

# Superconvergence of numerical gradient for weak Galerkin finite element methods on nonuniform Cartesian partitions in three dimensions

Dan Li<sup>a,1</sup>, Yufeng Nie<sup>b,2</sup>, Chunmei Wang<sup>c,\*,3</sup>

<sup>a</sup> Research Center for Computational Science, Northwestern Polytechnical University, Xi'an, Shannxi 710129, China

<sup>b</sup> Research Center for Computational Science, Northwestern Polytechnical University, Xi'an, Shannxi, 710129, China

<sup>c</sup> Department of Mathematics, Texas Tech University, Lubbock, TX 79409, USA

## ARTICLE INFO

### Article history:

Received 18 October 2018

Received in revised form 23 February 2019

Accepted 5 March 2019

Available online 28 March 2019

### Keywords:

Weak Galerkin finite element method

Superconvergence

Non-uniform cubic partitions

Second order elliptic problem

Three dimensions

## ABSTRACT

A superconvergence error estimate for the gradient approximation of the second order elliptic problem in three dimensions is analyzed by using weak Galerkin finite element scheme on the uniform and non-uniform cubic partitions. Due to the loss of the symmetric property from two dimensions to three dimensions, this superconvergence result in three dimensions is not a trivial extension of the recent superconvergence result in two dimensions Li et al. (0000) from rectangular partitions to cubic partitions. The error estimate for the numerical gradient in the  $L^2$ -norm arrives at a superconvergence order of  $\mathcal{O}(h^r)$  ( $1.5 \leq r \leq 2$ ) when the lowest order weak Galerkin finite elements consisting of piecewise linear polynomials in the interior of the elements and piecewise constants on the faces of the elements are employed. A series of numerical experiments are illustrated to confirm the established superconvergence theory in three dimensions.

© 2019 Elsevier Ltd. All rights reserved.

## 1. Introduction

Superconvergence is a phenomenon where the numerical solution converges to the exact solution at a rate faster than generally expected. Superconvergence has been widely used in post-processing techniques to yield a more accurate approximation [1]. Superconvergence has also been employed by the mesh refinement and adaptivity [2,3] to yield a posterior error estimator [4–8]. There has been a variety of research work in superconvergence based on finite difference methods [9,10], finite element methods [5,11–17], discontinuous Galerkin methods [18], hybridized discontinuous Galerkin methods [19], smoothed finite element methods [20], and weak Galerkin finite element methods [21–26].

In this paper, we are concerned with new developments of superconvergence of weak Galerkin finite element method for second order elliptic boundary value problem (BVP) in three dimensions. To this end, we consider the second order elliptic problem in three dimensions: Find  $u = u(x, y, z)$  satisfying

$$\begin{aligned} -\nabla \cdot (A \nabla u) &= f, & \text{in } \Omega, \\ u &= g, & \text{on } \partial\Omega, \end{aligned} \quad (1.1)$$

\* Corresponding author.

E-mail addresses: [yfnie@nwpu.edu.cn](mailto:yfnie@nwpu.edu.cn) (Y. Nie), [chunmei.wang@ttu.edu](mailto:chunmei.wang@ttu.edu) (C. Wang).

<sup>1</sup> The research of Dan Li was supported in part by National Natural Science Foundation of China grant number 11471262.

<sup>2</sup> The research of Yufeng Nie was supported in part by National Natural Science Foundation of China grants 11471262.

<sup>3</sup> The research of Chunmei Wang was partially supported by National Science Foundation Awards DMS-1849483 and DMS-1905195.

where  $\Omega$  is an open bounded domain in  $\mathbb{R}^3$  with Lipschitz continuous boundary  $\partial\Omega$ ;  $f = f(x, y, z) \in H^{-1}(\Omega)$  and  $g = g(x, y, z) \in H^{\frac{1}{2}}(\partial\Omega)$  are given functions defined on  $\Omega$  and the boundary  $\partial\Omega$ , respectively. We assume that the diffusive coefficient tensor  $A = \{a_{ij}\}_{3 \times 3}$  is uniformly bounded, symmetric, and positive definite in  $\Omega$ .

The weak formulation of the second order elliptic model problem (1.1) using the usual integration by parts is as follows: Find  $u \in H^1(\Omega)$  satisfying  $u = g$  on  $\partial\Omega$ , such that

$$(A\nabla u, \nabla v) = (f, v), \quad \forall v \in V, \quad (1.2)$$

where  $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$ .

Superconvergence for the gradient of the finite element approximation for the second order elliptic boundary value problem has been an active research topic for many years [12,27–31]. There have been various numerical methods for solving the second order elliptic equations (1.1), such as finite element methods, finite volume methods, and finite difference methods etc. We shall focus on a newly-developed numerical method named “weak Galerkin finite element method (WG-FEM)” which is a natural extension of the classical Galerkin finite element methods. WG-FEM has several advantages over the classical Galerkin finite element methods: (1) WG-FEM is flexible to use discontinuous functions with interior information and boundary information; (2) WG-FEM is flexible to use polygons in two dimensions or polyhedra in three dimensions in mesh generation; (3) WG-FEM is stable and preserves the physical properties. WG-FEM has been widely applied to solve various partial differential equations such as elliptic interface problem [32], Maxwell’s equations [33–35], the Helmholtz equation [36], wave equation [37], Stokes equations [23,38], the div-curl system [39], the biharmonic problem [40,41], the Cahn–Hilliard equation [42], the singularly perturbed convection–diffusion–reaction problems [43] etc. Recently, the primal–dual weak Galerkin finite element method has been successfully developed to solve challenging problems such as the second order elliptic equation in non-divergence form [44], the Fokker–Planck equation [45] and the elliptic Cauchy problems [46,47].

Some superconvergence results were observed in the numerical experiments of WG-FEM method on uniform meshes for the gradient approximation for the elliptic equation in three dimensions (see  $\|\nabla_d e_h\|$  in Table 4.11 [48]) and the wave equation (see  $\|\nabla_w(e_h)\|$  in Table II [37]). The numerical results in [44] showed the superconvergence rate  $\mathcal{O}(h^4)$  in the discrete  $L^2$ -norm on uniform triangular partitions. Recently, the superconvergence theory based on WG scheme has been developed and analyzed on non-uniform rectangular partitions for the second order elliptic problem [22] and Stokes equation [23], respectively. In [26], a superconvergence in  $L^2$ -norm was proved between the  $L^2$  projection of the exact solution and its numerical approximation. In [21], the authors studied the  $H^1$ -superconvergence of the WG-FEM method by  $L^2$  projections introduced in [49], and derived a superconvergence rate  $\mathcal{O}(h^{1.5})$  or better by using the lowest order weak Galerkin element approximations for the elliptic problem.

There are some superconvergence results in three-dimensions in the literature [10,11,29,50–53]. The difficulty in the analysis of superconvergence for problems in three-dimensions lies in the loss of orthogonality and/or symmetry compared with the analysis for problems in two dimensions. In this paper, we shall extend the superconvergence result in [22] for the second order elliptic problem (1.1) from two dimensions to three dimensions. This is a non-trivial extension of [22] in both the analysis and numerical experiments. The main difficulty in this paper compared with [22] lies in that the symmetric property for the rectangular partitions in two dimensions is not available for the cubic partition in three dimensions. The innovative contribution in this paper is to develop the superconvergence order  $\mathcal{O}(h^r)$  ( $1.5 \leq r \leq 2$ ) for the numerical gradient for the second order problem in three dimensions.

The rest of this paper is organized as follows. In Section 2, we simply review the weak gradient operator as well as its discrete version. Section 3 is devoted to reviewing the WG-FEM finite element scheme for the second order elliptic problem (1.1) in three dimensions. A simplified WG-FEM scheme is derived in Section 4. The error equation for the simplified WG scheme is developed in Section 5. In Section 6, some technical results are provided which are useful in the analysis of the superconvergence of WG method. Superconvergence theory is established in Section 7. In Section 8, a variety of numerical experiments are demonstrated to verify the established superconvergence theory.

## 2. Weak gradient and discrete weak gradient

The classical gradient operator is the differential operator used in the weak formulation (1.2) of the second order elliptic model problem (1.1). In this section, we will briefly review the weak gradient operator as well as its discrete version which were first introduced in [26,54].

Let  $T$  be any polyhedral domain with boundary  $\partial T$ . Denote by  $v = \{v_0, v_b\}$  a weak function on  $T$ , where the first and second components  $v_0$  and  $v_b$  represent the information of  $v$  in the interior and on the boundary of  $T$ , respectively. Note that  $v_b$  may not necessarily be related to the trace of  $v_0$  on the boundary  $\partial T$ . However, it is feasible to take  $v_b$  as the trace of  $v_0$  on  $\partial T$ .

We introduce the space of the weak functions on  $T$ , denoted by  $W(T)$ ; i.e.,

$$W(T) = \{v = \{v_0, v_b\} : v_0 \in L^2(T), v_b \in L^2(\partial T)\}.$$

The weak gradient of  $v \in W(T)$ , denoted by  $\nabla_w v$ , is defined as a linear functional in the dual space of  $[H^1(T)]^3$  satisfying

$$\langle \nabla_w v, \psi \rangle_T = -(v_0, \nabla \cdot \psi)_T + \langle v_b, \psi \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \psi \in [H^1(T)]^3, \quad (2.1)$$

where  $\mathbf{n}$  is the unit outward normal direction to  $\partial T$ .

Denote by  $P_r(T)$  the set of polynomials on  $T$  with total degree no more than  $r$ . A discrete version of  $\nabla_w v$  for any  $v \in W(T)$ , denoted by  $\nabla_{w,r,T} v$ , is defined as the unique vector-valued polynomial in  $[P_r(T)]^3$  satisfying

$$(\nabla_{w,r,T} v, \boldsymbol{\psi})_T = -(v_0, \nabla \cdot \boldsymbol{\psi})_T + \langle v_b, \boldsymbol{\psi} \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \boldsymbol{\psi} \in [P_r(T)]^3. \quad (2.2)$$

### 3. Weak Galerkin finite element scheme

Let  $\mathcal{T}_h$  be a cubic partition of the domain  $\Omega \subset \mathbb{R}^3$ . Denote by  $\mathcal{E}_h$  the set of all flat faces in  $\mathcal{T}_h$ , and  $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$  the set of all interior flat faces. Denote by  $h_T$  the size of the element  $T \in \mathcal{T}_h$  and  $h = \max_{T \in \mathcal{T}_h} h_T$  the mesh size of the partition  $\mathcal{T}_h$ .

Let  $k \geq 1$  be a given integer. We introduce the local discrete weak finite element space on each element  $T \in \mathcal{T}_h$ , denoted by  $V(T, k)$ ; i.e.,

$$V(T, k) = \{v = \{v_0, v_b\}, v_0 \in P_k(T), v_b \in P_{k-1}(F), F \subset \partial T\}.$$

Patching  $V(T, k)$  over all the elements  $T \in \mathcal{T}_h$  through a common value  $v_b$  on the interior interface  $\mathcal{E}_h^0$  gives rise to a global weak finite element space  $V_h$ ; i.e.,

$$V_h = \{\{v_0, v_b\} : \{v_0, v_b\}|_T \in V(T, k), v_b \text{ is single-valued on } \mathcal{E}_h\}.$$

We further introduce the subspace of  $V_h$  with vanishing boundary values, denoted by  $V_h^0$ ; i.e.,

$$V_h^0 = \{\{v_0, v_b\} \in V_h, v_b|_F = 0, F \subset \partial\Omega\}.$$

For any  $v \in V_h$ , denote by  $\nabla_w v$  the discrete weak gradient  $\nabla_{w,k-1,T} v$  computed by using (2.2) on each element  $T$ ; i.e.,

$$(\nabla_{w,k-1} v)|_T = \nabla_{w,k-1,T}(v|_T).$$

For simplicity of notation and without confusion, we shall use  $\nabla_d$  to denote  $\nabla_{w,k-1}$ ; i.e.,

$$\nabla_d v = \nabla_{w,k-1} v, \quad \forall v \in V_h.$$

For any  $u = \{u_0, u_b\}$  and  $v = \{v_0, v_b\}$  in  $V_h$ , we introduce the following two bilinear forms; i.e.,

$$\begin{aligned} (A \nabla_d u, \nabla_d v)_h &= \sum_{T \in \mathcal{T}_h} (A \nabla_d u, \nabla_d v)_T, \\ s(u, v) &= \rho h^{-1} \sum_{T \in \mathcal{T}_h} \langle Q_b u_0 - u_b, Q_b v_0 - v_b \rangle_{\partial T}, \end{aligned}$$

where  $\rho > 0$  is a parameter, and  $Q_b$  is the usual  $L^2$  projection operator from  $L^2(F)$  onto  $P_{k-1}(F)$ .

We are in a position to review the weak Galerkin finite element method for the second order elliptic model problem (1.1) based on the weak formulation (1.2) [22,26,55].

**Weak Galerkin algorithm 1.** Find  $u_h = \{u_0, u_b\} \in V_h$  satisfying  $u_b = \tilde{Q}_b g$  on  $\partial\Omega$  such that

$$(A \nabla_d u_h, \nabla_d v_h)_h + s(u_h, v_h) = (f, v_0), \quad \forall v_h \in V_h^0, \quad (3.1)$$

where  $\tilde{Q}_b g$  is a suitably-chosen projection operator of the Dirichlet boundary data  $g$  onto the space of polynomials of degree  $k-1$ .

The approximate boundary data  $\tilde{Q}_b g$  may be chosen as

$$\tilde{Q}_b g := Q_b g + \varepsilon_b, \quad (3.2)$$

where  $\varepsilon_b$  is a small perturbation of the  $L^2$  projection  $Q_b g$ . A special example of the perturbation term is given by  $\varepsilon_b = 0$  such that  $\tilde{Q}_b g = Q_b g$ . However, a non-zero perturbation  $\varepsilon_b$  is necessary in the analysis of the superconvergence of the weak gradient approximation.

Note that the coefficient matrix of (3.1) is symmetric and positive definite for any  $\rho > 0$ . Thus, the system (3.1) is solvable.

### 4. Simplified weak Galerkin algorithm

In what follows of this paper, we shall focus on the lowest order of WG finite element, i.e.,  $k = 1$ . More precisely, the WG finite element  $u_h$  is a piecewise linear polynomial in the interior and a piecewise constant on the boundary. The discrete weak gradient  $\nabla_d u_h$  is a piecewise vector-valued constant.

A weak function  $v = \{v_0, v_b\} \in V_h$  can be rewritten as

$$v = \{v_0, 0\} + \{0, v_b\},$$

which, for simplicity of notation and without confusion, will be denoted by  $v = v_0 + v_b$ . Denote by  $V_0 = \{v_0 = \{v_0, 0\} \in V_h\}$  the interior space, and  $V_b = \{v_b = \{0, v_b\} \in V_h\}$  the boundary space, respectively. It is easy to check that  $\nabla_d v_0 = 0$  from the definition of discrete weak gradient (2.2). Thus, the weak Galerkin algorithm (3.1) can be simplified as follows: Find  $u_h = \{u_0, u_b\} \in V_h$  satisfying  $u_b = \tilde{Q}_b g$  on  $\partial\Omega$  such that

$$(\nabla_d u_b, \nabla_d v_b)_h + s(u_h, v_h) = (f, v_0), \quad \forall v_h \in V_h^0. \quad (4.1)$$

We introduce an extension operator  $\mathcal{S}$  mapping  $v_b \in P_0(\partial T)$  to a function in  $P_1(T)$  such that

$$\langle \mathcal{S}(v_b), Q_b \psi \rangle_{\partial T} = \langle v_b, \psi \rangle_{\partial T}, \quad \forall \psi \in P_1(T). \quad (4.2)$$

This implies

$$\begin{aligned} \langle Q_b u_0 - u_b, Q_b \mathcal{S}(v_b) - v_b \rangle_{\partial T} &= \langle -u_b, Q_b \mathcal{S}(v_b) - v_b \rangle_{\partial T} \\ &= \langle Q_b \mathcal{S}(u_b) - u_b, Q_b \mathcal{S}(v_b) - v_b \rangle_{\partial T}. \end{aligned} \quad (4.3)$$

Letting  $v_h = \{\mathcal{S}(v_b), v_b\} \in V_h^0$  in (4.1), and using (4.3), we obtain a simplified weak Galerkin finite element scheme.

**Simplified weak Galerkin algorithm 1.** Find  $u_b \in V_b^g$  satisfying

$$(\nabla_d u_b, \nabla_d v_b)_h + \rho h^{-1} \sum_{T \in \mathcal{T}_h} \langle Q_b \mathcal{S}(u_b) - u_b, Q_b \mathcal{S}(v_b) - v_b \rangle_{\partial T} = (f, \mathcal{S}(v_b)), \quad (4.4)$$

for any  $v_b \in V_b^0$ . Here,  $V_b^0 = \{v_b \in V_b : v_b|_{\partial\Omega} = 0\}$ , and  $V_b^g = \{v_b \in V_b : v_b|_{\partial\Omega} = \tilde{Q}_b g\}$ .

## 5. Error equations

In this section, we will derive an error equation for the simplified weak Galerkin finite element algorithm (4.4), which will play an important role in the analysis of the superconvergence error estimates in Section 7. For the convenience of analysis, we assume the coefficient tensor  $A$  in the model problem (1.1) is a piecewise matrix-valued constant with respect to the finite element partition  $\mathcal{T}_h$ . However, the results can be generalized to the variable coefficient tensor  $A$  without any difficulty, provided that the coefficient tensor  $A$  is piecewise smooth.

On each element  $T \in \mathcal{T}_h$ , denote by  $Q_0$  and  $Q_b$  the usual  $L^2$  projection operators onto  $P_1(T)$  and  $P_0(F)$ , respectively. Denote by  $Q_h$  the usual  $L^2$  projection operator onto  $[P_0(T)]^3$ . The  $L^2$  projection operators  $Q_b$  and  $Q_h$  satisfy the commutative property [22,26]:

$$\nabla_d Q_b w = Q_h \nabla w, \quad \forall w \in H^1(T). \quad (5.1)$$

Denote by  $e_b = Q_b u - u_b$  the error function between the WG solution and the  $L^2$  projection of the exact solution of the model problem (1.1). For the convenience of analysis, we introduce the flux variable  $\mathbf{q} = A \nabla u$ .

**Lemma 5.1.** The error function  $e_b$  satisfies the following error equation

$$(\nabla_d e_b, \nabla_d v_b)_h + \rho h^{-1} \sum_{T \in \mathcal{T}_h} \langle Q_b \mathcal{S}(e_b) - e_b, Q_b \mathcal{S}(v_b) - v_b \rangle_{\partial T} = \zeta_u(v_b), \quad (5.2)$$

for any  $v_b \in V_b^0$ , where

$$\begin{aligned} \zeta_u(v_b) &= \sum_{T \in \mathcal{T}_h} \langle (\mathbf{q} - Q_h \mathbf{q}) \cdot \mathbf{n}, \mathcal{S}(v_b) - v_b \rangle_{\partial T} \\ &\quad + \rho h^{-1} \sum_{T \in \mathcal{T}_h} \langle Q_b \mathcal{S}(Q_b u) - Q_b u, Q_b \mathcal{S}(v_b) - v_b \rangle_{\partial T} \end{aligned} \quad (5.3)$$

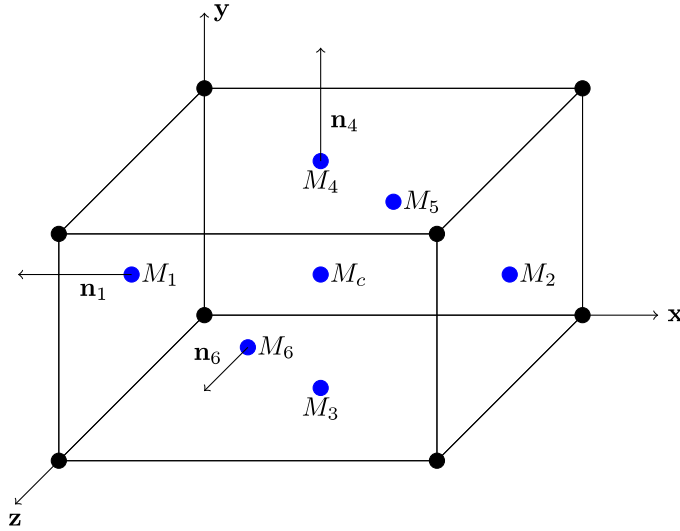
is a linear functional on  $V_b$ .

**Proof.** The proof is similar to the proof of Lemma 5.1 in [22], and therefore the details are omitted here.  $\square$

## 6. Technical estimates

We consider the second order elliptic model problem (1.1) on the unit cubic domain  $\Omega = (0, 1)^3$ . Let the domain  $\Omega$  be partitioned into cubic elements as the Cartesian product of three partitions  $\Delta_x$ ,  $\Delta_y$  and  $\Delta_z$  on the unit interval  $(0, 1)$ :

$$\begin{aligned} \Delta_x : 0 &= x_0 < x_1 < x_2 < \cdots < x_i < \cdots < x_{n-1} < x_n = 1, \\ \Delta_y : 0 &= y_0 < y_1 < y_2 < \cdots < y_j < \cdots < y_{m-1} < y_m = 1, \\ \Delta_z : 0 &= z_0 < z_1 < z_2 < \cdots < z_s < \cdots < z_{q-1} < z_q = 1. \end{aligned}$$

Fig. 6.1. A cubic element  $T \in \mathcal{T}_h$ .

Let  $T = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{s-1}, z_s] \in \mathcal{T}_h$  be a cubic element for  $i = 1, \dots, n, j = 1, \dots, m$  and  $s = 1, \dots, q$  (see Fig. 6.1 for reference). Denote by  $|e_x|$ ,  $|e_y|$  and  $|e_z|$  the length of the edge of the cubic element  $T$  in the  $x$ -,  $y$ - and  $z$ -direction, respectively. Denote by  $|T|$  the volume of the element  $T$ . Denote by  $|F_p|$  the area of the flat face  $F_p$  for  $p = 1, \dots, 6$  such that  $|F_1| = |F_2|$ ,  $|F_3| = |F_4|$  and  $|F_5| = |F_6|$ . Denote by  $M_c = (x_c, y_c, z_c)$  the center of the cubic element  $T$ , and  $M_p = (x_p^*, y_p^*, z_p^*)$  the center of the flat face  $F_p$  for  $p = 1, \dots, 6$ , respectively. The unit outward normal directions to the flat faces  $F_p$  for  $p = 1, \dots, 6$  are given by  $\mathbf{n}_1 = (-1, 0, 0)'$ ,  $\mathbf{n}_2 = (1, 0, 0)'$ ,  $\mathbf{n}_3 = (0, -1, 0)'$ ,  $\mathbf{n}_4 = (0, 1, 0)'$ ,  $\mathbf{n}_5 = (0, 0, -1)'$ , and  $\mathbf{n}_6 = (0, 0, 1)'$ , respectively.

On the element  $T$ , denote by  $v_{bp}$  the value of  $v_b$  on the face  $F_p$ ,  $p = 1, \dots, 6$ . Using (2.2), we have

$$(\nabla_d v_b, \boldsymbol{\psi})_T = \langle v_b, \boldsymbol{\psi} \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \boldsymbol{\psi} \in [P_0(T)]^3,$$

which gives

$$\nabla_d v_b = \left( \frac{v_{b2} - v_{b1}}{|e_x|}, \frac{v_{b4} - v_{b3}}{|e_y|}, \frac{v_{b6} - v_{b5}}{|e_z|} \right)'. \quad (6.1)$$

For any linear function  $\psi \in P_1(T)$ , it is easy to see that  $Q_b \psi = \psi(M_p)$  on each face  $F_p$ . It thus follows from (4.2) that

$$\sum_{p=1}^6 |F_p| S(v_b)(M_p) \psi(M_p) = \sum_{p=1}^6 |F_p| v_{bp} \psi(M_p), \quad \forall \psi \in P_1(T). \quad (6.2)$$

**Lemma 6.1.** Assume a cubic element  $T = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{s-1}, z_s] \in \mathcal{T}_h$ . Let the extension function  $S(v_b) \in P_1(T)$  be defined in (6.2). There holds

$$\begin{aligned} (S(v_b) - v_b)(M_1) &= (S(v_b) - v_b)(M_2) \\ &= \frac{|F_3|(v_{b3} + v_{b4}) + |F_5|(v_{b5} + v_{b6}) - (|F_3| + |F_5|)(v_{b1} + v_{b2})}{2(|F_1| + |F_3| + |F_5|)}, \end{aligned} \quad (6.3)$$

$$\begin{aligned} (S(v_b) - v_b)(M_3) &= (S(v_b) - v_b)(M_4) \\ &= \frac{|F_1|(v_{b1} + v_{b2}) + |F_5|(v_{b5} + v_{b6}) - (|F_1| + |F_5|)(v_{b3} + v_{b4})}{2(|F_1| + |F_3| + |F_5|)}, \end{aligned} \quad (6.4)$$

$$\begin{aligned} (S(v_b) - v_b)(M_5) &= (S(v_b) - v_b)(M_6) \\ &= \frac{|F_1|(v_{b1} + v_{b2}) + |F_3|(v_{b3} + v_{b4}) - (|F_1| + |F_3|)(v_{b5} + v_{b6})}{2(|F_1| + |F_3| + |F_5|)}. \end{aligned} \quad (6.5)$$

Furthermore, there holds

$$|F_1|(S(v_b) - v_b)(M_1) + |F_3|(S(v_b) - v_b)(M_3) + |F_5|(S(v_b) - v_b)(M_5) = 0. \quad (6.6)$$

**Proof.** From the definition of the extension function  $S(v_b)$ , we have

$$S(v_b) = c_1 + c_2(x - x_c) + c_3(y - y_c) + c_4(z - z_c). \quad (6.7)$$

Letting  $\psi = 1$  in (6.2) gives

$$\sum_{p=1}^6 |F_p| \left( c_1 + c_2(x_p^* - x_c) + c_3(y_p^* - y_c) + c_4(z_p^* - z_c) \right) = \sum_{p=1}^6 |F_p| v_{bp},$$

which leads to

$$c_1 = \frac{|F_1|(v_{b1} + v_{b2}) + |F_3|(v_{b3} + v_{b4}) + |F_5|(v_{b5} + v_{b6})}{2(|F_1| + |F_3| + |F_5|)}.$$

Similarly, setting  $\psi = x - x_c$ ,  $\psi = y - y_c$ , and  $\psi = z - z_c$  in (6.2) yields

$$c_2 = \frac{v_{b2} - v_{b1}}{|e_x|}, \quad c_3 = \frac{v_{b4} - v_{b3}}{|e_y|}, \quad c_4 = \frac{v_{b6} - v_{b5}}{|e_z|}.$$

Therefore, (6.7) can be rewritten as follows

$$\begin{aligned} \mathcal{S}(v_b) &= \frac{|F_1|(v_{b1} + v_{b2}) + |F_3|(v_{b3} + v_{b4}) + |F_5|(v_{b5} + v_{b6})}{2(|F_1| + |F_3| + |F_5|)} + \frac{v_{b2} - v_{b1}}{|e_x|}(x - x_c) \\ &\quad + \frac{v_{b4} - v_{b3}}{|e_y|}(y - y_c) + \frac{v_{b6} - v_{b5}}{|e_z|}(z - z_c). \end{aligned}$$

Next, we compute  $\mathcal{S}(v_b) - v_b$  at the center  $M_p$  of each flat face  $F_p$  for  $p = 1, \dots, 6$ . At the center  $M_1$  of the flat face  $F_1$ , we have

$$\begin{aligned} (\mathcal{S}(v_b) - v_b)(M_1) &= \frac{|F_1|(v_{b1} + v_{b2}) + |F_3|(v_{b3} + v_{b4}) + |F_5|(v_{b5} + v_{b6})}{2(|F_1| + |F_3| + |F_5|)} - \frac{(v_{b2} - v_{b1})|e_x|}{2} - v_{b1} \\ &= \frac{|F_3|(v_{b3} + v_{b4}) + |F_5|(v_{b5} + v_{b6}) - (|F_3| + |F_5|)(v_{b1} + v_{b2})}{2(|F_1| + |F_3| + |F_5|)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} (\mathcal{S}(v_b) - v_b)(M_2) &= \frac{|F_3|(v_{b3} + v_{b4}) + |F_5|(v_{b5} + v_{b6}) - (|F_3| + |F_5|)(v_{b1} + v_{b2})}{2(|F_1| + |F_3| + |F_5|)}, \\ (\mathcal{S}(v_b) - v_b)(M_3) &= \frac{|F_1|(v_{b1} + v_{b2}) + |F_5|(v_{b5} + v_{b6}) - (|F_1| + |F_5|)(v_{b3} + v_{b4})}{2(|F_1| + |F_3| + |F_5|)}, \\ (\mathcal{S}(v_b) - v_b)(M_4) &= \frac{|F_1|(v_{b1} + v_{b2}) + |F_5|(v_{b5} + v_{b6}) - (|F_1| + |F_5|)(v_{b3} + v_{b4})}{2(|F_1| + |F_3| + |F_5|)}, \\ (\mathcal{S}(v_b) - v_b)(M_5) &= \frac{|F_1|(v_{b1} + v_{b2}) + |F_3|(v_{b3} + v_{b4}) - (|F_3| + |F_1|)(v_{b5} + v_{b6})}{2(|F_1| + |F_3| + |F_5|)}, \\ (\mathcal{S}(v_b) - v_b)(M_6) &= \frac{|F_1|(v_{b1} + v_{b2}) + |F_3|(v_{b3} + v_{b4}) - (|F_3| + |F_1|)(v_{b5} + v_{b6})}{2(|F_1| + |F_3| + |F_5|)}. \end{aligned}$$

This completes the proof of the Lemma.  $\square$

We now focus on the two terms on the right-hand side of the error Eq. (5.2), where the first term  $\sum_{T \in \mathcal{T}_h} ((\mathbf{q} - \mathbb{Q}_h \mathbf{q}) \cdot \mathbf{n}, \mathcal{S}(v_b) - v_b)_{\partial T}$  is critical in the analysis. Lemma 6.1 indicates that  $\mathcal{S}(v_b) - v_b$  has the same value at the center of the flat faces  $F_1$  and  $F_2$ . Moreover,  $\mathcal{S}(v_b) - v_b$  has the same directional derivative along the flat faces  $F_1$  and  $F_2$  which are  $\frac{\partial(\mathcal{S}(v_b) - v_b)}{\partial y} = \nabla_d v_b \cdot \mathbf{n}_4$  and  $\frac{\partial(\mathcal{S}(v_b) - v_b)}{\partial z} = \nabla_d v_b \cdot \mathbf{n}_6$ , respectively. Hence,  $\mathcal{S}(v_b) - v_b$  has the same value along the flat faces  $F_1$  and  $F_2$  at the symmetric points  $(x_{i-1}, y, z)$  and  $(x_i, y, z)$ . Likewise,  $\mathcal{S}(v_b) - v_b$  has the same value along the flat faces  $F_3$  and  $F_4$  at the symmetric points  $(x, y_{j-1}, z)$  and  $(x, y_j, z)$ , and has the same value along the flat faces  $F_5$  and  $F_6$  at the symmetric points  $(x, y, z_{s-1})$  and  $(x, y, z_s)$ , respectively. It thus follows that

$$(\mathbb{Q}_h \mathbf{q} \cdot \mathbf{n}, \mathcal{S}(v_b) - v_b)_{\partial T} = 0. \quad (6.8)$$

Since  $\mathcal{S}(v_b) - v_b$  has the same value at the symmetric points  $(x_{i-1}, y, z)$  and  $(x_i, y, z)$ , this boundary function on the flat faces  $F_1$  and  $F_2$  can be extended to the cubic element  $T$  by assigning the value  $(\mathcal{S}(v_b) - v_b)(x_{i-1}, y, z)$  along each face parallel to the flat face  $F_1$  (or  $F_2$ ). Denote this extension by  $\chi_1$ ; i.e.,

$$\chi_1(x, y, z) := (\mathcal{S}(v_b) - v_b)(x_{i-1}, y, z), \quad (x, y, z) \in T. \quad (6.9)$$

Similarly, denote by  $\chi_2$  the extension function of  $\mathcal{S}(v_b) - v_b$  to the cubic element  $T$  by assuming the value  $(\mathcal{S}(v_b) - v_b)(x, y_{j-1}, z)$  along each face parallel to the flat face  $F_3$  (or  $F_4$ ); i.e.,

$$\chi_2(x, y, z) := (\mathcal{S}(v_b) - v_b)(x, y_{j-1}, z), \quad (x, y, z) \in T. \quad (6.10)$$

Likewise, we define

$$\chi_3(x, y, z) := (S(v_b) - v_b)(x, y, z_{s-1}), \quad (x, y, z) \in T. \quad (6.11)$$

It follows from Lemma 6.1 and (6.1) that

$$\begin{aligned} \partial_x \chi_1 &= 0, \quad \partial_y \chi_1 = \nabla_d v_b \cdot \mathbf{n}_4, \quad \partial_z \chi_1 = \nabla_d v_b \cdot \mathbf{n}_6, \\ \partial_y \chi_2 &= 0, \quad \partial_x \chi_2 = \nabla_d v_b \cdot \mathbf{n}_2, \quad \partial_z \chi_2 = \nabla_d v_b \cdot \mathbf{n}_6, \\ \partial_z \chi_3 &= 0, \quad \partial_x \chi_3 = \nabla_d v_b \cdot \mathbf{n}_2, \quad \partial_y \chi_3 = \nabla_d v_b \cdot \mathbf{n}_4, \\ |F_1| \chi_1(M_1) + |F_3| \chi_2(M_3) + |F_5| \chi_3(M_5) &= 0. \end{aligned} \quad (6.12)$$

**Lemma 6.2.** Assume  $u \in H^3(\Omega)$  is a given function. Let  $\mathcal{T}_h = \Delta_x \times \Delta_y \times \Delta_z$  be a cubic partition. On each element  $T \in \mathcal{T}_h$ , for any  $v_b \in V_b^0$ , there holds

$$\begin{aligned} &\langle (\mathbf{q} - \mathbb{Q}_h \mathbf{q}) \cdot \mathbf{n}, S(v_b) - v_b \rangle_{\partial T} \\ &= \chi_1(M_1) \int_T q_{1x} dT + \chi_2(M_3) \int_T q_{2y} dT + \chi_3(M_5) \int_T q_{3z} dT + R_1(T), \end{aligned} \quad (6.13)$$

where  $\chi_i$  ( $i = 1, 2, 3$ ) are the extension functions defined in (6.9)–(6.11), and  $\mathbf{q} = A \nabla u = (q_1, q_2, q_3)'$ ,  $q_1 = a_{11}u_x + a_{12}u_y + a_{13}u_z$ ,  $q_2 = a_{21}u_x + a_{22}u_y + a_{23}u_z$ ,  $q_3 = a_{31}u_x + a_{32}u_y + a_{33}u_z$ ,  $q_{1x} = \frac{\partial q_1}{\partial x}$ ,  $q_{2y} = \frac{\partial q_2}{\partial y}$ ,  $q_{3z} = \frac{\partial q_3}{\partial z}$ , respectively. The remainder term  $R_1(T)$  satisfies the following estimate

$$\sum_{T \in \mathcal{T}_h} |R_1(T)| \leq Ch^2 \|\mathbf{q}\|_2 \|\nabla_d v_b\|_0. \quad (6.14)$$

**Proof.** Using the definition of  $\chi_i$  in (6.9)–(6.11), (6.8), and the usual integration by parts, we obtain

$$\begin{aligned} &\langle (\mathbf{q} - \mathbb{Q}_h \mathbf{q}) \cdot \mathbf{n}, S(v_b) - v_b \rangle_{\partial T} \\ &= \langle \mathbf{q} \cdot \mathbf{n}, S(v_b) - v_b \rangle_{\partial T} \\ &= - \int_{F_1} q_1 \chi_1 dF + \int_{F_2} q_1 \chi_1 dF - \int_{F_3} q_2 \chi_2 dF \\ &\quad + \int_{F_4} q_2 \chi_2 dF - \int_{F_5} q_3 \chi_3 dF + \int_{F_6} q_3 \chi_3 dF \\ &= \int_T q_{1x} \chi_1 dT + \int_T q_{2y} \chi_2 dT + \int_T q_{3z} \chi_3 dT. \end{aligned} \quad (6.15)$$

Since  $\chi_1$  is linear in both the  $y$ -direction and the  $z$ -direction, and is constant in the  $x$ -direction, we have

$$\chi_1(y, z) = \chi_1(M_1) + \partial_y \chi_1(y - y_c) + \partial_z \chi_1(z - z_c),$$

which, together with the usual integration by parts, gives

$$\begin{aligned} \int_T q_{1x} \chi_1 dT &= \int_T q_{1x} \chi_1(M_1) dT + \int_T q_{1x} \partial_y \chi_1(y - y_c) dT + \int_T q_{1x} \partial_z \chi_1(z - z_c) dT \\ &= \int_T q_{1x} \chi_1(M_1) dT + \int_T q_{1xy} \partial_y \chi_1 E_{31}(y) dT + \int_T q_{1xz} \partial_z \chi_1 E_{32}(z) dT, \end{aligned}$$

where  $E_{31}(y) = \frac{1}{8}|e_y|^2 - \frac{1}{2}(y - y_c)^2$  and  $E_{32}(z) = \frac{1}{8}|e_z|^2 - \frac{1}{2}(z - z_c)^2$ .

Similarly, there holds

$$\begin{aligned} \int_T q_{2y} \chi_2 dT &= \int_T q_{2y} \chi_2(M_3) dT + \int_T q_{2yx} \partial_x \chi_2(x - x_c) dT + \int_T q_{2yz} \partial_z \chi_2(z - z_c) dT \\ &= \int_T q_{2y} \chi_2(M_3) dT + \int_T q_{2yx} \partial_x \chi_2 E_{41}(x) dT + \int_T q_{2yz} \partial_z \chi_2 E_{42}(z) dT, \end{aligned}$$

where  $E_{41}(x) = \frac{1}{8}|e_x|^2 - \frac{1}{2}(x - x_c)^2$ ,  $E_{42}(z) = \frac{1}{8}|e_z|^2 - \frac{1}{2}(z - z_c)^2$ .

Likewise, we have

$$\begin{aligned} \int_T q_{3z} \chi_3 dT &= \int_T q_{3z} \chi_3(M_5) dT + \int_T q_{3zx} \partial_x \chi_3(x - x_c) dT + \int_T q_{3zy} \partial_y \chi_3(y - y_c) dT \\ &= \int_T q_{3z} \chi_3(M_5) dT + \int_T q_{3zx} \partial_x \chi_3 E_{51}(x) dT + \int_T q_{3zy} \partial_y \chi_3 E_{52}(y) dT, \end{aligned}$$

where  $E_{51}(x) = \frac{1}{8}|e_x|^2 - \frac{1}{2}(x - x_c)^2$ , and  $E_{52}(y) = \frac{1}{8}|e_y|^2 - \frac{1}{2}(y - y_c)^2$ .

Substituting the above three identities into (6.15) gives

$$\begin{aligned} & \langle (\mathbf{q} - \mathbb{Q}_h \mathbf{q}) \cdot \mathbf{n}, \mathcal{S}(v_b) - v_b \rangle_{\partial T} \\ &= \chi_1(M_1) \int_T q_{1x} dT + \chi_2(M_3) \int_T q_{2y} dT + \chi_3(M_5) \int_T q_{3z} dT + R_1(T), \end{aligned}$$

where the remainder term  $R_1(T)$  is given by

$$\begin{aligned} R_1(T) &= \int_T q_{1xy} \partial_y \chi_1 E_{31}(y) dT + \int_T q_{1xz} \partial_z \chi_1 E_{32}(z) dT \\ &\quad + \int_T q_{2yx} \partial_x \chi_2 E_{41}(x) dT + \int_T q_{2yz} \partial_z \chi_2 E_{42}(z) dT \\ &\quad + \int_T q_{3zx} \partial_x \chi_3 E_{51}(x) dT + \int_T q_{3zy} \partial_y \chi_3 E_{52}(y) dT. \end{aligned} \quad (6.16)$$

Using the Cauchy–Schwarz inequality and (6.12), there holds

$$\begin{aligned} \left| \sum_{T \in \mathcal{T}_h} \int_T q_{1xy} \partial_y \chi_1 E_{31}(y) dT \right| &\leq Ch^2 \left( \sum_{T \in \mathcal{T}_h} \|\nabla^2 q_1\|_T^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} \|\partial_y \chi_1\|_T^2 \right)^{\frac{1}{2}} \\ &\leq Ch^2 \|q_1\|_2 \|\nabla_d v_b\|_0. \end{aligned}$$

Each of the rest five terms in the remainder term  $R_1(T)$  in (6.16) can be estimated in a similar way. This completes the proof of the Lemma.  $\square$

The following Lemma shall provide an estimate for the second term on the right-hand side of the error Eq. (5.2).

**Lemma 6.3.** Under the assumptions of Lemma 6.2, there holds

$$\begin{aligned} & \rho h^{-1} \langle Q_b \mathcal{S}(Q_b u) - Q_b u, Q_b \mathcal{S}(v_b) - v_b \rangle_{\partial T} \\ &= -A_1 \rho h^{-1} (|e_x| \chi_1(M_1) \int_T u_{xx} dT + |e_y| \chi_2(M_3) \int_T u_{yy} dT \\ &\quad + |e_z| \chi_3(M_5) \int_T u_{zz} dT) + R_2(T), \end{aligned} \quad (6.17)$$

where  $A_1 = \frac{1}{6}$  and the remainder term  $R_2(T)$  satisfies

$$\sum_{T \in \mathcal{T}_h} |R_2(T)| \leq Ch^2 \|u\|_3 \|\mathcal{S}(v_b) - v_b\|_{\varepsilon_h}. \quad (6.18)$$

Here we define

$$\|\mathcal{S}(v_b) - v_b\|_{\varepsilon_h}^2 := \rho h^{-1} \sum_{T \in \mathcal{T}_h} \langle Q_b \mathcal{S}(v_b) - v_b, Q_b \mathcal{S}(v_b) - v_b \rangle_{\partial T}. \quad (6.19)$$

**Proof.** Using (4.2), (6.9)–(6.11), Lemma 6.1, and (6.12), we have

$$\begin{aligned} & \rho h^{-1} \langle Q_b \mathcal{S}(Q_b u) - Q_b u, Q_b \mathcal{S}(v_b) - v_b \rangle_{\partial T} \\ &= -\rho h^{-1} \langle Q_b u, \mathcal{S}(v_b) - v_b \rangle_{\partial T} \\ &= -\rho h^{-1} \left( |F_1| \chi_1(M_1) Q_b u(M_1) + |F_2| \chi_1(M_2) Q_b u(M_2) \right. \\ &\quad + |F_3| \chi_2(M_3) Q_b u(M_3) + |F_4| \chi_2(M_4) Q_b u(M_4) \\ &\quad + |F_5| \chi_3(M_5) Q_b u(M_5) + |F_6| \chi_3(M_6) Q_b u(M_6) \left. \right) \\ &= -\rho h^{-1} |F_1| \chi_1(M_1) (Q_b u(M_1) + Q_b u(M_2) - Q_b u(M_5) - Q_b u(M_6)) \\ &\quad - \rho h^{-1} |F_3| \chi_2(M_3) (Q_b u(M_3) + Q_b u(M_4) - Q_b u(M_5) - Q_b u(M_6)). \end{aligned} \quad (6.20)$$

Note that  $Q_b u|_{F_i} = \frac{1}{|F_i|} \int_{F_i} u dF$  is the average of  $u$  on the flat face  $F_i$ . Using the Euler–MacLaurin formula gives

$$\begin{aligned}
 & |F_1| \left( Q_b u(M_1) + Q_b u(M_2) - Q_b u(M_5) - Q_b u(M_6) \right) \\
 &= \int_{F_1} u(x_{i-1}, y, z) dF + \int_{F_2} u(x_i, y, z) dF \\
 &\quad - \frac{|F_1|}{|F_5|} \left( \int_{F_5} u(x, y, z_{s-1}) dF + \int_{F_6} u(x, y, z_s) dF \right) \\
 &= \frac{1}{|e_x|} \left( 2 \int_T u(x, y, z) dT + A_1 |e_x|^2 \int_T u_{xx} dT + \frac{1}{24} |e_x|^3 \int_T u_{xxx} E_1(x) dT \right) \\
 &\quad - \frac{1}{|e_z|} \cdot \frac{|F_1|}{|F_5|} \left( 2 \int_T u(x, y, z) dT + A_1 |e_z|^2 \int_T u_{zz} dT \right. \\
 &\quad \left. + \frac{1}{24} |e_z|^3 \int_T u_{zzz} E_3(z) dT \right) \\
 &= A_1 \left( |e_x| \int_T u_{xx} dT - \frac{|e_z|^2}{|e_x|} \int_T u_{zz} dT \right) \\
 &\quad + \frac{1}{24} \left( |e_x|^2 \int_T u_{xxx} E_1(x) dT - \frac{|e_z|^3}{|e_x|} \int_T u_{zzz} E_3(z) dT \right),
 \end{aligned} \tag{6.21}$$

where  $E_1(x)$  and  $E_3(z)$  are the cubic polynomials in both the  $x$ -direction and the  $z$ -direction.

Similarly, we arrive at

$$\begin{aligned}
 & |F_3| \left( Q_b u(M_3) + Q_b u(M_4) - Q_b u(M_5) - Q_b u(M_6) \right) \\
 &= A_1 |e_y| \int_T u_{yy} dT - A_1 \frac{|e_z|^2}{|e_y|} \int_T u_{zz} dT \\
 &\quad + \frac{|e_y|^2}{24} \int_T u_{yyy} E_2(y) dT - \frac{|e_z|^3}{24 |e_y|} \int_T u_{zzz} E_3(z) dT,
 \end{aligned} \tag{6.22}$$

where  $E_2(y)$  is the cubic polynomial in the  $y$ -direction.

Substituting (6.21)–(6.22) into (6.20) and using (6.12), we have

$$\begin{aligned}
 & \rho h^{-1} \langle Q_b S(Q_b u) - Q_b u, Q_b S(v_b) - v_b \rangle_{\partial T} \\
 &= -\rho h^{-1} \chi_1(M_1) \left( A_1 |e_x| \int_T u_{xx} dT - A_1 \frac{|e_z|^2}{|e_x|} \int_T u_{zz} dT \right. \\
 &\quad \left. + \frac{|e_x|^2}{24} \int_T u_{xxx} E_1(x) dT - \frac{|e_z|^3}{24 |e_x|} \int_T u_{zzz} E_3(z) dT \right) \\
 &\quad - \rho h^{-1} \chi_2(M_3) \left( A_1 |e_y| \int_T u_{yy} dT - A_1 \frac{|e_z|^2}{|e_y|} \int_T u_{zz} dT \right. \\
 &\quad \left. + \frac{|e_y|^2}{24} \int_T u_{yyy} E_2(y) dT - \frac{|e_z|^3}{24 |e_y|} \int_T u_{zzz} E_3(z) dT \right) \\
 &= -A_1 \rho h^{-1} \left( |e_x| \chi_1(M_1) \int_T u_{xx} dT + |e_y| \chi_2(M_3) \int_T u_{yy} dT \right. \\
 &\quad \left. + |e_z| \chi_3(M_5) \int_T u_{zz} dT \right) + R_2(T),
 \end{aligned}$$

where the remainder  $R_2(T)$  is given by

$$\begin{aligned}
 R_2(T) = & -\frac{1}{24} \rho h^{-1} \left( |e_x|^2 \chi_1(M_1) \int_T u_{xxx} E_1(x) dT \right. \\
 & \left. + |e_y|^2 \chi_2(M_3) \int_T u_{yyy} E_2(y) dT + |e_z|^2 \chi_3(M_5) \int_T u_{zzz} E_3(z) dT \right).
 \end{aligned}$$

Similar to the proof of (6.14), it is easy to arrive at (6.18). This completes the proof of the Lemma.  $\square$

## 7. Superconvergence

In this section, we shall establish the superconvergence error estimates for the simplified weak Galerkin finite element scheme (4.4) for solving the three dimensional second order model problem (1.1) on the cubic partitions.

**Theorem 7.1.** Assume that  $u \in H^3(\Omega)$  is the exact solution of the second order elliptic model problem (1.1) in three dimensions. Let  $u_b \in V_b$  be the weak Galerkin finite element solution arising from the simplified WG scheme (4.4) satisfying the boundary condition  $u_b = Q_b u$  on  $\partial\Omega$ . On each cubic element  $T = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{s-1}, z_s] \in \mathcal{T}_h$ , we define  $w_b \in V_b$  as follows

$$w_b = \begin{cases} \frac{1}{12} \rho^{-1} h^{-1} (\rho h^{-1} (|e_y|^2 Q_b u_{yy}|_{F_1} + |e_z|^2 Q_b u_{zz}|_{F_1}) - 6(|e_y| Q_b q_{2y}|_{F_1} + |e_z| Q_b q_{3z}|_{F_1})), & \text{on } F_1, \\ \frac{1}{12} \rho^{-1} h^{-1} (\rho h^{-1} (|e_y|^2 Q_b u_{yy}|_{F_2} + |e_z|^2 Q_b u_{zz}|_{F_2}) - 6(|e_y| Q_b q_{2y}|_{F_2} + |e_z| Q_b q_{3z}|_{F_2})), & \text{on } F_2, \\ \frac{1}{12} \rho^{-1} h^{-1} (\rho h^{-1} (|e_x|^2 Q_b u_{xx}|_{F_3} + |e_z|^2 Q_b u_{zz}|_{F_3}) - 6(|e_x| Q_b q_{1x}|_{F_3} + |e_z| Q_b q_{3z}|_{F_3})), & \text{on } F_3, \\ \frac{1}{12} \rho^{-1} h^{-1} (\rho h^{-1} (|e_x|^2 Q_b u_{xx}|_{F_4} + |e_z|^2 Q_b u_{zz}|_{F_4}) - 6(|e_x| Q_b q_{1x}|_{F_4} + |e_z| Q_b q_{3z}|_{F_4})), & \text{on } F_4, \\ \frac{1}{12} \rho^{-1} h^{-1} (\rho h^{-1} (|e_y|^2 Q_b u_{yy}|_{F_5} + |e_x|^2 Q_b u_{xx}|_{F_5}) - 6(|e_y| Q_b q_{2y}|_{F_5} + |e_x| Q_b q_{1x}|_{F_5})), & \text{on } F_5, \\ \frac{1}{12} \rho^{-1} h^{-1} (\rho h^{-1} (|e_y|^2 Q_b u_{yy}|_{F_6} + |e_x|^2 Q_b u_{xx}|_{F_6}) - 6(|e_y| Q_b q_{2y}|_{F_6} + |e_x| Q_b q_{1x}|_{F_6})), & \text{on } F_6. \end{cases}$$

Let  $\tilde{e}_b = (Q_b u - u_b) + h^2 w_b$  be the modified error function. For any  $v_b \in V_b^0$ , the error function  $\tilde{e}_b$  satisfies

$$\begin{aligned} (A \nabla_d \tilde{e}_b, \nabla_d v_b)_h + \rho h^{-1} \sum_{T \in \mathcal{T}_h} \langle Q_b S(\tilde{e}_b) - \tilde{e}_b, Q_b S(v_b) - v_b \rangle_{\partial T} \\ = h^2 (A \nabla_d w_b, \nabla_d v_b)_h + R_4(v_b), \end{aligned} \quad (7.1)$$

where  $R_4(v_b)$  is the remainder satisfying

$$|R_4(v_b)| \leq Ch^2 \|u\|_3 \|S(v_b) - v_b\|_{\varepsilon_h}. \quad (7.2)$$

**Proof.** It follows from (5.3), Lemmas 6.2 and 6.3 that

$$\begin{aligned} \zeta_u(v_b) = & - \sum_{T \in \mathcal{T}_h} A_1 \rho h^{-1} |e_x| \chi_1(M_1) \int_T u_{xx} dT + \sum_{T \in \mathcal{T}_h} \chi_1(M_1) \int_T q_{1x} dT \\ & - \sum_{T \in \mathcal{T}_h} A_1 \rho h^{-1} |e_y| \chi_2(M_3) \int_T u_{yy} dT + \sum_{T \in \mathcal{T}_h} \chi_2(M_3) \int_T q_{2y} dT \\ & - \sum_{T \in \mathcal{T}_h} A_1 \rho h^{-1} |e_z| \chi_3(M_5) \int_T u_{zz} dT + \sum_{T \in \mathcal{T}_h} \chi_3(M_5) \int_T q_{3z} dT \\ & + \sum_{T \in \mathcal{T}_h} (R_1(T) + R_2(T)). \end{aligned} \quad (7.3)$$

Using the usual integration by parts yields

$$\begin{aligned} \int_T u_{xx} dT &= - \int_T u_{xy}(y - y_c) dT + \frac{1}{2} |e_y| \int_{F_3} u_{xx} dF + \frac{1}{2} |e_y| \int_{F_4} u_{xx} dF \\ &= - \int_T u_{xy}(y - y_c) dT + \frac{1}{2} |e_y| |F_3| Q_b u_{xx}|_{F_3} + \frac{1}{2} |e_y| |F_4| Q_b u_{xx}|_{F_4}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \int_T u_{yy} dT &= - \int_T u_{yz}(z - z_c) dT + \frac{1}{2} |e_z| |F_5| Q_b u_{yy}|_{F_5} + \frac{1}{2} |e_z| |F_6| Q_b u_{yy}|_{F_6}, \\ \int_T u_{zz} dT &= - \int_T u_{zx}(x - x_c) dT + \frac{1}{2} |e_x| |F_1| Q_b u_{zz}|_{F_1} + \frac{1}{2} |e_x| |F_2| Q_b u_{zz}|_{F_2}. \end{aligned}$$

Likewise, there holds

$$\begin{aligned} \int_T q_{1x} dT &= - \int_T q_{1xy}(y - y_c) dT + \frac{1}{2} |e_y| |F_3| Q_b q_{1x}|_{F_3} + \frac{1}{2} |e_y| |F_4| Q_b q_{1x}|_{F_4}, \\ \int_T q_{2y} dT &= - \int_T q_{2yz}(z - z_c) dT + \frac{1}{2} |e_z| |F_5| Q_b q_{2y}|_{F_5} + \frac{1}{2} |e_z| |F_6| Q_b q_{2y}|_{F_6}, \\ \int_T q_{3z} dT &= - \int_T q_{3zx}(x - x_c) dT + \frac{1}{2} |e_x| |F_1| Q_b q_{3z}|_{F_1} + \frac{1}{2} |e_x| |F_2| Q_b q_{3z}|_{F_2}. \end{aligned}$$

Substituting the above identities into (7.3) gives rise to

$$\begin{aligned}
 \zeta_u(v_b) = & -\frac{A_1}{2} \sum_{T \in \mathcal{T}_h} \rho h^{-1} |e_x| |e_y| |F_3| \chi_1(M_1) (Q_b u_{xx}|_{F_3} + Q_b u_{xx}|_{F_4}) \\
 & -\frac{A_1}{2} \sum_{T \in \mathcal{T}_h} \rho h^{-1} |e_y| |e_z| |F_5| \chi_2(M_3) (Q_b u_{yy}|_{F_5} + Q_b u_{yy}|_{F_6}) \\
 & -\frac{A_1}{2} \sum_{T \in \mathcal{T}_h} \rho h^{-1} |e_z| |e_x| |F_1| \chi_3(M_5) (Q_b u_{zz}|_{F_1} + Q_b u_{zz}|_{F_2}) \\
 & +\frac{1}{2} \sum_{T \in \mathcal{T}_h} |e_y| |F_3| \chi_1(M_1) (Q_b q_{1x}|_{F_3} + Q_b q_{1x}|_{F_4}) \\
 & +\frac{1}{2} \sum_{T \in \mathcal{T}_h} |e_z| |F_5| \chi_2(M_3) (Q_b q_{2y}|_{F_5} + Q_b q_{2y}|_{F_6}) \\
 & +\frac{1}{2} \sum_{T \in \mathcal{T}_h} |e_x| |F_1| \chi_3(M_5) (Q_b q_{3z}|_{F_1} + Q_b q_{3z}|_{F_2}) + \sum_{T \in \mathcal{T}_h} R_3(T),
 \end{aligned} \tag{7.4}$$

where the remainder term  $R_3(T)$  is given by

$$\begin{aligned}
 R_3(T) = & A_1 \rho h^{-1} |e_x| \chi_1(M_1) \int_T u_{xy}(y - y_c) dT - \chi_1(M_1) \int_T q_{1xy}(y - y_c) dT \\
 & + A_1 \rho h^{-1} |e_y| \chi_2(M_3) \int_T u_{yz}(z - z_c) dT - \chi_2(M_3) \int_T q_{2yz}(z - z_c) dT \\
 & + A_1 \rho h^{-1} |e_z| \chi_3(M_5) \int_T u_{zx}(x - x_c) dT - \chi_3(M_5) \int_T q_{3zx}(x - x_c) dT \\
 & + R_1(T) + R_2(T).
 \end{aligned}$$

From the definition of  $Q_b$  and the usual integration by parts, we arrive at

$$\begin{aligned}
 & \frac{1}{2} |e_x|^2 |F_5| \chi_3(M_5) (Q_b u_{xx}|_{F_3} + Q_b u_{xx}|_{F_4}) \\
 = & \frac{|e_x|^2 |F_5|}{|e_y| |F_3|} \chi_3(M_5) \left( \int_T u_{xx} dT + \int_T u_{xy}(y - y_c) dT \right) \\
 = & \frac{|e_x|^2 |F_5|}{2} \chi_3(M_5) (Q_b u_{xx}|_{F_5} + Q_b u_{xx}|_{F_6}) \\
 & + \frac{|e_x|^2}{|e_z|} \chi_3(M_5) \left( \int_T u_{xy}(y - y_c) dT - \int_T u_{xz}(z - z_c) dT \right).
 \end{aligned} \tag{7.5}$$

Similarly, we have

$$\begin{aligned}
 & \frac{1}{2} |e_x| |F_5| \chi_3(M_5) (Q_b q_{1x}|_{F_3} + Q_b q_{1x}|_{F_4}) \\
 = & \frac{|e_x| |F_5|}{|e_y| |F_3|} \chi_3(M_5) \left( \int_T q_{1x} dT + \int_T q_{1xy}(y - y_c) dT \right) \\
 = & \frac{|e_x| |F_5|}{2} \chi_3(M_5) (Q_b q_{1x}|_{F_5} + Q_b q_{1x}|_{F_6}) \\
 & + \frac{|e_x|}{|e_z|} \chi_3(M_5) \left( \int_T q_{1xy}(y - y_c) dT - \int_T q_{1xz}(z - z_c) dT \right),
 \end{aligned} \tag{7.6}$$

$$\begin{aligned}
 & \frac{1}{2} |e_y|^2 |F_1| \chi_1(M_1) (Q_b u_{yy}|_{F_5} + Q_b u_{yy}|_{F_6}) \\
 = & \frac{|e_y|^2 |F_1|}{|e_z| |F_5|} \chi_1(M_1) \left( \int_T u_{yy} dT + \int_T u_{yz}(z - z_c) dT \right) \\
 = & \frac{|e_y|^2 |F_1|}{2} \chi_1(M_1) (Q_b u_{yy}|_{F_1} + Q_b u_{yy}|_{F_2}) \\
 & + \frac{|e_y|^2}{|e_x|} \chi_1(M_1) \left( \int_T u_{yz}(z - z_c) dT - \int_T u_{yx}(x - x_c) dT \right),
 \end{aligned} \tag{7.7}$$

$$\begin{aligned}
& \frac{1}{2}|e_z|^2|F_3|\chi_2(M_3)(Q_b u_{zz}|_{F_1} + Q_b u_{zz}|_{F_2}) \\
&= \frac{|e_z|^2|F_3|}{|e_x||F_1|}\chi_2(M_3)\left(\int_T u_{zz}dT + \int_T u_{zzx}(x - x_c)dT\right) \\
&= \frac{|e_z|^2|F_3|}{2}\chi_2(M_3)(Q_b u_{zz}|_{F_3} + Q_b u_{zz}|_{F_4}) \\
&\quad + \frac{|e_z|^2}{|e_y|}\chi_2(M_3)\left(\int_T u_{zzx}(x - x_c)dT - \int_T u_{zzy}(y - y_c)dT\right),
\end{aligned} \tag{7.8}$$

$$\begin{aligned}
& \frac{1}{2}|e_y||F_1|\chi_1(M_1)(Q_b q_{2y}|_{F_5} + Q_b q_{2y}|_{F_6}) \\
&= \frac{|e_y||F_1|}{2}\chi_1(M_1)(Q_b q_{2y}|_{F_1} + Q_b q_{2y}|_{F_2}) \\
&\quad + \frac{|e_y|}{|e_x|}\chi_1(M_1)\left(\int_T q_{2yz}(z - z_c)dT - \int_T q_{2yx}(x - x_c)dT\right),
\end{aligned} \tag{7.9}$$

$$\begin{aligned}
& \frac{1}{2}|e_z||F_3|\chi_2(M_3)(Q_b q_{3z}|_{F_1} + Q_b q_{3z}|_{F_2}) \\
&= \frac{|e_z||F_3|}{2}\chi_2(M_3)(Q_b q_{3z}|_{F_3} + Q_b q_{3z}|_{F_4}) \\
&\quad + \frac{|e_z|}{|e_y|}\chi_2(M_3)\left(\int_T q_{3zx}(x - x_c)dT - \int_T q_{3zy}(y - y_c)dT\right).
\end{aligned} \tag{7.10}$$

Using (6.12) and (7.5)–(7.10), (7.4) can be rewritten as

$$\begin{aligned}
& \zeta_u(v_b) \\
&= \frac{A_1}{2} \sum_{T \in \mathcal{T}_h} \rho h^{-1}|F_1|\chi_1(M_1) \left( |e_y|^2(Q_b u_{yy}|_{F_1} + Q_b u_{yy}|_{F_2}) \right. \\
&\quad \left. + |e_z|^2(Q_b u_{zz}|_{F_1} + Q_b u_{zz}|_{F_2}) \right) \\
&\quad + \frac{A_1}{2} \sum_{T \in \mathcal{T}_h} \rho h^{-1}|F_3|\chi_2(M_3) \left( |e_x|^2(Q_b u_{xx}|_{F_3} + Q_b u_{xx}|_{F_4}) \right. \\
&\quad \left. + |e_z|^2(Q_b u_{zz}|_{F_3} + Q_b u_{zz}|_{F_4}) \right) \\
&\quad + \frac{A_1}{2} \sum_{T \in \mathcal{T}_h} \rho h^{-1}|F_5|\chi_3(M_5) \left( |e_y|^2(Q_b u_{yy}|_{F_5} + Q_b u_{yy}|_{F_6}) \right. \\
&\quad \left. + |e_x|^2(Q_b u_{xx}|_{F_5} + Q_b u_{xx}|_{F_6}) \right) \\
&\quad - \frac{1}{2} \sum_{T \in \mathcal{T}_h} |F_1|\chi_1(M_1) \left( |e_y|(Q_b q_{2y}|_{F_1} + Q_b q_{2y}|_{F_2}) \right. \\
&\quad \left. + |e_z|(Q_b q_{3z}|_{F_1} + Q_b q_{3z}|_{F_2}) \right) \\
&\quad - \frac{1}{2} \sum_{T \in \mathcal{T}_h} |F_3|\chi_2(M_3) \left( |e_x|(Q_b q_{1x}|_{F_3} + Q_b q_{1x}|_{F_4}) \right. \\
&\quad \left. + |e_z|(Q_b q_{3z}|_{F_3} + Q_b q_{3z}|_{F_4}) \right) \\
&\quad - \frac{1}{2} \sum_{T \in \mathcal{T}_h} |F_5|\chi_3(M_5) \left( |e_x|(Q_b q_{1x}|_{F_5} + Q_b q_{1x}|_{F_6}) \right. \\
&\quad \left. + |e_y|(Q_b q_{2y}|_{F_5} + Q_b q_{2y}|_{F_6}) \right) + \sum_{T \in \mathcal{T}_h} R_4(T),
\end{aligned} \tag{7.11}$$

where the remainder term  $R_4(T)$  is given by

$$\begin{aligned} R_4(T) = & A_1 \frac{\rho h^{-1} |e_x|^2}{|e_z|} \chi_3(M_5) \left( \int_T u_{xyy}(y - y_c) dT - \int_T u_{xxz}(z - z_c) dT \right) \\ & + A_1 \frac{\rho h^{-1} |e_y|^2}{|e_x|} \chi_1(M_1) \left( \int_T u_{yyz}(z - z_c) dT - \int_T u_{yyx}(x - x_c) dT \right) \\ & + A_1 \frac{\rho h^{-1} |e_z|^2}{|e_y|} \chi_2(M_3) \left( \int_T u_{zzx}(x - x_c) dT - \int_T u_{zzy}(y - y_c) dT \right) \\ & - \frac{|e_x|}{|e_z|} \chi_3(M_5) \left( \int_T q_{1xy}(y - y_c) dT - \int_T q_{1xz}(z - z_c) dT \right) \\ & - \frac{|e_y|}{|e_x|} \chi_1(M_1) \left( \int_T q_{2yz}(z - z_c) dT - \int_T q_{2yx}(x - x_c) dT \right) \\ & - \frac{|e_z|}{|e_y|} \chi_2(M_3) \left( \int_T q_{3zx}(x - x_c) dT - \int_T q_{3zy}(y - y_c) dT \right) \\ & + R_3(T). \end{aligned}$$

Letting

$$w_b = \begin{cases} \frac{1}{2} \rho^{-1} h^{-1} (\rho h^{-1} A_1 (|e_y|^2 Q_b u_{yy}|_{F_1} + |e_z|^2 Q_b u_{zz}|_{F_1}) - |e_y| Q_b q_{2y}|_{F_1} - |e_z| Q_b q_{3z}|_{F_1}), & \text{on } F_1, \\ \frac{1}{2} \rho^{-1} h^{-1} (\rho h^{-1} A_1 (|e_y|^2 Q_b u_{yy}|_{F_2} + |e_z|^2 Q_b u_{zz}|_{F_2}) - |e_y| Q_b q_{2y}|_{F_2} - |e_z| Q_b q_{3z}|_{F_2}), & \text{on } F_2, \\ \frac{1}{2} \rho^{-1} h^{-1} (\rho h^{-1} A_1 (|e_x|^2 Q_b u_{xx}|_{F_3} + |e_z|^2 Q_b u_{zz}|_{F_3}) - |e_x| Q_b q_{1x}|_{F_3} - |e_z| Q_b q_{3z}|_{F_3}), & \text{on } F_3, \\ \frac{1}{2} \rho^{-1} h^{-1} (\rho h^{-1} A_1 (|e_x|^2 Q_b u_{xx}|_{F_4} + |e_z|^2 Q_b u_{zz}|_{F_4}) - |e_x| Q_b q_{1x}|_{F_4} - |e_z| Q_b q_{3z}|_{F_4}), & \text{on } F_4, \\ \frac{1}{2} \rho^{-1} h^{-1} (\rho h^{-1} A_1 (|e_y|^2 Q_b u_{yy}|_{F_5} + |e_x|^2 Q_b u_{xx}|_{F_5}) - |e_x| Q_b q_{1x}|_{F_5} - |e_y| Q_b q_{2y}|_{F_5}), & \text{on } F_5, \\ \frac{1}{2} \rho^{-1} h^{-1} (\rho h^{-1} A_1 (|e_y|^2 Q_b u_{yy}|_{F_6} + |e_x|^2 Q_b u_{xx}|_{F_6}) - |e_x| Q_b q_{1x}|_{F_6} - |e_y| Q_b q_{2y}|_{F_6}), & \text{on } F_6. \end{cases}$$

Thus, (7.11) is rewritten as

$$\begin{aligned} \zeta_u(v_b) = & \rho h \sum_{T \in \mathcal{T}_h} \langle w_b, Q_b S(v_b) - v_b \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} R_4(T) \\ = & -\rho h \sum_{T \in \mathcal{T}_h} \langle Q_b S(w_b) - w_b, Q_b S(v_b) - v_b \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} R_4(T), \end{aligned} \quad (7.12)$$

where we use (4.2) on the last line.

Substituting (7.12) into (5.2) we obtain

$$\begin{aligned} & (A \nabla_d e_b, \nabla_d v_b)_h + \rho h^{-1} \sum_{T \in \mathcal{T}_h} \langle Q_b S(e_b) - e_b, Q_b S(v_b) - v_b \rangle_{\partial T} \\ = & -\rho h \sum_{T \in \mathcal{T}_h} \langle Q_b S(w_b) - w_b, Q_b S(v_b) - v_b \rangle_{\partial T} + R_4(v_b), \end{aligned} \quad (7.13)$$

where  $R_4(v_b) = \sum_{T \in \mathcal{T}_h} R_4(T)$ .

Letting  $\tilde{e}_b = e_b + h^2 w_b$ , we arrive at

$$\begin{aligned} & (A \nabla_d \tilde{e}_b, \nabla_d v_b)_h + \rho h^{-1} \sum_{T \in \mathcal{T}_h} \langle Q_b S(\tilde{e}_b) - \tilde{e}_b, Q_b S(v_b) - v_b \rangle_{\partial T} \\ = & h^2 (A \nabla_d w_b, \nabla_d v_b)_h + R_4(v_b). \end{aligned}$$

It is easy to see from the definition of  $w_b$  that

$$\|\nabla_d w_b\|_0 \leq C \|u\|_3, \quad (7.14)$$

from which, (7.2) is obtained in a similar way of the proof of (6.14). This completes the proof of the theorem.  $\square$

For the second order elliptic problem (1.1) in three dimensions with the homogeneous Dirichlet boundary value, we have  $w_b \in V_b^0$ . Letting  $v_b = \tilde{e}_b$  in (7.1) gives a superconvergence estimate

$$\left( \sum_{T \in \mathcal{T}_h} \|\nabla_d \tilde{e}_b\|_T^2 \right)^{\frac{1}{2}} \leq Ch^2 \|u\|_3.$$

For the second order elliptic problem (1.1) in three dimensions with the nonhomogeneous Dirichlet boundary condition, we have  $w_b \notin V_b^0$ . Thus,  $\tilde{e}_b = (Q_b u - u_b) + h^2 w_b \notin V_b^0$ . In order to obtain a superconvergence estimate, we enforce a computational solution  $u_b$  satisfying the following condition:

$$u_b = Q_b g + h^2 w_b, \quad \text{on } \partial\Omega. \quad (7.15)$$

The above boundary condition is able to be implemented if  $w_b|_{\partial\Omega}$  is computable without any prior knowledge of the exact solution  $u$ . The following theorem and corollary assume that  $w_b|_{\partial\Omega}$  is computable.

**Theorem 7.2.** Assume that  $u \in H^3(\Omega)$  is the exact solution of the second order elliptic model problem (1.1) in three dimensions and  $u_b \in V_b^g$  is the numerical solution of the simplified weak Galerkin finite element scheme (4.4). Let  $w_b \in V_b$  be a given function defined in Theorem 7.1. Denote by  $\tilde{e}_b = (Q_b u - u_b) + h^2 w_b$  the modified error function. The following superconvergence estimate holds true:

$$\left( \sum_{T \in \mathcal{T}_h} \|\nabla_d \tilde{e}_b\|_T^2 \right)^{\frac{1}{2}} + \|S(\tilde{e}_b) - \tilde{e}_b\|_{\varepsilon_h} \leq Ch^2 \|u\|_3. \quad (7.16)$$

**Proof.** Letting  $v_b = \tilde{e}_b \in V_b^0$  in (7.1) gives

$$\begin{aligned} & (A \nabla_d \tilde{e}_b, \nabla_d \tilde{e}_b)_h + \rho h^{-1} \sum_{T \in \mathcal{T}_h} \langle Q_b S(\tilde{e}_b) - \tilde{e}_b, Q_b S(\tilde{e}_b) - \tilde{e}_b \rangle_{\partial T} \\ &= h^2 (A \nabla_d w_b, \nabla_d \tilde{e}_b)_h + R_4(\tilde{e}_b). \end{aligned}$$

It follows from (7.2), (7.14), (6.19) and the Cauchy–Schwarz inequality that

$$\begin{aligned} & (A \nabla_d \tilde{e}_b, \nabla_d \tilde{e}_b)_h + \rho h^{-1} \sum_{T \in \mathcal{T}_h} \|Q_b S(\tilde{e}_b) - \tilde{e}_b\|_{\partial T}^2 \\ & \leq Ch^2 \|u\|_3 (\|\nabla_d \tilde{e}_b\|_0 + \|S(\tilde{e}_b) - \tilde{e}_b\|_{\varepsilon_h}) \\ & \leq Ch^4 \|u\|_3^2 + \frac{\|A\|_\infty}{2} \|\nabla_d \tilde{e}_b\|_0^2 + \frac{1}{2} \|S(\tilde{e}_b) - \tilde{e}_b\|_{\varepsilon_h}^2, \end{aligned}$$

which leads to

$$\sum_{T \in \mathcal{T}_h} \|\nabla_d \tilde{e}_b\|_T^2 + \|S(\tilde{e}_b) - \tilde{e}_b\|_{\varepsilon_h}^2 \leq Ch^4 \|u\|_3^2.$$

This completes the proof of the theorem.  $\square$

We can see from (7.15) that the standard  $L^2$  projection of Dirichlet data  $g$  is perturbed by

$$\varepsilon_b := \frac{1}{12} (|e_y|^2 Q_b g_{yy} + |e_z|^2 Q_b g_{zz} - 6\rho^{-1} h (|e_y| Q_b q_{2y} + |e_z| Q_b q_{3z}))$$

on boundary faces parallel to the flat face  $F_1$  (or  $F_2$ ),

$$\varepsilon_b := \frac{1}{12} (|e_x|^2 Q_b g_{xx} + |e_z|^2 Q_b g_{zz} - 6\rho^{-1} h (|e_x| Q_b q_{1x} + |e_z| Q_b q_{3z}))$$

on boundary faces parallel to the flat face  $F_3$  (or  $F_4$ ), and

$$\varepsilon_b := \frac{1}{12} (|e_x|^2 Q_b g_{xx} + |e_y|^2 Q_b g_{yy} - 6\rho^{-1} h (|e_x| Q_b q_{1x} + |e_y| Q_b q_{2y}))$$

on boundary faces parallel to the flat face  $F_5$  (or  $F_6$ ). For the Dirichlet boundary value problem with a diagonal diffusive tensor  $A = (a_{11}, 0, 0; 0, a_{22}, 0; 0, 0, a_{33})$ , the perturbation  $\varepsilon_b$  is computable using the boundary data  $g$  and thus the mixed partial derivatives of  $u$  are not needed. The following superconvergence estimate is particularly for the second order elliptic problem in three dimensions with a diagonal diffusive tensor  $A$ .

**Corollary 7.3.** Assume that  $u \in H^3(\Omega)$  is the exact solution of the model problem (1.1) in three dimensions with a diagonal diffusive tensor  $A = (a_{11}, 0, 0; 0, a_{22}, 0; 0, 0, a_{33})$ . Let  $u_b \in V_b$  be the weak Galerkin finite element solution arising from the

scheme (4.4) with the boundary values specified as follows: on the boundary faces which are parallel to the flat face  $F_1$  (or  $F_2$ ), let

$$u_b = Q_b g + \frac{1}{12} \left( |e_y|(|e_y| - \frac{6}{\rho} h a_{22}) Q_b g_{yy} + |e_z|(|e_z| - \frac{6}{\rho} h a_{33}) Q_b g_{zz} \right) \quad (7.17)$$

on the boundary faces which are parallel to the flat face  $F_3$  (or  $F_4$ ), let

$$u_b = Q_b g + \frac{1}{12} \left( |e_x|(|e_x| - \frac{6}{\rho} h a_{11}) Q_b g_{xx} + |e_z|(|e_z| - \frac{6}{\rho} h a_{33}) Q_b g_{zz} \right) \quad (7.18)$$

and on the boundary faces which are parallel to the flat face  $F_5$  (or  $F_6$ ), let

$$u_b = Q_b g + \frac{1}{12} \left( |e_y|(|e_y| - \frac{6}{\rho} h a_{22}) Q_b g_{yy} + |e_x|(|e_x| - \frac{6}{\rho} h a_{11}) Q_b g_{xx} \right). \quad (7.19)$$

There holds

$$\left( \sum_{T \in \mathcal{T}_h} \|Q_h \nabla u - \nabla_d u_b\|_T^2 \right)^{1/2} \leq Ch^2 \|u\|_3. \quad (7.20)$$

**Proof.** Note that the perturbation  $\varepsilon_b$  is of the order  $\mathcal{O}(h^2)$ . It follows from Theorem 7.2 that

$$\left( \sum_{T \in \mathcal{T}_h} \|\nabla_d Q_b u - \nabla_d u_b\|_T^2 \right)^{1/2} \leq Ch^2 \|u\|_3,$$

which combined with the commutative property (5.1), yields

$$\left( \sum_{T \in \mathcal{T}_h} \|Q_h \nabla u - \nabla_d u_b\|_T^2 \right)^{1/2} \leq Ch^2 \|u\|_3.$$

This completes the proof of the corollary.  $\square$

For the model problem (1.1) in three dimensions with arbitrary diffusive coefficient  $A$ , the following superconvergence error estimate holds true.

**Theorem 7.4.** Assume that  $u \in H^3(\Omega)$  is the exact solution of the second order elliptic model problem (1.1) in three dimensions. Let  $u_b \in V_b$  be the weak Galerkin finite element solution arising from the scheme (4.4) with the boundary value  $u_b = Q_b g$  on  $\partial\Omega$ . There holds

$$\left( \sum_{T \in \mathcal{T}_h} \|\nabla_d e_b\|_T^2 \right)^{\frac{1}{2}} \leq Ch^{1.5} (\|u\|_3 + \|\nabla^2 u\|_{0,\partial\Omega}).$$

**Proof.** The proof is the similar to the proof of Theorem 6.7 in [22], and therefore the details are omitted here.  $\square$

**Corollary 7.5.** Let  $u_b \in V_b$  such that  $u_b = Q_b g$  on  $\partial\Omega$  be the weak Galerkin finite element solution of the model problem (1.1) in three dimensions arising from the scheme (4.4). Assume that (1) the exact solution  $u \in H^3(\Omega)$  of the model problem (1.1) satisfies  $u_{xx} = u_{yy} = u_{zz}$ ; (2)  $\mathcal{T}_h$  is a uniform cubic partition  $\Omega$  with  $|e_x| = |e_y| = |e_z|$ ; and (3) the diffusive tensor  $A = (a_{11}, 0, 0, 0, a_{22}, 0, 0, 0, a_{33})$  satisfies  $a_{11} = a_{22} = a_{33}$ . Denote by  $e_b = Q_b u - u_b$ . The following superconvergence result holds true:

$$\left( \sum_{T \in \mathcal{T}_h} \|\nabla_d e_b\|_T^2 \right)^{\frac{1}{2}} \leq Ch^2 \|u\|_3.$$

**Proof.** Using Theorem 7.1, Lemma 6.1 and the error Eq. (7.13) concludes the corollary. Details are omitted here due to page limitation.  $\square$

## 8. Numerical experiments

In this section, a series of numerical tests will be demonstrated for the simplified WG algorithm (4.4) for solving the second order elliptic problem (1.1) in three dimensions to verify the superconvergence error estimates established in the previous sections.

**Table 8.1**

Test Case 1: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \sin(\pi x)\sin(\pi y)\sin(\pi z)$ , uniform cubic partitions, stabilization parameter  $\rho = 6$ .

| Meshe                    | $\ u - \mathcal{S}(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|---------------------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $4 \times 4 \times 4$    | 2.4845e-02                            | 1.9393e-02  | 1.8494e-01           | 4.1467e-02                          | 1.6637e-01      |
| $8 \times 8 \times 8$    | 6.4194e-03                            | 4.6306e-03  | 4.8626e-02           | 1.1850e-02                          | 4.3758e-02      |
| $16 \times 16 \times 16$ | 1.6069e-03                            | 1.1415e-03  | 1.2310e-02           | 3.0582e-03                          | 1.1079e-02      |
| $32 \times 32 \times 32$ | 4.0164e-04                            | 2.8433e-04  | 3.0872e-03           | 7.7058e-04                          | 2.7784e-03      |
| Rate                     | 2.00                                  | 2.01        | 2.00                 | 1.99                                | 2.00            |

**Table 8.2**

Test Case 1: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \sin(\pi x)\sin(\pi y)\sin(\pi z)$ , non-uniform cubic partitions, stabilization parameter  $\rho = 6$ , and  $h = (|e_x|^2 + |e_y|^2 + |e_z|^2)^{\frac{1}{2}}$ .

| Meshe                    | $\ u - \mathcal{S}(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|---------------------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $3 \times 4 \times 5$    | 3.0558e-02                            | 2.3666e-02  | 2.1210e-01           | 5.2931e-02                          | 1.9037e-01      |
| $6 \times 8 \times 10$   | 6.3404e-03                            | 5.6370e-03  | 5.5494e-02           | 1.3847e-02                          | 4.9893e-02      |
| $12 \times 16 \times 20$ | 1.5721e-03                            | 1.3928e-03  | 1.4036e-02           | 3.5264e-03                          | 1.2625e-02      |
| $24 \times 32 \times 40$ | 3.9192e-04                            | 3.4718e-04  | 3.5192e-03           | 8.8605e-04                          | 3.1660e-03      |
| Rate                     | 2.00                                  | 2.00        | 2.00                 | 1.99                                | 2.00            |

The numerical tests are based on the lowest order (i.e.,  $k = 1$ ) of weak functions on the uniform and non-uniform cubic partitions of the unit cube  $\Omega = (0, 1)^3$ . More precisely, the local weak finite element space is given by  $V(T, 1) = \{v = \{v_0, v_b\}, v_0 \in P_1(T), v_b \in P_0(F), F \subset \partial T\}$ , and  $\nabla_d v|_T \in [P_0(T)]^3$ .

Let  $u$  be the exact solution of the model problem (1.1). The error function  $e_h$  is given by  $e_h = Q_h u - u_h = \{e_0, e_b\}$  where  $e_0 = Q_0 u - \mathcal{S}(u_b)$  and  $e_b = Q_b u - u_b$ . The error functions are measured in various norms as follows:

$$\text{Discrete } L^\infty\text{-norm : } \|u - \mathcal{S}(u_b)\|_{\infty,*} = \max_{T \in \mathcal{T}_h} |(u - \mathcal{S}(u_b))(M_c)|,$$

$$L^2\text{-norm : } \|e_0\|_0 = \left( \sum_{T \in \mathcal{T}_h} \int_T |Q_0 u - \mathcal{S}(u_b)|^2 dT \right)^{1/2},$$

$$H^1\text{-norm : } \|\nabla_d e_b\|_0 = \left( \sum_{T \in \mathcal{T}_h} \int_T |\nabla_d(Q_b u - u_b)|^2 dT \right)^{1/2},$$

$$\text{Discrete } W^{1,1}\text{-norm : } \|\nabla_d u_b - \nabla u\|_{1,*} = \left( \sum_{T \in \mathcal{T}_h} \int_T |\nabla_d u_b - \nabla u(M_c)|^2 dT \right)^{1/2},$$

$$W^{1,1}\text{-seminorm : } \|e_0\|_{1,1} = \left( \sum_{T \in \mathcal{T}_h} \int_T |\nabla(Q_0 u - \mathcal{S}(u_b))|^2 dT \right)^{1/2},$$

where  $M_c$  is the center of the cubic element  $T$ .

### 8.1. Numerical experiments for constant diffusion tensor $A$

**Test Case 1 (Homogeneous BVP).** In this set of tests, the diffusive coefficient tensor  $A$  is an identity matrix and the exact solution is  $u = \sin(\pi x)\sin(\pi y)\sin(\pi z)$ . This is a homogeneous boundary value problem on the domain  $\Omega = (0, 1)^3$ .

Tables 8.1–8.4 illustrate the numerical results on the uniform and non-uniform cubic partitions with the stabilization parameters  $\rho = 6$  and  $\rho = 1$ , respectively. These results show that the convergence rate for the error  $e_b$  in  $H^1$ -norm is of order  $\mathcal{O}(h^2)$ , which is consistent with the conclusion in Corollary 7.3. We also compute the convergence rates for the error functions in  $\|\cdot\|_{1,*}$  norm and  $\|\cdot\|_{1,1}$  norm which seem to be in the superconvergence order of  $\mathcal{O}(h^2)$ , although there are not any corresponding theories available in this paper.

It is interesting to see from Tables 8.1 and 8.3 that the numerical results corresponding to stabilization parameters  $\rho = 6$  and  $\rho = 1$  are very close to each other. Furthermore, we compute the numerical results for some other stabilization parameter  $\rho = 0.01, 0.1, 2, 5$ , and we found the numerical results are still very close to the results in Table 8.1. Due to page limitation, we shall not demonstrate the numerical results for  $\rho = 0.01, 0.1, 2, 5$  in this paper. Interested readers are welcome to draw their own conclusions for this phenomenon.

**Test Case 2 (Nonhomogeneous BVP).** In this set of tests, the exact solution is  $u = \cos(x)\sin(y)\cos(z)$  and the coefficient matrix  $A$  is an identity matrix. This is a non-homogeneous boundary value problem on the domain  $\Omega = (0, 1)^3$ .

In Tables 8.5 and 8.6, we employ the perturbed  $L^2$  projection defined in (7.17)–(7.19) and stabilization parameter  $\rho = 1$ . The numerical results demonstrate that  $\|\nabla_d e_b\|_0$  converges in the superconvergence order of  $\mathcal{O}(h^2)$ , which perfectly consists with Corollary 7.3. In Tables 8.7 and 8.8, we use the usual  $L^2$  projection (i.e., the perturbation term  $\varepsilon_b = 0$ ) and

**Table 8.3**

Test Case 1: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \sin(\pi x)\sin(\pi y)\sin(\pi z)$ , uniform cubic partitions, stabilization parameter  $\rho = 1$ .

| Mesheres                 | $\ u - S(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|-----------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $4 \times 4 \times 4$    | 2.4845e-02                  | 1.9393e-02  | 1.8494e-01           | 4.1467e-02                          | 1.6637e-01      |
| $8 \times 8 \times 8$    | 6.4194e-03                  | 4.6306e-03  | 4.8626e-02           | 1.1850e-02                          | 4.3758e-02      |
| $16 \times 16 \times 16$ | 1.6069e-03                  | 1.1415e-03  | 1.2310e-02           | 3.0582e-03                          | 1.1079e-02      |
| $32 \times 32 \times 32$ | 4.0164e-04                  | 2.8433e-04  | 3.0872e-03           | 7.7058e-04                          | 2.7784e-03      |
| Rate                     | 2.00                        | 2.01        | 2.00                 | 1.99                                | 2.00            |

**Table 8.4**

Test Case 1: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \sin(\pi x)\sin(\pi y)\sin(\pi z)$ , non-uniform cubic partitions, stabilization parameter  $\rho = 1$ , and  $h = \max(|e_x|, |e_y|, |e_z|)$ .

| Mesheres                 | $\ u - S(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|-----------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $3 \times 4 \times 5$    | 2.2605e-02                  | 2.5271e-02  | 2.1817e-01           | 6.9417e-02                          | 1.9998e-01      |
| $6 \times 8 \times 10$   | 3.4472e-03                  | 6.6425e-03  | 6.1177e-02           | 2.8983e-02                          | 5.7273e-02      |
| $12 \times 16 \times 20$ | 7.3558e-04                  | 1.6886e-03  | 1.5931e-02           | 8.3053e-03                          | 1.5017e-02      |
| $24 \times 32 \times 40$ | 1.7500e-04                  | 4.2391e-04  | 4.0277e-03           | 2.1491e-03                          | 3.8034e-03      |
| Rate                     | 2.07                        | 1.99        | 1.98                 | 1.95                                | 1.98            |

**Table 8.5**

Test Case 2: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \cos(x)\sin(y)\cos(z)$ , uniform cubic partitions, stabilization parameter  $\rho = 1$ , and perturbed  $L^2$  projection of the Dirichlet boundary data  $g$  by (7.17)–(7.19).

| Mesheres                 | $\ u - S(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|-----------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $4 \times 4 \times 4$    | 2.8491e-02                  | 1.6837e-02  | 3.5512e-02           | 3.0139e-02                          | 3.4796e-02      |
| $8 \times 8 \times 8$    | 7.7591e-03                  | 4.2216e-03  | 8.9019e-03           | 7.5565e-03                          | 8.7222e-03      |
| $16 \times 16 \times 16$ | 2.0045e-03                  | 1.0576e-03  | 2.2294e-03           | 1.8930e-03                          | 2.1845e-03      |
| $32 \times 32 \times 32$ | 5.0788e-04                  | 2.6457e-04  | 5.5769e-04           | 4.7357e-04                          | 5.4645e-04      |
| Rate                     | 1.98                        | 2.00        | 2.00                 | 2.00                                | 2.00            |

**Table 8.6**

Test Case 2: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \cos(x)\sin(y)\cos(z)$ , non-uniform cubic partitions, stabilization parameter  $\rho = 1$ ,  $h = \max(|e_x|, |e_y|, |e_z|)$ , and perturbed  $L^2$  projection of the Dirichlet boundary data  $g$  by (7.17)–(7.19).

| Mesheres                 | $\ u - S(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|-----------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $3 \times 4 \times 5$    | 4.5196e-02                  | 2.5473e-02  | 5.1833e-02           | 4.5731e-02                          | 5.1101e-02      |
| $6 \times 8 \times 10$   | 1.2338e-02                  | 6.3927e-03  | 1.2987e-02           | 1.1459e-02                          | 1.2803e-02      |
| $12 \times 16 \times 20$ | 3.1901e-03                  | 1.6029e-03  | 3.2538e-03           | 2.8716e-03                          | 3.2078e-03      |
| $24 \times 32 \times 40$ | 8.0847e-04                  | 4.0115e-04  | 8.1411e-04           | 7.1856e-04                          | 8.0261e-04      |
| Rate                     | 1.98                        | 2.00        | 2.00                 | 2.00                                | 2.00            |

stabilization parameter  $\rho = 1$ . Table 8.7 shows the convergence rate for  $\|\nabla_d e_b\|_0$  is in a superconvergence order of  $\mathcal{O}(h^2)$ , which is in great consistency with Corollary 7.5, since the exact solution satisfies  $u_{xx} = u_{yy} = u_{zz}$ , the coefficient matrix  $A$  is an identity matrix, and the partitions are uniform. Table 8.8 shows that the superconvergence order in  $H^1$ - norm for  $e_b$  is higher than our theory  $r = 1.5$  in Theorem 7.4. Tables 8.5–8.8 show that a perturbed  $L^2$  projection does provide a better numerical solution than the usual  $L^2$  projection.

Tables 8.9–8.11 show the numerical results for the stabilization parameter  $\rho = 6$ . Note that the perturbed  $L^2$  projection defined in (7.17)–(7.19) turns to be the usual  $L^2$  projection for the stabilization parameter  $\rho = 6$  on the uniform partitions. Table 8.9 shows the convergence rate for  $\|\nabla_d e_b\|_0$  for the usual  $L^2$  projection on the uniform partitions is in the superconvergence order of  $\mathcal{O}(h^2)$  which is consistent with Corollary 7.3. In Table 8.10, the perturbed  $L^2$  projection on the non-uniform partitions is used and the numerical results show the superconvergence order for  $\|\nabla_d e_b\|_0$  is  $\mathcal{O}(h^2)$ , which consists with Corollary 7.3. In Table 8.11, the usual  $L^2$  projection on non-uniform partitions is employed and it seems that the convergence rate for  $\|\nabla_d e_b\|_0$  is in the order of  $\mathcal{O}(h^{1.95})$ , which is higher than  $\mathcal{O}(h^{1.5})$  in Theorem 7.4.

**Test Case 3 (Nonhomogeneous BVP).** In this group of numerical tests, the coefficient tensor  $A$  is an identity matrix, the stabilization parameter is  $\rho = 1$ , and the exact solution is  $u = \cos(\pi x)\cos(\pi y)\exp(z)$ . This is a nonhomogeneous boundary value problem.

Figs. 8.1 and 8.2 compare the performance on uniform cubic partitions when the perturbed  $L^2$  projection and the usual  $L^2$  projection are used, respectively. Fig. 8.1 presents the superconvergence order for  $\|\nabla_d e_b\|_0$  for the usual  $L^2$  projection is better than the theory  $\mathcal{O}(h^{1.5})$  in Theorem 7.4. Fig. 8.2 demonstrates that the superconvergence rate for  $\|\nabla_d e_b\|_0$  is  $\mathcal{O}(h^2)$  for the perturbed  $L^2$  projection, which consists perfectly with Corollary 7.3.

**Table 8.7**

Test Case 2: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \cos(x)\sin(y)\cos(z)$ , uniform cubic partitions, stabilization parameter  $\rho = 1$ , and  $L^2$  projection of the Dirichlet boundary data  $g$ .

| Meshe                    | $\ u - \mathcal{S}(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|---------------------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $4 \times 4 \times 4$    | 9.6021e-03                            | 1.7217e-03  | 1.6445e-03           | 5.3190e-03                          | 1.6266e-03      |
| $8 \times 8 \times 8$    | 2.5944e-03                            | 4.3709e-04  | 4.0413e-04           | 1.3353e-03                          | 4.0262e-04      |
| $16 \times 16 \times 16$ | 6.6871e-04                            | 1.1006e-04  | 1.0087e-04           | 3.3482e-04                          | 1.0093e-04      |
| $32 \times 32 \times 32$ | 1.6933e-04                            | 2.7576e-05  | 2.5230e-05           | 8.3791e-05                          | 2.5280e-05      |
| Rate                     | 1.98                                  | 2.00        | 2.00                 | 2.00                                | 2.00            |

**Table 8.8**

Test Case 2: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \cos(x)\sin(y)\cos(z)$ , non-uniform cubic partitions, stabilization parameter  $\rho = 1$ ,  $h = \max(|e_x|, |e_y|, |e_z|)$ , and  $L^2$  projection of the Dirichlet boundary data  $g$ .

| Meshe                    | $\ u - \mathcal{S}(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|---------------------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $3 \times 4 \times 5$    | 1.0491e-02                            | 1.7126e-03  | 9.4802e-03           | 1.1082e-02                          | 9.5562e-03      |
| $6 \times 8 \times 10$   | 3.1746e-03                            | 4.4825e-04  | 3.4760e-03           | 3.7710e-03                          | 3.4972e-03      |
| $12 \times 16 \times 20$ | 8.7237e-04                            | 1.1627e-04  | 1.0666e-03           | 1.1282e-03                          | 1.0715e-03      |
| $24 \times 32 \times 40$ | 2.3618e-04                            | 2.9556e-05  | 3.0668e-04           | 3.2021e-04                          | 3.0782e-04      |
| Rate                     | 1.89                                  | 1.98        | 1.80                 | 1.82                                | 1.80            |

**Table 8.9**

Test Case 2: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \cos(x)\sin(y)\cos(z)$ , uniform cubic partitions, stabilization parameter  $\rho = 6$ , and  $L^2$  projection of the Dirichlet boundary data  $g$ .

| Meshe                    | $\ u - \mathcal{S}(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|---------------------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $4 \times 4 \times 4$    | 9.6682e-03                            | 1.7514e-03  | 1.5820e-03           | 5.3468e-03                          | 1.5844e-03      |
| $8 \times 8 \times 8$    | 2.6003e-03                            | 4.4054e-04  | 4.0144e-04           | 1.3398e-03                          | 4.0227e-04      |
| $16 \times 16 \times 16$ | 6.6911e-04                            | 1.1033e-04  | 1.0080e-04           | 3.3520e-04                          | 1.0104e-04      |
| $32 \times 32 \times 32$ | 1.6935e-04                            | 2.7594e-05  | 2.5229e-05           | 8.3818e-05                          | 2.5291e-05      |
| Rate                     | 1.98                                  | 2.00        | 2.00                 | 2.00                                | 2.00            |

**Table 8.10**

Test Case 2: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \cos(x)\sin(y)\cos(z)$ , non-uniform cubic partitions, stabilization parameter  $\rho = 6$ ,  $h = (|e_x|^2 + |e_y|^2 + |e_z|^2)^{\frac{1}{2}}$ , and perturbed  $L^2$  projection of the Dirichlet boundary data  $g$  by (7.17)–(7.19).

| Meshe                    | $\ u - \mathcal{S}(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|---------------------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $3 \times 4 \times 5$    | 4.5849e-03                            | 1.1666e-03  | 5.8119e-03           | 7.9357e-04                          | 5.0500e-03      |
| $6 \times 8 \times 10$   | 1.2310e-03                            | 2.8277e-04  | 1.4518e-03           | 1.9305e-04                          | 1.2608e-03      |
| $12 \times 16 \times 20$ | 3.1661e-04                            | 7.0116e-05  | 3.6286e-04           | 4.8178e-05                          | 3.1510e-04      |
| $24 \times 32 \times 40$ | 8.0124e-05                            | 1.7493e-05  | 9.0708e-05           | 1.2045e-05                          | 7.8768e-05      |
| Rate                     | 1.98                                  | 2.00        | 2.00                 | 2.00                                | 2.00            |

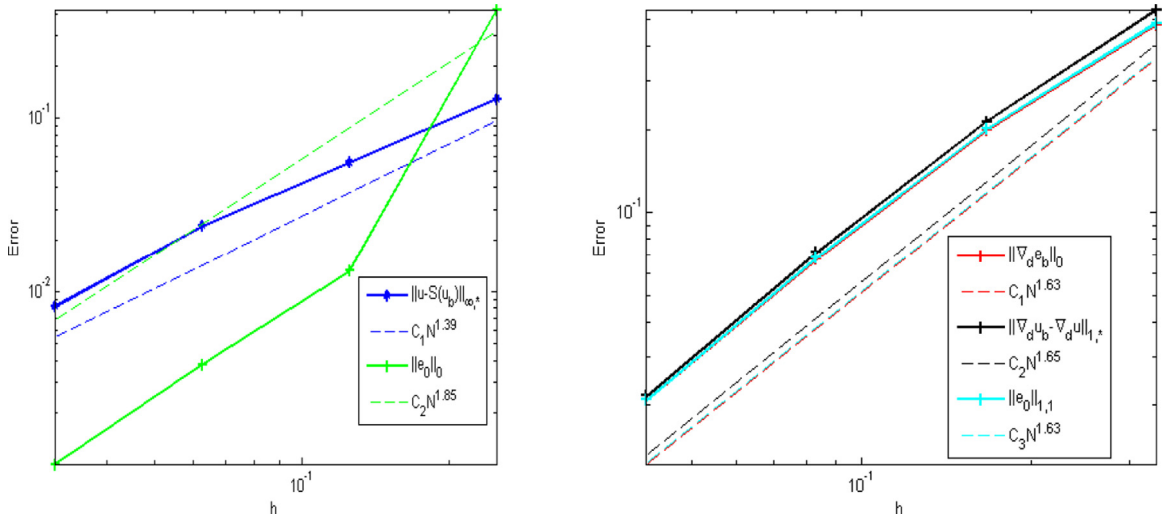
**Table 8.11**

Test Case 2: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \cos(x)\sin(y)\cos(z)$ , non-uniform cubic partitions, stabilization parameter  $\rho = 6$ ,  $h = (|e_x|^2 + |e_y|^2 + |e_z|^2)^{\frac{1}{2}}$ , and  $L^2$  projection of the Dirichlet boundary data  $g$ .

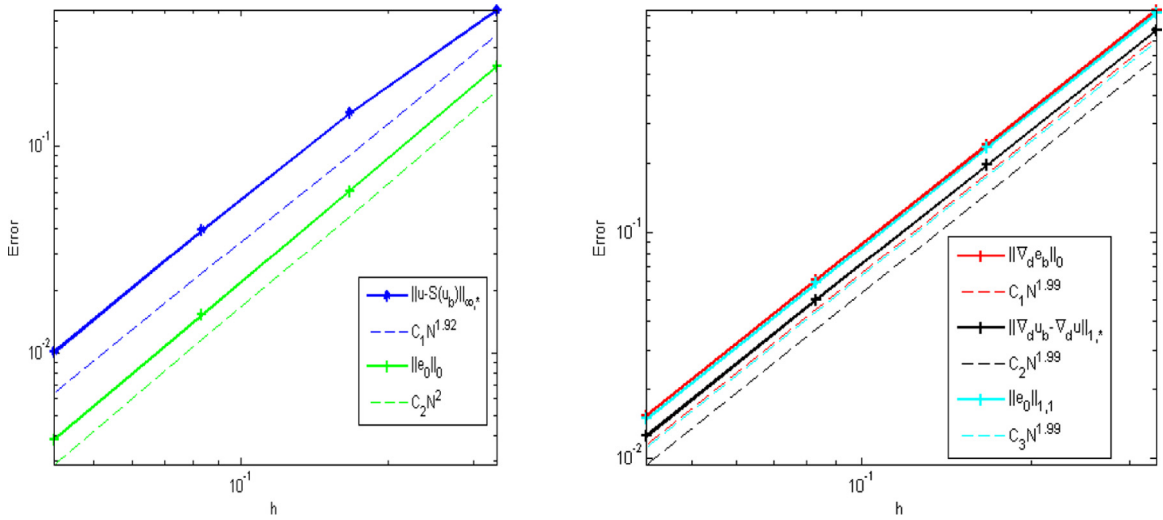
| Meshe                    | $\ u - \mathcal{S}(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|---------------------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $3 \times 4 \times 5$    | 1.0575e-02                            | 1.8212e-03  | 2.0566e-03           | 6.1443e-03                          | 1.9932e-03      |
| $6 \times 8 \times 10$   | 2.8577e-03                            | 4.5857e-04  | 5.2697e-04           | 1.5434e-03                          | 5.1248e-04      |
| $12 \times 16 \times 20$ | 7.3831e-04                            | 1.1496e-04  | 1.3610e-04           | 3.8768e-04                          | 1.3270e-04      |
| $24 \times 32 \times 40$ | 1.8753e-04                            | 2.8764e-05  | 3.5157e-05           | 9.7348e-05                          | 3.4342e-05      |
| Rate                     | 1.98                                  | 2.00        | 1.95                 | 1.99                                | 1.95            |

Figs. 8.3 and 8.4 compare the performance on non-uniform cubic partitions for the perturbed  $L^2$  projection and the usual  $L^2$  projection, respectively. The convergence order shown in Fig. 8.3 is in good consistency with the theory  $\mathcal{O}(h^2)$ . Moreover, from Fig. 8.4, we can see that the convergence rate is greater than the theoretical result  $\mathcal{O}(h^{1.5})$  for the error function  $\nabla_d e_b$ . It should be pointed out that the convergence rate is obtained based on a least square fitting technique as specified in [56].

**Test Case 4 (Nonhomogeneous BVP).** In this test, the exact solution is  $u = \sin(x)\sin(y)\sin(z)$ , the diffusive tensor is  $A = \begin{bmatrix} 10 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ , and the stabilization parameter is  $\rho = 1$ . The numerical results shown in Table 8.12 are based on the non-uniform cubic partitions and the usual  $L^2$  projection. The numerical performance in Table 8.12 is in great consistency with Theorem 7.4.



**Fig. 8.1.** Test Case 3: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \cos(\pi x)\cos(\pi y)\exp(z)$ , uniform cubic partitions, stabilization parameter  $\rho = 1$ , and  $L^2$  projection of the Dirichlet boundary data  $g$ .



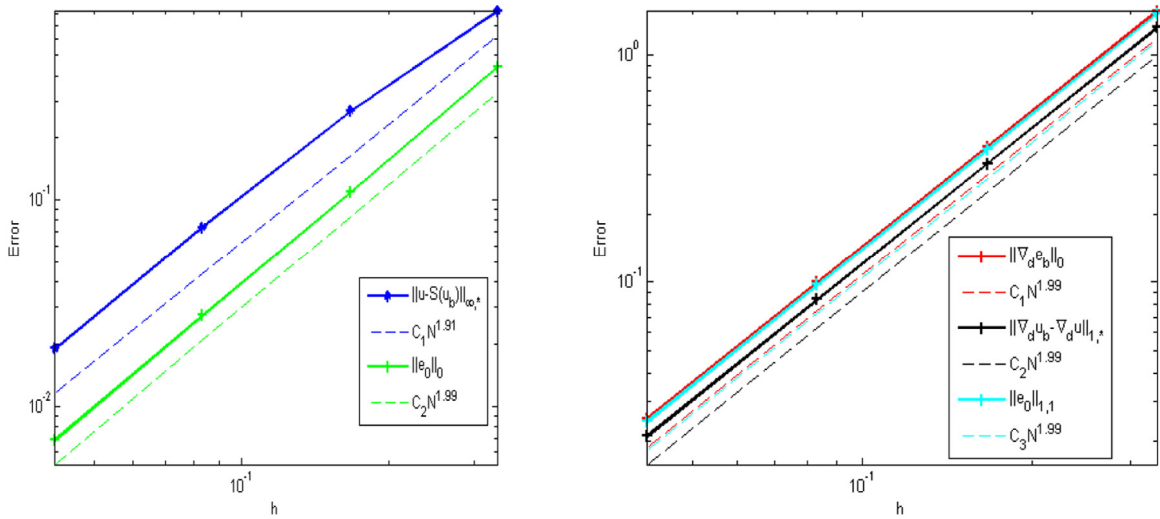
**Fig. 8.2.** Test Case 3: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \cos(\pi x)\cos(\pi y)\exp(z)$ , uniform cubic partitions, stabilization parameter  $\rho = 1$ , and perturbed  $L^2$  projection of the Dirichlet boundary data  $g$  by (7.17)–(7.19).

**Table 8.12**

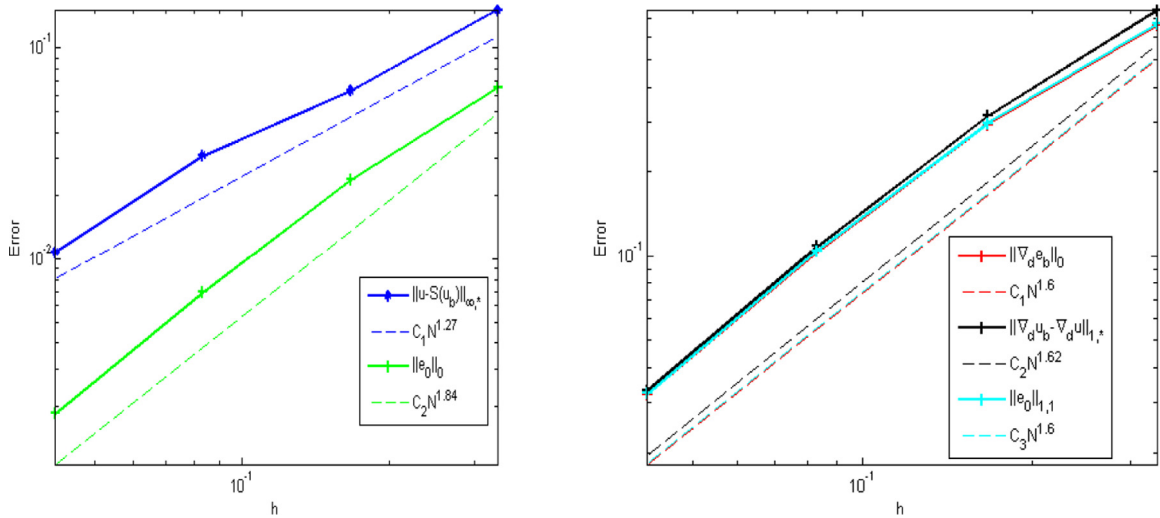
Test Case 4: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \sin(x)\sin(y)\sin(z)$ , non-uniform cubic partitions, stabilization parameter  $\rho = 1$ ,  $h = \max(|e_x|, |e_y|, |e_z|)$ , and  $L^2$  projection of the Dirichlet boundary data  $g$ . The coefficient matrix is  $a_{11} = 10$ ,  $a_{12} = 3$ ,  $a_{13} = 1$ , and  $a_{22} = 2$ ,  $a_{23} = 1$ ,  $a_{33} = 2$ .

| Meshe                    | $\ u - \mathcal{S}(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla_d u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|---------------------------------------|-------------|----------------------|---------------------------------------|-----------------|
| $3 \times 4 \times 5$    | 1.3451e-02                            | 7.2295e-03  | 5.5726e-02           | 5.5713e-02                            | 5.5722e-02      |
| $6 \times 8 \times 10$   | 8.6020e-03                            | 3.4567e-03  | 2.9073e-02           | 2.9050e-02                            | 2.9071e-02      |
| $12 \times 16 \times 20$ | 4.2236e-03                            | 1.2471e-03  | 1.1590e-02           | 1.1583e-02                            | 1.1589e-02      |
| $24 \times 32 \times 40$ | 1.5369e-03                            | 3.6709e-04  | 3.9243e-03           | 3.9227e-03                            | 3.9242e-03      |
| Rate                     | 1.46                                  | 1.76        | 1.56                 | 1.56                                  | 1.56            |

**Test Case 5 (Nonhomogeneous BVP).** This test is in the following configuration: (1) The coefficient tensor  $A$  is an identity matrix; (2) the stabilization parameter  $\rho = 1$ ; (3) the exact solution is  $u = \cos(\pi x)\sin(\pi y)\cos(\pi z)$ . We first construct an initial non-uniform  $4 \times 4 \times 4$  cubic partition generated by perturbing a uniform  $4 \times 4 \times 4$  cubic partitions with a random noise. More precisely, for the element  $T = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_s, z_{s+1}]$  in an initial uniform  $4 \times 4 \times 4$  cubic partitions,  $x_{i+1}$ ,  $y_{j+1}$  and  $z_{s+1}$  are adjusted as follows:  $x_{i+1}^* = x_{i+1} + 0.2(\text{rand}(1) - 0.5)/N$ ,  $y_{j+1}^* = y_{j+1} + 0.2(\text{rand}(1) - 0.5)/N$ ,



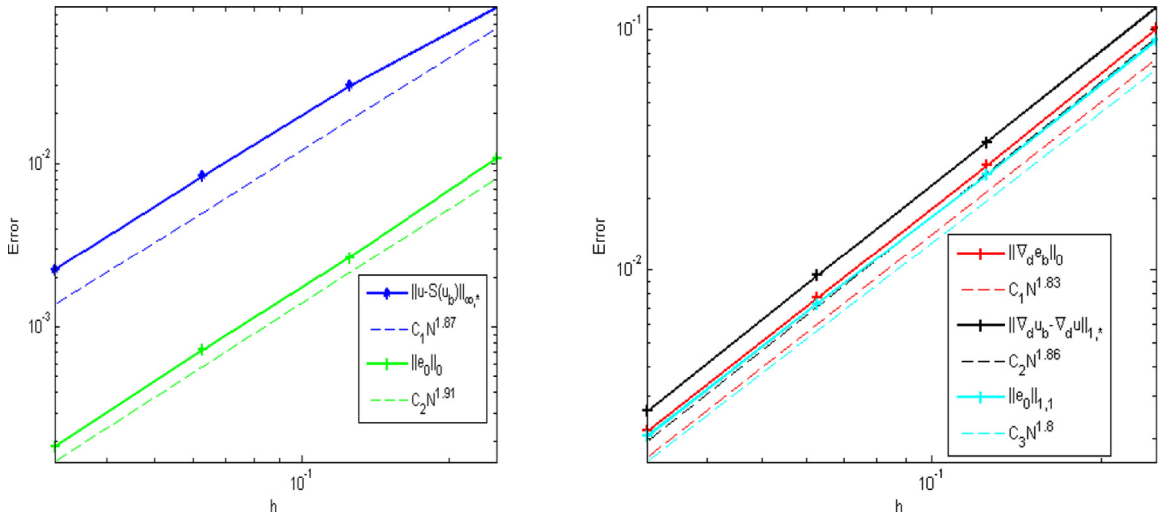
**Fig. 8.3.** Test Case 3: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \cos(\pi x)\cos(\pi y)\exp(z)$ , non-uniform cubic partitions, stabilization parameter  $\rho = 1$ ,  $h = \max(|e_x|, |e_y|, |e_z|)$ , and perturbed  $L^2$  projection of the Dirichlet boundary data  $g$  by (7.17)–(7.19).



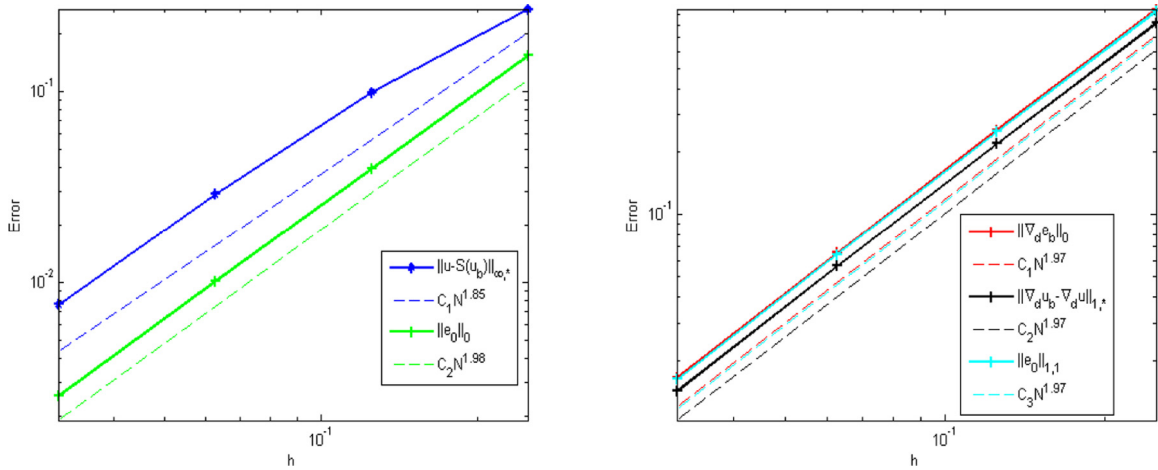
**Fig. 8.4.** Test Case 3: Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \cos(\pi x)\cos(\pi y)\exp(z)$ , non-uniform cubic partitions, stabilization parameter  $\rho = 1$ ,  $h = \max(|e_x|, |e_y|, |e_z|)$ , and  $L^2$  projection of the Dirichlet boundary data  $g$ .

and  $z_{s+1}^* = z_{s+1} + 0.2(\text{rand}(1) - 0.5)/N$ , where  $N = 4$  and  $\text{rand}(1)$  is the Matlab function which returns to a single uniformly distributed random number in  $(0, 1)$ . The random numbers  $\text{rand}(1) = \{0.141886, 0.933993, 0.031833\}$ ,  $\text{rand}(1) = \{0.959492, 0.392227, 0.823457\}$ , and  $\text{rand}(1) = \{0.421761, 0.678735, 0.276922\}$  are used in the  $x$ -,  $y$ - and  $z$ -directions, respectively. The next level of mesh is then generated by uniformly refining each of the non-uniform cubic elements into 8 sub-cubes. The numerical results where the usual  $L^2$  projection and the perturbed  $L^2$  projection are employed are presented in Figs. 8.5 and 8.6, respectively. Fig. 8.5 demonstrates that the superconvergence order is much better than our theory for the usual  $L^2$  projection. Fig. 8.6 shows that the superconvergence rate is in good consistency with the theory for the perturbed  $L^2$  projection.

**Test Case 6 (Nonhomogeneous BVP).** This test has the following configuration: (1) The coefficient tensor  $A$  is an identity matrix and the stabilization parameter is  $\rho = 1$ ; (2) the exact solution is  $u = \cos(\pi x)\sin(\pi y)\cos(\pi z)$ ; (3) The cubic partitions are obtained as follows: firstly, the domain  $\Omega = (0, 1)^3$  is uniformly partitioned into  $2^N \times 2^N \times 2^N$  ( $N = 1, 2, 3, 4$ ) sub-cubes; secondly, the perturbation method used in Test Case 5 is employed on the uniform  $2^N \times 2^N \times 2^N$  meshes to generate the non-uniform  $2^N \times 2^N \times 2^N$  meshes; thirdly, the  $2^{N+1} \times 2^{N+1} \times 2^{N+1}$  meshes are then derived by uniformly refining each cubic element in the non-uniform  $2^N \times 2^N \times 2^N$  meshes into 8 sub-cubic elements. Table 8.13 shows that the convergence order for  $\|\nabla_d e_b\|_0$  with the usual  $L^2$  projection outperforms the result  $\mathcal{O}(h^{1.5})$  in Theorem 7.4. Table 8.14



**Fig. 8.5.** Test Case 5: Convergence of the lowest order WG-FEM on the unit cubic domain  $(0, 1)^3$  with exact solution  $u = \cos(\pi x) \sin(\pi y) \cos(\pi z)$ , non-uniform cubic partitions, the coefficient matrix is identity, stabilization parameter  $\rho = 1$ , and  $L^2$  projection of the Dirichlet boundary data  $g$ .



**Fig. 8.6.** Test Case 5: Convergence of the lowest order WG-FEM on the unit cubic domain  $(0, 1)^3$  with exact solution  $u = \cos(\pi x) \sin(\pi y) \cos(\pi z)$ , non-uniform cubic partitions, the coefficient matrix is identity, stabilization parameter  $\rho = 1$ , and perturbed  $L^2$  projection of the Dirichlet boundary data  $g$  by (7.17)–(7.19).

shows that the superconvergence order for  $\|\nabla_d e_b\|_0$  is in the order  $\mathcal{O}(h^2)$  with the perturbed  $L^2$  projection, which is in great consistency with Corollary 7.3.

## 8.2. Numerical experiments for piecewise constant diffusion tensor $A$

**Test Case 7 (Nonhomogeneous BVP).** The domain  $\Omega = (0, 1)^3$  is divided into two subdomains by a flat face  $x = 1/2$ , where  $\alpha_i^{x,0,0}$   
 $\Omega_1 = (0, 1/2) * (0, 1) * (0, 1)$  and  $\Omega_2 = (1/2, 1) * (0, 1) * (0, 1)$ . The diffusive coefficient tensor is  $A_i = \begin{bmatrix} 0, \alpha_i^y, 0 \\ 0, 0, \alpha_i^z \end{bmatrix}$ , and the exact solution is  $u_i = \alpha_i \cos(\pi x) \sin(\pi y) \cos(\pi z)$  for the subdomain  $\Omega_i$ , where the coefficients  $\alpha_i^x, \alpha_i^y, \alpha_i^z, \alpha_i$  are specified in Table 8.15 for  $i = 1, 2$ . The stabilization parameter is  $\rho = 1$ . Table 8.16 presents that the convergence rate for  $\|\nabla_d e_b\|_0$  on the non-uniform partition with the usual  $L^2$  projection is better than the theory  $\mathcal{O}(h^{1.5})$ .

## 8.3. Numerical experiments for variable diffusive tensor $A$

**Test Case 8 (Nonhomogeneous BVP).** We consider a nonhomogeneous boundary value problem with the exact solution  $u = \sin(x) \sin(y) \sin(z)$ . The coefficient tensor  $A$  is a symmetric and positive definite matrix with  $a_{11} = 1 + x^2$ ,  $a_{12} = xy/4$ ,

**Table 8.13**

Test Case 6: Convergence of the lowest order WG-FEM on the unit cubic domain  $(0, 1)^3$  with exact solution  $u = \cos(\pi x) \sin(\pi y) \cos(\pi z)$ , non-uniform cubic partitions, stabilization parameter  $\rho = 1$ , and  $L^2$  projection of the Dirichlet boundary data  $g$ .

| Mesher                   | $\ u - \mathcal{S}(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|---------------------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $2 \times 2 \times 2$    | 1.3717e-01                            | 4.8600e-02  | 2.7327e-01           | 4.3071e-01                          | 2.4263e-01      |
| $4 \times 4 \times 4$    | 8.4609e-02                            | 1.0223e-02  | 9.0252e-02           | 1.1798e-01                          | 8.0640e-02      |
| Rate                     | 0.70                                  | 2.25        | 1.60                 | 1.87                                | 1.60            |
| $4 \times 4 \times 4$    | 8.8874e-02                            | 1.0715e-02  | 1.0106e-01           | 1.2318e-01                          | 9.0524e-02      |
| $8 \times 8 \times 8$    | 2.9802e-02                            | 2.6654e-03  | 2.7306e-02           | 3.4129e-02                          | 2.5044e-02      |
| Rate                     | 1.58                                  | 2.01        | 1.89                 | 1.85                                | 1.85            |
| $8 \times 8 \times 8$    | 2.6820e-02                            | 2.4725e-03  | 2.6568e-02           | 3.3013e-02                          | 2.4314e-02      |
| $16 \times 16 \times 16$ | 7.5071e-03                            | 6.4926e-04  | 7.1617e-03           | 8.8805e-03                          | 6.6747e-03      |
| Rate                     | 1.84                                  | 1.93        | 1.89                 | 1.89                                | 1.87            |
| $16 \times 16 \times 16$ | 8.1166e-03                            | 7.4520e-04  | 7.6548e-03           | 9.5440e-03                          | 7.2359e-03      |
| $32 \times 32 \times 32$ | 2.1534e-03                            | 1.9459e-04  | 2.2590e-03           | 2.6909e-03                          | 2.1752e-03      |
| Rate                     | 1.91                                  | 1.94        | 1.76                 | 1.83                                | 1.73            |

**Table 8.14**

Test Case 6: Convergence of the lowest order WG-FEM on the unit cubic domain  $(0, 1)^3$  with exact solution  $u = \cos(\pi x) \sin(\pi y) \cos(\pi z)$ , non-uniform cubic partitions, stabilization parameter  $\rho = 1$ , and perturbed  $L^2$  projection of the Dirichlet boundary data  $g$  by (7.17)–(7.19).

| Mesher                   | $\ u - \mathcal{S}(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|---------------------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $2 \times 2 \times 2$    | 3.3996e-01                            | 6.0584e-01  | 3.2721e+00           | 2.7915e+00                          | 3.2143e+00      |
| $4 \times 4 \times 4$    | 2.4166e-01                            | 1.4017e-01  | 8.7948e-01           | 7.4759e-01                          | 8.6207e-01      |
| Rate                     | 0.49                                  | 2.11        | 1.90                 | 1.90                                | 1.90            |
| $4 \times 4 \times 4$    | 2.6832e-01                            | 1.5400e-01  | 9.6036e-01           | 8.2564e-01                          | 9.4258e-01      |
| $8 \times 8 \times 8$    | 9.8770e-02                            | 3.9531e-02  | 2.5404e-01           | 2.1932e-01                          | 2.4937e-01      |
| Rate                     | 1.44                                  | 1.96        | 1.92                 | 1.91                                | 1.92            |
| $8 \times 8 \times 8$    | 9.0683e-02                            | 3.7438e-02  | 2.4135e-01           | 2.0704e-01                          | 2.3677e-01      |
| $16 \times 16 \times 16$ | 2.6625e-02                            | 9.5967e-03  | 6.2081e-02           | 5.3436e-02                          | 6.0922e-02      |
| Rate                     | 1.77                                  | 1.96        | 1.96                 | 1.95                                | 1.96            |
| $16 \times 16 \times 16$ | 2.9695e-02                            | 1.0408e-02  | 6.7150e-02           | 5.8436e-02                          | 6.5996e-02      |
| $32 \times 32 \times 32$ | 7.8290e-03                            | 2.6270e-03  | 1.6961e-02           | 1.4781e-02                          | 1.6672e-02      |
| Rate                     | 1.92                                  | 1.99        | 1.99                 | 1.98                                | 1.98            |

**Table 8.15**

Test Case 7: Parameters for the diffusive coefficients and the exact solution.

|                     |                     |
|---------------------|---------------------|
| $\alpha_1^x = 1000$ | $\alpha_2^x = 1$    |
| $\alpha_1^y = 100$  | $\alpha_2^y = 0.1$  |
| $\alpha_1^z = 10$   | $\alpha_2^z = 0.01$ |
| $\alpha_1 = 0.01$   | $\alpha_2 = 10$     |

**Table 8.16**

Test Case 7: Convergence of the lowest order WG-FEM on  $(0, 1)^3$  with exact solution  $u = \alpha_i \cos(\pi x) \sin(\pi y) \cos(\pi z)$ , piecewise constant diffusive tensor, non-uniform cubic partitions, stabilization parameter  $\rho = 1$ ,  $h = \max(|e_x|, |e_y|, |e_z|)$ , and  $L^2$  projection of the boundary data  $g$ .

| Mesher                   | $\ u - \mathcal{S}(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|---------------------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $3 \times 4 \times 5$    | 5.9145e-01                            | 5.1023e-01  | 6.8587e+00           | 1.1085e+01                          | 6.7852e+00      |
| $6 \times 8 \times 10$   | 5.6868e-01                            | 2.0684e-01  | 3.4145e+00           | 3.3386e+00                          | 3.4065e+00      |
| $12 \times 16 \times 20$ | 1.5697e-01                            | 4.9439e-02  | 9.9702e-01           | 9.8059e-01                          | 9.9530e-01      |
| $24 \times 32 \times 40$ | 3.9016e-02                            | 1.3197e-02  | 2.8390e-01           | 2.8028e-01                          | 2.8352e-01      |
| Rate                     | 2.01                                  | 1.91        | 1.81                 | 1.81                                | 1.81            |

$a_{13} = xz/4$ ,  $a_{22} = 1 + y^2$ ,  $a_{23} = yz/4$ ,  $a_{33} = 1 + z^2$ . The non-uniform cubic partitions and the usual  $L^2$  projection are used in this test with the stabilized parameter  $\rho = 1$ . Table 8.17 shows that the convergence rate for  $\|\nabla_d e_b\|_0$  outperforms the result  $\mathcal{O}(h^{1.5})$  in Theorem 7.4.

**Test Case 9 (Reaction–diffusion equation).** Consider the reaction–diffusion model:

$$\begin{aligned} -\Delta u + cu &= f \quad \text{in } \Omega = (0, 1)^3, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

**Table 8.17**

Test Case 8 : Convergence of the lowest order WG-FEM on the unit cubic domain with exact solution  $u = \sin(x)\sin(y)\sin(z)$ , non-uniform cubic partitions, stabilization parameter  $\rho = 1$ ,  $h = \max(|e_x|, |e_y|, |e_z|)$ , and  $L^2$  projection of the Dirichlet boundary data  $g$ . The coefficient matrix is  $a_{11} = 1 + x^2$ ,  $a_{12} = xy/4$ ,  $a_{13} = xz/4$ ,  $a_{22} = 1 + y^2$ ,  $a_{23} = yz/4$ , and  $a_{33} = 1 + z^2$ .

| Mesheres                 | $\ u - S(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|-----------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $3 \times 4 \times 5$    | 4.8515e-03                  | 9.3955e-04  | 8.4062e-03           | 8.8736e-03                          | 8.3704e-03      |
| $6 \times 8 \times 10$   | 1.3571e-03                  | 3.1310e-04  | 3.7875e-03           | 3.8168e-03                          | 3.7760e-03      |
| $12 \times 16 \times 20$ | 3.9676e-04                  | 9.4737e-05  | 1.1738e-03           | 1.1752e-03                          | 1.1707e-03      |
| $24 \times 32 \times 40$ | 1.1258e-04                  | 2.5404e-05  | 3.1457e-04           | 3.1448e-04                          | 3.1379e-04      |
| Rate                     | 1.82                        | 1.90        | 1.90                 | 1.90                                | 1.90            |

**Table 8.18**

Test Case 9: Convergence of the lowest order WG-FEM on the  $(0, 1)^3$  with exact solution  $u = x(1-x)y(1-2y)z(1-3z)$ , non-uniform cubic partitions, stabilization parameter  $\rho = 1$ ,  $h = \max(|e_x|, |e_y|, |e_z|)$ , and  $L^2$  projection of the boundary data  $g$ .

| Mesheres                 | $\ u - S(u_b)\ _{\infty,*}$ | $\ e_0\ _0$ | $\ \nabla_d e_b\ _0$ | $\ \nabla_d u_b - \nabla u\ _{1,*}$ | $\ e_0\ _{1,1}$ |
|--------------------------|-----------------------------|-------------|----------------------|-------------------------------------|-----------------|
| $3 \times 4 \times 5$    | 2.1502e-02                  | 7.5361e-03  | 7.2672e-02           | 7.4019e-02                          | 7.2672e-02      |
| $6 \times 8 \times 10$   | 1.1389e-02                  | 1.9955e-03  | 2.6789e-02           | 2.7184e-02                          | 2.6789e-02      |
| $12 \times 16 \times 20$ | 3.7672e-03                  | 5.3715e-04  | 8.6059e-03           | 8.6998e-03                          | 8.6059e-03      |
| $24 \times 32 \times 40$ | 1.0559e-03                  | 1.4113e-04  | 2.6083e-03           | 2.6290e-03                          | 2.6083e-03      |
| Rate                     | 1.84                        | 1.93        | 1.72                 | 1.73                                | 1.72            |

where the reaction coefficient is  $c = 2$ . The stabilizer parameter is  $\rho = 1$ , and the exact solution is  $u = x(1-x)y(1-2y)z(1-3z)$ . The non-uniform cubic partitions and the usual  $L^2$  projection are taken in the test. Table 8.18 indicates that the convergence order for  $\|\nabla_d e_b\|_0$  is higher than  $\mathcal{O}(h^{1.5})$ . It should be pointed out that the reaction–diffusion model is not the second order elliptic model for which the superconvergence theory is established in the paper. However, the numerical results demonstrate a good computational performance of the WG finite element method for the reaction–diffusion model.

In summary, the superconvergence theory established in this paper is well verified by various numerical experiments. It is exciting that the convergence rate for  $\|\nabla_d e_b\|_0$  is higher than the conclusion  $\mathcal{O}(h^{1.5})$  in Theorem 7.4 when the usual  $L^2$  projection is taken. The numerical results show that the numerical solution related to the perturbed  $L^2$  projection does perform better than the numerical solution related to the usual  $L^2$  projection. Furthermore, the numerical solution for the reaction–diffusion equation shows a superconvergence error estimate by using the weak Galerkin scheme.

## Acknowledgment

The authors would like to gratefully acknowledge Dr. Junping Wang in NSF for his invaluable discussion and suggestion for this paper.

## References

- [1] R.E. Ewing, M. Liu, J. Wang, A new superconvergence for mixed finite element approximations, *SIAM J. Numer. Anal.* 40 (6) (2002) 2133–2150.
- [2] H. Wei, L. Chen, B. Zheng, Adaptive mesh refinement and superconvergence for two-dimensional interface problems, *SIAM J. Sci. Comput.* 36 (4) (2014) A1478–A1499.
- [3] O. Zienkiewicz, J. Zhu, The superconvergence patch recovery (SPR) and adaptive finite element refinement, *Comput. Methods Appl. Mech. Engrg.* 101 (1–3) (1992) 207–224.
- [4] M. Ainsworth, J.T. Oden, *A Posteriori Error Estimation in Finite Element Analysis*, Wiley Interscience, New York, 2000.
- [5] H. Chen, J. Wang, An interior estimate of superconvergence for finite element solutions for second-order elliptic problems on quasi-uniform meshes by local projections, *SIAM J. Numer. Anal.* 41 (4) (2003) 1318–1338.
- [6] G. Fairweather, Q. Lin, Y. Lin, J. Wang, S. Zhang, Asymptotic expansions and richardson extrapolation of approximate solutions for second order elliptic problems on rectangular domains by mixed finite element methods, *SIAM J. Numer. Anal.* 44 (3) (2006) 1122–1149.
- [7] O. Zienkiewicz, J. Zhu, The superconvergence patch recovery and a posteriori error estimates, Part 1, *Internat. J. Numer. Methods Engrg.* 33 (1992) 1331–1364.
- [8] O. Zienkiewicz, J. Zhu, The superconvergence patch recovery and a posteriori error estimates, Part 2, *Internat. J. Numer. Methods Engrg.* 33 (1992) 1365–1382.
- [9] J.A. Ferreira, R.D. Grigorieff, Supraconvergence and supercloseness of a scheme for elliptic equations on nonuniform grids, *Numer. Funct. Anal. Optim.* 27 (2006) 539–564.
- [10] R. He, X. Feng, Z. Chen,  $H^1$ -superconvergence of a difference finite element method based on the  $P_1 - P_1$ -conforming element on non-uniform meshes for the 3D Poisson equation, *Math. Comp.* 87 (312) (2018) 1659–1688.
- [11] J.H. Brandts, M. Křížek, History and future of superconvergence in three-dimensional finite element methods, in: *Proceedings of the Conference on Finite Element Methods: Three-Dimensional Problems*, in: GAKUTO Internat. Ser. Math. Sci. Appl., vol. 15, Gakkōtoshō, Tokyo, 2001, pp. 24–35.
- [12] I. Babuška, T. Strouboulis, C.S. Upadhyay, S.K. Gangaray, Computer-based proof of existence of superconvergence points in the finite element method: superconvergence of derivatives in finite element solutions of Laplace's, Poisson's and the elasticity equations, *Numer. Methods Partial Differential Equations* 12 (1996) 347–392.
- [13] C.M. Chen, Y.Q. Huang, *High Accuracy Theory of Finite Elements*, Hunan Science Press, Changsha, China, 1995 (in Chinese).
- [14] C. Wang, Superconvergence of Ritz–Galerkin finite element approximations for second order elliptic problems, *Numer. Methods Partial Differential Equations* 34 (2018) 838–856.

- [15] Q.D. Zhu, Q. Lin, Superconvergence Theory of the Finite Element Method, Hunan Science Press, China, Changsha, 1989.
- [16] M. Zlamal, Some superconvergence results in the finite element method, in: A. Dold, B. Eckmann (Eds.), Mathematical Aspects of Finite Element Methods, in: Springer Lecture Notes in Mathematics, vol. 606, 1975.
- [17] M. Krizek, P. Neittaanmäki, On superconvergence techniques, Acta Appl. Math. 9 (1987) 175–198.
- [18] W. Cao, C. Shu, Y. Yang, Z. Zhang, Superconvergence of discontinuous Galerkin methods for two-dimensional hyperbolic equations, SIAM J. Numer. Anal. 53 (4) (2015) 1651–1671.
- [19] K. Mustapha, M. Nour, B. Cockburn, Convergence and superconvergence analyses of HDG methods for time fractional diffusion problems, Adv. Comput. Math. 42 (2) (2016) 377–393.
- [20] G.R. Liu, T. Nguyen-Thoi, Smoothed Finite Element Methods, CRC press, 2016.
- [21] A. Harris, S. Harris, Superconvergence of weak Galerkin finite element approximation for second order elliptic problems by L2-projections, Appl. Math. Comput. 227 (2014) 610–621.
- [22] D. Li, C. Wang, J. Wang, Superconvergence of the gradient approximation for weak Galerkin finite element methods on nonuniform rectangular partitions, <https://arxiv.org/pdf/1804.03998v2.pdf>.
- [23] Y. Liu, J. Wang, Simplified weak Galerkin and finite difference schemes for the Stokes equation, <https://arxiv.org/pdf/1803.0012v1.pdf>.
- [24] Y. Liu, J. Wang, A simplified weak Galerkin finite element method: algorithm and error estimates, <https://arxiv.org/pdf/1808.08667v2.pdf>.
- [25] R. Wang, R. Zhang, X. Zhang, Z. Zhang, Superconvergence analysis and polynomial preserving recovery for a class of weak Galerkin methods, Numer. Methods Partial Differential Equations 34 (1) (2018) 317–335.
- [26] J. Wang, X. Ye, A weak Galerkin finite element method for second-order elliptic problems, J. Comput. Appl. Math. 307 (2013) 103–115.
- [27] B. Heimsund, X. Tai, J. Wang, Superconvergence for the gradient of finite element approximations by L2-projections, SIAM J. Numer. Anal. 40 (4) (2002) 1263–1280.
- [28] M. Krizek, P. Neittaanmä, Superconvergence phenomenon in the finite element method arising from averaging gradients, Numer. Math. 45 (1984) 105–116.
- [29] V. Kantchev, P.D. Lazarov, Superconvergence of the gradient of linear finite elements for 3D poisson equation, optimal algorithms, Publ. Bulg. Acad. Sci. Sofia (1986) 172–182.
- [30] A.H. Schatz, I.H. Sloan, L.B. Wahlbin, Superconvergence in finite element methods and meshes that are locally symmetric with respect to a point, SIAM J. Numer. Anal. 33 (1996) 505–521.
- [31] L. Wahlbin, Superconvergence in Galerkin Finite Element Methods, Springer, 2006.
- [32] L. Mu, J. Wang, X. Ye, S. Zhao, A new weak Galerkin finite element method for elliptic interface problems, J. Comput. Phys. 325 (2016) 157–173.
- [33] C. Wang, New discretization schemes for time-harmonic Maxwell equations by weak Galerkin finite element methods, J. Comput. Appl. Math. 341 (2018) 127–143.
- [34] S. Shields, J. Li, E.A. Machorro, Weak Galerkin methods for time-dependent Maxwell's equations, Comput. Math. Appl. 74 (2017) 2106–2124.
- [35] L. Mu, J. Wang, X. Ye, S. Zhao, A weak Galerkin finite element method for the Maxwell equations, J. Sci. Comput. 65 (2015) 363–386.
- [36] L. Mu, J. Wang, X. Ye, S. Zhao, Numerical studies on the weak Galerkin method for the Helmholtz equation with large wave number, Commun. Comput. Comput. Phys. 15 (2014) 1461–1474.
- [37] Y. Huang, J. Li, D. Li, Developing weak Galerkin finite element method for the wave equation, Numer. Methods Partial Differential Equations 33 (3) (2017) 868–884.
- [38] J. Wang, X. Ye, A weak Galerkin finite element method for the Stokes equations, Adv. Comput. Math. 42 (2016) 155–174.
- [39] C. Wang, J. Wang, Discretization of div-curl systems by weak Galerkin finite element methods on polyhedral partitions, J. Sci. Comput. 68 (2016) 1144–1171.
- [40] C. Wang, J. Wang, A hybridized formulation for weak Galerkin finite element methods for biharmonic equation on polygonal or polyhedral meshes, Int. J. Numer. Anal. Model. 12 (2015) 302–317.
- [41] C. Wang, J. Wang, An efficient numerical scheme for the biharmonic equation by weak Galerkin finite element methods on polygonal or polyhedral meshes, J. Comput. Math. Appl. 68 (12) (2014) 2314–2330.
- [42] J. Wang, Q. Zhai, R. Zhang, S. Zhang, A weak Galerkin finite element scheme for the Cahn-Hilliard equation, Math. Comp. 88 (315) (2018) 211–235.
- [43] R. Lin, X. Ye, S. Zhang, P. Zhu, A weak Galerkin finite element method for singularly perturbed by convection-diffusion-reaction problems, SIAM J. Numer. Anal. 56 (3) (2018) 1482–1497.
- [44] C. Wang, J. Wang, A primal-dual weak Galerkin finite element method for second order elliptic equations in non-divergence form, Math. Comp. 87 (2018) 515–545.
- [45] C. Wang, J. Wang, A primal-dual weak Galerkin finite element method for Fokker-Planck type equations, SIAM J. Numer. Anal. <https://arxiv.org/pdf/170405606pdf>, accepted for publication.
- [46] C. Wang, J. Wang, Primal-dual weak Galerkin finite element methods for elliptic Cauchy problems, <https://arxiv.org/pdf/1806.01583.pdf>.
- [47] C. Wang, A new primal-dual weak Galerkin finite element method for ill-posed elliptic Cauchy problems, <https://arxiv.org/pdf/1809.04697v1.pdf>.
- [48] L. Mu, J. Wang, Y. Wang, X. Ye, A computational study of the weak Galerkin method for second-order elliptic equations, Numer. Algorithms 63 (2013) 753–777.
- [49] J. Wang, A superconvergence analysis for finite element solutions by the least-squares surface fitting on irregular meshes for smooth problems, J. Math. Study 33 (3) (2000) 229–243.
- [50] A. Hannukainen, S. Korotov, M. Krizek, Nodal  $\mathcal{O}(h^4)$ -superconvergence in 3D by averaging piecewise linear, bilinear, and trilinear FE approximations, J. Comput. Math. 28 (1) (2010) 1–10.
- [51] M. Krizek, Superconvergence phenomenon on three-dimensional meshes, Int. J. Numer. Anal. Model. 2 (1) (2015) 43–56.
- [52] R. Lin, Z. Zhang, Natural superconvergence points in three-dimensional finite elements, SIAM J. Numer. Anal. 46 (2008) 1281–1297.
- [53] Z. Zhang, Derivative superconvergence points in finite element solutions of Poisson's equation for the serendipity and intermediate families—a theoretical justification, Math. Comp. 67 (1998) 541–552.
- [54] J. Wang, X. Ye, A weak Galerkin mixed finite element method for second-order elliptic problems, Math. Comp. 83 (2014) 2101–2126.
- [55] J. Wang, C. Wang, Weak Galerkin finite element methods for elliptic PDEs, Sci. China 45 (2015) 1061–1092.
- [56] L. Chen, J. Wang, X. Ye, A posteriori error estimates for weak Galerkin finite element methods for second order elliptic problems, J. Sci. Comput. 59 (2014) 496–511.