MORE DEFINABLE COMBINATORICS AROUND THE FIRST AND SECOND UNCOUNTABLE CARDINAL

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Abstract. Assume ZF + AD. The following two continuity results for functions on certain subsets of P $(!_1)$ and P $(!_2)$ will be shown:

For every $<$ $!_1$ and function : $[!_1]$! $!_1$, there is a club C $!_1$ and a $<$ so that for all f; g 2 [C], if f = g and sup(f) = sup(g), then (f) = (g).

For every $<$ $!_2$ and function : $[!_2]$! $!_2$, there is an !-club C $!_2$ and a $<$ so that for all f; g 2 [C], if f = g and sup(f) = sup(g), then (f) = (g).

The previous two continuity results will be used to distinguish cardinalities below P $(!_2)$: $j[!_1]^! j < j[!_1]^{<!_1} j$. $j[!_2]^! j < j!_2]^{<!_1} j < j[!_2]^{!_1} j < j[!_2]^{<!_2} j$. :$(j[!_1]^{<!_1} j$ $[!_2]^! j)$. :$(j[!_1]^{!_1} j$ $([!_2]^{<!_1} j)$.

$[!_1]^!$ has the Jonsson property: That is, for every : $^{<!}([!_1]^!)$! $[!_1]^!$, there is an X $[!_1]^!$ with jXj = $j[!_1]^! j$ so that $[^{<!} X] = [!_1]^!$.

1. Introduction

Under the axiom of determinacy, AD, the cardinality of sets have a very rich and non-linear structure. The cardinality of wellorderable sets are called cardinals. $!_1$ and $!_2$ refer to the rst and second uncountable cardinal, respectively. This article will distinguish the cardinality of some important subsets of P $(!_1)$ (the power set of $!_1$) and P $(!_2)$ (the power set of $!_2$) under AD. Since cardinalities are compared through injections, a deep understanding of the behavior of functions between the relevant sets will be necessary. This will be obtained through a complete analysis of the continuity properties of functions of the form : $[!_1]$! $!_1$ when $<$ $!_1$ and functions of the form : $[!_2]$! $!_2$ when $<$ $!_2$. The arguments in this article are entirely combinatorial and should be accessible with minimal knowledge of determinacy. The necessary combinatorial consequences of determinacy such as the partition relations on $!_1$ and $!_2$, the ultrapower representation of $!_2$, and some combinatorial tools to handles this ultrapower such as Kunen functions and sliding arguments will be reviewed.

Descriptive set theory have studied the denable cardinalities of quotients of equivalence relations on Polish spaces through denable reductions. If E is an equivalence relation on R, then let R=E denote the set of equivalence classes of E. If E and F are two equivalence relations on R, then a reduction between E and F is a function : R ! R so that for all x; y 2 R, x E y if and only if (x) F (y). The reduction between E and F induces an injection : R=E ! R=F. Motivated by this, an injection : R=E ! R=F is said to be a Borel denable injection if and only if is induced by a Borel reduction : R ! R between E and F.

There are several important dichotomy results of descriptive set theory which elucidate the structure of the quotient of Borel equivalence relations under Borel denable injections. Silver ([17]) showed that if E is a Borel (or even coanalytic) equivalence relation, then either

- E has countably many classes or
- there is a Borel reduction of the equality equivalence relation = on R into E.

Thus the quotient of a Borel equivalence relation E is either countable or there is a Borel denable injection of R into R=E. Let E_0 be the equivalence relation on $^!2$ of eventual equality dened by x E_0 y if and only if (9m)(8n m)(x(n) = y(n)). Harrington, Kechris, and Louveau [9] showed that for any Borel equivalence relation E, either

- there is a Borel reduction of E into the equality relation = or
- there is a Borel reduction of E_0 into E.

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Thus for any Borel equivalence relation E, either there is a Borel denable injection of $\mathbb{R}/E$ into $\mathbb{R}$ (which is in bijection with $\mathscr{P}(\omega)$) or there is a Borel denable injection of $\mathbb{R}/E_0$ into $\mathbb{R}/E$.

With the axiom of choice, this nice structure for the denable cardinalities under denable injections is not the structure of the true cardinalities of these sets. The axiom of determinacy, AD, asserts that every two player game where each player takes turns playing a natural number has a winning strategy for one of the two players. Determinacy axioms allow the structure of the true cardinality of sets (which are surjective images of $\mathbb{R}$) to possess a structure that resembles the structure of Borel denable cardinalities and this structure is established through techniques that have a descriptive set theoretic avor.

The two dichotomy results for Borel reduction mentioned above are proved by using the Gandy-Harrington forcing of lightface Σ^1_1 subsets of $\mathbb{R}$ developed in [10]. In an extension of AD called AD^+, highly absolute deni-tions for equivalence relations called ∞-Borel codes exists. The Vopenka forcing of ordinal denable (relative to the ∞-Borel code) subsets of $\mathbb{R}$ can be used to extend Silver's dichotomy and the E_0-dichotomy into true cardinality dichotomies in AD^+. Generalizing Silver's dichotomy, the Woodin's perfect set dichotomy ([3], [1]) states that if E is an equivalence relation on $\mathbb{R}$, then either

- $\mathbb{R}/E$ is wellorderable (that is, injects into an ordinal) or
- $\mathbb{R}$ injects into $\mathbb{R}/E$.

Since all sets which are surjective images of $\mathbb{R}$ are in bijection with a quotient of an equivalence relation on $\mathbb{R}$, this can be reformulated to say that for all sets X which are surjective image of $\mathbb{R}$, either X is wellorderable or $\mathbb{R}$ injects into X. In $L(\mathbb{R}) \models \mathsf{AD}$, Caicedo and Ketchersid [1] extended these results further by showing every set $X \in L(\mathbb{R})$ is either wellorderable or $\mathbb{R}$ injects into X. Generalizing the E_0-dichotomy, Hjorth's E_0-dichtomy ([11]) states that if E is an equivalence relation on $\mathbb{R}$, then either

- $\mathbb{R}/E$ injects into $\mathscr{P}(\kappa)$ for some ordinal κ or
- $\mathbb{R}/E_0$ injects into $\mathbb{R}/E$.

The rst two authors have recently obtain additional new cardinality results for quotients of equivalence relations on $\mathbb{R}$ in $L(\mathbb{R}) \models \mathsf{AD}$. Borrowing a term from classical descriptive set theory, an equivalence relation E on $\mathbb{R}$ is strongly smooth if and only if $\mathbb{R}/E$ is in bijection with $\mathbb{R}$. In $L(\mathbb{R}) \models \mathsf{AD}$, many subsets of $\mathscr{P}(\omega_1)$ are in bijection with an ω_1-length disjoint union of quotients of strongly smooth equivalence relations on $\mathbb{R}$; however, only one cardinality can be represented in this way if each equivalence relation have only countable equivalence classes: Combining ideas from the Woodin perfect set dichotomy and Hjorth's E_0-dichotomy, [5] Theorem 5.8 showed that in $L(\mathbb{R}) \models \mathsf{AD}$, if $\langle E_\alpha : \alpha < \omega_1 \rangle$ is a sequence of strongly smooth equivalence relations on $\mathbb{R}$ so that each E_α has all countable equivalence classes, then the disjoint union $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ is in bijection with $\mathbb{R} \times \omega_1$.

Another classical cardinality result under AD is the perfect set property which asserts that every subset of $\mathbb{R}$ is either countable or contains a perfect subset (a nonempty closed set with no isolated points). Since $\mathbb{R}$ is in bijection with $\mathscr{P}(\omega)$, this result completely characterizes the cardinalities of sets below $\mathscr{P}(\omega)$ by establishing a suitable form of the continuum hypothesis: All subsets of $\mathscr{P}(\omega)$ are either countable or in bijection with $\mathscr{P}(\omega)$. This article and other recent work of the authors seek to understand the structure of the cardinalities below $\mathscr{P}(\omega_1)$ and $\mathscr{P}(\omega_2)$.

By the Moschovakis coding lemma, $\mathbb{R}$ surjects onto $\mathscr{P}(\omega_1)$ and $\mathscr{P}(\omega_2)$. Thus every subset of $\mathscr{P}(\omega_1)$ and $\mathscr{P}(\omega_2)$ is in bijection with a quotient of an equivalence relation on $\mathbb{R}$. Rather than viewing these sets as quotients of equivalence relations, the approach of this paper will be to consider these sets as increasing sequences of ordinals and use an important consequence of determinacy known as the partition relations on ω_1 and ω_2. The following will summarize the results of this paper and its context within determinacy.

Let A and B be two sets. If there is an injection from A into B, then write $|A| \le |B|$. Denote $|A| < |B|$ if $|A| \le |B|$ but $\neg(|B| \le |A|)$. If there is a bijection between A and B, then one writes $|A| = |B|$. By the Cantor-Schröder-Bernstein theorem (proved in ZF), $|A| = |B|$ if and only $|A| \le |B|$ and $|B| \le |A|$. In the absence of choice, the cardinality of A, referred to as $|A|$, is the equivalence class of A under the bijection relation.

To understand cardinalities and injections, one will need to study functions between sets under determinacy. One such classical result concerns continuity for functions from $\mathbb{R}$ to $\mathbb{R}$. Assuming AD, every function $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous on a comeager subset of $\mathbb{R}$. As customary in descriptive set theory, thinking of $\mathbb{R}$ as ${}^\omega\omega$ (the collection of functions from ω into ω), continuity can be understood using the following example:

(f)(0), the rst bit of (f), a priori could require global information about all of f . Continuity on a comea-ger set implies that if f belongs to this comeager set, then (f)(0) only depends on a local behavior of f . That is, there is some n 2 ! so that for all g which belong to this appropriate comeager set, if g n = f n, then (g)(0) = (f)(0). Continuity of on this comeager set means this property holds for the k^{th} bit of (f) for each k 2 ! and f belonging to the suitable comeager set.

This paper will be interested in the collection of subsets of $!_1$ and $!_2$ which have cardinality less than $!_1$ and $!_2$, respectively. Identifying subsets of $!_1$ or $!_2$ by their increasing enumeration, one will prefer to work with the collection of increasing sequences through $!_1$ and $!_2$ (primarily because the partition properties are formulated for these sets). If are two ordinals, then [] is the collection of increasing functions f : ! . Let $[]^{<} = \bigcup_{<}[]$. This paper will be particular interested in $[!_1]^{!}$, $[!_1]^{<!_1}$, $[!_2]^{!}$, $[!_2]^{!_1}$, and $[!_1]^{\ 2}$.

This article will study the short functions on $!_1$ and $!_2$ (i.e. functions : $[!_1]$! $!_1$ when < $!_1$ or : $[!_2]$! $!_2$ when < $!_2$). The continuity phenomenon for full functions on $!_1$ (i.e. : $[!_1]^{!_1}$! $!_1$) is investigated in [6], and the techniques there are quite dierent than what is used here. The rst two authors [6] showed that for every function : $[!_1]^{!_1}$! $!_1$, there is a club C $!_1$ with the property that for all f 2 $[C]^{!_1}$, there exists an < $!_1$ so that for all g 2 $[C]^{!_1}$, if g = f , then (f) = (g). ($[C]^{!_1}$ is the collection of increasing functions from $!_1$ into C of the correct type, which will be dened in Denition 2.1.) The authors [6] also showed an even stronger version that for every function : $[!_1]^{!_1}$! $^{!_1}!_1$, there is a club C $!_1$ so that for all f 2 $[C]^{!_1}$ and < $!_1$, there exists an < $!_1$ so that for all g 2 $[C]^{!_1}$, if g = f , then (g) = (f) . Note that this latter continuity property is just the standard notion of continuity where the domain and range spaces are given the topology generated by sets of the form N = ff 2 $[!_1]^{!_1}$: fg where 2 $[!_1]^{<!_1}$ (or N = ff 2 $^{!_1}!_1$: fg where 2 $^{<!_1}!_1$) as a basis.

As a consequence of Martin's result that $!_1$ is a strong partition cardinal, the lter $^{!_1}$ on $[!_1]^{!_1}$ dened by X 2 $^{!_1}$ if and only if there exists a club C $!_1$ so that $[C]^{!_1}$ X is a countably complete measure on $!_1$. Thus in the above two continuity results, the notion of largeness given by comeagerness for classical continuity on R is replaced with largeness on $[!_1]^{!_1}$ given by the ultralter $^{!_1}$. The continuity property for functions mentioned in the previous paragraph can be used to show that if hX : < $!_1$i is a sequence of subsets of $[!_1]^{!_1}$ so that $[!_1]^{!_1}$ = $\bigcup_{<!}$ X, then there is an < $!_1$ such that jXj = $j[!_1]^{!_1}j$. This result can then be used to show that $j[!_1]^{<!_1}j$ < $j[!_1]^{!_1}j$. (See Fact 3.30 for a dierent argument using measures and certain inner models of ZFC.)

This article will be concerned with continuity phenomenon for functions : $[!_1]$! $!_1$ where < $!_1$. The partition measure on $[!_1]$ will serve as the notion of largeness for subsets of $[!_1]$. However, continuity in the sense described above for the functions from $[!_1]^{!_1}$ into $[!_1]^{!_1}$ is impossible by the following example. Consider the function : $[!_1]^{!}$! $!_1$ dened by (f) = sup(f). There is no club C $!_1$ so that for all f 2 $[C]^{!}$, there is an n < ! such that whenever g 2 $[C]^{!}$ and f n = g n, (f) = (g). However, does satisfy a particular continuity phenomenon in the sense that (f) depend only on one piece of information, namely sup(f). That is (by denition of), for any f; g 2 $[!_1]^{!}$, if sup(f) = sup(g), then (f) = (g). The rst main result is that this is a general occurrence that holds for any function : $[!_1]$! $!_1$ when < $!_1$. For each f 2 $[!_1]$ and , let bound(f;) = supff() : < g. Note that bound(f; 0) = 0 and bound(f;) = sup(f).

Theorem 2.14. Assume ZF + AD. Let < $!_1$ and : $[!_1]$! $!_1$. Then there is a decreasing se-quence of ordinals which are less than or equal to , ($_i$: i n), with $_n$ = 0 and a club C $!_1$ so that if f; g 2 [C] has the property that bound(f; $_i$) = bound(g; $_i$) for all i n, then (f) = (g).

This result is a continuity property which states that for any such function , (f) depends only on local behaviors of f at certain nitely many places for -almost all f . The following is a more coarse but useful consequence of the above result which states that for every function , there is a < so that (f) depends only on the -length initial segment of f and sup(f).

Theorem 2.15. Assume ZF + AD. Let < $!_1$ and : $[!_1]$! $!_1$. Then there is a < and some club C $!_1$ so that for all f; g 2 [C] with f = g and sup(f) = sup(g), (f) = (g).

$[!_1]^!$ and $[!_1]^{<!_1}$ are two distinguished subsets of $P(!_1)$. One natural question is whether these two sets are dierent in terms of cardinality. Woodin [18] studied the cardinals below $[!_1]^{<!_1}$ under $ZF + AD_R + DC$. From the dichotomy results in [18], it was known to Woodin that $j[!_1]^!j < j[!_1]^{<!_1}j$. Moreover, Woodin isolated a subset of $[!_1]^{<!_1}$ called S_1 dened by $S_1 = f f \; 2 \; [!_1]^{<!_1} : \sup(f) = !^{L[f]}_1 g$. It is implicit in [18] that jS_1j is incomparable with $[!_1]^!$ and hence one can concludes that $j[!_1]^!j < j[!_1]^{<!_1}j$.

The proofs of some of the main properties of S_1 (assuming $ZF + AD + DC_R$ and all sets of reals have 1-Borel codes) can be found [4] and [5]. Assuming just $ZF + AD$, one can show that $jRj \; jS_1j$ and $:(!_1 \; jS_1j)$ (see [5] Fact 6.3). The main property of S_1 shown in [4] is that there is no injection of S_1 into ${}^!ON$ assuming $ZF + AD + DC_R$ and all sets of reals have 1-Borel codes. From this, one can conclude that $jRj < jS_1j$ and $:(jS_1j \; j[!_1]^!j)$. The argument for the main property of S_1 in [4] goes roughly as follows: Suppose such an injection exists. Using 1-Borel codes, one can nd an inner model M of ZFC that \absorbs" some fragment of this injection in a suitable sense. Let $< !^V_1$ be an inaccessible cardinal of M. Since $Coll(!; <)$ is countable in the real world satisfying AD, one can nd a $G \; Coll(!; <)$ which is $Coll(!; <)$-generic over M. One can show that G adds a $g \; 2 \; S_1$ such that $M[G] = M[g]$. Since M \absorbs" , $(g) \; 2 \; M[g]$. Since is an injection, one can argue that $M[g] = M[[g]]$. However, (g) is an !-sequence of ordinals. By a crucial property of the Levy collapse, there is a $<$ so that $(g) \; 2 \; M[G \;]$. Then one has that $M[G] = M[g] = M[(g)] = M[G \;]$. This is impossible.

The authors know very little about the cardinality properties of S_1 in the absence of 1-Borel codes. S_1 is a set whose denition is based upon the notion of constructibility. The two sets $[!_1]^!$ and $[!_1]^{<!_1}$ are very concrete combinatorial objects. There should be no need to employ AD^+ concepts to distinguish these two cardinals. Using the continuity properties for short functions mentioned above, one can distinguish these two sets within $ZF + AD$ using combinatorial arguments.

Theorem 2.16. Assume $ZF + AD$. $j[!_1]^!j < j[!_1]^{<!_1}j$.

Recently, the authors have used Theorem 2.16 as a backbone for more general results concerning injections of $[!_1]^{<!_1}$. For example, [7] showed under just $ZF + AD$ that there is no injection of $[!_1]^{<!_1}$ into ${}^!(!_!)$, the set of !-sequences into $!_!$. Moreover with the additional of DC_R, [7] proved in $ZF + AD + DC_R$ that there is no injection of $[!_1]^{<!_1}$ into ${}^!ON$, the class of !-sequences of ordinals. These results use a variety of combinatorial and descriptive set theory consequences of determinacy to reduce back to Theorem 2.16.

Next, one will consider various subsets of $P(!_2)$. Of particular interests are $[!_2]^!$, $[!_2]^{<!_1}$, $[!_2]^{!_1}$, $[!_2]^{<!_2}$, and $[!_2]^{!_2}$. One would like to distinguish the cardinality of these sets from each other as well as from the cardinality of the subsets of $P(!_1)$ considered earlier such as $[!_1]^!$, $[!_1]^{<!_1}$, and $[!_1]^{!_1}$.

Martin showed that $!_2$ is a weak partition cardinal and hence measurable. Using the same technique mentioned above (for showing $j[!_1]^{<!_1}j < j[!_1]^{!_1}j$) involving using a measure and going into an appropriate inner model of ZFC, one can show $j[!_2]^{<!_2}j < j[!_2]^{!_2}j$ under just $ZF + AD$.

Similar to the study of $!_1$, one needs to establish the analogous continuity property for $!_2$.

Theorem 3.21. Assume $ZF + AD$. Let $< !_2$ and $: [!_2] \; ! \; !_2$. Then there is a decreasing se-quence of ordinals less than or equal to , $(_i : i \; n)$, with $_n = 0$ and an !-club $B \; !_2$ so that if $F; G \; 2 \; [B]$ has the property that $bound(F;_i) = bound(G;_i)$ for all $i \; n$, then $(F) = (G)$.

Theorem 3.22. Assume $ZF + AD$. Let $< !_2$ and $: [!_2] \; ! \; !_2$. Then there is a $<$ and an !-club $B \; !_2$ so that for all $F; G \; 2 \; [B]$ with $F \; = G$ and $\sup(F) = \sup(G)$, $(F) = (G)$.

Using these continuity results, one can establish the following cardinality relations:

Theorem 3.23. Assume $ZF + AD$. $j[!_2]^!j < j[!_2]^{<!_1}j$.

Theorem 3.24. Assume $ZF + AD$. $j[!_2]^{<!_1}j < j[!_2]^{!_1}j$.

Theorem 3.26. Assume $ZF + AD$. $j[!_2]^{!_1}j < j[!_2]^{<!_2}j$.

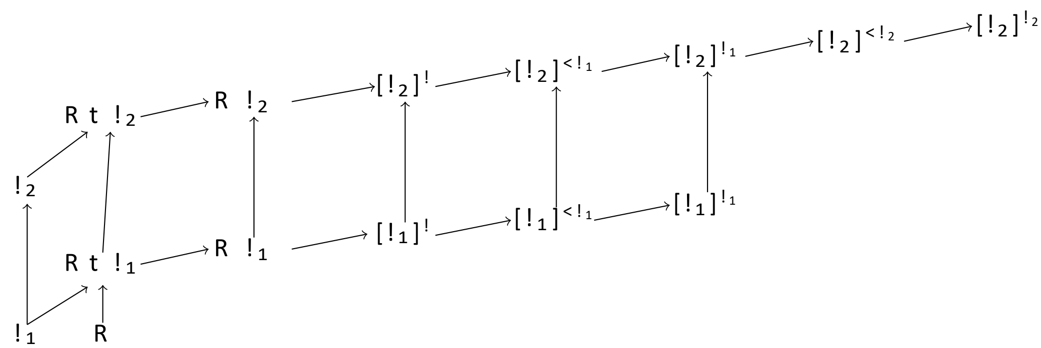
It should be mentioned that these results concerning !2 are proved in ZF + AD and the arguments provided here are the only proofs presently known to the authors. That is, the authors do not know of an AD$^+$ style proof involving some analog of S_1. In the proof that S_1 does not inject into $^!$ON sketched above, one considered the forcing Coll(!; <) where < !^{V} is an$_1$inaccessible cardinal of an inner model M of ZFC. In that case, one was able to nd, in the real world, a generic over M since the forcing is countable in the real world. One may attempt to make analogs of S_1 to handle results at !2. However, the naturally associated forcing appears to be uncountable even in the real world, and one can no longer be certain that generics for such forcings exist in the real world.

To give a more complete picture of the relations between cardinalities, one also has the following results.

Theorem 3.29. Assume ZF + AD. :(j[!1]$^{<!_1}$j j[!2]$^!$j). Thus :(j[!1]$^{!_1}$j [!2]$^!$).

Theorem 3.31. Assume ZF + AD. Then :(j[!1]$^{!_1}$j j[!2]$^{<!_1}$j).

From the result mentioned throughout the paper, one has the following diagram depicting all the cardinal relationships between the uncountable cardinals below P(!2) which will be discussed in this paper. An arrow between A and B indicates jAj < jBj. All relations among these cardinals are those derivable by compositions of the arrows on the diagram. Of course, there are other cardinals below P(!2) which are not on the diagram, for instance [!1]$^{<!_1}$ t [!2]$^!$ and [!1]$^{!_1}$ [!2]$^{<!_1}$. With additional determinacy assumptions such as AD , the set S_1 can be proved to be distinct from all of these.



The main technique used in this paper involves Kunen functions for !1. Let be the club measure on !1. Using the Kunen tree analysis, one can show that for any function f : !1 ! !1, there is a function : !1 !1 ! !1 so that for -almost all , f() < supf(;) : < g and f(;) : < g is an ordinal (not just a set of ordinals). This function will be called a Kunen function for f. allows for a uniform way of selecting a representative for any g < f, i.e. there is a < !1 so that the function : !1 ! !1 dened by () = (;) is -almost equal to g. Using these Kunen functions and sliding arguments, Martin proved an ultrapower representation for !2 = Q$_{!_1}$!1= and showed the weak partition property on !2.

The ultrapower representation is important for studying the continuity property at !2 in this paper. In fact, these continuity properties for functions : [!2] ! !2 expressed in Theorem 3.21 and Theorem 3.22 when < !2 and has uncountable conality are exceptionally remarkable and unique to !2. For instance, one can show under AD that the ultrapower of !2 by the club measure on !1, Q$_{!_1}$!2=, is !3. Dene : [!2]$^{!_1}$! !3 by (f) = [f] (where [f] is the element of this ultrapower represented by f). There is no !-club B, ordinal < so that if f; g 2 [B]$^{!_1}$ with f = g and sup(f) = sup(g), then (f) = (g). This example shows that the continuity property expressed in Theorem 3.22 fails if one considers functions whose range is larger than !2. For partition cardinals greater than !2, the failure of the continuity property at of uncountable conality is even worse. !$_{!+1}$ is the next strong partition cardinal after !1 under AD. The ultrapower of !$_{!+1}$ by the club measure on !1, Q$_{!_1}$!$_{!+1}$=, is !$_{!+1}$. Dene : [!$_{!+1}$]$^{!_1}$! !$_{!+1}$ by (f) = [f]. For the same reason as before, the continuity property expressed in Theorem 3.22 fails. These continuity results at ω_2 are largely possible due to the combinatorial tool available from the ultrapower representation of ω_2.

The basic facts about partition properties and Kunen functions can be found in [3]. These arguments are well known and due to Jackson, Kunen, and Martin. (See [13], [14], and [15].) However, the article will follow [3] which develops the minimal notation and theory necessary for the results in this paper.

The final section of this paper will study functions on tuples in $[\omega_1]^\omega$ using partition properties to establish a simple combinatorial property called the Jónsson property for $[\omega_1]^\omega$. Let X be a set. Let $[X]^n_= = \{f \in {}^n X : (\forall i < j < n)(f(i) \neq f(j))\}$. Let $[X]^{<\omega}_= = \bigcup_{n \in \omega} [X]^n_=$. X is n-Jonsson if and only if for every $: [X]^n_= \to X$, there exists a $Y \subseteq X$ with $|Y| = |X|$ and $[[Y]^n_=] \neq X$. X is Jónsson if and only if for every $: [X]^{<\omega}_= \to X$, there is a $Y \subseteq X$ with $|Y| = |X|$ and $[[Y]^{<\omega}_=] \neq X$.

Under AD, Kleinberg [16] showed that ω_n is Jónsson for all $n \in \omega$. Jackson, Ketchersid, Schlutzenberg, and Woodin [12] showed that under ZF + AD + V = L(R) (and also ZF + AD^+) that every cardinal $<$ is Jonsson. Holshouser and Jackson showed that R and R for $<$ are Jónsson. The first author [2] showed in fact that for all ordinals , R is Jonsson. Holshouser and Jackson showed that ${}^\omega 2 / E_0$ is 2-Jonsson. The first author and Meehan [8] showed that ${}^\omega 2 / E_0$ is not 3-Jonsson and hence not Jonsson. The final result of this paper shows $[\omega_1]^\omega$ has the Jónsson property:

Theorem 4.12. Assume ZF + AD. $[\omega_1]^\omega$ is Jonsson.

2. Continuity of Short Functions on ω_1

For the rest of the paper, assume ZF + AD. (Not even $\mathrm{DC}_\mathbb{R}$ will be implicitly assumed.)

If are ordinals, then [] is the collection of increasing functions $f : \to$.

Definition 2.1. ([14]) Let be an ordinal and . A function $f : \to$ has uniform cofinality ω if and only if there is a function $g : \times \omega \to$ with the following two properties:
(a) For all $<$ and $n \in \omega$, $g(; n) < g(; n + 1)$. (b)
For all $<$, $f() = \sup\{g(; n) : n \in \omega\}$.

A function $f : \to$ is discontinuous at if and only if $f() > \sup\{f() : < \}$.

A function $f : \to$ is of the correct type if and only if f has uniform cofinality ω and f is discontinuous everywhere.

Let $A \subseteq$, [A] denote the collection of all increasing functions $f : \to A$ of the correct type.

The collection of increasing functions and the collection of increasing functions of the correct type have the same cardinality. In the following, one may use either sets for purpose of distinguishing cardinality.

Fact 2.2. Let be a cardinal. Let . [] [].

Proof. Let $H : \to$ be any increasing function of the correct type. Define $: [] \to []$ by $(f) = H \circ f$. Then is an injection. The two sets are in bijection by the Cantor-Schröder-Bernstein theorem.

Definition 2.3. Let be an ordinal and . One write $\to ()_2$ to indicate that for every $P : [] \to 2$, there is some club $C \subseteq \omega_1$ and an $i \in 2$ so that for all $f \in [C]$, $(f) = i$.

If $\to ()_2$, then one says that is a strong partition cardinal.

If $\to ()_2$ for all $<$, then is said to be a weak partition cardinal.

Fact 2.4. ([3] Section 2 and 4, [16] Chapter II, [15] Theorem 7.3 and 12.2.) (Solovay) The club measure on ω_1 is a countably complete normal measure on ω_1. (Martin) ω_1 is a strong partition cardinal.

Definition 2.5. Let denote the club measure on ω_1. For each $\leq \omega_1$, let be a filter on $[\omega_1]$ defined by $K \in$ if and only if there is a club $C \subseteq \omega_1$ so that $[C] \subseteq K$. Since ω_1 is a strong partition cardinal, one has that is a countably complete ultrafilter for all $\leq \omega_1$.

If ' is a formula, then one write $(\forall f)'(f)$ to indicate that the set $\{f \in [\omega_1] : '(f)\} \in$.

Definition 2.6. ([3] Section 5) Let be a club measure on ω_1.

Let $: \omega_1 \times \omega_1 \to \omega_1$. For each $< \omega_1$, let $= \sup\{(;) : < \}$. Let $: \to$ be defined by $() = (;)$.

is a Kunen function for f with respect to if and only if K = $_f$ f < !$_1$: f() ^ is a surjectiong 2 . K$_f$ is the set of on which provides a bounding for f.

For < !$_1$, let : !$_1$! !$_1$ be dened by () = (;) where > and 0 otherwise.

Fact 2.7. ([3] Section 5, [14] Lemma 4.1) (Kunen) For every f : !$_1$! !$_1$, there is a Kunen function for f with respect to .

Denition 2.8. Let < !$_1$ and f 2 [!$_1$]. Let bound(f;) = supff() : < g, where sup(;) is dened to be 0.

If A !$_1$ with jAj = !$_1$, then let enum$_A$: !$_1$! A denote the increasing enumeration of A.

Let C !$_1$ be a club. Let next$_C$() denote !th element of C above .

Fact 2.9. Let < !$_1$. For all : [!$_1$] ! !$_1$, there exists a unique b so that b is the largest so (8 f)(bound(f;) (f)).

Proof. For each < !$_1$, let A be the set of f so that is the largest so that (f) bound(f;). [!$_1$] = S A. Since is a countably complete ultralter on [!$_1$], there is a b so that A$_b$ 2 .

Lemma 2.10. Let < !$_1$. Let : [!$_1$] ! !$_1$. Then there are club subsets of !$_1$, C and D, so that for all f 2 [D], (f) < next$^!_C$(bound(f; b)).

Proof. Let be a new symbol. Dene a linear ordering L on [fg by x y if and only if (a) x; y 2 and x < y
(b) x = , y 2 , and y b (c) x
2 , y = , and x < b.

Note that L is a wellordering of ordertype less than !$_1$. If f : L ! !$_1$ is an increasing function, then let main(f) : ! !$_1$ be dened by main(f)() = f(). Let extra(f) 2 !$_1$ be dened by extra(f) = f().

Dene a partition P : [!$_1$]^L ! 2 by P(g) = 0 , (main(g)) < extra(g). By the weak partition property of !$_1$, there is some C !$_1$ which is homogeneous for this partition. By intersecting with an appropriate club, one may assume that for all f 2 [C], b is the largest so that (f) bound(f;). Therefore if b < , (f) < f(b).

The claim is that C is homogeneous for P taking value 0: Let D = f 2 C : enum$_C$() = g which is the club set of closure point of C. Let f 2 D. In the case that b < , since bound(f; b) (f) < f(b) and f(b) 2 D, the !th-element of C above (f) is below f(b). In all cases, let g : L ! C be dened by g() = f() for all 2 and g() = next$^!_C$((f)). Using any function witnessing that f has uniform conality !, one can show that g has uniform conality !. g is discontinuous everywhere. So g 2 [C]^L and (main(g)) = (f) < = extra(g). Thus P(g) = 0 and hence C must have been homogeneous for P taking value 0. The establishes the claim.

Now suppose f 2 [D]. In the case that b < , since bound(f; b) (f) < f(b) and f(b) 2 D, next$^!_C$(bound(f; b)) < f(b). In all cases, let g : L ! C be dened by g() = f() if < and g() = next$^!_C$(bound(f; b)). As before, g is a function of the correct type in [C]^L. P(g) = 0 implies that (f) = (main(g)) < extra(g) = next$^!_C$(bound(f; b)). This completes the proof.

Lemma 2.11. Let < !$_1$ and : [!$_1$] ! !$_1$ be such that b = 0. Then there is some club D !$_1$, some Kunen function : !$_1$!$_1$! !$_1$, and some ^{0} : [!$_1$] ! !$_1$ so that for all f 2 [D]$^{!_1}$, (f) = (bound(f; b); ^{0}(f)) where b$_0$ < b.

Proof. By Lemma 2.10, there are clubs C and D$_1$ so that for all f 2 [D$_1$], (f) < next$^!_C$(bound(f; b)). Let be a Kunen function for next$_C$: !$_1$!! !$_1$. Since K$_{next^!}$ 2 , let D$_2$ K$_{next^!}$ be a club subset of !$_1$. Let D$_3$ = D$_1$ \ D$_2$. Thus for all f 2 [D$_3$], (f) < next$^!_C$(bound(f; b)) < $^C_{bound(f;b\ )}$. Let ^{0} : [D$_3$]^C ! !$_1$ be dened by (f) is the least < bound(f; b) so that (f) = (bound(f; b);). Thus one has that for all f 2 [D$_3$], (f) = (bound(f; b); ^{0}(f)). Also (8f)(^{0}(f) < bound(f; b)) implies that b$_0$ < b as long as b = 0.

Denition 2.12. Let < !$_1$ and : [!$_1$] ! !$_1$.

A representation for is a tuple ($_0$; :::; $_{n\ 1}$; $_0$; :::; $_n$;) with the following properties (a) n 2 !. If n = 0, then no appears.

(b) $_0 > {}_1 > ::: > {}_{n\ 1} > {}_n = 0$ is a sequence of strictly decreasing ordinals less than or equal to . $< !_1$.
(c) Each $_i : !_1\ \ !_1 !\ \ !_1$ is a Kunen function (for some function with respect to).
(c) Let $_n(f) =$. Suppose for $0 <\ i\ \ n$, $_i$ has been dened, then let $_{i\ 1}(f) = {}_{i\ 1}(\text{bound}(f; {}_{i\ 1}); {}_i(f))$. One has that $(8f)(_0(f) = (f))$.

Theorem 2.13. Let $< !_1$. Every $: [!_1] !\ \ !_1$ has a representation.

Proof. Let T be the tree of decreasing sequences $= (_0; :::; _k)$ in $+ 1$ ordered by reverse string extension with the property that there exists some Kunen functions $_0; :::; _{k\ 1}$ and functions $_0; :::; _k$ with the property that
(i) $_0 =$. (ii) $_i$
$= b$.
(iii) $(8f)(_i(f) = {}_i(\text{bound}(f; {}_i); {}_{i+1}(f)))$ for all $i < k$.

The claim is that there there is some $= (_0; :::; _n)$ 2 T so that $_n = 0$.

To see this: Suppose not. Let $= (_0; :::; _k)$ 2 T with $_k = 0$. Let $_0; :::; _{k\ 1}$ and $_0; :::; _k$ witness that 2 T. (ii) implies that $b\ \ = {}_k > 0$. Lemma 2.11 implies that there is some $_k$ and ^{0} so that $(8f)(_k(f) = {}_k(\text{bound}(f; b\ ); {}^0(f)))$ with $b_0 < {}_k b\ \ = {}_k$. Let $_{k+1} = {}^0$. Let $_{k+1} = b_0$. Let $^0 = \hat{}_{k+1}$. Then $_0; :::; _{k+1}$ and $_0; :::; _k$ witness that ^{0} 2 T.

It has been shown that any 2 T can be extending to some ^{0} 2 T. T is a tree on $+ 1$ with no dead branches. Since is a wellordering, T must have an innite branch. This is impossible since an innite branch is an innite descending sequence of ordinals.

The claim has been shown. So let $= (_0; :::; _n)$ 2 T be such that $_n = 0$. Let $_0; :::; _{n\ 1}$ and $_0; :::; _n$ be witnesses to 2 T. Since $b\ \ = {}_n = 0$, one has that for -almost all f, $\text{bound}(f; 0) = 0\ \ {}_n(f) < f(0)$. So for -almost all f, $_n(f)$ is a constant function taking value some 2 $!_1$. This implies that $(_0; :::; _{n\ 1}; {}_0; :::; {}_n;)$ is a representation of .

The theorem implies a -almost everywhere continuity result for function $: [!_1] !\ \ !_1$.

Theorem 2.14. Let $< !_1$ and $: [!_1] !\ \ !_1$. Then there is a decreasing sequence of ordinals which are less than or equal to , $(_i : i\ \ n)$, with $_n = 0$ and a club C $!_1$ so that if $f; g$ 2 [C] has the property that $\text{bound}(f; {}_i) = \text{bound}(g; {}_i)$ for all i n, then $(f) = (g)$.

The following is an even coarser form of continuity:

Theorem 2.15. Let $< !_1$ and $: [!_1] !\ \ !_1$. Then there is a $<$ and some club C $!_1$ so that for all $f; g$ 2 [C] with f = g and $\sup(f) = \sup(g)$, $(f) = (g)$.

Proof. If $n = 0$, then is a constant function so this immediately true. If $n = 1$, then let $= {}_0$ if $_0 <$ and $= 0$ if $_0 =$. If $n > 1$, then let $= {}_1$.

Woodin [18] has observed the conclusion of the next theorem at least under ZF + DC + AD_R or ZF + AD^+. The following gives a combinatorial proof in AD.

Theorem 2.16. $j[!_1]^! j < j[!_1]^{<!_1} j$.

Proof. Observe that $[!_1]^!\ \ [!_1]^!$ and $[!_1]^{<!_1}\ \ [!_1]^{<!_1}$. So suppose there is an injection $: [!_1]^{<!_1}$! $[!_1]^!$.

For each $< !_1$ and n 2 !, let $_n : [!_1] !\ \ !_1$ be dened by $_n(f) = (f)(n)$. By Theorem 2.15, there is some $_n <$ so that there is some C $!_1$ club with the property that for all $f; g$ 2 [C], $\sup(f) = \sup(g)$ and f = g implies that $_n(f) = {}_n(g)$.

For each n 2 !, dene $_n : !_1 !\ \ !_1$ by $_n(') =$. Each $_n$ is a regressive function. Therefore, using AC_R, let C_n and $_n$ be such that for all 2 C_n, $_n() = {}_n$. Let $C = \bigcap_{n 2 !} C_n$. Let $= \sup\{_n : n 2 !\}$. Since $!_1$ is regular, $< !_1$.

Now let $>$ be some limit ordinal with 2 C. Using AC , let D_n $!_1$ be clubs so that for all $f; g$ 2 $[D_n]$, $\sup(f) = \sup(g)$ and f = g imply that $(f) = (g)$. Let $D = \bigcap_{n 2 !} D_n$.

Now pick $f; g$ 2 [D] so that f = g, $\sup(f) = \sup(g)$, and f = g. Since for all n 2 !, $_n =$ and 2 C_n, one has that $(f) = (g)$. This contradicts being an injection.

3. Continuity of Short Functions on $!_2$

First, one will review the notations and basic tools needed to analyze $!_2$ under AD. See [3] Section 5 and 6 for more details and the proofs of the following results.

Let denote the club lter on $!_1$. An important application of the Kunen function for functions f : $!_1$! $!_1$ is the existence of a uniform procedure to select representative of the ultrapower $\prod_{!_1} !_1 =$.

Fact 3.1. Let be the club measure on $!_1$. Suppose f : $!_1$! $!_1$ and possesses a Kunen function with respect to . Suppose G 2 $\prod_{2 !_1}$ f()=. Then there is a < $!_1$ so that [] = G

As a consequence, one can show that $\prod_{!_1} !_1 =$ is wellfounded even without DC_R.

Fact 3.2. Let f : $!_1$! $!_1$ and possesses a Kunen function with respect to . Then $\prod_{2 ! _1}$ f()=, i.e. the initial segment of $\prod_{!_1} !_1 =$ determined by [f], is a wellordering.

$\prod_{!_1} !_1 =$ is wellfounded.

For each F 2 $\prod_{!_1} !_1 =$, F < $!_2$. Thus $\prod_{!_1} !_1 = !_2$.

Fact 3.3. (Martin) Assume just ZF. Let be a strong partition cardinal.

If is a measure on , then $\prod$ = is a cardinal.

If is a normal -complete measure on , then $\prod$ = is a regular cardinal.

Corollary 3.4. (Martin) Let be the club measure on $!_1$. $!_2 = \prod_{!_1} !_1 =$ and $!_2$ is a regular cardinal.

Denition 3.5. Let be the club measure on $!_1$. Let h : $!_1$! $!_1$. Suppose h possesses a Kunen function with respect to . An ordinal < $!_1$ is a minimal code (relative to) if and only if for all < , : (=). Let J be the collection of which are minimal codes and < h. Dene an ordering on J by if and only if < . By Fact 3.1, for every G < [h], there is a unique 2 J so that 2 G (i.e. [] = G). In this way, one says that is a minimal code for G or for any g 2 G with respect to . Thus (J;) has the same ordertype as [h]. By Fact 3.2, [h] is a wellordering. Let 2 ON denote the ordertype of ([h]; <) which is equal to the ordertype of (J;). Let : ! (J;) be the unique order-preserving isomorphism.

Note that the objects J , , , and depend on and h. However, within this section, one will only work with a single and h at a given time. It should be clear in context that these object depend on this xed and h.

Denition 3.6. Let be the club measure on $!_1$. Let h : $!_1$! $!_1$ be a function so that h() > 0 -almost everywhere. Let be a Kunen function for h with respect to . Let = [h] = ot(J;) which are dened relative to and h.

Let T^h = f(;) 2 $!_1$ $!_1$: < h()g. Let T^h = (T^h; @) where @ is the lexicographic ordering. Note that ot(T^h) = $!_1$.

Suppose F : T^h ! $!_1$ is an order-preserving function. Let g 2 $!_1$! $!_1$ be such that g < h. Let A^g = f : g() < h()g. Let F^g : $!_1$! $!_1$ be dened by

$$F^g() = \begin{cases} F(; g()) & 2\ A^g \\ F(; 0) & \text{otherwise} \end{cases}$$

Note that if $g_1 < g_2 <$ h, then $F^{g_1} < F^{g_2}$.

If 2 , then let $F^{()} = F^{()}$. Let funct(F) : ! ON be dened by funct(F)() = [$F^{()}$].

Fact 3.7. Let be the club measure on $!_1$. Let h : $!_1$! $!_1$ be a function possessing a Kunen function with respect to . Suppose $F_0; F_1$ 2 $[!_1]^{T^h}$ have the property that $F_0^{()} = F_1^{()}$ for all < . Then for -almost all , $F_0(;) = F_1(;)$ for all < h().

Suppose < $!_2$ and F : ! $!_2$. Let h : $!_1$! $!_1$ be such that [h] = . Let be a Kunen function for h. Via a "sliding argument", one can nd an increasing function F : T^h ! $!_1$ so that for all < , [$F^{()}$] = F(). Hence one can study functions F : ! $!_2$ by using the strong partition property of $!_1$ on partitions of functions in $[!_1]^{T^h}$. See [3] Section 5 on the statement of the sliding lemma and how it can be used to prove the following results:

Theorem 3.8. (Martin-Paris) Let be the club measure on $!_1$. Then for all < $!_2$, the partition relation $!_2$! $(!_2)_2$ holds. That is, $!_2$ is a weak partition cardinal.

As a consequence of the weak partition property on $!_1$, one can completely characterize the normal measures on $!_2$.

Corollary 3.9. Let $W_!^{!_2}$ and $W_2^{!_2}$ denote the !-club and $!_1$-club lter, respectively.
$W_!^{!_2}$ and $W_{!_1}^{!_2}$ are the only two $!_2$-complete normal ultralter on $!_2$.

The next two results show that club subsets and !-club subsets of $!_2$ are lift (in a certain sense) of some club subsets of $!_1$.

Fact 3.10. Let be the club measure on $!_1$. If C $!_1$ is a club subset of $!_1$, then $[C]^{!_1}$= is a club subset of $!_2$.
If D $!_2$ is club, then there is a club C $!_1$ so that $[C]^{!_1}$= D.

Fact 3.11. Let be the club measure on $!_1$. Let C $!_1$ be a club. Then $[C]^{!_1}$= is an !-club subset. Moreover, for every !-club D $!_2$, there is a club C $!_1$ so that $[C]^1$= D.

Fact 3.12. Let denote the club measure on $!_1$. Let C $!_1$ be club. Let B = $[C]^{!_1}$= which is an !-club subset of $!_2$.
Let < $!_2$. Let h : $!_1$! $!_1$ with h() > 0 for all < $!_1$ and [h] = . Let be a Kunen function for h. Let F 2 [B] (be of correct type). Then there is an F 2 $[C]^{T^h}$ so that for all < , $[F^{()}]$ = F().

Denition 3.13. Let denote the club measure on $!_1$. Let = $W^{!_2}$ denote the !-club measure on $!_2$. Let < $!_2$. Dene as follows: for all A $[!_2]$, A 2 if and only if there is a !-club B $!_2$ so that [B] A. is an $!_2$-complete measure on $[!_2]$ by the weak partition property of $!_2$.
Let F 2 $[!_2]$. For , let bound(F;) = supfF() : < g.
Let : $[!_2]$! $!_2$. Let b be dened so that for -almost all F 2 $[!_2]$, b is the largest so that (F) bound(F;).
Let h 2 $!_1$! $!_1$ with h() > 0 be such that [h] = . Let be a Kunen function for h with respect to . Suppose F 2 $[!_1]^{T^h}$ and . Dene Bound(F)() = supf$F^{()}$() : < g. Note that although may be uncountable, for each , this is a supremum of a set containing at most jh()j = $@_0$ many elements.

For the next several results, assume the setting of Denition 3.13.

The next results states that if F 2 $[!_2]$ and F 2 $[!_1]^{T^h}$ is a lifted representation of F, then Bound(F) is a lifted representation of bound(F;).

Fact 3.14. Let . Let F 2 $[!_2]$. Let F 2 $[!_1]^{T^h}$ be such that for all < , $[F^{()}]$ = F(). Then bound(F;) = [Bound(F)].

Proof. First observe that for any F, there is an F with the above property by Fact 3.12.
Let < bound(F;). Then there is some < so that < F(). So < $[F^{()}]$. Hence < [Bound(F)].
Now suppose that < [Bound]. Let ' : $!_1$! $!_1$ be such that ['] = . Then for -almost all , '() < supf$F^{()}$() : < g. Therefore, for -almost all , there is a < h() and, in fact, if < , there is a < $^{()}$() so that '() < F(;). Let : $!_1$! $!_1$ be dened so that for the set of -almost all with the previous property, () is the least such with '() < F(;). There is some < so that = $^{()}$. Thus ' < F = $F^{()}$ = $F^{()}$. Hence < F() where < . This shows that [Bound] < bound(F;).

Denition 3.15. Let . Let C $!_1$ be a club subset of $!_1$.
For each F 2 $[!_1]^{T^h}$, dene $\mathrm{Fnext}_{;C}$(F)() = $\mathrm{next}^!_C$(Bound(F)()).
Using either Fact 3.7 or Fact 3.14, if $F_0; F_1$ 2 $[!_1]^{T^h}$ have the property that for all , $F^{()}$ =$_0$ $F^{()}_1$, then $\mathrm{Fnext}_{;C}(F_0)$ = $\mathrm{Fnext}_{;C}(F_1)$.
Therefore the following is well dened: if F 2 $[!_2]$, let $\mathrm{fnext}_{;C}$(F) = [$\mathrm{Fnext}_{;C}$(F)], for any F 2 $[!_1]^{T^h}$ such that for all < , $[F^{()}]$ = F().

Lemma 3.16. Assume the setting of Denition 3.13. There is a club C $!_1$ and an !-club B $!_2$ so that for all F 2 [B], (F) < $\mathrm{fnext}_{b;C}$(F).

Proof. For each $< !_1$, one will dene a wellordering L: Let be a distinct new object. The underlying domain of L is h() [fg.

First assume b < . Dene the linear ordering by x y if and only if (a) x; y 2 h() and x < y.
(b) x = and y 2 h(), and y $^{(b)}$(). (c) x 2 h(), y = , and x < $^{(b)}$().

If b = , then dene x y if and only if (a) x; y 2 h() ^ x < y.
(b) x 2 h() and y = .

Let L = (L;) be a linear ordering on L = f(; x) : 2 $!_1$ ^ x 2 Lg where is the lexicographic ordering on L with on the th-coordinate. Note that L has ordertype $!_1$.

In the case the b = , let h = h() + 1. By initially choosing large enough, one may assume that is also a Kunen function for h with respect to . Note that L is order isomorphic to T $\tilde{}$.h

Suppose K 2 $[!_1]^L$. Dene main(K) : $[!_1]^{T^h}$! $!_1$ by main(K)(;) = K(;). Dene extra(K) : $!_1$! $!_1$ by extra(K)() = K(;).

Let P : $[!_1]^L$! 2 be dened by P(K) = 0 , (funct(main(K))) < [extra(K)]. By $!_1$! $(!_1)$ $^1_2$, there is a club C $!_1$ which is homogeneous for P.

Claim 1: C is homogeneous for P taking value 0.

By denition of b, there is an !-club B^0 $!_2$ so that all F 2 [B], b is the largest so that (F) bound(F;). By Fact 3.14, there is a club C^0 so that $[C^0]^{!_1}$ = B. By intersecting with C^0, assume that C C^0.

(Case I) b < .

Let D = f 2 C : enum$_C$() = g be the closure points of C. Let B = $[D]^{!_1}$. Pick any F 2 [B]. By Fact 3.12, there is some F 2 $[D]^{T^h}$ so that for all < , [$F^{()}$] = F(). Let f : $!_1$! $!_1$ be such that [f] = (F). By Fact 3.14, bound(F; b) = [Bound$_b$ (F)]. Since b is the least so that (F) bound(F;), one has that the set A of 's so that Bound$_b$ (F)() f() < $F^{(b)}$() belongs to . Dene K 2 $[C]^L$ by

$$= \begin{cases} F(; z) & z \, 2 \, h() \;\; K(; z) \\ \mathrm{next}^!_C(f()) & 2 \, A \wedge z = \; : \\ \mathrm{next}^!_C(\mathrm{Bound}_b(F)()) & 2 \, A \wedge z = \end{cases}$$

Note that since F(; $^{(b)}$) 2 D, K(;) < K(; $^{(b)}$) for all . Thus K : L ! C is indeed an increasing function. Since F is a function of the correct type, one can check that K is also of the correct type.

Note that main(K) = F and for -almost all , extra(K)() = next$^!_C$(f()) > f(). Thus (funct(main(K))) = (F) = [f] < [extra(K)]. Thus P(K) = 0. However since C is homogeneous for P and K 2 $[C]^L$, one has that C is homogeneous for P taking value 0.

(Case II) b = .

Let B = $[C]^{!_1}$. Pick any F 2 [B]. Let f : $!_1$! $!_1$ be such that [f] = (F). Let g() = next$^!_C$(f()).

Let G 2 $[B]^{+1}$ be dened by

$$G() = \begin{cases} F() & < \\ [g] & = \end{cases}$$

By Fact 3.12, there is some K 2 $[C]^{T^h}$ = $[C]^L$ so that for all < + 1, $K^{()}$ = G().

Then one has that (funct(main(K))) = (F) = [f] < [g] = [extra(K)]. Thus P(K) = 0. Since K 2 $[C]^L$, C is homogeneous for P taking value 0.

The claim has now been established.

Let D = f 2 C : enum$_C$() = g. Let B = $[D]^{!_1}$. Now suppose F 2 [B]. By Fact 3.12, pick any F 2 $[D]^{T^h}$ so that for all < , [$F^{()}$] = F(). Now dene K 2 $[C]^L$ by

$$K(; z) = \begin{cases} F(; z) & z \, 2 \, h() \\ \mathrm{next}^!_C(\mathrm{Bound}_b(F)()) & z = \end{cases} :$$

Since C is homogeneous for P taking value 0, one has P (K) = 0. This implies (F) = (funct(main(K))) < [extra(K)] = [Fnext$_{b;C}$(F)] = fnext$_{b;C}$(F). This completes the proof.

Denition 3.17. Suppose : !$_1$!$_1$! !$_1$.

Suppose f$_0$: !$_1$! !$_1$ and f$_1$: !$_1$! !$_1$. Let v$_{f_0;f_1}$: !$_1$! !$_1$ be dened by v$_{f_0;f_1}$ () = (f$_0$(); f$_1$()). Note that if f^0 = f$_0$ and f^0 = f$_1$, then v$_{f_0;f_1}$ = v$_{f^0;f^0}$.

Therefore, dene : !$_2$!$_2$! !$_2$ by (;) = [v$_{f;f}$], where f; f : !$_1$! !$_1$ are such that [f] = and [f] = .

Lemma 3.18. Suppose b > 0. Then there is a Kunen function : !$_1$!$_1$! !$_1$ and a function ^{0}: [!$_2$] ! !$_2$ so that for -almost all F, (F) = (bound(F; b); ^{0}(F)) where b$_0$ < b.

Proof. Let B !$_2$ be the !-club and C !$_1$ be the club from Lemma 3.16.

Pick any F 2 [B]. Let F 2 [!$_1$]$^{T\ h}$ be so that for all < !$_1$, [F$^{()}$] = F(). Let f : !$_1$! !$_1$ be such that [f] = (F). By Lemma 3.16, for -almost all , f() < next$^!_C$(Bound$_b$ (F)()). Let : !$_1$!$_1$! !$_1$ be a Kunen function for next$_C$. For -almost all , let v$_{f;F}$ () be the least < Bound$_b$(F)() so that f() = (Bound$_b$ (F)();). Observe that if g = f and G 2 [!$_1$]^T ^{h} is such that G$^{()}$ = F$^{()}$ for all < , then v$_{f;F}$ = v$_{g;F}$. Therefore, dene ^{0}(F) = [v$_{f;F}$]. Note by construction, (F) = [f] = (bound(F; b); [v$_{f;F}$]) = (bound(F; b); ^{0}(F)). Since ^{0}(F) < bound(F; b), one has that b$_0$ < b if b > 0.

Denition 3.19. Let < !$_2$ and : [!$_2$] ! !$_2$.

A representation for is a tuple ($_0$; :::; $_{n\ 1}$; $_0$; :::; $_n$;) with the following properties (a) n 2 !. If n = 0, then no appears.

(b) $_0$ > $_1$ > ::: > $_{n\ 1}$ > $_n$ = 0 is a sequence of strictly decreasing ordinals less than or equal to . < !$_2$.

(c) Each $_i$: !$_1$!$_1$! !$_1$.

(d) Let $_n$(F) = . Suppose for 0 < i n, $_i$ has been dened, then let $_{i\ 1}$(F) = $_i$(bound(F; $_{i\ 1}$); $_i$(F)). One has that for -almost all F, $_0$(F) = (F).

Theorem 3.20. Let < !$_2$. Every : [!$_2$] ! !$_2$ has a representation.

Proof. The proof is analogous to the proof of Theorem 2.13 using the !$_2$ version of the analogous lemmas.

Now one has the analogous continuity result for functions : [!$_2$] ! !$_2$ where < !$_2$.

Theorem 3.21. Let < !$_2$ and : [!$_2$] ! !$_2$. Then there is a decreasing sequence of ordinals less than or equal to , ($_i$: i n), with $_n$ = 0 and an !-club B !$_2$ so that if F; G 2 [B] has the property that bound(F; $_i$) = bound(G; $_i$) for all i n, then (F) = (G).

Theorem 3.22. Let < !$_2$ and : [!$_2$] ! !$_2$. Then there is a < and an !-club B !$_2$ so that for all F; G 2 [B] with F = G and sup(F) = sup(G), (F) = (G).

Now one has some new cardinality results:

Theorem 3.23. j[!$_2$]$^!$j < j[!$_2$]$^{<!_1}$j.

Proof. Suppose : [!$_2$]$^{<!_1}$! [!$_2$]$^!$ is a function. For each < !$_1$ and each n 2 !, let $_n$: [!$_2$] ! !$_2$ be dened be (F) = (F)(n). By Theorem 3.22, there is some < so that (F) = (G) for -almost all F and G so that F = G and sup(F) = sup(G). Let be the least such . The function $_n$: !$_1$! !$_1$ dened by $_n$() = is a regressive function. Using AC$_R$, there is a $_n$ < !$_1$ and A$_n$ 2 so that for all 2 A$_n$, $_n$() = $_n$. Let A = $_{n\,2\,!}$ A$_n$ 2 and = sup$_{n2!}$ $_n$ < !$_1$. Pick a limit ordinal 2 A with > . By AC , let B$_n$ be an !-club subset of !$_2$ so that for all F; G 2 [B$_n$], if sup(F) = sup(G) and F $_n$ = G $_n$, then $^!_n$(F) = $_n$(G). Since is !$_2$-complete, B = T B$_n$ 2 . Thus pick some F; G 2 [B] with F = G, sup(F) = sup(G), and F = G . Then for all n 2 !, (F)$_n$= (G). So (F) = (G). can not be an injection.

Theorem 3.24. j[!$_2$]$^{<!_1}$j < j[!$_2$]$^{!_1}$j.

Proof. Let $: [!_2]^{!_1}$! $[!_2]^{<!_1}$ be a function. Let $: [!_2]^{!_1}$! $!_1$ be length , where length(F) = if F : ! $!_2$. Since is $!_2$-complete, there is a B 2 and an < $!_1$ so that for all F 2 $[B]^{!_1}$, (F) = . In other words, for all F 2 $[B]^{!}$, (F) 2 $[!_2]$.

Let < . Let (F) = (F)(). By Theorem 3.22 and AC , there are < $!_1$ and !-club B $!_2$ so that for all F; G 2 $[B]^{!_1}$, if F = G and sup(F) = sup(G), then (F) = (G).

Now let U = $\bigcap_{<}$ B 2 since is $!_2$-complete. Let = supf : < g. Note that < $!_1$ since $!_1$ is regular. Pick F; G 2 $[U]^{!_1}$ with F = G, F = G , sup(F) = sup(G). Since F; G 2 $[B]^{!_1}$, (F) and (G) both have length . By choice, (F)() = (F) = (G) = (G)() for all < . So (F) = (G). is not an injection.

Previously, one only needed $AC^R_!$ to make a countable selection of subsets of $!_1$ or $!_2$. For the next theorem, one will need to make an $!_1$-length selection of club subset of $!_1$. The following fact ensures this can be done.

Fact 3.25. ([3] Section 4) Let hA : < $!_1$i be such that each A is a nonempty -downward closed collection of clubs subsets of $!_1$. Then there is a sequence hC : < $!_1$i with each C $!_1$ a club subset of $!_1$ and C 2 A.

Theorem 3.26. $j[!_2]^{!_1}j < j[!_2]^{<!_2}j$.

Proof. Let $: [!_2]^{<!_2}$! $[!_2]^{!_1}$ be a function. For each < $!_2$ and < $!_1$, let : $[!_2]$! $!_2$ be dened by (F) = (F)(). By Theorem 3.22, there is a minimal < so that for -almost all F; G 2 $[!_2]$, if F = G and sup(F) = sup(G), then (F) = (G).

For each < $!_1$, let : $!_2$! $!_2$ be dened by () = . Since is a normal measure on $!_2$ and is a regressive function, there is a minimal < $!_2$ so that for -almost all , () = . By Fact 3.11, for every B 2 , there is a C $!_1$ club so that $[C]^{!_1}$= B. Let A be the collection of all club C $!_1$ so that for all 2 $[C]^{!_1}$=, () = . A is clearly -downward closed. Apply Fact 3.25 to obtain a sequence hC : < $!_1$i so that C 2 A. Let B = $\bigcap$ $[C]^{!_1}$= which belong to as is $!_2$-complete. Let = supf : < $!_1$g < $!_2$ since $!_2$ is regular. Now pick a limit ordinal > with 2 B.

For < $!_1$, let A^0 be the collection of club C $!_1$ so that if D = $[C]^{!_1}$=, then D has the property that for all F; G 2 [D] , if F = G and sup(F) = sup(G), then (F) = (G). A^0 is a -downward closed nonempty collection of club subsets of $!_1$. Apply Fact 3.25 to obtain a collection hC^0 : < $!_1$i of club subsets of $!_1$ with the property that for all < $!_1$, C^0 2 A^0 . Let B^0 = $\bigcap_{<!_1}$ $[C^0]^{!}$= which belongs to since is $!_2$-complete. Now pick F; G 2 $[B^0]$ with F = G , sup(F) = sup(G), and F = G. Note that for all < $!_1$, (F)() = (F) = (G) = (G)(). Thus (F) = (G). is not an injection.

Theorem 3.27. $j[!_2]^{!}j < j[!_2]^{<!_1}j < j[!_2]^{!_1}j < j[!_2]^{<!_2}j$.

Proof. Given the previous theorems, one needs only to show that the appriopriate injections exists. The only one that is not immediately clear is the injection from $[!_2]^{<!_1}$ into $[!_2]^{!_1}$.

Let add : $!_2$ $[!_2]^{<!_1}$! $[!_2]^{<!_1}$ be dened by if F 2 $[!_2]$ for some < $!_1$, then add(; F) 2 $[!_2]$ be dened by add(; F)() = + F().

If F 2 $[!_2]^{<!_1}$, then let ll(F) 2 $[!_2]^{!_1}$ is dened by appending onto F the next $!_1$-many ordinals after sup(F).

Let $: [!_2]^{<!_1}$! $[!_2]^{!_1}$ be dened by (F) = ll(length(F)^add(length(F); F)). In words, (F) starts with length(F), then shifts up all the values of F by length(F), and ll in the rest with successive ordinals until one reaches length $!_1$. One can check that is an injection.

Fact 3.28. $!_2$ does not inject into $[!_1]^{!_1}$. Thus $[!_2]^{!}$ does not inject into $[!_1]^{!_1}$.

Proof. This is a consequence of the measurability of $!_2$ in the same way the fact that there are no uncountable wellordered sequences of reals follows from the measurability of $!_1$. The details follow:

Let be an $!_2$-complete measure on $!_2$. Suppose hf : < $!_2$i is an injection of $!_2$ into $[!_1]^{!_1}$. Let F = rang(f). Then hF : < $!_2$i is an $!_2$-sequence of distinct subsets of $!_1$.

For each < $!_1$, let A = f < $!_2$: 2 Fg and A = f < $!_2$: 2 Fg. Since is a measure, there is some i 2 2 so that A^i 2 .

By the ω_2-completeness of , $\bigcap_{2 \omega} A^i \in$. Let $_{0;1} \in \bigcap_{2\omega} A^i$. Let $F \subseteq \omega_1$ be defined $\in F$, $i = 1$. Then $F = F = F$. This contradicts the fact that $\langle F : < \omega_2 \rangle$ is a sequence of distinct subsets of ω_1.

Like the original argument for the cardinal relation $|[\omega_1]^{\omega}| < |[\omega_1]^{<\omega_1}|$, the argument that $[\omega_1]^{<\omega_1}$ does not inject into $[\omega_2]^{\omega}$ passes through the set S_1 using 1-Borel code and forcing arguments. This originally was proved under ZF + AD . The following gives a purely descriptive set theoretic proof using just AD.

Theorem 3.29. $\neg(|[\omega_1]^{<\omega_1}| \leq |[\omega_2]^{\omega}|)$. Thus $\neg(|[\omega_1]^{\omega_1}| \leq |[\omega_2]^{\omega}|)$.

Proof. Suppose $: [\omega_1]^{<\omega_1} \to [\omega_2]^{\omega}$ is an injection.

For each $< \omega_1$ and $f \in [\omega_1]^{\omega_1}$, let $\mathrm{tail}(f;) \in [\omega_1]^{\omega_1}$ be defined by $\mathrm{tail}(f;)() = f(+)$. Note that for all $<$ ω_1 and $f \in [\omega_1]^{\omega_1}$, $f = (f \quad)^\frown \mathrm{tail}(f;)$. Let denote the club measure on ω_1.

For each $< \omega_1$, let $P : [\omega_1]^{\omega_1} \to 2$ by defined by $P(f) = 0$ if and only if $\sup((f \quad)) < [\mathrm{tail}(f;)]$. (Recall that $\prod_{\omega_1} \omega_1 / = \omega_2$.)

Let $C \subseteq \omega_1$ be a club which is homogeneous for P. The claim is that C is homogeneous for P taking value 0. Suppose otherwise, then pick any $\in [C]$. For any $g \in [C]^{\omega_1}$ with $\min(g) > \sup()$, define $^g \in [C]^{\omega_1}$ by $^\frown g$. Then $P(^g) = 1$ implies that $[g] = \mathrm{tail}(^g;) \leq \sup((^g \quad)) = \sup(())$. This impossible since is fixed, $[C]^{\omega_1} / = \omega_2$, and g can be any member of $[C]^{\omega_1}$ with $\min(g) > \sup()$.

It has been shown that C is homogeneous for P taking value 0. Let $\ell \in [C]^{\omega_1}$ and let $= [\ell]$. Note that for all $< \omega_1$, $\ell = \mathrm{tail}(\ell;)$. Let $\in [C]$. Let be the least so that $\ell() > \sup()$. Define $f = {}^\frown \mathrm{tail}(\ell;)$. Note that $f \in [C]^{\omega_1}$. Thus $P(f) = 0$ implies that $\sup(()) = \sup((f \quad)) < [\mathrm{tail}(f;)] = [\mathrm{tail}(\ell;)] = [\ell] =$. That is, maps $[C]$ into $[]^{\omega}$.

For each $< \omega_1$, let be the least $< \omega_2$ so that there exists a club $C \subseteq \omega_1$ with the property that for all $\in [C]$, $\sup(()) <$. This defines a sequence $\langle : < \omega_1 \rangle$. Let $= \sup\{ : < \omega_1\}$. Since ω_2 is regular, $< \omega_2$.

For $< \omega_1$, let A be the collection of clubs $C \subseteq \omega_1$ so that for all $\in [C]$, $\sup(()) <$. This defines a sequence $\langle A : < \omega_1 \rangle$. Note that for all $< \omega_1$, A is a nonempty -downward closed collection of club subsets of ω_1. By Fact 3.25, let $\langle C : < \omega_1 \rangle$ be a sequence so that $C \in A$ for all $\in \omega_1$. So for any $< \omega_1$, if $\in [C]$, then $\sup(()) <$.

Note that $\bigcup_{<\omega_1} [C] \subseteq [\omega_1]^{<\omega_1}$. Observe that

$$\left[\bigcup [C] \right] \subseteq []^{\omega} : < \omega_1 .$$

Hence induces an injection of $[\omega_1]^{<\omega_1}$ into $[]^{\omega} \approx [\omega_1]^{\omega}$ since $< \omega_2$. By Theorem 2.16, this is impossible.

Fact 3.30. $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$.

Proof. There is a purely descriptive set theoretic proof of this result in the flavor of the continuity argument used throughout this paper in [6]. However, the requisite continuity property is more challenging to establish than the analogous continuity properties in this paper. However, there is a very simple set theoretic proof of this result:

Suppose there was an injection $: [\omega_1]^{\omega_1} \to [\omega_1]^{<\omega_1}$. Let $L[] \models \mathrm{ZFC}$ be the Gödel constructible universe built relative to as a predicate.

Note that ω_1^V is inaccessible in $L[]$: Suppose $< \omega_1^V$ and $|P()^{L[]}|^{L[]} \geq \omega_1^V$. Since $L[] \models \mathrm{AC}$, $P()^{L[]}$ is a wellorderable collection of subsets of of cardinality ω_1^V. In the real world V, is a countable ordinal and hence there is a bijection of with ω. Using this bijection, one can obtain an ω_1^V-length sequence of distinct reals from $P()^{L[]}$. This is impossible under AD by a simple form of the argument in Fact 3.28. Thus $|P()^{L[]}|^{L[]} < \omega_1^V$. This implies ω_1^V is inaccessible in $L[]$.

Since $L[] \models \mathrm{ZFC}$, Cantor's theorem assert that $L[] \models |[\omega_1^V]^{\omega_1^V}| = |2^{\omega_1^V}| \geq (\omega_1^V)^+$. Also since $L[] \models \mathrm{ZFC}$ and ω_1^V is inaccessible in $L[]$, $L[] \models |[\omega_1^V]^{<\omega_1^V}| = |2^{<\omega_1^V}| = \omega_1^V$. By absoluteness, $L[] \models$ is an injection. It is impossible that $L[]$ thinks that is an injection of $2^{\omega_1^V}$ into ω_1^V.

A very similar argument can be used to show that $|[\omega_2]^{<\omega_2}| < |[\omega_2]^{\omega_2}|$. See [4] Section 4.

Theorem 3.31. Then :(j[!1]^{!1} j j[!2]^{<!1} j).

Proof. Let T = (!1 2;) where is the lexicographic ordering. (Note that ot(T) = !1.) If F 2 [!1]^T and i 2 2, let F_i 2 [!1]^{!1} be dened by F_i() = F(; i).

Now suppose : [!1]^{!1} ! [!2]^{<!1} is an injection. Dene a partition P : [!1]^T ! 2 by P(F) = 0 if and only if sup((F_0)) sup((F_1)). Let C !1 be a club homogeneous subset for P. The claim is C is homogeneous for P taking value 0.

Suppose C was homogeneous for P taking value 1. Let g_0(0) = next^!_C(0). Suppose g_k() has been dened, then let g_{k+1}() = next^!_C(g_k()). Suppose g_n() has been dened for all n 2 ! and < . Then let g_0() = next^!_C (supfg_n() : n 2 ! ^ < g).

For each n 2 !, g_n 2 [C]^{!1}. Dene for < !1 and i 2 2, G^n(; i) = g_{n+i}(). By the construction of hg_n : n 2 !i, one has that G^n 2 [C]^T.

Thus one has that P(G^n) = 1 for all n 2 !. This implies for all n 2 !.

sup((g_{n+1})) = sup((G_1)) ≮ sup((G_0)) =^n sup((g_n)):

It has been shown that hsup((g_n)) : n 2 !i is an innite decreasing sequence of ordinals. This contradicts the wellfoundedness of the ordinals.

One must have that C is homogeneous for P taking value 0. For the next part, take g_0, g_1, and g_2 from the sequence hg_n : n 2 !i constructed above. The important observation from above is that g_0() < g_1() < g_2() < g_0(+ 1) for all .

For each A 2 ^{!1}2, let h_A 2 [C]^{!1} be dened by h_A() = g_{A()}(). Let H^A 2 [C]^T be dened by

H^A(; i) = (h_A() i = 0 ; g_2() i = 1) :

Note that H^A_0 = h_A and H^A_1 = g_2. P(H^A) = 0 implies that sup((h_A)) = sup((H^A))_0 sup((H^A)) = _1 sup((g_2)). Let = sup((g_n)) which is some ordinal less than !2.

Dene : ^{!1}2 ! [!2]^{<!1} by (A) = (h_A). Note that is a injection. By the above, : ^{!1}2 ! []^{<!1}. Since ^{!1}2 P(!1) [!1]^{!1}, one has shown that there is an injection of [!1]^{!1} into []^{<!1} [!1]^{<!1}. This is not possible by Fact 3.30.

For the sake of completeness, one sketches the remaining well-known cardinal relations among the sets considered in this paper:

Fact 3.32. :(!1 jRj) and :(jRj !1).

Proof. By a simple form of the argument in the proof of Fact 3.28, there are no uncountable wellordered sequences of distinct reals. That is, !1 can not inject into R.

Under AD, R can not be wellordered. (For instance, a category argument can be used to show that a wellordered union of meager sets is meager under AD.) Hence R can not inject into !1.

Fact 3.33. Let be an ordinal. :(j[!1]^! j), :(j[!1]^! j R), :(j[!1]^! j jR t j), and :(j[!1]^! j jRj). Similarly, :(j[!2]^! j), :(j[!2]^! j R), :(j[!2]^! j jR t j), and :(j[!2]^! j jR j).

Proof. Since R injects into [!1]^! and R is not wellorderable, [!1]^! is not wellorderable. So [!1]^! can not inject into any ordinal .

Let : [!1]^! ! ^!2. For each n 2 !, dene P_n : [!1]^! ! 2 by P_n(f) = f(n). By AC_R^!, let C_n !1 be club homogeneous for P_n taking some value i_n 2 2. Let C = T_{n2!} C_n. Let r 2 ^!2 by r(n) = i_n. Note that for all [[C]^!] = frg. Thus is not an injection.

Now suppose : [!1]^! ! t R. Dene Q : [!1]^! ! 2 by

Q(f) = (0 (f) 2 1 ; (f) 2 R)

Let C !1 be club homogeneous for Q. If C is homogeneous for Q taking value 0, then maps [C]^! into . By the earlier argument, can not be an injection. If C is homogeneous for Q taking value 1, the maps [C] into R. Again by the earlier argument, can not be an injection.

Suppose $\Phi : [\omega_1]^\omega \to \mathbb{R} \times \omega_1$. Let $\pi_1 : \mathbb{R} \times \omega_1 \to \mathbb{R}$ be the projection onto the first coordinate. Then $\pi_1 \circ \Phi : [\omega_1]^{\omega_1} \to \mathbb{R}$. By the argument above, there is a club $C \subseteq \omega_1$ and an $r \in \mathbb{R}$ so that $(\pi_1 \circ \Phi)[[C]^\omega] = \{r\}$. Then $\Phi : [C]^\omega \to \{r\} \times \omega_1$. Since $\{r\} \times \omega_1$ is in bijection with ω_1, Φ can not be an injection by the earlier part of this proof.

The result for $[\omega_2]^\omega$ follows by the same argument using the weak partition property for ω_2.

The cardinal relations displayed in the diagram from the introduction follow from the work so far.

4. $[\omega_1]^\omega$ is Jonsson

Definition 4.1. Let X be a set. Define $[X]^n_{=} = \{f \in {}^nX : (\forall i < j < n)(f(i) = f(j))\}$. Let $[X]^{<\omega}_{=} = \bigcup_{n \in \omega} [X]^n_{=}$.

For $n < \omega$, X is n-Jonsson if and only if for every $\Phi : [X]^n_{=} \to X$, there is some $Z \subseteq X$ with $|Z| = |X|$ so that $\Phi[[Z]^n_{=}] \neq X$.

X is Jonsson if and only if for all $\Phi : [X]^{<\omega}_{=} \to X$, there is some $Z \subseteq X$ with $|Z| = |X|$ so that $\Phi[[X]^{<\omega}_{=}] \neq X$.

Definition 4.2. Let $f \in {}^{<\omega}([\omega_1]^\omega)$. The tuple-type of f, denoted type(f), is a 4-tuple (n, m, G, D) with the following properties:

(1) n is the length of the tuple f.

(2) Let $S = \{\sup(f_i) : i < n\}$. Then $m = |S|$.

Let $\text{rang}(f) = \bigcup_{i<n} \text{rang}(f_i)$. Note that m also has the property that $\text{ot}(\text{rang}(f)) = \omega \cdot m$. Let $\langle a_0, \dots, a_{m-1}\rangle$ be the increasing enumeration of S. Let $F : \omega \cdot m \to \text{rang}(f)$ be the increasing enumera-tion of $\text{rang}(f)$.

(3) $G : m \to \mathcal{P}(n)$ is defined by $G(i) = \{k \in n : \sup(f_k) = a_i\}$.

(4) Let $D : \omega \cdot m \to \mathcal{P}(n)$ be defined by $D(\alpha) = \{i \in n : F(\alpha) \in \text{rang}(f_i)\}$. If $Z \subseteq [\omega_1]^\omega$, then let $\text{type}(Z) = \{\text{type}(f) : f \in {}^{<\omega}Z\}$.

Example 4.3. Consider $f_0, f_1, f_2 \in [\omega_1]^\omega$ defined by

$$f_0(x) = \begin{cases} 0 & x = 0 \\ x + 1 & x \geq 1 \end{cases}; \quad f_1(x) = \begin{cases} x & x = 0, 1 \\ \omega + 2(x-1) & x \geq 2 \end{cases}$$

$$f_2(x) = \begin{cases} x & x = 0, 1 \\ \omega + (x-2) & x = 2, 3 \\ \omega + 2(x-3) + 1 & x \geq 4 \end{cases}$$

The first several values of f_0, f_1, and f_2 are the following:

$$f_0 = \langle 0, 2, 3, 4, 5, 6, 7, \dots\rangle \quad f_1 = \langle 0, 1, \omega + 2, \omega + 4, \omega + 6, \omega + 8, \omega + 10, \dots\rangle$$

$$f_2 = \langle 0, 1, \omega, \omega + 1, \omega + 3, \omega + 5, \omega + 7, \omega + 9, \omega + 11, \dots\rangle.$$

The picture looks as follows: There are $\omega \cdot 2$ many columns. Row 0, 1, and 2 indicate which values among $\omega \cdot 2$ are taken by f_0, f_1, and f_2, respectively.

0		0	0	0	0	···	|												
1	1						|			1		1		1		1		1	···
2	2						|	2	2		2		2		2		2		···

Then $\text{type}((f_0, f_1, f_2)) = (3, 2, G, D)$ where G and D are defined as follows: $G : 2 \to \mathcal{P}(3)$ is defined by $G(0) = \{0\}$ and $G(1) = \{1, 2\}$. The function $D : \omega \cdot 2 \to \mathcal{P}(3)$ can be read off the diagram above by

$$D(\alpha) = \begin{cases} \{0, 1, 2\} & \alpha = 0 \\ \{1, 2\} & \alpha = 1 \\ \{0\} & 2 \leq \alpha < \omega \\ \{2\} & \alpha = \omega, \omega + 1 \\ \{1\} & (\exists k \in \omega)[\alpha = \omega + 2(k+1)] \\ \{2\} & (\exists k \in \omega)[\alpha = \omega + 2(k+1) + 1] \end{cases}$$

With Denition 4.2 as the motivation, one makes the following abstract denition of a tuple-type:

Denition 4.4. A tuple-type t is a 4-tuple (n; m; G; D) with the following properties:
(1) n 2 ! and n > 0 which is called the length of tuple type.
(2) 1 m n which is called the arrangement number of the tuple type.
(3) G : m ! P(n) with the property that for all i < m, G(i) = ;, $S_{i 2 m}$ G(i) = n, and for all i < j < m, G(i) \ G(j) = ;. G is called the grouping order of the tuple-type.
(4) D : ! m ! P(n), which is called the distribution of the type, is a function with the following properties:
(a) For each i < m and l 2 !,

$$D(! \; i + l) \setminus @ \overset{0}{} \bigcup_{j<i} G(j) A \overset{1}{} = ;:$$

(b) For each k < n, if k 2 G(i), then fl 2 ! : k 2 D(! i + l)g is innite.
(c) For each k < n, if k 2 G(i), then for each j < i, fl 2 ! : k 2 D(! j + l)g is nite.

Observe that if f 2 $^{<!}([!_1]^!)$, then the tuple-type of f, type(f), is a tuple-type as dened in Denition 4.4.

Denition 4.5. Let t = (n; m; G; D) be a tuple-type. Let h : $[!_1]^{!m}$! $!_1$. For i < n, let $f_i^{t;h}$ be dened to be the increasing enumeration of fh() : < ! m ^ i 2 D()g. Note that the properties of the distibution imply that $f_i^{t;h}$ 2 $[!_1]^!$.

Dene extract(t; h) = ($f_0^{t;h}$; :::; $f_{n-1}^{t;h}$). This is the n-tuple extracted from h of tuple-type t. Note that type(extract(t; h)) = t.

Denition 4.6. Let X be any set and P : ! ! X. P is eventually periodic if and only if there exists k; p 2 ! and x_0; :::; x_{p-1} 2 X so that for all n > k, P(n) = x_i where i < p is such that n k is congruent to i mod p.

A tuple-type t = (n; m; G; D) is an eventually periodic tuple-type if and only if for each i < m, the function P_i : ! ! P(n) dened by P_i(k) = D(! i + k) is eventually periodic.

Note that there are only countably many eventually periodic tuple-types.

Denition 4.7. Let L be the collection of nite tuples (; n; $_0$; :::; $_n$) where < $!_1$, n 2 !, $_0$ < $_1$ < ::: < $_n$ < . Let be the lexicographic ordering on L. Let L = (L;). Note that ot(L) = $!_1$.

Let H 2 $[!_1]^L$, that is an order-preserving function of L into $!_1$.

Dene ^{H} : $[!_1]^!$! $[!_1]^!$ by (f)(k) = H(sup(f); k; f(0); :::; f(k)).

Lemma 4.8. ^{H} is an injection and type($^H[[!_1]^!]$) consists only of eventually periodic tuple-types.

Proof. Suppose f; g 2 $[!_1]^!$ with f = g.

(Case I) Suppose sup(f) = sup(g). Without loss of generality, suppose sup(f) < sup(g). Then ^{H}(f)(0) = H(sup(f); 0; f(0)) < H(sup(g); 0; g(0)) = ^{H}(g)(0). Therefore, ^{H}(f) = ^{H}(g).

(Case II) Suppose sup(f) = sup(g). f = g implies that there is a least k so that f(k) = g(k). Without loss of generality, suppose f(k) < g(k). Then ^{H}(f)(k) = H(sup(f); k; f(0); :::; f(k)) < H(sup(g); k; g(0); :::; g(k)) = ^{H}(g)(k). So ^{H}(f) = ^{H}(g).

It has been shown that ^{H} is an injection.

Now suppose f = (f_0; :::; f_{n-1}) 2 $^{<!}([!_1]^!)$. Let ^{H}(f) = (^{H}(f_0); :::; ^{H}(f_{n-1})). Let type(f) = (n; m; G; D). Suppose type(^{H}(f)) = (n^0; m^0; G^0; D^0).

For i < j < n, if sup(f_i) < sup(f_j), then

$$^H(f_i)(a) = H(sup(f_i); a; f_i(0); :::; f_i(a)) < H(sup(f_j); b; f_j(0); :::; f_j(b)) = {}^H(f_j)(b)$$

for any a; b 2 !. This implies that if sup(f_i) < sup(f_j), then sup(^{H}(f_i)) < sup(^{H}(f_j)). This shows that m^0 = m and G^0 = G.

Pick any i < m. Let P_i(k) = D^0(! i + k). Pick a ' 2 ! large enough so that for all a; b 2 G(i), if f_a = f_b, then there is some < ' so that f_a() = f_b().

Dene an preordering v on G(i) by a v b if and only if f_a ' = f_b ' or f_a ' is lexicographically less than f_b '. The v-preordering classes of G(i) are naturally linearly ordered. Note that P_i is eventually periodic by repeating the v-preordering classes of G(i) in this natural order.

It has been established that $\mathrm{type}({}^{H}(f))$ is an eventually periodic tuple-type.

Example 4.9. Let f_0, f_1, and f_2 be the functions from Example 4.3. Let $H : L \to \omega_1$ be any order-preserving function of the correct type. Let $\mathrm{type}((f_0; f_1; f_2)) = (3; 2; G; D)$. Let ^{H} be the associated function as dened above. Let $\mathrm{type}(({}^{H}(f_0); {}^{H}(f_1); {}^{H}(f_2))) = (3; 2; G; D^0)$, where D^0 is dened below:

Observe that in $L = (L;)$, the following objects are arranged as follows:

$$(\omega; 0; 0)\ (\omega; 1; 0; 2)\ (\omega; 2; 0; 2; 3)\ (\omega; 3; 0; 2; 3; 4)\ \ldots\ (\omega\ 2; 0; 0)\ (\omega\ 2; 1; 0; 1)$$

$$(\omega 2; 2; 0; 1; \omega)\ (\omega 2; 2; 0; 1; \omega+2)\ (\omega 2; 3; 0; 1; \omega; \omega+1)\ (\omega 2; 3; 0; 1; \omega+2; \omega+4)\ (\omega 2; 4; 0; 1; \omega; \omega+1; \omega+3)\ (\omega\ 2; 4; 0; 1; \omega + 2; \omega + 4; \omega + 6)\ (\omega\ 2; 5; 0; 1; \omega; \omega + 1; \omega + 3; \omega + 5)\ \ldots$$

This implies that

$$
\begin{aligned}
&{}^{H}(f_0)(0) < {}^{H}(f_0)(1) < {}^{H}(f_0)(2) < {}^{H}(f_0)(3) < {}^{H}(f_0)(4) < \ldots \\
&< {}^{H}(f_1)(0) = {}^{H}(f_2)(0) < {}^{H}(f_2)(1) = {}^{H}(f_1)(1) < {}^{H}(f_2)(2) \\
&< {}^{H}(f_1)(2) < {}^{H}(f_2)(3) < {}^{H}(f_1)(3) < {}^{H}(f_2)(4) < {}^{H}(f_1)(4) < {}^{H}(f_2)(5)
\end{aligned}
$$

From the example above, the diagram for D^0 is given below. In his diagram, $\hat{0}$, $\hat{1}$, and $\hat{2}$ represent ${}^{H}(f_0)$, ${}^{H}(f_1)$, and ${}^{H}(f_2)$:

$$
\begin{array}{l}
\hat{0}\ \hat{0}\ \hat{0}\ \hat{0}\ \hat{0}\ \hat{0}\ \cdots\ | \\
\qquad\qquad\qquad\quad |\ \hat{1}\ \hat{1}\quad \hat{1}\quad \hat{1}\quad \hat{1}\quad \hat{1}\quad \cdots \\
\qquad\qquad\qquad\quad |\ \hat{2}\ \hat{2}\ \hat{2}\quad \hat{2}\quad \hat{2}\quad \hat{2}\quad \hat{2}\ \cdots
\end{array}
$$

Explicitly, $D^0 : \omega\ 2\ \omega \to P(3)$ is

$$
D^0() = \begin{cases}
\{0\} & < \omega \\
\{1; 2\} & = \omega; \omega + 1 \\
\{2\} & (\exists k \in \omega)[= \omega + 2(k + 1)] \\
\{1\} & (\exists k \in \omega)[= \omega + 2(k + 1) + 1]
\end{cases}
$$

Note that $P_0(k) = D^0(k)$ is eventually periodic by repeating $\{0\}$ and $P_1(k) = D^0(\omega + k)$ is eventually periodic by eventually alternating between $\{1\}$ and $\{2\}$.

Fact 4.10. Let $: {}^{<\omega}([\omega_1]^{\omega}) \to [\omega_1]^{\omega}$ be a function. Let $t = (n; m; G; D)$ be a tuple-type. Let denote the club measure on ω_1. Let ${}^{t;k} : [\omega_1]^{\omega m} \to \omega_1$ be dened ${}^{t;k}(h) = (\mathrm{extract}(t; h))(k)$.

If for ${}^{\omega m}$-almost all h, ${}^{t;k}(h) < h(0)$, then for ${}^{\omega m}$-almost all h, ${}^{t;k}(h)$ take a constant value $c^{;t}_{k}$.

Proof. This follows from the countable additivity of ${}^{\omega m}$.

Denition 4.11. Assume the setting of fact 4.10. Let $d^{;t}$ be the least k if it exists so that ${}^{t;k}(h) \geq h(0)$ for ${}^{\omega m}$-almost all h. Otherwise, let $d^{;t} = \omega$.

Let $\mathrm{stem}^{;t} : d^{;t} \to \omega_1$ be dened by $\mathrm{stem}^{;t}(j) = c^{;t}_{j}$, where $j < d^{;t}$

Thus for ${}^{\omega m}$-almost all h, $\mathrm{stem}^{;t} \subseteq (\mathrm{extract}(t; h))$ and if $d^{;t} < \omega$, then $(\mathrm{extract}(; t))(d^{;t}) \geq h(0)$.

Theorem 4.12. $[\omega_1]^{\omega}$ is Jonsson.

Proof. A slightly stronger version of the Jonsson property will be shown: Let $: {}^{<\omega}([\omega_1]^{\omega}) \to [\omega_1]^{\omega}$.

Using AC^{R}_{ω} and the discussion in Denition 4.11, for each (of the countably many) eventually periodic tuple-type t, let $C_t \subseteq \omega_1$ be a club so that for all $h \in [C_t]^{\omega}$, $\mathrm{stem}^{;t} \subseteq (\mathrm{extract}(t; h))$ and if $d^{;t} < \omega$, then $(\mathrm{extract}(; t))(d^{;t}) \geq h(0)$.

Let be the supremum of $\sup(\mathrm{stem}^{;t})$ as t ranges over the countable set of eventually periodic tuple-types. As ω_1 is regular, $< \omega_1$. Let C be the intersection of all C_t as t ranges over all eventually periodic tuple-type. By removing an initial segment of C, one may assume that $< \min(C) + 1$.

Let $H : L \to C$ be any order-preserving function of the correct type. Note that ${}^{H}(f) \in [\omega_1]^{\omega}$, i.e. is also a function of the correct type for any $f \in [\omega_1]^{\omega}$.

Let $Z = {}^{H}[[\omega_1]^{\omega}]$. Since ^{H} is an injection by Lemma 4.8, $Z \subseteq [\omega_1]^{\omega}$.

Now suppose $f = (f_0; :::; f_n\ _1)\ 2\ {}^{<!}Z$. By Lemma 4.8, $t =$ type$(f) = (n; m; G; D)$ is an eventually periodic tuple-type. There is a unique $h\ 2\ [C]^{!m}$ so that extract$(t; h) = f$. In particular, since $h\ 2\ [C_t]^{!m}$, stem$^{;t}$ (f) and if $d^{;t} < !$, $(f)(d^{;t})$ $h(0)$ min$(C) >$. This and the denition of imply that 2 rang((f)).

It has been shown that for all $f\ 2\ {}^{<!}Z$, 2 rang((f)). In particular, $[{}^{<!}Z] = [!_1]^!$.

As was arbitrary, this implies that $[!_1]^!$ is Jasson.

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