Using textbook word problems to enhance mathematical reasoning and problem solving


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In this article the authors describe how standard textbook questions can be turned into open questions to promote problem solving and reasoning. Six example solutions are given to one problem showing how an open problem can cater to the diversity of students in your class.

Introduction

Mathematics education researchers and classroom teachers have been focused on mathematical problem solving as a main goal of school mathematics for decades. From Polya's *How to Solve It* (1945) to present, thousands of mathematics education researchers and even more practitioners worldwide have studied and struggled with ways to help students become more confident and able with their skill at applying mathematics to solve problems. This article describes one strategy for adapting common classroom practice in the form of textbook word problems to engage students in problem solving activities along with an example.

Open-ended problems and problem solving

A traditional approach to include problem solving in the school classroom has been the use of textbook word problems. Traditionally, students learn new mathematics and mathematical techniques, practice these skills through exercises, and then are assigned a few word problems from the textbook which help them apply those skills. But often those problems are unrealistically prescribed with all the needed information and only the needed information included. Furthermore, perhaps believing that students struggle with and dislike these word problems, teachers may opt to teach the students a set of procedural steps to complete to arrive at the needed answers. In a sense, the word problems have become simple skill exercises similar to the exercises practiced earlier in the textbook section. Students are not required to think critically by, for example, evaluating a variety of solution pathways. They simply mimic a set of steps and, as a result, their mathematical reasoning is not enhanced.

Open-ended tasks in the mathematics classroom are commonly considered those tasks where there are multiple approaches and sometimes even multiple correct answers. Students are given the opportunity to openly explore some mathematics problem using a multiplicity of strategies and then given flexibility in how they solve, display, and explain their solutions. Much discussion and research has centred around the use of these tasks with both younger students (Boaler, 1998; Varygiannes, 2014; Munroe, 2015) and older students (Sanchez, 2013; Sole, 2018). Although many traditional mathematics word problems may have only one correct answer, there may be ways to rewrite or repose the question to make it less procedural and to allow for more flexibility, engagement, and participation in the solution process.

Mathematics educators have long been aware of students' struggles with applying mathematics to real contexts to solve problems. The international PISA (Program for International Student Assessment) 2018 results (Organisation for Economic Cooperation and Development (OECD), 2019) note that only slightly more than half of those students tested (53.8%) have sufficient skills for "building a simple model or for selecting and applying simple problem-solving strategies." And only 10.9% can "develop and work with models for complex situations, identifying constraints and specifying assumptions." Boaler and Selling's longitudinal study (2017) compared two groups of students: one group which was taught mathematics traditionally and one which learned mathematics using a problem-solving approach. Eight years later the students who experienced the problem-based instruction were more likely to be in more highly technical and skilled jobs while the others were less confident with their abilities to use mathematics in their careers. Much recent writing has helped teachers explore how to integrate more problem solving and modelling activities in the classroom including work by Brown (2021) and Galbraith and Holton (2018). Such exposure to problem-solving in real-world contexts is crucial for preparing students to be successful in their future endeavours.

One example: The classic 'pipe around a corner' problem

One classic optimisation problem from an early calculus class is the 'pipe around a corner' problem. One key strategy of such an activity is to choose a problem which has multiple solution approaches and match with a variety of student backgrounds and preparation. This calculus problem can also easily be solved in an algebra, geometry, trigonometry, or pre-calculus class or by using an exploratory approach with graphing technology. The standard problem typically is stated like this:

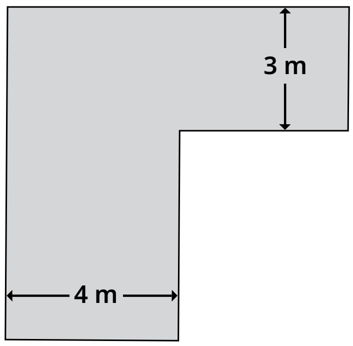


Figure 1. Available hallway space for 'pipe around a corner' problem.

A pipe is being carried horizontally down a 4-metre wide hallway. At the end of the hallway there is a 90° turn into a 3-metre wide hallway. What is the length of the longest pipe which can be carried horizontally through the hallway? (See Figure 1.)

The statement of this textbook word problem is simplistic. The situation is described with all the measures given and the question is posed so that the mathematics can be completed. Yet it may not be very engaging to students who likely don't need to carry a pipe through a hallway. One first adjustment that could be made for classroom presentation is to change the context by determining what might be interesting to students in the school. Does the theatre tech crew need to carry part of a stage set through the hallway? Does the marching band or cheer-leading team need to carry a large banner? The problem can be posed in a way that is more interesting to students without initially including the measurements. A more relatable context can better engage the students in seeking out the needed information and exploring a method of solution.

In addition to presenting the problem in a more relatable context, having the students move into the hallway and take measurements themselves can help to make the problem even more realistic and engaging. Once students are challenged with the task of carrying the item (which we'll continue to call a pipe) through the hallway, they should be asked to determine where the main obstacles might be; thus, in addition to "problem solving", students are responsible for "problem identification", a layer of complexity that requires higher-order thinking. Although identifying the tightest corners may be intuitive to the teacher, some students may need to actually take a walk through the hallways to think about this. The next task is to measure the hallways (the hallway entering the turn and the hallway after the turn). If possible, the teacher could provide string, measuring tapes, or other materials to allow students to estimate the maximum length of the pipe that will fit.

Although the task is to find the longest pipe that will fit, the problem is often solved by finding the minimum distance through which the pipe will need to fit, so this may be a good time to explore that while standing in the hallway. Scaffolding may be required to help students recognise that the maximum pipe length cannot exceed the minimum distance around the corner. This also suggests that the minimum distance around the corner and the maximum pipe length have the same value. Students may solve for the maximum pipe length or the minimum distance around the corner to solve the problem, but different solution methods are required depending on which variable they are trying to find. Spending time recognising that the problem can be solved by finding either the maximum pipe length or the minimum distance around the corner can help support students as they look for a possible solution method and develop their problem identification skills. At this point students have become involved in the context of the problem, have thought about and determined necessary measurements, and have collected measurements to consider reasonable answers. It is time to mathematise the problem by creating a model to represent the problem and then explore methods of solution.

Another alteration to the presentation of this problem to make it both realistic and engaging is to justify the constraints given in the problem. Although these constraints are given to make the problem more straightforward and limit ambiguity for students, sometimes the constraints seem counterproductive to a real-world context. One constraint that some students may recognise is that carrying the pipe completely horizontally means that the maximum pipe length is going to be shorter than if the pipe is carried at an angle. Teachers will recognise that this is true but allowing tipping makes the problem more complex. Depending on the classroom, solving this problem without the constraint of carrying the pipe horizontally may provide an interesting extension, but for many classrooms this approach may be outside the scope of the course. In order to make this problem more realistic, suggesting that the pipe is too heavy to be carried and tilted around the corner and needs to be rolled on carts supporting each end provides a more authentic representation of the constraint. For the solution methods provided, this constraint is assumed.

Possible solutions

Although there is only one correct final answer for any given corner in the school building, this particular problem has a richness of solution methods depending on the background of students and student choice of strategies and some of these are summarised below. Again, the goal is to encourage students to create their own solution method. One approach is to split the students into groups (3 to 5 students) and give them the first task of drawing the picture to illustrate the situation similar to the one in Figure 1 with all the dimensions labelled (the dimensions given in the original problem statement will be used in this writing for illustration purposes). Then the student groups can brainstorm about how to proceed.

An initial goal may be to find an expression for the length of the pipe P which then needs to be maximised (again, the minimum distance through which the pipe will need to fit) and the teacher may pose this as a scaffolding question if needed. The amount of scaffolding and guided discourse involved will depend on students' experience with such types of mathematical reasoning. As student problem solving skills develop with continued experience with this kind of true problem and as student confidence grows, it is the goal to provide as little direction as possible to maximise student exploration and reasoning development. Multiple solution paths may evolve as follows.

Solution 1: An algebra student who has not yet experienced trigonometry or calculus might approach this problem using Cartesian coordinates and the Pythagorean Theorem as shown in Figure 2.

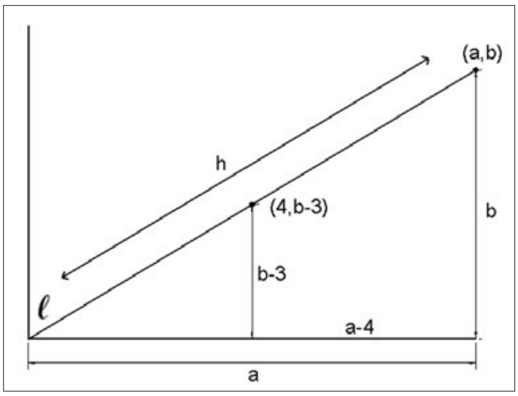


Figure 2. Pipe problem using Cartesian coordinates.

The line sketched on the coordinate plane is the pipe P. The origin of the coordinate plane (0, 0) is where the pipe hits the left wall. The pipe hits the corner at (4, b–3) when the pipe meets the opposite wall at the point (a, b). We want to find the length of the pipe P which is represented by length h in the diagram. The work below eliminates one of the unknowns (a) by writing it in terms of b using knowledge about the slope m of the line.

$$m = \frac{b}{a} = \frac{b-3}{4}$$
$$4b = ab - 3a = a(b-3)$$
$$\frac{4b}{b-3} = a$$

Now using the Pythagorean Theorem on the right triangle with the vertices (0, 0), (a, 0), and (a,b) and substituting for a yields:

$$h^2 = a^2 + b^2$$
$$h^2 = \left(\frac{4b}{b-3}\right)^2 + b^2$$
$$h^2 = \frac{16b^2}{(b-3)^2} + b^2$$
$$h(b) = \sqrt{\frac{16b^2}{(b-3)^2} + b^2}$$

The graph of this function can be found using technology (Figure 3) and the minimum is approximated as b ≈ 6.634 m and the length of the pipe h(b) = 9.866 m. Thus, a pipe of 986 centimetres can be carried through the hallway (to the nearest possible centimetre).

Solution 1b: Of course, rather than graphing the above function, this function can also be optimised using calculus with the more advanced student. In this case, the solution is elementary but a bit tedious for high school students as it requires finding the root of the derivative either by factoring a cubic or using graphing technology to estimate:

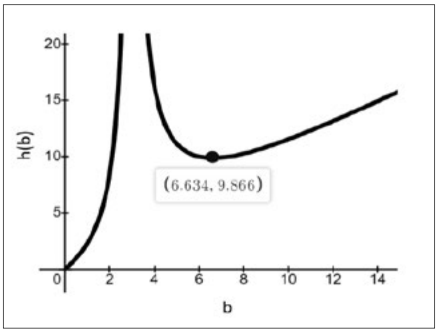


Figure 3. Graph of h(b).

$$h = \sqrt{\frac{16b^2}{(b-3)^2} + b^2}$$
$$= \sqrt{\frac{16b^2}{(b-3)^2} + \frac{b^2(b-3)^2}{(b-3)^2}}$$
$$= \frac{b}{b-3}\left(16 + (b-3)^2\right)^{\frac{1}{2}}$$
$$\frac{dh}{db} = \frac{b}{b-3}\left(\frac{1}{2}\right)\left(16 + (b-3)^2\right)^{-\frac{1}{2}}\left(2(b-3)\right) + \left(16 + (b-3)^2\right)^{\frac{1}{2}}\left(\frac{-3}{(b-3)^2}\right)$$
$$= \frac{b}{\sqrt{16 + (b-3)^2}} + \frac{-3\left(\sqrt{16 + (b-3)^2}\right)}{(b-3)^2}$$

These denominators are zero only at *b* = 3 which is unreasonable for the pipe and the given hallway. Getting a common denominator and setting the numerator equal to zero yields:

$$b(b-3)^2 - 3\left(16 + (b-3)^2\right) = 0 \text{ or}$$
$$b^3 - 9b^2 + 27b - 75 = 0$$

This leads to an opportunity to use The Rational Root Theorem and Descartes' Rule of Signs to analyse and classify the roots to this cubic. A graph of this cubic (Figure 4) yields the root $b \approx 6.634$ and plugging this into our function h(b) from above gives the similar estimate of $h(b) \approx 9.866$ m for the length of the pipe.

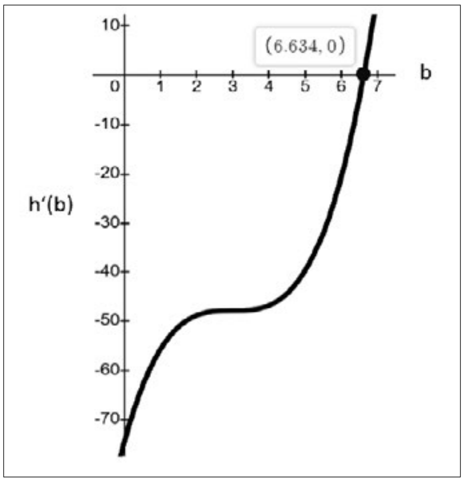
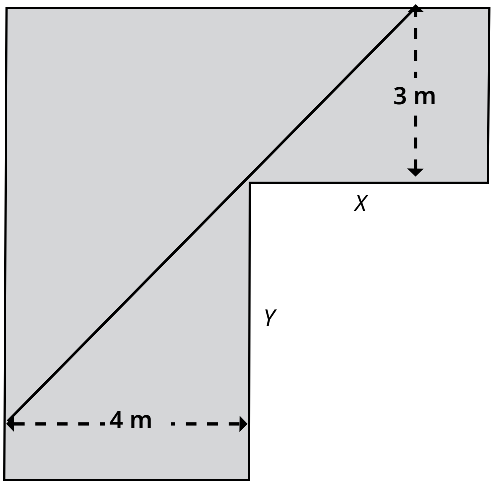


Figure 4. Estimating root of the derivative h'(b).

Solution 2: Geometry students might solve this problem by setting up similar triangles in the following way.



Here the following proportion can be set up from the similar triangles:

$$\frac{4}{y} = \frac{x}{3}$$
$$xy = 12$$
$$y = \frac{12}{x}$$

Using the Pythagorean Theorem twice, the length of the pipe then becomes:

$$P(x) = \sqrt{\left(\frac{12}{x}\right)^2 + 16} + \sqrt{x^2 + 9}$$

Figure 5. Similar triangle approach.

The graph of this function yields the optimised solution as before:

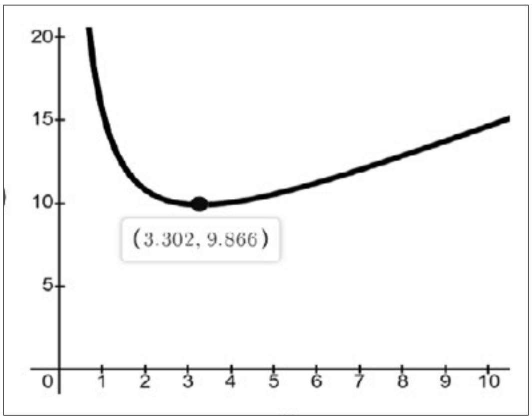


Figure 6. Graph of P(x).

Solution 2b: Again, Solution 2 can be completed using a calculus approach:

$$P = \sqrt{\left(\frac{12}{x}\right)^2 + 16} + \sqrt{x^2+9}$$
$$= \left(\left(\frac{12}{x}\right)^2 + 16\right)^{\frac{1}{2}} + (x^2+9)^{\frac{1}{2}}$$

and $$\frac{dP}{dx} = \frac{1}{2}\left(\left(\frac{12}{x}\right)^2 + 16\right)^{-\frac{1}{2}}\left(2\left(\frac{12}{x}\right)\left(-\frac{12}{x^2}\right)\right) + \frac{1}{2}(x^2+9)^{-\frac{1}{2}}(2x)$$

This derivative is only undefined at *x* = 0 (which again makes no sense in this context). Setting this expression equal to zero yields:

$$\frac{-144(\sqrt{x^2+9}) + x^4\sqrt{\frac{144+16x^2}{x^2}}}{x^3\sqrt{\frac{144}{x^2+16}}\sqrt{x^2+9}} = 0$$

or $$144\sqrt{x^2+9} = x^3\sqrt{144+16x^2} = x^3\sqrt{16(9+x^2)} = 4x^3\sqrt{9+x^2}$$

Setting the first and last expression equal to each other yields:

$$x^3 = 36 \text{ or } x \approx 3.3019$$

Substituting this into the equation above gives $P \approx 9.866$ m.

Solution 3: Another possible solution path involves consideration of the angle theta which is formed between the pipe and one wall and using triangle trigonometric ratios to write length $P = A + B$ in terms of theta as shown in Figure 7.

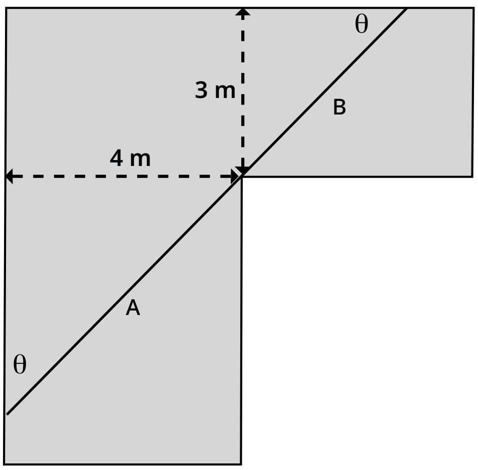


Figure 7. Hallway with pipe and angle θ.

$$\sin\theta = \frac{4}{A} \rightarrow A\sin\theta = 4 \rightarrow A = \frac{4}{\sin\theta} = 4\csc\theta$$
$$\cos\theta = \frac{3}{B} \rightarrow B\cos\theta = 3 \rightarrow B = \frac{3}{\cos\theta} \rightarrow = 3\sec\theta$$
$$P = A + B = \frac{4}{\sin\theta} + \frac{3}{\cos\theta} = 4\csc\theta + 3\sec\theta$$

P represents the minimum distance around the corner that the pipe will need to pass through or the place of "tightest fit". This function can then be graphed on a graphing device to approximate the minimum point as in Figure 8.

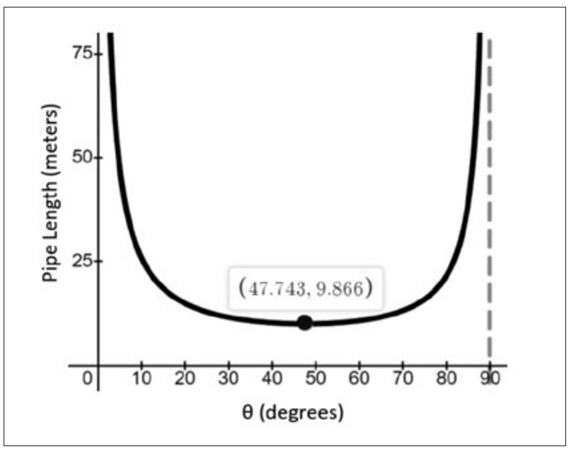


Figure 8. Graph of P(θ).

So, it appears from the graph as if the minimum distance around the corner and therefore the largest pipe *P* that can be carried through the hallway has measure approximately $P = 9.866$ m.

Solution 3b: Again, an alternate path for those with some calculus background is to create the same diagram and function as in Solution 3 but to minimise that function on a reasonable interval using calculus techniques. The solution below does just that.

$$P = A + B = 4\csc\theta + 3\sec\theta$$
$$\frac{dP}{d\theta} = -4(\csc\theta)\cot\theta + 3(\sec\theta)\tan\theta$$

$$\frac{-4\cos\theta}{\sin^2\theta} + \frac{3\sin\theta}{\cos^2\theta} = 0\ (\sin^2\theta \neq 0, \cos^2\theta \neq$$
$$-4\cos^3\theta + 3\sin^3\theta = 0$$
$$\sin^3\theta = 4\cos^3\theta$$
$$\frac{\sin^3\theta}{\cos^3\theta} = \frac{4}{3}$$
$$\tan^3\theta = \frac{4}{3}$$
$$\tan\theta = \sqrt[3]{\frac{4}{3}} \approx 1.10064$$
$$\theta \approx \tan^{-1}(1.10064) \approx .8333 \text{ radians}$$
$$P \approx 4\csc(0.8333) + 3\sec(0.8333)$$
$$= \frac{4}{\sin(0.8333)} + \frac{3}{\cos(0.8333)}$$
$$\approx 9.866 \text{ metres}$$

This approach again looks for the solution by minimising the distance around the corner and using that information to inform the maximum pipe length. So, the largest pipe which can be carried through the hallway would be about 9.866 m or 986 centimetres (to the nearest possible centimetre).

Other possibilities and summary

The previous six solutions in no way present all the possibilities of how the students might approach this problem. For example, the original diagram presumed that the hallway turns to the right. If the student considers a hallway turning to the left, the diagrams, angles, and Cartesian coordinate equations would vary accordingly. Rather than considering the angle that the pipe makes with one wall, the student might consider the angle which is complementary to that (the angle that the pipe makes to a line perpendicular to a wall) and the solution process will change. The richness of this problem lies in the multiple solution strategies and choices which allow each group of students to approach it in their own way. In addition, the students may need to carry the pipe around multiple corners (e.g. a 4 m x 3 m corner followed by a 4.5 m x 2.5 m corner) which could lead to some rich extensions.

Other extensions might include conversations about what might happen if the pipe is tilted (as discussed previously) or how the actual dimensions of the pipe (or other object) might change the solution. Students may recognise that many pipes, depending on the material, may have some degree of slack or bending and therefore could make it around a tighter corner. Conversely, it may be beneficial to emphasise that if the pipe is completely rigid, having the pipe length exactly the same length as the minimum distance around the corner may result in the pipe scratching the walls. This leads to a discussion of how to eliminate this issue without reducing the length of the pipe more than is needed to account for buffer space. And finally, students should compare their solutions with their estimates from when they originally measured in the hallway at the beginning of the activity.

A valuable part of the solution process is to have groups share their solutions during a debriefing session so that students experience that real mathematics problems often don't have only one correct method of solution.

The success of activities as described above depend on the teacher's ability to facilitate the classroom mathematics community and process student collaboration during group-work. Recent work related to this by Tabach and Schwarz (2018), Haataja, et al. (2021), and Hofmann and Mercer (2016) provide insights and demonstrate that this is an ongoing area of study. Practical guides such as *Strength in Numbers: Collaborative Learning in Secondary Mathematics* (Horn, 2012) can be good resources for classroom implementation strategies. Likewise, Smith, Steele, and Sherin's *The 5 Practices in Practice: Successfully Orchestrating Mathematics Discussions in Your High School Classrooms* (2020) provides suggestions to help the teacher facilitate the group conversations as well as the debriefing process. These are necessary skills which the teacher can grow and perfect as the group problem-solving tasks are implemented over time.

Many classic textbook word problems can be adjusted to create more engaging and authentic activities for students. The goal is that students are given the opportunity to work on problem solving in this way where all tasks and steps are not defined but they are supported to reason and explore mathematically to create models and approaches they understand. When students are routinely given these rich experiences, initially with much scaffolding and guidance and later with less, they can become better at seeing mathematical applications and gain confidence in their ability to use mathematics to explore real problems.

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