ (\mathbf{f},\mathbf{v}_0)+\langle(\mathbf{v}_0-\mathbf{v}_b)\times\mathbf{n},\nabla\times(\mathcal{Q}_h^{k-2}-I)((\nabla\times)^2\mathbf{u})\rangle_{\partial T}\\ &+\langle(\nabla\times\mathbf{v}_0-\mathbf{v}_n)\times\mathbf{n},(\mathcal{Q}_h^{k-2}-I)((\nabla\times)^2\mathbf{u})\rangle_{\partial T}+s_1(\mathbf{Q}_h\mathbf{u},\mathbf{v}_h).\end{aligned}$$

Subtracting (4.7) from the above equation gives (5.5).

Note that $q_h=\{q_0,q_b\}$. To derive (5.6), taking $q=q_0$ in (5.2) and using the usual integration by parts, we have

$$0=-\sum_{T\in\mathcal{T}_h}(\mathbf{u},\nabla q_0)+\sum_{T\in\mathcal{T}_h}\langle\mathbf{u}\cdot\mathbf{n},q_0-q_b\rangle_{\partial T},\tag{5.11}$$

where we used $\sum_{T\in\mathcal{T}_h}\langle\mathbf{u}\cdot\mathbf{n},q_b\rangle_{\partial T}=0$. Using (3.3) and the usual integration by parts gives

$$\begin{aligned}-b(\mathbf{Q}_h\mathbf{u},q_h)&=-\sum_{T\in\mathcal{T}_h}(\mathbf{Q}_0\mathbf{u},\nabla_wq_h)_T\\ &=\sum_{T\in\mathcal{T}_h}(q_0,\nabla\cdot(\mathbf{Q}_0\mathbf{u}))_T-\langle q_b,\mathbf{Q}_0\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T}\\ &=\sum_{T\in\mathcal{T}_h}-(\nabla q_0,\mathbf{Q}_0\mathbf{u})_T+\langle q_0-q_b,\mathbf{Q}_0\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T}\\ &=\sum_{T\in\mathcal{T}_h}-(\nabla q_0,\mathbf{u})_T+\langle q_0-q_b,\mathbf{Q}_0\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T}\\ &=\sum_{T\in\mathcal{T}_h}\langle q_0-q_b,(\mathbf{Q}_0-I)\mathbf{u}\cdot\mathbf{n}\rangle_{\partial T},\end{aligned}\tag{5.12}$$

where we used (5.11) on the last line.

Subtracting (4.8) from the above equation completes the proof of (5.6).

This completes the proof of the lemma. □

6. Error estimates

For any $\mathbf{v}\in V_h^0$, we define the energy norm $|||\mathbf{v}|||$ as follows

$$|||\mathbf{v}|||^2=\sum_{T\in\mathcal{T}_h}\|(\nabla\times)^2_w\mathbf{v}\|_T^2+s_1(\mathbf{v},\mathbf{v}).\tag{6.1}$$

It is easy to check that $|||\cdot|||$ is a semi-norm in V_h^0. We further introduce a norm in V_h^0; i.e.,

$$|||\mathbf{v}|||_1=|||\mathbf{v}|||+\Big(\sum_{T\in\mathcal{T}_h}\|\nabla\cdot\mathbf{v}_0\|_T^2\Big)^{\frac12}+\Big(\sum_{e\in\mathcal{E}_h^0}h_T^{-1}\|[\![\mathbf{v}_0\cdot\mathbf{n}]\!]\|_e^2\Big)^{\frac12}.\tag{6.2}$$

For any $q\in W_h^0$, we define the following norm

$$|||q|||_0=(s_2(q,q))^{\frac12}.$$

Recall that $\mathcal{T}_h$ is a shape-regular finite element partition of the domain Ω. For any $T \in \mathcal{T}_h$ and $\varphi \in H^1(T)$, the following trace inequality holds true [19]:

$$\|\varphi\|_{\partial T}^2 \leq C(h_T^{-1}\|\varphi\|_T^2 + h_T\|\varphi\|_{1,T}^2). \tag{6.3}$$

Furthermore, if φ is a polynomial on T, the standard inverse inequality yields

$$\|\varphi\|_{\partial T}^2 \leq Ch_T^{-1}\|\varphi\|_T^2. \tag{6.4}$$

Lemma 6.1. *Let $k \geq 2$, and $s \in [1, k]$. Suppose $\mathbf{u} \in [H^{k+1}(\Omega)]^3$ and $(\nabla\times)^2\mathbf{u} \in [H^k(\Omega)]^3$. Then, for $(\mathbf{v}, q) \in V_h^0 \times W_h^0$, the following estimates hold true; i.e.,*

$$|s_1(\mathbf{Q}_h\mathbf{u}, \mathbf{v})| \leq Ch^{s-1}\|\mathbf{u}\|_{s+1}s_1(\mathbf{v}, \mathbf{v})^{\frac{1}{2}}, \tag{6.5}$$
$$|\ell_1(\mathbf{u}, \mathbf{v})| \leq Ch^{s-1}\|(\nabla\times)^2\mathbf{u}\|_{s-1}s_1(\mathbf{v}, \mathbf{v})^{\frac{1}{2}}, \tag{6.6}$$
$$|\ell_2(\mathbf{u}, q)| \leq Ch^{s-1}\|\mathbf{u}\|_{s+1}|\!|\!|q|\!|\!|_0, \tag{6.7}$$
$$|s_2(Q_h p, q)| = 0. \tag{6.8}$$

Proof. Using the Cauchy–Schwarz inequality, the trace inequality (6.3), gives

$$\begin{aligned}
|s_1(\mathbf{Q}_h\mathbf{u}, \mathbf{v})| = \Big| &\sum_{T\in\mathcal{T}_h} h_T^{-3}\langle(\mathbf{Q}_0\mathbf{u} - \mathbf{Q}_b\mathbf{u}) \times \mathbf{n}, (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n}\rangle_{\partial T} \\
&+ h_T^{-1}\langle\nabla \times \mathbf{Q}_0\mathbf{u} \times \mathbf{n} - \mathbf{Q}_n(\nabla \times \mathbf{u}) \times \mathbf{n}, \nabla \times \mathbf{v}_0 \times \mathbf{n} - \mathbf{v}_n \times \mathbf{n}\rangle_{\partial T}\Big| \\
\leq\ & \{\big(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|\mathbf{Q}_0\mathbf{u} - \mathbf{u}\|_{\partial T}^2\big)^{\frac{1}{2}} + \big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\nabla \times (\mathbf{Q}_0\mathbf{u} - \mathbf{u})\|_{\partial T}^2\big)^{\frac{1}{2}}\}s_1(\mathbf{v}, \mathbf{v})^{\frac{1}{2}} \\
\leq\ & \{\big(\sum_{T\in\mathcal{T}_h} h_T^{-4}\|\mathbf{Q}_0\mathbf{u} - \mathbf{u}\|_T^2 + h_T^{-2}\|\mathbf{Q}_0\mathbf{u} - \mathbf{u}\|_{1,T}^2\big)^{\frac{1}{2}} \\
&+ \big(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|\nabla \times (\mathbf{Q}_0\mathbf{u} - \mathbf{u})\|_T^2 + \|\nabla \times (\mathbf{Q}_0\mathbf{u} - \mathbf{u})\|_{1,T}^2\big)^{\frac{1}{2}}\}s_1(\mathbf{v}, \mathbf{v})^{\frac{1}{2}} \\
\leq\ & Ch^{s-1}\|\mathbf{u}\|_{s+1}s_1(\mathbf{v}, \mathbf{v})^{\frac{1}{2}}.
\end{aligned}$$

Using the Cauchy–Schwarz inequality, the trace inequality (6.3), gives

$$\begin{aligned}
&\ell_1(\mathbf{u}, \mathbf{v}) \\
=& \sum_{T\in\mathcal{T}_h}\langle(\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n}, \nabla \times (\mathcal{Q}_h^{k-2} - I)((\nabla\times)^2\mathbf{u})\rangle_{\partial T} \\
&+ \langle(\nabla \times \mathbf{v}_0 - \mathbf{v}_n) \times \mathbf{n}, (\mathcal{Q}_h^{k-2} - I)((\nabla\times)^2\mathbf{u})\rangle_{\partial T} \\
\leq& \{\big(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla \times (\mathcal{Q}_h^{k-2} - I)((\nabla\times)^2\mathbf{u})\|_{\partial T}^2\big)^{\frac{1}{2}} \\
&+ \big(\sum_{T\in\mathcal{T}_h} h_T\|(\mathcal{Q}_h^{k-2} - I)((\nabla\times)^2\mathbf{u})\|_{\partial T}^2\big)^{\frac{1}{2}}\}s_1(\mathbf{v}, \mathbf{v})^{\frac{1}{2}} \\
\leq& \{\big(\sum_{T\in\mathcal{T}_h} h_T^2\|\nabla \times (\mathcal{Q}_h^{k-2} - I)((\nabla\times)^2\mathbf{u})\|_T^2 + h_T^4\|\nabla \times (\mathcal{Q}_h^{k-2} - I)((\nabla\times)^2\mathbf{u})\|_{1,T}^2\big)^{\frac{1}{2}} \\
&+ \big(\sum_{T\in\mathcal{T}_h} \|(\mathcal{Q}_h^{k-2} - I)((\nabla\times)^2\mathbf{u})\|_T^2 + h_T^2\|(\mathcal{Q}_h^{k-2} - I)((\nabla\times)^2\mathbf{u})\|_{1,T}^2\big)^{\frac{1}{2}}\}s_1(\mathbf{v}, \mathbf{v})^{\frac{1}{2}} \\
\leq& Ch^{s-1}\|(\nabla\times)^2\mathbf{u}\|_{s-1}s_1(\mathbf{v}, \mathbf{v})^{\frac{1}{2}}.
\end{aligned}$$

Similarly, using the Cauchy–Schwarz inequality, the trace inequality (6.3) gives

$$\begin{aligned}
\ell_2(\mathbf{u}, q) &= \sum_{T\in\mathcal{T}_h}\langle q_0 - q_b, (I - \mathbf{Q}_0)\mathbf{u} \cdot \mathbf{n}\rangle_{\partial T} \\
&\leq \Big(\sum_{T\in\mathcal{T}_h} h_T^3\|q_0 - q_b\|_{\partial T}^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|(I - \mathbf{Q}_0)\mathbf{u} \cdot \mathbf{n}\|_{\partial T}^2\Big)^{\frac{1}{2}}
\end{aligned}$$

$$\le \Big(\sum_{T\in\mathcal{T}_h} h_T^{-4}\|(I-\mathbf{Q}_0)\mathbf{u}\cdot\mathbf{n}\|_T^2 + h_T^{-2}\|(I-\mathbf{Q}_0)\mathbf{u}\cdot\mathbf{n}\|_{1,T}^2\Big)^{\frac{1}{2}} |||q|||_0$$
$$\le Ch^{s-1}\|\mathbf{u}\|_{s+1}|||q|||_0.$$

Since $p=0$, it is easy to obtain $s_2(Q_hp,q)=0$. □

Theorem 6.2. *Let $k\ge 2$. Suppose that $\mathbf{u}\in[H^{k+1}(\Omega)]^3$. The following error estimate holds*

$$|||\mathbf{e}_h||| + |||\epsilon_h|||_0 \le Ch^{k-1}(\|\mathbf{u}\|_{k+1} + \|(\nabla\times)^2\mathbf{u}\|_{k-1}). \tag{6.9}$$

Proof. Letting $\mathbf{v}_h=\mathbf{e}_h$ in (5.5) and $q_h=\epsilon_h$ in (5.6) and adding the two equations, we have

$$|||\mathbf{e}_h|||^2 + |||\epsilon_h|||_0^2 = s_1(\mathbf{Q}_h\mathbf{u},\mathbf{e}_h) + s_2(Q_hp,\epsilon_h) + \ell_1(\mathbf{u},\mathbf{e}_h) - \ell_2(\mathbf{u},\epsilon_h).$$

Using Lemma 6.1 completes the proof of the theorem. □

7. L^2 error estimates

We consider an auxiliary problem of finding $(\boldsymbol{\phi};\xi)$ such that

$$\begin{aligned}
(\nabla\times)^4\boldsymbol{\phi} + \nabla\xi &= \mathbf{e}_0, && \text{in } \Omega,\\
\nabla\cdot\boldsymbol{\phi} &= 0, && \text{in } \Omega,\\
\boldsymbol{\phi}\times\mathbf{n} &= 0, && \text{on } \partial\Omega,\\
\nabla\times\boldsymbol{\phi} &= 0, && \text{on } \partial\Omega,\\
\xi &= 0, && \text{on } \partial\Omega.
\end{aligned} \tag{7.1}$$

Let $t_0=\min\{k,3\}$. We assume the regularity property holds true in the sense that $\boldsymbol{\phi}$ and ξ satisfy

$$\|\boldsymbol{\phi}\|_{t_0+1} + \|(\nabla\times)^2\boldsymbol{\phi}\|_{t_0-1} + \|\xi\|_1 \le C\|\mathbf{e}_0\|. \tag{7.2}$$

Theorem 7.1. *Let $k\ge 2$ and $t_0=\min\{k,3\}$. Suppose that $\mathbf{u}\in[H^{k+1}(\Omega)]^3$. The following estimate holds*

$$\|\mathbf{e}_0\| \le Ch^{t_0+k-2}(\|\mathbf{u}\|_{k+1} + \|(\nabla\times)^2\mathbf{u}\|_{k-1}). \tag{7.3}$$

In other words, we have a sub-optimal order of convergence for $k=2$ and an optimal order of convergence for $k\ge 3$.

Proof. Using the usual integration by parts, letting $\mathbf{u}=\boldsymbol{\phi}$ and $\mathbf{v}_h=\mathbf{e}_h$ in (5.7), letting $\mathbf{v}_h=\mathbf{Q}_h\boldsymbol{\phi}$ in (5.5), letting $\mathbf{u}=\boldsymbol{\phi}$ and $q_h=\epsilon_h$ in (5.12), letting $q_h=Q_h\xi$ in (4.16) and (5.6), we have

$$\begin{aligned}
&\|\mathbf{e}_0\|^2\\
&= \sum_{T\in\mathcal{T}_h}((\nabla\times)^4\boldsymbol{\phi}+\nabla\xi,\mathbf{e}_0)_T\\
&= \sum_{T\in\mathcal{T}_h}((\nabla\times)^2\boldsymbol{\phi},(\nabla\times)^2\mathbf{e}_0)_T + \langle\nabla\times\mathbf{e}_0,\mathbf{n}\times(\nabla\times)^2\boldsymbol{\phi}\rangle_{\partial T}\\
&\quad + \langle\mathbf{e}_0,\mathbf{n}\times(\nabla\times)^3\boldsymbol{\phi}\rangle_{\partial T} + (\mathbf{e}_0,\mathcal{Q}_h^k\nabla\xi)_T\\
&= \sum_{T\in\mathcal{T}_h}((\nabla\times)_w^2\mathbf{Q}_h\boldsymbol{\phi},(\nabla\times)_w^2\mathbf{e}_h)_T - \langle(\mathbf{e}_0-\mathbf{e}_b)\times\mathbf{n},\nabla\times\mathcal{Q}_h^{k-1}((\nabla\times)^2\boldsymbol{\phi})\rangle_{\partial T}\\
&\quad - \langle(\nabla\times\mathbf{e}_0-\mathbf{e}_n)\times\mathbf{n},\mathcal{Q}_h^{k-1}((\nabla\times)^2\boldsymbol{\phi})\rangle_{\partial T}\\
&\quad + \langle\nabla\times\mathbf{e}_0,\mathbf{n}\times(\nabla\times)^2\boldsymbol{\phi}\rangle_{\partial T} + \langle\mathbf{e}_0,\mathbf{n}\times(\nabla\times)^3\boldsymbol{\phi}\rangle_{\partial T} + (\nabla_wQ_h\xi,\mathbf{e}_0)_T\\
&= -s_1(\mathbf{e}_h,\mathbf{Q}_h\boldsymbol{\phi}) - b(\mathbf{Q}_h\boldsymbol{\phi},\epsilon_h) + s_1(\mathbf{Q}_h\mathbf{u},\mathbf{Q}_h\boldsymbol{\phi}) + \ell_1(\mathbf{u},\mathbf{Q}_h\boldsymbol{\phi})\\
&\quad + \sum_{T\in\mathcal{T}_h} -\langle(\mathbf{e}_0-\mathbf{e}_b)\times\mathbf{n},\nabla\times(\mathcal{Q}_h^{k-1}-I)((\nabla\times)^2\boldsymbol{\phi})\rangle_{\partial T}\\
&\quad - \langle(\nabla\times\mathbf{e}_0-\mathbf{e}_n)\times\mathbf{n},(\mathcal{Q}_h^{k-1}-I)((\nabla\times)^2\boldsymbol{\phi})\rangle_{\partial T}\\
&\quad + s_2(\epsilon_h,Q_h\xi) - s_2(Q_hp,Q_h\xi) + \ell_2(\mathbf{u},Q_h\xi)
\end{aligned}$$

$$
\begin{aligned}
&= - s_1(\mathbf{e}_h, \mathbf{Q}_h\boldsymbol{\phi}) + s_1(\mathbf{Q}_h\mathbf{u}, \mathbf{Q}_h\boldsymbol{\phi}) + \ell_1(\mathbf{u}, \mathbf{Q}_h\boldsymbol{\phi}) \\
&\quad + \sum_{T\in\mathcal{T}_h} \langle \epsilon_0 - \epsilon_b, (\mathbf{Q}_0 - I)\boldsymbol{\phi}\cdot\mathbf{n}\rangle_{\partial T} \\
&\quad - \langle (\mathbf{e}_0 - \mathbf{e}_b)\times\mathbf{n}, \nabla\times(\mathcal{Q}_h^{k-1} - I)((\nabla\times)^2\boldsymbol{\phi})\rangle_{\partial T} \\
&\quad - \langle (\nabla\times\mathbf{e}_0 - \mathbf{e}_n)\times\mathbf{n}, (\mathcal{Q}_h^{k-1} - I)((\nabla\times)^2\boldsymbol{\phi})\rangle_{\partial T} \\
&\quad + s_2(\epsilon_h, Q_h\xi) - s_2(Q_hp, Q_h\xi) + \ell_2(\mathbf{u}, Q_h\xi),
\end{aligned}
\tag{7.4}
$$

where we used $\mathbf{e}_b\times\mathbf{n} = 0$ and $\mathbf{e}_n\times\mathbf{n} = 0$ on $\partial\Omega$.

Next, we shall estimate the terms on the last line of (7.4) one by one.

Recall that $t_0 = \min\{k, 3\}$. Using (6.5) with $\mathbf{v} = \mathbf{e}_h$ and $\mathbf{u} = \boldsymbol{\phi}$, we have

$$
|s_1(\mathbf{e}_h, \mathbf{Q}_h\boldsymbol{\phi})| \le Ch^{t_0-1}\|\boldsymbol{\phi}\|_{t_0+1} s_1(\mathbf{e}_h, \mathbf{e}_h)^{\frac{1}{2}} \le Ch^{t_0-1}\|\boldsymbol{\phi}\|_{t_0+1}|||\mathbf{e}_h|||. \tag{7.5}
$$

Using (6.5) with $\mathbf{v} = \mathbf{Q}_h\boldsymbol{\phi}$, we have

$$
|s_1(\mathbf{Q}_h\mathbf{u}, \mathbf{Q}_h\boldsymbol{\phi})| \le Ch^{k-1}\|\mathbf{u}\|_{k+1} s_1(\mathbf{Q}_h\boldsymbol{\phi}, \mathbf{Q}_h\boldsymbol{\phi})^{\frac{1}{2}}. \tag{7.6}
$$

Note that

$$
\begin{aligned}
& s_1(\mathbf{Q}_h\boldsymbol{\phi}, \mathbf{Q}_h\boldsymbol{\phi})^{\frac{1}{2}} \\
\le & C\sum_{T\in\mathcal{T}_h} h_T^{-3}\|\mathbf{Q}_0\boldsymbol{\phi}\times\mathbf{n} - \mathbf{Q}_b\boldsymbol{\phi}\times\mathbf{n}\|_{\partial T}^2 \\
& + h_T^{-1}\|\nabla\times\mathbf{Q}_0\boldsymbol{\phi}\times\mathbf{n} - \mathbf{Q}_n(\nabla\times\boldsymbol{\phi})\times\mathbf{n}\|_{\partial T}^2 \\
\le\ & C\sum_{T\in\mathcal{T}_h} h_T^{-4}\|\mathbf{Q}_0\boldsymbol{\phi} - \boldsymbol{\phi}\|_T^2 + h_T^{-2}\|\mathbf{Q}_0\boldsymbol{\phi} - \boldsymbol{\phi}\|_{1,T}^2 \\
& + h_T^{-2}\|\mathbf{Q}_0\boldsymbol{\phi} - \boldsymbol{\phi}\|_{1,T}^2 + \|\mathbf{Q}_0\boldsymbol{\phi} - \boldsymbol{\phi}\|_{2,T}^2 \\
\le\ & Ch^{t_0-1}\|\boldsymbol{\phi}\|_{t_0+1}.
\end{aligned}
\tag{7.7}
$$

where we used trace inequality (6.3). Substituting (7.7) into (7.6) gives

$$
|s_1(\mathbf{Q}_h\mathbf{u}, \mathbf{Q}_h\boldsymbol{\phi})| \le Ch^{k-1}\|\mathbf{u}\|_{k+1}h^{t_0-1}\|\boldsymbol{\phi}\|_{t_0+1}. \tag{7.8}
$$

Using (6.6) with $\mathbf{v} = \mathbf{Q}_h\boldsymbol{\phi}$ and (7.7), we have

$$
|\ell_1(\mathbf{u}, \mathbf{Q}_h\boldsymbol{\phi})| \le Ch^{k-1}\|(\nabla\times)^2\mathbf{u}\|_{k-1} s_1(\mathbf{Q}_h\boldsymbol{\phi}, \mathbf{Q}_h\boldsymbol{\phi})^{\frac{1}{2}} \le Ch^{k-1}\|(\nabla\times)^2\mathbf{u}\|_{k-1}h^{t_0-1}\|\boldsymbol{\phi}\|_{t_0+1}. \tag{7.9}
$$

Using (6.7) with $\mathbf{u} = \boldsymbol{\phi}$ and $q_h = \epsilon_h$, we have

$$
\Big|\sum_{T\in\mathcal{T}_h} \langle \epsilon_0 - \epsilon_b, (\mathbf{Q}_0 - I)\boldsymbol{\phi}\cdot\mathbf{n}\rangle_{\partial T}\Big| \le Ch^{t_0-1}\|\boldsymbol{\phi}\|_{t_0+1}|||\epsilon_h|||_0, \tag{7.10}
$$

Using (6.6) with $\mathbf{u} = \boldsymbol{\phi}$ and $\mathbf{v} = \mathbf{e}_h$, we have

$$
\begin{aligned}
& \Big|\sum_{T\in\mathcal{T}_h} \langle (\mathbf{e}_0 - \mathbf{e}_b)\times\mathbf{n}, \nabla\times(\mathcal{Q}_h^{k-1} - I)((\nabla\times)^2\boldsymbol{\phi})\rangle_{\partial T} \\
& + \langle (\nabla\times\mathbf{e}_0 - \mathbf{e}_n)\times\mathbf{n}, (\mathcal{Q}_h^{k-1} - I)((\nabla\times)^2\boldsymbol{\phi})\rangle_{\partial T}\Big| \\
\le & Ch^{t_0-1}\|(\nabla\times)^2\boldsymbol{\phi}\|_{t_0-1} s_1(\mathbf{e}_h, \mathbf{e}_h)^{\frac{1}{2}} \le\ Ch^{t_0-1}\|(\nabla\times)^2\boldsymbol{\phi}\|_{t_0-1}|||\mathbf{e}_h|||.
\end{aligned}
\tag{7.11}
$$

Using the Cauchy–Schwarz inequality and trace inequality (6.3), we have

$$
\begin{aligned}
s_2(\epsilon_h, Q_h\xi) &= \sum_{T\in\mathcal{T}_h} h_T^3\langle \epsilon_0 - \epsilon_b, Q_0\xi - Q_b\xi\rangle_{\partial T} \\
&\le\ C|||\epsilon_h|||_0\Big(\sum_{T\in\mathcal{T}_h} h_T^3\|Q_0\xi - \xi\|_{\partial T}^2\Big)^{\frac{1}{2}} \\
&\le\ C|||\epsilon_h|||_0\Big(\sum_{T\in\mathcal{T}_h} h_T^2\|Q_0\xi - \xi\|_T^2 + h_T^4\|Q_0\xi - \xi\|_{1,T}^2\Big)^{\frac{1}{2}} \\
&\le\ Ch^2\|\xi\|_1|||\epsilon_h|||_0.
\end{aligned}
\tag{7.12}
$$

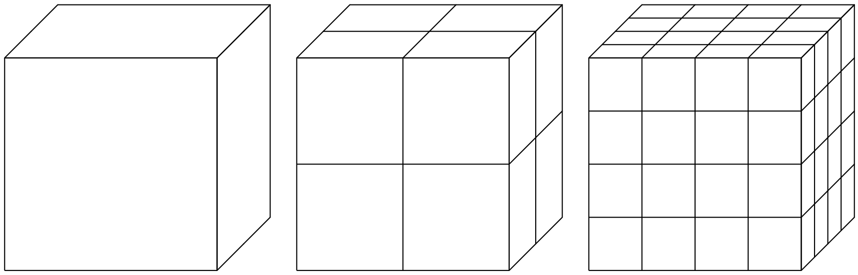

Fig. 8.1. The first three levels of uniform cubic grids used in Table 8.1.

Using (6.8) with $q = Q_h\xi$, we have

$$s_2(Q_hp, Q_h\xi) = 0. \tag{7.13}$$

Using (6.7) with $q = Q_h\xi$ and the trace inequality (6.3), we have

$$\begin{aligned}
\ell_2(\mathbf{u}, Q_h\xi) &\le Ch^{k-1}\|\mathbf{u}\|_{k+1}|||Q_h\xi|||_0 \\
&\le Ch^{k-1}\|\mathbf{u}\|_{k+1}\Big(\sum_{T\in\mathcal{T}_h} h_T^3\|Q_0\xi - Q_b\xi\|_{\partial T}^2\Big)^{\frac{1}{2}} \\
&\le Ch^{k-1}\|\mathbf{u}\|_{k+1}\Big(\sum_{T\in\mathcal{T}_h} h_T^3\|Q_0\xi - \xi\|_{\partial T}^2\Big)^{\frac{1}{2}} \\
&\le Ch^{k-1}\|\mathbf{u}\|_{k+1}\Big(\sum_{T\in\mathcal{T}_h} h_T^2\|Q_0\xi - \xi\|_T^2 + h_T^4\|Q_0\xi - \xi\|_{1,T}^2\Big)^{\frac{1}{2}} \\
&\le Ch^{k-1}\|\mathbf{u}\|_{k+1}h^2\|\xi\|_1.
\end{aligned} \tag{7.14}$$

Substituting (7.5)–(7.14) into (7.4) and using the regularity assumption (7.2) and the error estimate (6.9) gives

$$\begin{aligned}
\|\mathbf{e}_0\|^2 \le& Ch^{t_0-1}\|\boldsymbol{\phi}\|_{t_0+1}|||\mathbf{e}_h||| + Ch^{k-1}\|\mathbf{u}\|_{k+1}h^{t_0-1}\|\boldsymbol{\phi}\|_{t_0+1} \\
&+ Ch^{k-1}\|(\nabla\times)^2\mathbf{u}\|_{k-1}h^{t_0-1}\|\boldsymbol{\phi}\|_{t_0+1} + Ch^{t_0-1}\|\boldsymbol{\phi}\|_{t_0+1}|||\epsilon_h|||_0 \\
&+ Ch^{t_0-1}\|(\nabla\times)^2\boldsymbol{\phi}\|_{t_0-1}|||\mathbf{e}_h||| + Ch^2\|\xi\|_1|||\epsilon_h|||_0 \\
&+ Ch^{k-1}\|\mathbf{u}\|_{k+1}h^2\|\xi\|_1,
\end{aligned}$$

which yields

$$\|\mathbf{e}_0\|^2 \le Ch^{t_0+k-2}(\|\mathbf{u}\|_{k+1} + \|(\nabla\times)^2\mathbf{u}\|_{k-1})\|\mathbf{e}_0\|.$$

This completes the proof of the theorem. □

8. Numerical tests

In this section, we present some numerical results for the WG finite element method for solving the quad-curl problem analyzed in the previous sections. To this end, we shall solve the following quad-curl problem with non-homogeneous boundary conditions on an unit cube domain $\Omega = (0,1)^3$: Find an known $\mathbf{u}$ such that

$$\begin{aligned}
(\nabla\times)^4\mathbf{u} &= \mathbf{f}, && \text{in} \quad \Omega, \\
\nabla\cdot\mathbf{u} &= 0, && \text{in} \quad \Omega, \\
\mathbf{u}\times\mathbf{n} &= \mathbf{g}_1, && \text{on} \quad \partial\Omega, \\
\nabla\times\mathbf{u}\times\mathbf{n} &= \mathbf{g}_2, && \text{on} \quad \partial\Omega,
\end{aligned} \tag{8.1}$$

where $\mathbf{f}$, $\mathbf{g}_1$ and $\mathbf{g}_2$ are calculated by the exact solution

$$\mathbf{u} = \begin{pmatrix} -2x^2y^2z \\ 2x^2y^3z \\ -xy^2z^2(3x-2) \end{pmatrix}.$$

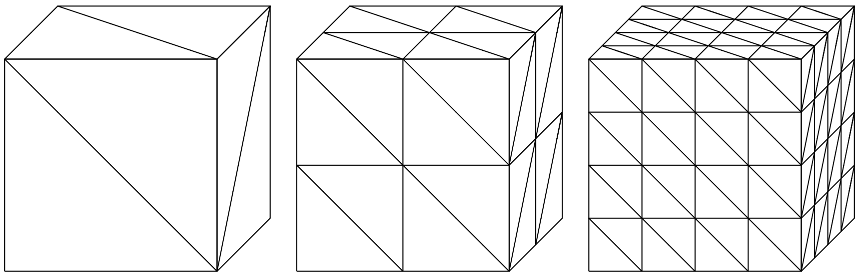

Fig. 8.2. The first three levels of uniform tetrahedral grids used in Table 8.2.

Table 8.1
Error profiles and convergence rates on uniform cubic grids shown in Fig. 8.1 for (8.1).

Level	$\|\mathbf{Q}_h\mathbf{u}-\mathbf{u}_h\|$	Rate	$\|\|\|\mathbf{Q}_h\mathbf{u}-\mathbf{u}_h\|\|\|$	Rate	$\|p_h\|$	Rate
	by the $\{P_2, P_2, P_1\} - P_1$ with $\{P_2, P_2\} - P_2$ WG method					
2	0.5263E+00	3.8	0.4078E+01	1.7	0.1742E−01	3.9
3	0.3345E−01	4.0	0.1237E+01	1.7	0.1947E−02	3.2
4	0.2151E−02	4.0	0.3739E+00	1.7	0.2231E−03	3.1
	by the $\{P_3, P_3, P_2\} - P_2$ with $\{P_3, P_3\} - P_3$ WG method					
2	0.8522E−01	5.3	0.1512E+01	2.2	0.1117E−01	4.0
3	0.2190E−02	5.3	0.2365E+00	2.7	0.4298E−03	4.7
4	0.6360E−04	5.1	0.3444E−01	2.8	0.1841E−04	4.5
	by the $\{P_4, P_4, P_3\} - P_3$ with $\{P_4, P_4\} - P_4$ WG method					
2	0.9472E−02	6.3	0.4145E+00	3.4	0.1476E−02	4.2
3	0.1841E−03	5.7	0.3102E−01	3.7	0.1760E−04	6.4
4	0.2992E−05	5.9	0.2162E−02	3.8	0.3430E−06	5.7
	by the $\{P_5, P_5, P_4\} - P_4$ with $\{P_5, P_5\} - P_5$ WG method					
1	0.1010E−02	0.0	0.5724E−01	0.0	0.3493E−03	0.0
2	0.1684E−04	5.9	0.3565E−02	4.0	0.5073E−05	6.1
3	0.2580E−06	6.0	0.2204E−03	4.0	0.9438E−07	5.7

Table 8.2
Error profiles and convergence rates on tetrahedral grids shown in Fig. 8.2 for (8.1).

Level	$\|\mathbf{Q}_h\mathbf{u}-\mathbf{u}_h\|$	Rate	$\|\|\|\mathbf{Q}_h\mathbf{u}-\mathbf{u}_h\|\|\|$	Rate	$\|p_h\|$	Rate
	by the $\{P_2, P_2, P_1\} - P_1$ with $\{P_2, P_2\} - P_2$ WG method					
2	0.444E+00	3.9	0.357E+01	1.8	0.288E−01	3.8
3	0.279E−01	4.0	0.117E+01	1.6	0.193E−02	3.9
4	0.172E−02	4.0	0.451E+00	1.4	0.304E−03	2.7
	by the $\{P_3, P_3, P_2\} - P_2$ with $\{P_3, P_3\} - P_3$ WG method					
1	0.187E+01	0.0	0.480E+01	0.0	0.181E+00	0.0
2	0.551E−01	5.1	0.724E+00	2.7	0.764E−02	4.6
3	0.163E−02	5.1	0.128E+00	2.5	0.257E−03	4.9
	by the $\{P_4, P_4, P_3\} - P_3$ with $\{P_4, P_4\} - P_4$ WG method					
1	0.265E+00	0.0	0.888E+00	0.0	0.308E−01	0.0
2	0.411E−02	6.0	0.738E−01	3.6	0.632E−03	5.6
3	0.693E−04	5.9	0.721E−02	3.4	0.139E−04	5.5

We first compute the solution of (8.1) by the P_k weak Galerkin finite element method (4.7)–(4.8) on uniform cubic grids shown in Fig. 8.1. For simplicity of notations, we denote the WG finite element solution $(\mathbf{u}_h; p_h)$ by $\{P_k, P_k, P_{k-1}\} - P_{k-1}$ with $\{P_k, P_k\} - P_k$. In Table 8.1, we list the errors in various norms and the computed orders of convergence for P_2, P_3, P_4 and P_5 finite element solutions on uniform cubic grids. It seems we do have one order superconvergence in most cases in Table 8.1.

Next we compute the solution of (8.1) again by the P_k weak Galerkin finite element method but on uniform tetrahedral grids shown in Fig. 8.2. In Table 8.2, we list the errors in various norms and the computed orders of convergence for P_2, P_3 and P_4 finite element solutions on uniform tetrahedral grids. It seems we do have one order superconvergence in most cases in Table 8.2.

Data availability

Data will be made available on request.

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