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Incarnations of XXX sl_N Bethe ansatz equations and integrable hierarchies

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In memory of Boris Dubrovin (1950-2019)

Abstract. We consider the space of solutions of the Bethe ansatz equations

of the sl_N XXX quantum integrable model, associated with the trivial representation of sl_N. We construct a family of commuting flows on this space and identify the flows with the flows of coherent rational Ruijesenaars-Schneider systems. For that we develop in full generality the spectral transform for the rational Ruijesenaars-Schneider system.

Contents

1. Introduction

In the Gaudin model associated with a Lie algebra one considers a commutative family of linear operators (Hamiltonians) acting on a tensor product of representations of the Lie algebra. To find common eigenvectors of Hamiltonians one considers a suitable system of Bethe ansatz equations, and then assigns an eigenvector to each solution of the system. That construction is called the Bethe ansatz method.

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It turns out that the set of solutions of the $\widehat{\ }$ Bethe ansatz equations is an interesting object. For example, for the affine Lie algebra sl_N and its trivial representation the associated system of the Bethe ansatz equations has the form $^{-1}$

$$\sum_{i'\neq i} \frac{2}{u_i^{(n)} - u_{i'}^{(n)}} - \sum_{i'=1}^{k_{n+1}} \frac{1}{u_i^{(n)} - u_{i'}^{(n+1)}} - \sum_{i'=1}^{k_n} \frac{1}{u_i^{(n)} - u_{i'}^{(n-1)}} = 0 \tag{1.1}$$
,

where $n = 1,...,N$ and $i = 1,...,k_n$. The system itself depends on the choice of nonnegative integers $k_1,...,k_N$, which must satisfy the equation

$$\sum_{j=1}^{N} \frac{(k_j - k_{j+1})^2}{2} - \sum_{j=1}^{N} k_j = 0 \tag{1.2}$$
.

Here we adopt the notations $k_{N+n} = k_n$ and $u_i^{(N+n)} = u_i^{(n)}$ for all i,n. The set of solutions of such a system forms one cell or an empty set. In [**VWr**] a family of commuting flows, acting on such a cell, was constructed. The family of flows was identified with the flows of the N mKdV integrable hierarchy.

The initial goal of this paper was to extend these results to the $\widehat{sl_N}$ XXX this case

the Bethe ansatz equations take the form

quantum integrable model, associated with the trivial representation ofsl_N. In

$$\begin{aligned} &\prod_{\ell=1}^{k_{n-1}} (u_i^{(n)} - u_\ell^{(n-1)} + 1) \prod_{\ell=1}^{k_n} (u_i^{(n)} - u_\ell^{(n)} - 1) \prod_{\ell=1}^{k_{n+1}} (u_i^{(n)} - u_\ell^{(n+1)}) \\ &+ \prod_{\ell=1}^{k_{n-1}} (u_i^{(n)} - u_\ell^{(n-1)}) \prod_{\ell=1}^{k_n} (u_i^{(n)} - u_\ell^{(n)} + 1) \prod_{\ell=1}^{k_{n+1}} (u_i^{(n)} - u_\ell^{(n+1)} - 1) = 0 \end{aligned} \tag{1.3}$$
,

where $n = 1,...,N$, $i = 1,...,k_n$, and the parameters $k_1,...,k_N$ still satisfy equation (1.2).

It turns out that we can do much more than just simple identification with a proper discrete analog of the N mKdV hierarchy. Roughly speaking we explicitly solve equations (1.3) using interplay with the theory of finite-dimensional integrable systems of particles, which are known to be equivalent to the theory of rational solutions of basic hierarchies considered in the framework of the theory of integrable partial differential, differential-difference and difference-difference equations. One way to write any solution of the Bethe ansatz equations (1.3) is to start with a suitable matrix y_n $\prod_{i=1}$ $-$ i $n=1$
A and write the polynomials ((x) = $^{k_i}(x\, u^{(n)}))^N$ as discrete Wronskians of some auxiliary

polynomials in x associated with A, see Theorem 7.9. Another way to write any solution is to start with a suitable flag in some infinite-dimensional vector space and write these polynomials ($y_n(x))_{n=1}^{N}$ as discrete Wronskians of some auxiliary polynomials in x associated with the flag, see Corollary 10.11.

In the remarkable paper [**AMM**] it was observed that the dynamics of poles of the elliptic (rational or trigonometric) solutions of the Korteweg–de Vries equation (KdV) can be described in terms of commuting flows of the elliptic (rational or trigonometric) Calogero-Moser (CM) system restricted to the space of stationary points of the CM system. In [**K3**] and [**K6**] this constrained correspondence between the theory of the elliptic CM system and the theory of the elliptic solutions of the KdV equation was extended to a similar construction of solutions of the KP equation in terms of the flows of the Calogero-Moser system. Moreover it was discovered for the first time that this correspondence of solutions can be established at the level of *auxiliary linear* problems.

In the rational case, which we consider in this paper, the corresponding result is as follows: the linear equation

(1.4) $$(\partial_t - \partial_x{}^2 + u(x,t))\psi(x,t) = 0$$

with a *rational in x potential* $u(x,t)$ vanishing as infinity, $u(x,t) \to 0$ as $x \to \infty$, has a rational in x solution if and only if the potential $u(x,t)$ is of the form

(1.5) $$u(x,t) = 2\sum_{i=1}^{k}(x-u_i(t))^{-2} = -2\partial_x^2 \ln y(x,t),$$

and its poles $u_i(t)$ (a.k.a. the zeros of the polynomial $y(x,t)$) as functions of t satisfy the equations of motion of the rational CM system.

Recall, that the rational CM system with k particles is a Hamiltonian system with coordinates $u = (u_1,...,u_k)$, momentums $p = (p_1,...,p_k)$, the canonical Poisson brackets $\{u_i,p_j\} = \delta_{ij}$, and the Hamiltonian

(1.6) $$H = \frac{1}{2}\sum_{i=1}^{k} p_i^2 + \sum_{i\neq j}\frac{1}{(u_i-u_j)^2}.$$

The corresponding equations of motion,

(1.7) $$u_i = 2\sum_{j\neq i}\frac{1}{(u_i-u_j)^3}, \qquad i = 1,\dots,k, \quad ..$$

admit the Lax presentation $\dot{L} = [M,L]$ with

(1.8) $$L_{ij} = p_i\delta_{ij} + 2\frac{1-\delta_{ij}}{u_i-u_j}, \qquad p_i = \dot{u}_i.$$

The commuting flows, generated by the integrals $H_k = k^{-1}\operatorname{tr}L^k$, are called the *hierarchy of the rational CM system*. Note that the Hamiltonian H equals H_2. It was shown in [**KZ**] that the linear equation

(1.9) $$\partial_t\psi(x,t) = \psi(x+1,t) + w(x,t)\psi(x,t)$$

with

$$w(x,t) = \partial_t \ln\left(\frac{y(x+1,t)}{y(x,t)}\right), \tag{1.10}$$

where $y(x,t)$ is a polynomial in x, has a solution rational in x if and only if the zeros $u_i(t)$ of $y(x,t)$ satisfy the equations of motion of the rational Ruijesenaars-Schneider (RS) system.

The rational RS system with k particles is a Hamiltonian system with coordinates $u = (u_1,...,u_k)$, momentums $p = (p_1,...,p_k)$, the canonical Poisson brackets $\{u_i,p_j\} = \delta_{ij}$, and the Hamiltonian

$$H(u,p) = \sum_{i=1}^{k} \gamma_i \tag{1.11}$$

where

$$\gamma_i := e^{p_i} \prod_{j\neq i} \left(\frac{(u_i - u_j - 1)(u_i - u_j + 1)}{(u_i - u_j)^2}\right)^{1/2}. \tag{1.12}$$

It is a completely integrable Hamiltonian system, whose equations of motion,

$$\dot{u}_i = \gamma_i, \qquad i = 1,...,k, \tag{1.13}$$

$$\dot{\gamma}_i = \sum_{j\neq i} \gamma_i\gamma_j \left(\frac{1}{u_i - u_j - 1} + \frac{1}{u_i - u_j + 1} - \frac{2}{u_i - u_j}\right), \tag{1.14}$$

admit the Lax representation $\dot{L} = [M,L]$, where

$$L_{ij}(u,\gamma) = \frac{\gamma_i}{u_i - u_j - 1}, \qquad i,j = 1,\dots,k, \tag{1.15}$$

$$M_{ij} = \left(\sum_{\ell\neq i} \frac{\gamma_\ell}{u_i - u_\ell} + \sum_{\ell} \frac{\gamma_\ell}{u_i - u_\ell + 1}\right)\delta_{ij} + (1-\delta_{ij})\frac{\gamma_i}{u_i - u_j}. \tag{1.16}$$

The functions $H_m = \mathrm{tr} L^m$ are integrals of the system. Note that the Hamiltonian H of the system equals H_1. These integrals are in involution, and hence generate commuting flows called the *rational RS hierarchy*.

A scheme, in which an integrable system of particles arises as a condition for a linear equation with elliptic (trigonometric, rational) coefficients to have a double Bloch solution (trigonometric, rational), was called in [**KZ**] a *generating linear problems scheme*.

The next step had been done in [**KLWZ**]. There the system of linear equations

$$\psi_{n+1}(x) = \psi_n(x+1) - v_n(x)\psi_n(x), \qquad n \in \mathbb{Z}, \tag{1.17}$$

with respect to unknown functions $(\psi_n(x))_{n\in\mathbb{Z}}$ was considered with

$$v_n(x) = \frac{y_n(x)y_{n+1}(x+1)}{y_n(x+1)y_{n+1}(x)},$$

where $(y_n(x))_{n\in\mathbb{Z}}$ is a given sequence of polynomials. It was shown that system (1.17) has a solution $(\psi_n(x))_{n\in\mathbb{Z}}$ rational in x with the poles of $\psi_n(x)$ only at the zeros of $y_n(x)$, if and only if the zeros $(u_i^{(n)})_{i=1}^{k_n}$ of $y_n(x)$ satisfy the Bethe ansatz equation (1.3).

We stress that in [**KLWZ**] the Bethe ansatz equations were considered for sequences of polynomials without the periodicity assumption that $y_n(x) = y_{n+N}(x)$ for some N.

Remark. In [**K7**] and [**K8**] all three linear problems with $y(x,t)$ being an *entire* function in x were used for the proof of the remarkable Welter's trisecant conjecture on the characterization of the Jacobians of smooth algebraic curves.

In this paper we apply these ideas to relate solutions of the N-periodic Bethe ansatz equations (1.3) with the equations of motion in the N-tuple of coherent rational Ruijesenaars-Schneider systems with respectively $k_1,...,k_N$ particles.

The paper is organized as follows. In Section 2 we reformulate the Bethe ansatz equations (1.3) and prove formula (1.2). In Section 3 we describe the procedure of generation of new solutions of the system of Bethe ansatz equations, if one solution is given. Theorem 3.4 says that all solutions are obtained from the single solution, namely, from the solution corresponding to the case of $k_1 = \cdots = k_N = 0$.

In Section 4 we start using the generating linear problem (1.17) and its interplay with two other generating linear problems. Having a solution of the Bethe ansatz equations we construct a family of solutions $(\psi_n(x,z))$ of (1.17) parameterized by a complex parameter z, see Theorem 4.2. The construction reveals an unexpected connection with the theory of the RS system. Namely, one of the steps in the proof of Theorem 4.2 can be seen as a map from the space of N-tuples of polynomials $(y_n(x))$ representing solutions of the Bethe ansatz equations to the product of N phase spaces of the rational RS systems with respectively $k_1,...,k_N$ particles, i.e. as the map

$$(y_n) \longmapsto (u^{(n)}, \gamma^{(n)}), \qquad n = 1, \dots, N, \tag{1.18}$$

where $\gamma_i^{(n)}$ are defined in (4.4). On each of these phase spaces we define commuting flows with some times $t = (t_1,t_2,...)$. That definition induces commuting flows with times t on the product of the phase spaces. One of our main results is the statement that the image of this map is invariant under these commuting flows on the product of the phase spaces, see Theorem 7.10.

In Section 5 we consider the functions $(\psi_n(x,z))$, constructed in Theorem 4.2, and study their analytic properties with respect to the *spectral* parameter z. In this way we identify the functions $(\psi_n(x,z))$ with a particular case of more general notion of the so-called Baker-Akhiezer functions. The results of Section 5 can be seen as a construction of the direct spectral transform for the rational RS system. To our surprise we were unable to find in the literature such a construction in its full generality.

The analogous result for the rational CM system was obtained in [**W**]. Our construction of the direct spectral transform is different from the one in [**W**]. It is pure algebraic and does not require the use of infinite dimensional Grassmanians, whose definition involves elements of real analysis, in particular, of the theory of Fredholm operators.

In Section 6 we write equations for zeros of the polynomials obtained by the construction of the Baker-Akhiezer functions corresponding to the spectral data of the rational RS systems.

In Section 7.1 we identify the spectral data corresponding to solutions of the N-periodic Bethe ansatz equations. The rest of Section 7 is on the inverse spectral transform. First we construct a family of solutions of the generating linear problem starting from a certain matrix A, see Theorem 7.4. That is done without any Nperiodicity assumptions. Then in Section 7.6 we describe the matrices A that give N-periodic answers. Theorem 7.9 can be seen as one of our main results.

For completeness in Section 8 we briefly present the integrable hierarchy, whose rational solutions describe the commuting flows on the space of solutions of the Bethe ansatz equations. We call it the *discrete N mKdV hierarchy*. Section 8.6 contains a short remark of discrete Miura opers.

Section 9 we discuss combinatorial data that will be used in Section 10. In Section 10 we identify solutions of the Bethe ansatz equations with points of a suitable infinite dimensional Grassmannian. We introduce a family of commuting flows on the Grassmannian and identify the flows induced on the space of solutions of the Bethe ansatz equations with the flows of the discrete N mKdV hierarchy, introduced in Section 8.

2. Incarnations of the Bethe ansatz equations

2.1. Bethe ansatz equations. Let $N > 2$ be a positive integer, $\vec{k} = (k_1, \dots, k_N) \in \mathbb{Z}^N_{\geqslant 0}$. Denote $k := k_1 + \cdots + k_N$. Consider $\mathbb{C}^k$ with coordinates u collected into N groups, the n-th group consists of k_n variables,

$$u = (u^{(1)}, \dots, u^{(N)}), \qquad u^{(n)} = (u_1^{(n)}, \dots, u_{k_n}^{(n)}).$$

We adopt the notations $k_{N+n} = k_n$ and $u_i^{(N+n)} = u_i^{(n)}$ for all i,n.

The *Bethe ansatz equations* is the following system of k equations:

$$\begin{aligned} (2.1)\qquad & \prod_{\ell=1}^{k_{n-1}} (u_i^{(n)} - u_\ell^{(n-1)} + 1) \prod_{\ell=1}^{k_n} (u_i^{(n)} - u_\ell^{(n)} - 1) \prod_{\ell=1}^{k_{n+1}} (u_i^{(n)} - u_\ell^{(n+1)}) \\ & + \prod_{\ell=1}^{k_{n-1}} (u_i^{(n)} - u_\ell^{(n-1)}) \prod_{\ell=1}^{k_n} (u_i^{(n)} - u_\ell^{(n)} + 1) \prod_{\ell=1}^{k_{n+1}} (u_i^{(n)} - u_\ell^{(n+1)} - 1) = 0 , \end{aligned}$$

where $n = 1,\dots,N$, $i = 1,\dots,k_n$.

These are the Bethe ansatz equations associated with the XXX quantum

intehighest weight. To study the associated Hamiltonians one assigns an eigenvector

grable model of the affine Lie algebra sl_N and the single representation with zero

of Hamiltonians to a solution of the Bethe ansatz equations. We will not discuss this topic in this paper. Different versions of the Bethe ansatz equations associated with Lie algebras see, for example in [**OW**,?MV2,**MV3**,**MV4**].

Remark. Equation (2.1) with $N = 2$ is the quasi-classical limit of the Bethe ansatz equations derived in [**AL**] for the Quantum Internal Long Wave model.

2.2. Polynomials representing a solution. Given $u = (u_i^{(n)}) \in \mathbb{C}^k$, introduce an N-tuple of polynomials $y = (y_1(x), \dots, y_N(x))$,

$$y_n(x) = c_n \prod_{i=1}^{k_n} (x - u_i^{(j)}), \qquad c_n \neq 0 . \tag{2.2}$$

We adopt the notations $y_{N+n}(x) = y_n(x)$ for any $n \in \mathbb{Z}$. Each polynomial is considered up to multiplication by a nonzero number. The N-tuple defines a point in the direct product $(\mathbb{P}(\mathbb{C}[x]))^N$, where $\mathbb{P}(\mathbb{C}[x])$ is the projective space associated with $\mathbb{C}[x]$. We say that the tuple y *represents* the point u.

We say that an N-tuple of polynomials $y = (y_1(x), \dots, y_N(x))$ is *generic* if for any n, the polynomial $y_n(x)$ has no common zeros with the polynomials $y_n(x+1)$, $y_{n-1}(x+1)$, $y_{n+1}(x)$. Denote

$$F_n(x) := \frac{y_{n-1}(x+1)y_{n+1}(x)}{y_n(x+1)y_n(x)}, \quad L_n(x) := \frac{y_n(x+1)y_{n+1}(x-1)}{y_n(x)y_{n+1}(x)} , \tag{2.3}$$

Lemma 2.1. Assume that an N-tuple of polynomials $y = (y_1(x), \dots, y_N(x))$ is *generic*. Then each equation in (2.1) can be reformulated as one of the following equations:

$$y_{n-1}(u_{(jn)} + 1)y_n(u_{(jn)} - 1)y_{n+1}(u_{(jn)}) \tag{2.4}$$

$$+y_{n-1}(u_{(jn)})y_n(u_{j(n)} + 1)y_{n+1}(u_{(jn)} - 1) = 0,$$

$$\mathrm{res}_{x=u_{i(n)}} (F_n(x) + F_n(x-1)) = 0, \tag{2.5}$$

$$\mathrm{res}_{x=u_i^{(n)}} (L_n(x) + L_{n-1}(x)) = 0 . \tag{2.6}$$

An important corollary of (2.6) is

Corollary 2.2. A generic N-tuple y represents a solution of the Bethe ansatz equations (2.1) if and only if the following equation holds:

$$L(x) := \sum_{n=1}^{N} L_n(x) = N. \tag{2.7}$$

This equation is a discrete version of "the new form" of the Bethe ansatz equations in the Gaudin model of an arbitrary Kac-Moody algebra, see [**MSTV**].

Proof. Equation (2.6) is equivalent to the condition that the function $L(x)$ defined in (2.7) has no poles. Each of the function $L_n(x)$ tends to 1 as $x \to \infty$. Hence, $L(x) = N$.

In its own turn Corollary 2.2 directly implies the following important statement. Consider the quadratic form

$$Q(k_1,\dots,k_N) = \sum_{j=1}^{N} k_j(k_j-1) - k_1k_2 - \cdots - k_{N-1}k_N - k_Nk_1$$
$$= \sum_{j=1}^{N} \frac{(k_j-k_{j+1})^2}{2} - \sum_{j=1}^{N} k_j\,,$$

introduced in [**MV3**].

Corollary 2.3. If a generic N-tuple of polynomials $(y_1,\dots,y_N)$ of degrees $(k_1,\dots,k_N)$ represents a solution of the Bethe ansatz equations (2.1), then

(2.8) $$Q(k_1,\dots,k_N) = 0.$$

Proof. Expanding at infinity, we observe that $L(x) - N = Q(k_1,\dots,k_N)x^{-2}$ $+O(x^{-3})$. □

Corollary 2.4. If a generic N-tuple of polynomials $(y_1,\dots,y_N)$ of degrees $k_1 = \cdots = k_N$ represents a solution of the Bethe ansatz equations (2.1), then $k_1 = \dots = k_N = 0$.

Remark. Equations (2.4), (2.5), (2.7) can be thought of as incarnations of the Bethe ansatz equations (2.1).

3. Generation of solutions of Bethe ansatz equations

3.1. Discrete Wronskian. For arbitrary functions $f_1(x),\dots,f_m(x)$ introduces the *discrete Wronskian* by the formula:

(3.1) $$\widehat{W}(f_1,\dots,f_m) = \det_{i,j=1}^{m}\big(f_i(x+j-1)\big).$$

For example,

$$\widehat{W}(f_1,f_2) = f_1(x)f_2(x+1) - f_1(x+1)f_2(x).$$

Denote

(3.2)
$$\Delta f(x) = f(x+1) - f(x),$$
$$\Delta^{(n+1)} f(x) = \Delta(\Delta^{(n)} f)(x), \quad \Delta^{(0)} f(x) = f(x).$$

Then

(3.3) $$\widehat{W}(f_1,\dots,f_n) = \det_{i,j=1}^{n}(\Delta^{j-1} f_i(x)).$$

Lemma 3.1 ([?MV2]). We have

(3.4) $$\widehat{W}(1,f_1,\dots,f_n)(x) = \widehat{W}(\Delta f_1,\dots,\Delta f_n).$$

Lemma 3.2 ([?MV2, Lemma 9.4]). For functions $f_1(x),\dots,f_n(x)$, $g_1(x),g_2(x)$ we have

(3.5) $$\widehat{W}(\widehat{W}(f_1,\dots,f_n,g_1),\widehat{W}(f_1,\dots,f_n,g_2))(x) = \widehat{W}(f_1,\dots,f_n)(x)\,\widehat{W}(f_1,\dots,f_n,g_1,g_2)(x+1).$$

3.2. Elementary generation. Recall that an N-tuple of polynomials $y = (y_1(x), ..., y_N(x))$ is called *generic* if for any n, the polynomial $y_n(x)$ has no common zeros with the polynomials $y_n(x+1), y_{n-1}(x+1), y_{n+1}(x)$.

We say that an N-tuple of polynomials $y = (y_1(x), ..., y_N(x))$ is *fertile*, if for any n the first order difference equation

(3.6) $\widehat{W}(y_n, \tilde{y}_n) = y_{n-1}(x+1)y_{n+1}(x)$ with respect to $\tilde{y}_n(x)$ has a polynomial solution.

If $\tilde{y}_n(x)$ is a polynomial solution of (3.6), then all other polynomials solutions are of the form

$$\tilde{y}_n(x,c) = \tilde{y}_n(x) + cy_n(x)$$

for $c \in \mathbb{C}$. The tuples

(3.7) $y^{(n)}(c) := (y_1(x),...,\tilde{y}_n(x,c),...,y_N(x)) \in (\mathbb{P}(\mathbb{C}[x]))^N$ form a one-parameter family. This family is called the *generation of tuples from y in the n-th direction*. A tuple of this family is called an *immediate descendant* of y in the n-th direction.

For example, the N-tuple

(3.8) $$y^{\emptyset} = (1,...,1)$$

of constant polynomials is fertile, and $y^{\emptyset,(n)}(c) = (1,...,1,x+c,1...,1)$.

It is convenient to think that $y^{\emptyset}$ represents a solution of the Bethe ansatz equations with $k = 0$, see (2.5).

Theorem 3.3 ([?MV2], cf. [**MV1**]).

(i) A generic tuple $y = (y_1,...,y_N)$ represents a solution of the Bethe ansatz equations (2.1) if and only if y is fertile.

(ii) Let y represent a solution of the Bethe ansatz equations (2.1), $n \in \{1,...,N\}$, and $y^{(n)}(c)$ an immediate descendant of y, then $y^{(n)}(c)$ is fertile for any $c \in \mathbb{C}$.

(iii) If y is generic and fertile, then for almost all values of the parameter $c \in \mathbb{C}$ the corresponding n-tuple $y^{(n)}(c)$ is generic. The exceptions form a finite set in $\mathbb{C}$.

3.3. Degree increasing generation. For $n = 1,...,N$, let $k_n = \deg y_n$.

The polynomial ˜that the *generation is degree increasing*y_n in (3.6) is of degree $k_n$if or$\tilde{k}_n k \tilde{>}_n = k_n k$. In that case deg $\tilde{}_{n-1} + k_{n+1} + 1 - yk_{nn}$. We say$= \tilde{k}_n$ for all c.

If the generation is degree increasing, we will normalize the family (3.7) and construct a map $Y_{y,n} : \mathbb{C} \to (\mathbb{C}[x])^N$ as follows. First we multiply the polynomials $y_1,...,y_N$ by numbers to make them monic. Then we choose a monic polynomial

$y_{n,0}(x)$ satisfying the equation^{k_n} in $\tilde{y}_{n,}\widehat{W}(y_n, y_{n,0}{}_0(x)$ equals zero. We define) = const $y_{n-1}(x+1)y_{n+1}(x)$ and such that the coefficient of x

(3.9) $$\tilde{y}_n(x,c) = y_{n,0}(x) + cy_n(x)$$

and

(3.10) $$\begin{aligned} Y_{y,n} : \mathbb{C} &\to (\mathbb{C}[x])^N, \\ c &\mapsto y^{(n)}(c) = (y_1(x), \dots, \tilde{y}_n(x,c), \dots, y_N(x)) \end{aligned}$$

All polynomials of the tuple $y^{(n)}(c)$ are monic.

3.4. Degree-transformations and generation of vectors of integers. For $j = 1,\dots,N$, the degree-transformation

(3.11) $$\begin{aligned} \vec{k} &= (k_1, \dots, k_N) \\ &\mapsto \vec{k}^{(j)} = (k_1, \dots, k_{j-1}, k_{j-1} + k_{j+1} - k_j + 1, k_{j+1}, \dots, k_N) \end{aligned}$$

corresponds to the shifted action of the affine reflection $w_j \in W_{A_{N-1}}$, where $W_{A_{N-1}}$ is the affine Weyl group of type A_{N-1} and $w_1,\dots,w_N$ are its standard generators, see Lemma 3.11 in [**MV1**] for more detail.

We take formula (3.11) as the definition of *degree-transformations*:

(3.12) $$\begin{aligned} w_j : \vec{k} &= (k_1, \dots, k_N) \\ &\mapsto \vec{k}^{(j)} = (k_1, \dots, k_{j-1} + k_{j+1} - k_j + 1, \dots, k_N) \end{aligned}$$

for $j = 1,\dots,N$. The degree-transformations act on arbitrary vectors $\vec{k} = (k_1,\dots,k_N)$.

In this formula we consider the indices of the coordinates modulo N, that is, we have $k_{N+j} = k_j$ for all j.

We start with the vector $\vec{k}^{\emptyset} = (0, \dots, 0)$ and a sequence $J = (j_1, j_2, \dots, j_m)$ of integers, $1\ j_i \leqslant N$. We apply the corresponding degree transformations to the vector $\vec{k}^{\emptyset}$ and obtain a sequence of vectors $\vec{k}^{\emptyset}$, $\vec{k}^{(j_1)} := w_{j_1}\vec{k}^{\emptyset}$, $\vec{k}^{(j_1,j_2)} := w_{j_2}w_{j_1}\vec{k}^{\emptyset}$,...,

(3.13) $$\vec{k}^J := w_{j_m} \dots w_{j_2} w_{j_1} \vec{k}^{\emptyset}.$$

We say that the *vector* $\vec{k}^J$ *is generated from* $(0,\dots,0)$ *in the direction of* J.

We call the sequence J *degree increasing* if for every i the transformation w_{j_i} applied to $w_{j_{i-1}} \cdots w_{j_1}\vec{k}^{\emptyset}$ increases the j_i-th coordinate.

3.5. Multistep generation. Let $J = (j_1,\dots,j_m)$ be a degree increasing sequence of integers. Starting from $y^{\emptyset} = (1,\dots,1)$ and J, we construct, by induction on m, a map

$$Y^J : \mathbb{C}^m \to (\mathbb{C}[x])^N.$$

If $J = \emptyset$, the map $Y^{\emptyset}$ is the map $\mathbb{C}^0 = (pt) \mapsto y^{\emptyset}$. If $m = 1$ and $J = (j_1)$, the map $Y^{(j_1)} : \mathbb{C} \to (\mathbb{C}[x])^N$ is given by formula (3.10). More precisely,

$$Y^{(j_1)} : \mathbb{C} \mapsto (\mathbb{C}[x])^N, \qquad c \mapsto (1, \dots, 1, x + c, 1 \dots, 1),$$

where $x + c$ stands at the j_1-th position. By Theorem 3.3 all tuples in the image are fertile and almost all tuples are generic (in this example all tuples are generic).

Assume that for J we apply the generation procedure in the $\tilde{J} = (j_1,...,j_{m-1})$, the map $Y^{\tilde{J}}$ is already constructed. To obtain Y^J we apply the generation procedure in the j_m-th direction to every tuple of the image of $Y^{\tilde{J}}$. More precisely, if

(3.14) $$Y^{\tilde{J}} \; : \; \tilde{c} = (c_1, \ldots, c_{m-1}) \; \mapsto \; (y_1(x, \tilde{c}), \ldots, y_N(x, \tilde{c})).$$

Then

(3.15) $$Y^J : \mathbb{C}^m \;\; \mapsto \;\; (\mathbb{C}[x])^N,$$
$$(\tilde{c}, c_m) \mapsto (y_1(x, \tilde{c}), \ldots, y_{j_m,0}(x, \tilde{c}) \;\; + \;\; c_m y_{j_m}(x, \tilde{c}), \ldots, y_N(x, \tilde{c})),$$

see formula (3.9). The map Y^J is called the *generation of N-tuples from $y^\emptyset$ in the J-th direction*.

All tuples in the image of Y^J are fertile and almost all tuples are generic. For any $c \in \mathbb{C}^m$ the N-tuple $Y^J(c)$ consists of monic polynomials. The degree vector of this tuple equals k^J, see (3.13).

The set of all tuples $(y_1,...,y_N) \in (\mathbb{C}[x])^N$ obtained from $y^\emptyset = (1,...,1)$ by generations in all degree increasing directions will be called the *population of N-tuples* generated from $y^\emptyset$.

3.6. Population generated from $y^\emptyset$.

Theorem 3.4 ([**MV4**]). If an N-tuple of polynomials $y = (y_1,...,y_N)$ with degree vector k represents a solution of the Bethe ansatz equations (2.1), then y is a point of the population generated from $y^\emptyset$ by degree increasing generations, that is, there exist a degree increasing sequence $J = (j_1,...,j_m)$ and a point $c \in \mathbb{C}^m$ such that $y = Y^J(c)$.

Moreover, for any other N-tuple y, representing a solution of the Bethe ansatz equations (2.1) and having the same degree vector k, there is a point $c' \in \mathbb{C}^m$ such that $y' = Y^J(c')$.

By Theorem 3.4 the N-tuples y, representing solutions of the Bethe ansatz equations (2.1) with the same degree vector k, form one cell $\mathbb{C}^m$.

The proof of Theorem 3.4 is word by word the same as the proof of [**MV5**, Theorem 3.8], although the generation procedure in [**MV5**] is slightly different from the generation procedure in this paper. The key point of the proof is the equality $Q(\vec{k}) = 0$, which is proved in Corollary 2.8 for our generation procedure and was proved in the proof of [**MV5**, Theorem 3.8]. See also the proof of [**VW1**, Theorem 6.4].

Remark. The condition of fertility of an N-tuple y can be also thought of as another incarnation of the Bethe ansatz equations (2.1), see Theorem 3.3.

4. Generating linear problem

4.1. Non-periodic sequences of polynomials. In this section we consider sequences of polynomials $y = (y_n(x))_{n\in\mathbb{Z}}$, not assuming that the sequences are N-periodic. Let

$$y_n(x) \;=\; c_n \prod_{i=1}^{k_n} (x - u_i^{(n)}), \qquad c_n \neq 0 .$$

The system of the *Bethe ansatz equations* in this case is the infinite system of equations:

$$(4.1)\qquad \prod_{\ell=1}^{k_{n-1}} (u_i^{(n)} - u_\ell^{(n-1)} + 1) \prod_{\ell=1}^{k_n} (u_i^{(n)} - u_\ell^{(n)} - 1) \prod_{\ell=1}^{k_{n+1}} (u_i^{(n)} - u_\ell^{(n+1)})$$
$$+ \prod_{\ell=1}^{k_{n-1}} (u_i^{(n)} - u_\ell^{(n-1)}) \prod_{\ell=1}^{k_n} (u_i^{(n)} - u_\ell^{(n)} + 1) \prod_{\ell=1}^{k_{n+1}} (u_i^{(n)} - u_\ell^{(n+1)} - 1) = 0 ,$$

where $n \in \mathbb{Z}$, $i = 1,\dots,k_n$.

We say that the sequence y is *generic* if for any n the polynomial $y_n(x)$ has no common zeros with the polynomials $y_n(x + 1), y_{n-1}(x + 1), y_{n+1}(x)$.

As in the periodic case the system of the Bethe ansatz equations (4.1) can be reformulated as the infinite system of equations (2.4), or equations (2.5), or equations (2.6).

Remark. Let the degrees $(k_n)_{n\in\mathbb{Z}}$ of the polynomials $(y_n(x))_{n\in\mathbb{Z}}$ be all equal. Then for each n system (4.1) can be regarded as a system of equations for $(u_i^{(n+1)})$ with $(u_i^{(n)})$ and $(u_i^{(n-1)})$ given. Hence, system (4.1) can be seen as a second order discrete time dynamical system. In such a form these equations were introduced in [**NRK**] as an integrable time-discretization of the Ruijesenaars-Schneider system, which in its turn was introduced as a relativistic analog of the Calogero-Moser (CM) system.

In [**KLWZ**] for system (4.1) the discrete time Lax representation with a ”spectral parameter” was found with the help of a ”generating linear problem”, see Theorem 6.1 in [**KLWZ**]. The Hamitonian approach for this system was developed in [**K6**].

Proof.

Notice that the case of all $(k_n)_{n\in Z}$ being equal is not allowed in the periodic case by Corollary 2.4. This fact can be interpreted as the statement that *the timediscretization of the Ruijesenaars-Schneider system has no periodic orbits.*

Given a generic sequence of polynomials $y = (y_n(x))_{n\in Z}$ the associated *generating linear problem* is the infinite system of equations

$$\psi_{n+1}(x) = \psi_n(x+1) - v_n(x)\psi_n(x), \qquad n \in Z, \tag{4.2}$$

with respect to the unknown sequence of functions $\psi = (\psi_n(x))_{n\in Z}$ with $v = (v_n(x))$ given by the formulas

$$v_n(x) = \frac{y_n(x)\, y_{n+1}(x+1)}{y_n(x+1)\, y_{n+1}(x)}. \tag{4.3}$$

We say that a solution $\psi = (\psi_n(x))_{n\in Z}$ of system (4.2) is *admissible* if for any n the function $y_n(x)\psi_n(x)$ is holomorphic. Define the nonzero numbers

$$\gamma_i^{(n)} := \operatorname{res}_{x=u_i^{(n)}-1} v_n(x) = \frac{y_n(u_i^{(n)}-1)\, y_{n+1}(u_i^{(n)})}{\prod_{j\neq i}(u_i^{(n)}-u_j^{(n)})\, y_{n+1}(u_i^{(n)}-1)}, \tag{4.4}$$

where $i = 1,\ldots,k_n$, and nonzero numbers

$$\varepsilon_i^{(n)} := \operatorname{res}_{x=u_i^{(n+1)}} v_n(x) = \frac{y_n(u_i^{(n+1)})\, y_{n+1}(u_i^{(n+1)}+1)}{y_n(u_i^{(n+1)}+1)\prod_{j\neq i}(u_i^{(n+1)}-u_i^{(n+1)})}, \tag{4.5}$$

where $i = 1,\ldots,k_{n+1}$.

Lemma 4.1. The infinite system of equations

$$\gamma_i^{(n+1)} + \varepsilon_i^{(n)} = 0, \qquad n \in \mathbb{Z}, \quad i = 1,\ldots,k_{n+1}, \tag{4.6}$$

is equivalent to the infinite system of equations (2.4).

In its turn the property of the infinite system of equations (2.4) to have a solution y is equivalent to the property of y to represent a solution of the Bethe ansatz equations (4.1), see Lemma 2.1.

Theorem 4.2. Let $y = (y_n(x))_{n\in Z}$ be a generic sequence of polynomials. Then the system of equations (4.2) has an admissible solution $\psi = (\psi_n(x))_{n\in Z}$ if and only if y represents a solution of system (4.1). Moreover, if a generic sequence y represents a solution of system (4.1), then there exists a unique one-parameter family $\Psi(z) = (\Psi_n(x,z))$ of admissible solutions of system (4.2), which has the form

$$\Psi_n(x,z) = z^n(1+z)^x\left(1+\sum_{i=1}^{k_n}\xi_i^{(n)}(x)z^{-i}\right), \qquad n \in \mathbb{Z}, \tag{4.7}$$

where $\xi_i^{(n)}(x)$ are rational functions in x such that all the functions $y_n(x)\,\xi_i^{(n)}(x)$ are holomorphic in x.

Remark. The first statement of the theorem is an analog of Lemma 5.1 in [**K8**], and the second statement is a stronger version of Lemma 5.2 in [**K8**].

Remark. The equivalence in Theorem 4.2 of the existence of an admissible solution ψ of system (4.2) and the property of y to represent a solution of system (4.1) may be thought of as another incarnation of the Bethe ansatz equations.

Let ψ be an admissible solution of the generating linear problem equation (4.2). For any $n \in \mathbb{Z}$ and $i = 1,\dots,k_n$, consider the Laurent expansion of $\psi_n(x)$ at $x = u_i^{(n)}$,

(4.8)
$$\psi_n(x) = \frac{\alpha_i^{(n)}}{x - u_i^{(n)}} + \mathcal{O}(1)\,, \qquad \alpha_i^{(n)} \in \mathbb{C}.$$

The comparison of the residues of the left and right-hand sides of equation (4.2) at $x = u_i^{(n)} - 1$ and $x = u_j^{(n+1)}$ gives us the equations

(4.9) $\alpha_i^{(n)} = \gamma_i^{(n)}\,\psi_n(u_i^{(n)} - 1)$, (4.10) $\alpha_j^{(n+1)} = -\varepsilon_j^{(n)}\,\psi_n(u_j^{(n+1)})$,

respectively. We obtain the third set of equations

(4.11)
$$\psi_{n+1}(u_j^{(n+1)} - 1) = \psi_n(u_j^{(n+1)})\,, \qquad j = 1,\dots,k_{n+1},$$

by substituting $x = u_j^{(n+1)} - 1$ to equation (4.2) and taking into account that $v_n(u_j^{(n+1)} - 1) = 0$. Shifting the index $(n,i) \to (n+1,j)$ in (4.9) we obtain

(4.12)
$$\alpha_j^{(n+1)} = \gamma_j^{(n+1)}\,\psi_{n+1}(u_j^{(n+1)} - 1).$$

Using (4.10), (4.11), (4.12) we obtain equations $\gamma_i^{(n+1)} + \varepsilon_i^{(n)} = 0$ for $n \in \mathbb{Z}$ and $j = 1,\dots,k_{n+1}$, which are equations (4.6). By Lemma 4.1 this means that the sequence y represents a solution of the Bethe ansatz equations (4.1). That proves the "only if" part of the first statement of the theorem.

Now the goal is to construct the family $\psi(z)$ of admissible solutions of (4.2) assuming that y is generic and represents a solution of (4.1). The construction has two steps. First, we construct a certain sequence of functions $\psi(z)$ by using the generic y, but not using the fact that y satisfies (4.1). Then we prove that $\psi(z)$ has the form (4.7) and is a solution of (4.2), if y represents a solution of (4.1).

Lemma 4.3. Let y be a generic sequence of polynomials. Then for $n \in \mathbb{Z}$ there exists a unique function $\psi_n(x,z)$ of the form

Proof.

$$\psi_n(x,z) = z^n(1+z)^x\left(1+\sum_{i=1}^{k_n}\frac{C_i^{(n)}(z)}{x-u_i^{(n)}}\right) \tag{4.13}$$

such that the function

(4.14) $\phi_n(x,z) := \psi_n(x+1,z) - v_n(x)\psi_n(x,z)$ has no residues at

$x = u_i^{(n)} - 1$ for all $i = 1,\ldots,k_n$,

$$\mathrm{res}_{x=u_i^{(n)}-1}\,\varphi_n(x,z) = 0. \tag{4.15}$$

Remark. Notice that $C_i^{(n)}(z)$ are some functions in z. The proof shows that $C_i^{(n)}(z)$ are rational functions in z.

Remark. Notice that $\phi_n(x,z)$ would be equal to $\psi_{n+1}(x,z)$ if the sequence $(\psi_n(x,z))$ were a solution of the system of the generating linear problem equations (4.2).

Proof.

By (4.13) the function $\psi_n(x,z)$ is regular at $x = u_i^{(n)} - 1$. We also have

(4.16) $$\operatorname{res}_{x=u_i^{(n)}-1} \psi_n(x+1,z) = \operatorname{res}_{x=u_i^{(n)}} \psi_n(x,z).$$

Hence, equation (4.15) is equivalent to the equation

(4.17) $$\operatorname{res}_{x=u_i^{(n)}} \psi_n(x,z) - \gamma_i^{(n)} \psi_n(u_i^{(n)} - 1,z) = 0.$$

Let $C^{(n)}(z)$ be the k_n-vector with coordinates $C_i^{(n)}(z)$ appearing in (4.13). Let $\gamma^{(n)}$ be the k_n-vector with coordinates $\gamma_i^{(n)}$. Let $L^{(n)}(z)$ be the $k_n \times k_n$-matrix with entries

(4.18) $$L_{ii}^{(n)}(z) = 1 + z + \gamma_i^{(n)}, \qquad L_{ij}^{(n)}(z) = \frac{-\gamma_i^{(n)}}{u_i^{(n)} - u_j^{(n)} - 1}, \qquad i \neq j\,.$$

Then the substitution of (4.13) into (4.17) gives an inhomogenous linear equation

(4.19) $$L^{(n)}(z)C^{(n)}(z) = \gamma^{(n)}$$

with respect to $C^{(n)}(z)$. Indeed, the substitution gives us

$$(1+z)\,C_i^{(n)}(z) - \gamma_i^{(n)} \left(1 + \sum_{j=1}^{k_n} \frac{C_j^{(n)}(z)}{u_i^{(n)} - u_j^{(n)} - 1}\right) = 0,$$

which implies (4.19). It is clear that for generic z we have $\det L^{(n)}(z) \neq 0$ and equation (4.19) has a unique solution $C^{(n)}(z)$. The lemma is proved.

Below we give a determinant formula for $\psi_n(x,z)$. By Cramer's rule we have

(4.20) $$C_i^{(n)}(z) = \frac{\det L_i^{(n)}(z)}{\det L^{(n)}(z)},$$

where $L_i^{(n)}(z)$ is the matrix obtained from $L^{(n)}(z)$ by replacing the i-th column by the vector $\gamma^{(n)}$.

Define a $(k_n + 1) \times (k_n + 1)$ matrix $\widehat{L}^{(n)}(x,z)$, whose rows and columns are labeled by indices $0,\dots,k_n$ and entries are given by the formulas:

(4.21) $$\widehat{L}_{0,0}^{(n)} = 1\,, \qquad \widehat{L}_{0,j}^{(n)} = \frac{1}{x - u_j^{(n)}}\,, \qquad \widehat{L}_{i,0}^{(n)} = -\gamma_i^{(n)}$$
$$\widehat{L}_{i,j}^{(n)} = L_{i,j}^{(n)}\,, \qquad i,j = 1,\dots,k_n\,.$$

Using the determinant expansion of $\widehat{L}^{(n)}(z)$ relative to the 0-th row we obtain the formula

Proof.

(4.22)
$$\psi_n(x,z) = z^n(1+z)^x \frac{\det \widehat{L}^{(n)}(x,z)}{\det L^{(n)}(z)}.$$

Lemma 4.4. If y represents a solution of the Bethe ansatz equations (4.1), then the sequence $\Psi(z)$, constructed in Lemma 4.3, is an admissible solution of (4.2).

By definition of $\psi_n(x,z)$ and $\phi_n(x,z)$, the function

$$R_n(x,z) := \phi_n(x,z)z^{-n}(1+z)^{-x}$$

is a rational function of x with at most first order poles at the zeros of $y_{n+1}(x)$. Since $v_n(x) \to 1$ as $x \to \infty$, we have $R_n(x,z) \to 1 + z - 1 = z$ as $x \to \infty$. Hence,

the function $\phi_n(x,z)$ has the form

(4.23)
$$\varphi_n(x,z) = z^{n+1}(1+z)^x \left(1 + \sum_{i=1}^{k_{n+1}} \frac{D_i^{(n)}(z)}{x - u_i^{(n+1)}}\right)$$

with suitable functions $D_i^{(n)}(z)$.

Since the function $\psi_n(x+1,z)$ is regular at $x = u_i^{(n+1)}$, it follows from (4.14) that

(4.24)
$$\operatorname{res}_{x=u_i^{(n+1)}} \varphi_n(x,z) = -\varepsilon_i^{(n)} \psi_n(u_i^{(n+1)},z).$$

From the equation $v_n(u_i^{(n+1)} - 1) = 0$ it follows that

(4.25)
$$\varphi_n(u_i^{(n+1)} - 1, z) = \psi_n(u_i^{(n+1)}, z).$$

Hence

(4.26)
$$\operatorname{res}_{x=u_i^{(n+1)}} \varphi_n(x,z) + \varepsilon_i^{(n)} \varphi_n(u_i^{(n+1)} - 1, z) = 0.$$

Using equations (4.6) we rewrite this as

(4.27)
$$\operatorname{res}_{x=u_{i(n+1)}} \phi_n(x,z) - \gamma_{i(n+1)}\phi_n(u_{(in+1)} - 1, z) = 0.$$

By Lemma 4.3 the function $\psi_{n+1}(x,z)$ is uniquely determined by the equations

(4.28)
$$\operatorname{res}_{x=u_i^{(n+1)}} \psi_{n+1}(x,z) - \gamma_i^{(n+1)} \psi_{n+1}(u_i^{(n+1)} - 1, z) = 0.$$

Hence $\phi_n(x,z) = \psi_{n+1}(x,z)$ and the lemma is proved.

For any $n \in \mathbb{Z}$, let $q_n(z)$ be the monic polynomial of minimal degree such that $q_n(0) \neq 0$ and the function $q_n(z)\,\frac{\det \widehat{L}^{(n)}(x,z)}{\det L^{(n)}(z)}$ is a function in z holomorphic on $\mathbb{C} - \{0\}$. Clearly the polynomial $q_n(z)$ does exist, it divides the polynomial $\det L^{(n)}(z)$, and $\deg q_n(z) \leqslant k_n$.

Lemma 4.5. The polynomial $q_n(z)$ does not depend on $n \in \mathbb{Z}$. Proof.

Equation (4.2) implies

$$
\begin{aligned}
z\,\frac{\det \widehat{L}^{(n+1)}(x,z)}{\det L^{(n+1)}(z)} &= (1+z)\frac{\det \widehat{L}^{(n)}(x+1,z)}{\det L^{(n)}(z)} \\
&- v_n(x)\frac{\det \widehat{L}^{(n)}(x,z)}{\det L^{(n)}(z)}\,.
\end{aligned}
\tag{4.29}
$$

Given $\zeta \neq 0$, let d_n be the multiplicity of the root $z = \zeta$ of the polynomial $q_n(x)$. We need to show that $d_n = d_{n+1}$. Clearly the inequality $d_n < d_{n+1}$ contradicts equation (4.29). Now we assume that $d_n > d_{n+1}$ and also will obtain a contradiction. Namely, consider the expansions

$$
\begin{aligned}
\frac{\det \widehat{L}^{(n)}(x,z)}{\det L^{(n)}(z)} &= c_n(x)(z-\zeta)^{-d_n} + \mathcal{O}((z-\zeta)^{-d_n+1}) \\
&= (bx^a + \mathcal{O}(x^{a-1}))(z-\zeta)^{-d_n} + \mathcal{O}((z-\zeta)^{-d_n+1}),
\end{aligned}
$$

where the first equality is the Laurant expansion of $\frac{\det \widehat{L}^{(n)}(x,z)}{\det L^{(n)}(z)}$ at $z = \zeta$, and $c_n(x) = bx^a + O(x^{a-1})$ is the Laurent expansion of $c(x)$ at $x = \infty$. Here a is a suitable integer and b a nonzero number. We also have $v_n(x) = 1 + O(x^{-1})$ as $x \to \infty$. Considering the leading coefficients of these double expansions for each of the three summands in (4.29) we obtain the equation $0 = \zeta + 1 - 1$, which is impossible. The lemma is proved.

The n-independent polynomial $q_n(z)$ will be denoted by $q(z)$. Let κ be the degree of $q(z)$.

Introduce new functions

$$
\Psi_n(x,z) := \frac{q(z)}{z^\kappa}\psi_n(x,z) = z^n(1+z)^x\frac{q(z)}{z^\kappa}\left(1+\sum_{i=1}^{k_n}\frac{C_i^{(n)}(z)}{x-u_i^{(n)}}\right)
\tag{4.30}
$$

Clearly the sequence $(\Psi_n(x,z))$ is an admissible solution of (4.2) and

$$
\frac{q(z)}{z^\kappa}\left(1+\sum_{i=1}^{k_n}\frac{C_i^{(n)}(z)}{x-u_i^{(n)}}\right) = 1+\sum_{i=1}^{k_n}\xi_i^{(n)}(x)z^{-i},
\tag{4.31}
$$

where $\xi_i^{(n)}(x)$ are rational functions of x with at most first order poles at the zeros of $y_n(x)$. Thus the sequence of functions $(\Psi_n(x,z))$ has the properties listed in Theorem 4.2. Theorem 4.2 is proved.

4.2. Example. Consider the sequence $y^\emptyset = (y_n(x))_{n\in\mathbb{Z}}$, where $y_n(x) = 1$ for all n, see (3.8). As discussed in Section 3.2, this sequence represents a solution of the Bethe ansatz equations (4.1) with $k_n = 0$ for all n. In this case, the generating linear problem equations (4.2) take the form

$$
\psi_{n+1}(x) = \psi_n(x+1) - \psi_n(x), \qquad n \in \mathbb{Z},
\tag{4.32}
$$

Proof.
and the admissible solution $\Psi^{\emptyset}(z) = (\Psi^{\emptyset}_n(x,z))_{n\in\mathbb{Z}}$ of Theorem 4.2 is

$$\Psi^{\emptyset}_n(x,z) = z^n(1+z)^x, \qquad n \in \mathbb{Z}. \tag{4.33}$$

4.3. Solutions $\Psi(z)$ and the operation of generation. Let $y = (y_n(x))_{n\in\mathbb{Z}}$ be a generic sequence of polynomials, which represents a solution of the Bethe ansatz equations (4.1). Then there exists a unique one-parameter family $\Psi(z) = (\Psi_n(x,z))$ of solutions of the generating linear problem equations (4.2) given by Theorem 4.2.

Choose $m \in \mathbb{Z}$. Consider the one-parameter family $y^{(m)}(c) = (\tilde{y}_n(x,c))_{n\in\mathbb{Z}}$, obtained from y by generation in the m-th direction, see (3.7). Here $\tilde{y}_n(x,c) = y_n(x)$ for $n \neq m$ and the polynomial $\tilde{y}_m(x,c)$ satisfies the equation

$$\tilde{y}_m(x,c)y_m(x+1) - \tilde{y}_m(x+1,c)y_m(x) = y_{m-1}(x+1)y_{m+1}(x). \tag{4.34}$$

Choose the value $c = c_0$ so that the sequence $y^{(m)}(c_0)$ is generic. Then $y^{(m)}(c_0)$ represents a solution of the Bethe ansatz equations (4.1) by Theorem 3.3. Define the sequence $\tilde{y} = (\tilde{y}_n(x))_{n\in\mathbb{Z}}$ by the formula $\tilde{y} = y^{(m)}(c_0)$. Denote $\tilde{k}_n = \deg \tilde{y}_n(x)$ for $n \in \mathbb{Z}$.

Starting from $\tilde y$ define a sequence of rational functions $\tilde v = (\tilde v_n(x))$ by formula (4.3). We have $\tilde v_n(x) = v_n(x)$ if $n \neq m-1, m$ and

$$\tilde v_{m-1}(x) = \frac{y_{m-1}(x)\tilde y_m(x+1)}{y_{m-1}(x+1)\tilde y_m(x)}$$

$$\tilde v_m(x) = \frac{\tilde y_m(x) y_{m+1}(x+1)}{\tilde y_m(x+1) y_{m+1}(x)}. \tag{4.35}$$

Apply Theorem 4.2 to the sequence $\tilde v$ and obtain the unique one-parameter family $\tilde\Psi(z) = (\tilde\Psi_n(x,z))$ of admissible solutions of the generating linear problem equation (4.2) with the chosen sequence $\tilde v$,

$$\tilde\Psi_n(x,z) = z^n(1+z)^x \left(1 + \sum_{i=1}^{\tilde k_n} \tilde\xi_i^{(n)}(x) z^{-i}\right), \tag{4.36}$$

where $\tilde\xi_i^{(n)}(x)$ are rational functions in x with at most first order poles at the zeros of $\tilde y_n(x)$.

Theorem 4.6. We have $\tilde\Psi_n(x,z) = \Psi_n(x,z)$ for $n \neq m$ and

$$\tilde\Psi_m(x,z) = \Psi_m(x,z) + g(x)\Psi_{m-1}(x,z), \tag{4.37}$$

where

$$g(x) = \frac{y_{m-1}(x) y_{m+1}(x)}{y_m(x)\tilde y_m(x)}. \tag{4.38}$$

Proof.

Lemma 4.7. We have

$$\tilde v_m(x) g(x) = v_{m-1}(x) g(x+1), \tag{4.39}$$

$$\tilde v_m(x) - v_m(x) = g(x+1). \tag{4.40}$$

Remark. Equations (4.39) and (4.40) imply the equation

$$v_m(x)g(x) - v_{m-1}(x)g(x+1) + g(x)g(x+1) = 0. \tag{4.41}$$

This equation with respect to $g(x)$ is called the *discrete Riccati equation*, see [**MV3**]. This discrete Riccati equation has a rational solution $g(x)$, given by (4.38). On discrete Riccati equations with rational solutions see [**MV3**].

Proof. The proof of (4.39) is straightforward. We also have

$$\tilde v_m(x) - v_m(x) = \frac{\tilde y_m(x) y_{m+1}(x+1)}{\tilde y_m(x+1) y_{m+1}(x)} - \frac{y_m(x) y_{m+1}(x+1)}{y_m(x+1) y_{m+1}(x)}$$
$$= \frac{y_{m+1}(x+1)}{y_{m+1}(x)} \frac{\tilde y_m(x) y_m(x+1) - \tilde y_m(x+1) y_m(x)}{\tilde y_m(x+1)\tilde y_m(x+1)}$$
$$= \frac{y_{m+1}(x+1)}{y_{m+1}(x)} \frac{y_{m-1}(x+1) y_{m+1}(x)}{\tilde y_m(x+1)\tilde y_m(x+1)} = g(x+1).$$

Let us check that the functions $\Psi_{m+1}(x,z)$, $\Psi_m(x,z) + g(x)\Psi_{m-1}(x,z)$, $\Psi_{m-1}(x,z)$ satisfy equations (4.2) with $\tilde v_{m-1}(x), \tilde v_m(x)$. Indeed, we have

$$\Psi_m(x,z) + g(x)\Psi_{m-1}(x,z) = \Psi_{m-1}(x+1) - \tilde v_{m-1}(x)\Psi_{m-1}(x,z)$$

by formula (4.40) and

$$\begin{aligned}\Psi_{m+1}(x,z) &= \Psi_m(x+1,z) + g(x+1)\Psi_{m-1}(x+1,z)\\ &- \tilde v_m(x)\big(\Psi_m(x,z) + g(x)\Psi_{m-1}(x,z)\big)\end{aligned}$$

by formulas (4.39) and (4.40).

Lemma 4.8. We have

$$\Psi_m(x,z) + g(x)\Psi_{m-1}(x,z) = z^m(1+z)^x\left(1 + \sum_{i>0}\tilde r_i(x)z^{-i}\right), \tag{4.42}$$

where $\tilde r_i(x)$ are rational functions of x with at most first order poles at the zeros of $\tilde y_m(x)$.

Proof. It is enough to show that the left-hand side in (4.42) is regular at the roots of the polynomial $y_m(x)$. Indeed,

$$\begin{aligned}&\Psi_m(x,z) + g(x)\Psi_{m-1}(x,z)\\ &= \Psi_{m-1}(x+1,z) - v_{m-1}(x)\Psi_{m-1}(x,z) + g(x)\Psi_{m-1}(x,z)\\ &= \Psi_{m-1}(x+1,z) - (v_{m-1}(x) - g(x))\Psi_{m-1}(x,z)\\ &= \Psi_{m-1}(x+1,z) - \frac{g(x)}{g(x+1)}v_m(x)\Psi_{m-1}(x,z)\\ &= \Psi_{m-1}(x+1,z) - \frac{y_{m-1}(x)\tilde y_m(x+1)}{y_{m-1}(x+1)\tilde y_m(x)}\Psi_{m-1}(x,z)\,,\end{aligned}$$

and the last expression is regular at the roots of $y_m(x)$.

Theorem 4.6 is proved.

Remark. Let $y = (y_n(x))_{n\in\mathbb{Z}}$ be a generic N-periodic sequence of polynomials representing a solution of the Bethe ansatz equations (2.1). Let $\Psi(z) = (\Psi_n(x,z))_{n\in\mathbb{Z}}$ be the associated one-parameter family of admissible solutions determined by Theorem 4.2. By Theorem 3.4 the sequence $y = (y_n(x))_{n\in\mathbb{Z}}$ can be obtained from the sequence $y^\emptyset$ by the iterated generation procedure of Section 3. Theorem 4.6 shows how to obtain the family of admissible solutions $\Psi(z)$ from the family of admissible solutions $\Psi^\emptyset(z)$ in (4.33) by transformations of Theorem 4.6.

5. Spectral transforms for the rational RS system

5.1. Lax matrices. In Section 4 for any sequence of polynomials $(y_n(x))_{n\in\mathbb{Z}}$, whose roots satisfy the Bethe ansatz equations (4.1), we constructed solutions $(\psi_n(x,z))_{n\in\mathbb{Z}}$ of the generating linear problem equation (4.2) depending on the spectral parameter z. Formulas (4.18), (4.19) of that construction reveal a'priory unexpected connections of the construction with the theory of the rational RS system. In this section we develop the direct and inverse *spectral transforms* for the rational RS system.

We identify the phase space of the k-particle rational RS system with the subspace $\mathcal{P}_k \subset \mathbb{C}^k \times (\mathbb{C}^\times)^k$ of pairs of vectors $u = (u_1,\dots,u_k)$ and $\gamma = (\gamma_1,\dots,\gamma_k)$, such that

(5.1) $$u_i \neq u_j, \quad u_i \neq u_j + 1 \text{ for } i \neq j\,.$$

A point $(u,\gamma) \in \mathcal{P}_k$ defines the $k \times k$ *Lax matrix* $L(u,\gamma)$,

(5.2) $$L_{ij}(u,\gamma) = \frac{\gamma_i}{u_i - u_j - 1}\,, \qquad i,j = 1,\dots,k\,.$$

Notice that the Lax matrix has already appeared in (4.18), where

$$L_{(n)}(z) = 1 + z - L(u_{(n)},\gamma_{(n)}).$$

The matrix $L(u,\gamma)$ is a particular case of the *Cauchy matrix*. Its determinant equals

(5.3) $$\det L(u,\gamma) = \prod_{i=1}^k \gamma_i \prod_{i<j} \frac{(u_i-u_j)^2}{(u_i-u_j)^2-1}\,.$$

It satisfies, the so-called *displacement equation*

(5.4) $$[U, L(u,\gamma)] = L(u,\gamma) + \Gamma F\,,$$

where $U = \operatorname{diag}(u_1,\dots,u_k)$, $\Gamma = \operatorname{diag}(\gamma_1,\dots,\gamma_k)$, $F = (f_{ij})$ with $f_{ij} = 1$ for all i,j. Equation (5.4) can be easily checked directly. Let E be the $k \times k$ unit matrix. Denote

(5.5) $$L(z\,|\,u,\gamma) := (1+z)E - L(u,\gamma).$$

Let $\hat L(x,z\,|\,u,\gamma)$ be the $(k+1)\times(k+1)$-matrix, whose rows and columns are labeled by indices $0,\dots,k$ and entries are given by the formulas:

(5.6) $$\widehat{L}_{0,0} = 1\,, \qquad \widehat{L}_{0,j} = \frac{1}{x-u_j}\,, \qquad \widehat{L}_{i,0} = -\gamma_i\,, \qquad \widehat{L}_{i,j} = L_{i,j}(z\,|\,u,\gamma)\,, \qquad i,j = 1,\dots,k\,.$$

cf. formulas (4.21). Define the function $\psi(x,z\,|\,u,\gamma)$ by the formula

(5.7) $$\psi(x,z\,|\,u,\gamma) = (1+z)^x \frac{\det \hat L(x,z\,|\,u,\gamma)}{\det L(z\,|\,u,\gamma)}\,.$$

5.2. Direct transform in generic case. We define the direct spectral transform first for points (u,γ) of the following open subset $\mathcal{P}'_k \subset \mathcal{P}_k$.

Let $\mu = (\mu_1,\dots,\mu_k)$ be the set of eigenvalues of the matrix $L(u,\gamma)$. We have $\mu_j \neq 0$ for all j by formula (5.3). Hence $\mu \in (\mathbb{C}^\times)^k$.

Define

(5.8) $$\mathcal{P}'_k = \{(u,\gamma) \in \mathcal{P}_k \mid \mu_1,\dots,\mu_k \text{ are distinct}\}.$$

Clearly $\mathcal{P}'_k$ is nonempty, since for big distinct $u_1,\dots,u_k$ the matrix $L(u,\gamma)$ is close to the diagonal matrix $-\mathrm{diag}(\gamma_1,\dots,\gamma_k)$.

The function $\psi(x,z\,|u,\gamma)$ has at most simple pole at $z = \mu_j - 1$. Consider the Laurent expansion of $\psi(x,z\,|u,\gamma)$ at $z = \mu_j - 1$,

(5.9)
$$\psi(x,z\,|\,u,\gamma) = \frac{\varphi_j^{(0)}(x\,|\,u,\gamma)}{z-\mu_j+1} + \varphi_j^{(1)}(x\,|\,u,\gamma) + \mathcal{O}(z-\mu_j+1).$$

Theorem 5.1. For $(u,\gamma) \in \mathcal{P}'_k$, there exists a unique $a = (a_1,\dots,a_k) \in \mathbb{C}^k$ such that

(5.10) $\phi^{(1)}{}_j(x|u,\gamma) + a_j\phi^{(0)}{}_j(x|u,\gamma) = 0, \qquad j = 1,\dots,k.$

Proof. The function $\psi(x,z\,|u,\gamma)$ has the form

(5.11)
$$\psi(x,z\,|\,u,\gamma) = (1+z)^x\left(1+\sum_{i=1}^k \frac{C_i(z)}{x-u_i}\right),$$

cf. (4.13). The vector $C(z)$ with coordinates $C_i(z)$ is given by (4.20). The vector $C(z)$ solves equation (4.19). Consider the Laurent expansion of $C(z)$ at $z = \mu_j - 1$,

(5.12)
$$C(z) = \frac{c_j}{z-\mu_j+1} + d_j + \mathcal{O}(z-\mu_j+1),$$

where c_j, d_j are k-vectors with coordinates denoted by c_{ij}, d_{ij}, respectively. The substitution of (5.12) into (4.19) gives the relations:

(5.13),
(5.14)
$$\begin{aligned}(\mu_j - L)\,c_j &= 0\\ (\mu_j - L)\,d_j + c_j &= \gamma,\end{aligned}$$

where $L = L(u,\gamma)$.

Let $\tilde{c}_j$ be a nonzero eigenvector of L with eigenvalue μ_j. It is unique up to multiplication by a nonzero constant. Using (5.4) we get

(5.15) $(\mu_j - L)U\tilde{c}_j = [U,L]\tilde{c}_j = (L + \Gamma F)\tilde{c}_j = \mu_j\tilde{c}_j + \nu_j\gamma,$

where $\nu_j := \sum_{i=1}^k \tilde{c}_{ij}$. We have $\nu_j \neq 0$. Indeed, if $\nu_j = 0$, then (5.15) shows that L has a nontrivial Jordan block with eigenvalue μ_j. That contradicts to the assumption that $(\mu_1,\dots,\mu_k)$ are distinct. Since $\nu_j \neq 0$. We can uniquely define the vector $\tilde{c}_j$ by the normalization $\nu_j = -\mu_j$.

Lemma 5.2. The vector c_j defined in (5.12) is nonzero.

Proof. If $c_j = 0$, then (5.14) gives

$$(\mu_j - L)d_j = \gamma.$$

Formula (5.15) with $\nu_j = -\mu_j$ gives

$$\mu_j^{-1}(\mu_j - L)U\tilde{c}_j = \tilde{c}_j - \gamma .$$

Adding the two formula gives

$$(\mu_j - L)(\mu_j^{-1}U\tilde{c}_j + d_j) = \tilde{c}_j, \tag{5.16}$$

which means that L has a nontrivial Jordan block with eigenvalue μ_j. Contradiction.

Lemma 5.3. Let $\tilde{c}_j = (\tilde{c}_{ij})$, $\tilde{d}_j \in \mathbb{C}^k$ be a solution of the system of equations

$$(\mu_j - L)\tilde{c}_j = 0, \tag{5.17}$$

$$(\mu_j - L)\tilde{d}_j + \tilde{c}_j = \gamma, \tag{5.18}$$

such that $\tilde{c}_j \neq 0$. Then

$$\sum_{i=1}^{k} \tilde{c}_{ij} = -\mu_j, \qquad \tilde{d}_j = -\mu_j^{-1}U\tilde{c}_j - a_j\tilde{c}_j ,$$

for some $a_j \in \mathbb{C}$.

Corollary 5.4. The vectors c_j, d_j in (5.12) satisfy the equations

$$\sum_{i=1}^{k} c_{ij} = -\mu_j, \qquad d_j = -\mu_j^{-1}Uc_j - a_jc_j , \tag{5.19}$$

for some $a_j \in \mathbb{C}$.

Proof of Lemma 5.3. The vector $\tilde{c}_j$ is an eigenvector of L with eigenvalue μ_j. Fix $\tilde{c}_j$ by the condition $\sum_{i=1}^{k} \tilde{c}_{ij} = -\mu_j$. Then (5.15) and (5.14) show that $\tilde{c}_j$ and $\tilde{d}_j = -\mu_j^{-1}U\tilde{c}_j$ give a solution to the system of equations (5.17) and (5.18). For that $\tilde{c}_j$ the general solution of (5.18) has the form

$$\tilde{d}_j = -\mu_j^{-1}U\tilde{c}_j - a_j\tilde{c}_j, \tag{5.20}$$

where a_j is an arbitrary constant.

Let $(\tilde{c}_j, \tilde{d}_j)$ and $(\hat{c}_j, \hat{d}_j)$ be two solutions of system (5.17), (5.18). Then

$$(\mu_j - L)(\tilde{c}_j - \hat{c}_j) = 0,$$
$$(\mu_j - L)(\tilde{d}_j - \hat{d}_j) + (\tilde{c}_j - \hat{c}_j) = 0.$$

If $\tilde{c}_j - \hat{c}_j \neq 0$, then L has a nontrivial Jordan block with eigenvalue μ_j. This leads to contradiction. Hence $\tilde{c}_j = \hat{c}_j$. The lemma is proved.

By formula (4.13) the first two coefficients of the Laurent expansion of $\psi(x,z\,|u,\gamma)$ at $z = \mu_j - 1$ are

(5.21)
$$\varphi_j^{(0)}(x\,|\,u,\gamma) = \mu_j^x \left(\sum_{i=1}^k \frac{c_{ij}}{x-u_i}\right)$$

(5.22).
$$\varphi_j^{(1)}(x\,|\,u,\gamma) = \mu_j^x \left(1+\sum_{i=1}^k \frac{x\mu_j^{-1}c_{ij}+d_{ij}}{x-u_i}\right)$$

Using (5.19) we get

(5.23)
$$\varphi_j^{(1)}(x\,|\,u,\gamma) = \mu_j^x \left(1+\sum_{i=1}^k \frac{(\mu_j^{-1}(x-u_i)-a_j)c_{ij}}{x-u_i}\right)$$
$$= \mu_j^x \left(1+\sum_{i=1}^k \left(\mu_j^{-1}c_{ij} - a_j\frac{c_{ij}}{x-u_i}\right)\right) = -a_j\varphi_j^{(0)}(x\,|\,u,\gamma)$$

.

The theorem is proved.

Theorem 5.1 gives us the correspondence

(5.24) $$S\ :\ (u,\gamma)\ \mapsto\ (\mu,a),$$

where $(u,\gamma)\in\mathcal{P}'_k\subset\mathbb{C}^k\times(\mathbb{C}^\times)^k$ and $(\mu,a)\in(\mathbb{C}^\times)^k\times\mathbb{C}^k$.

Below we will need the following stronger version of Lemma 5.2.

Lemma 5.5. Let μ_j be an eigenvalue of $L(u,\gamma)$ (of any multiplicity). Then the function $\psi(x,z\,|u,\gamma)$ is not holomorphic at $z=\mu_j-1$.

Proof. The function $\psi(x,z\,|u,\gamma)$ has the form (5.11) with the vector $C(z)$ that solves equation (4.19). If $\psi(x,z\,|u,\gamma)$ is holomorphic at $z=\mu_j-1$, then the vector $C(z)$ is holomorphic at $z=\mu_j-1$ as well. Let $d_j:=C(\mu_j-1)$, then (4.19) gives the relation:

(5.25) $$(\mu_j-L)d_j=\gamma\,,$$

where $L=L(u,\gamma)$.

Let $c_{j,s},\ s=1,\dots,\ell$, be a Jordan chain of the operator $L=L(u,\gamma)$ of maximal length with eigenvalue μ_j. Let $c_{j,s,i}$ be coordinates of the vector $c_{j,s}$. Then

(5.26) $$(\mu_j-L)c_{j,1}=0,\qquad (\mu_j-L)c_{j,s}=c_{j,s-1}\,.$$

Using the displacement equation (5.4) we get

(5.27) $$(\mu_j-L)Uc_{j,\ell}=\mu_jc_{j,l}-c_{j,\ell-1}+\nu_{j,l}\gamma\,,$$

where $\nu_{j,\ell}:=\sum_i c_{j,\ell,i}$. Using equations 5.25, (5.26) with $s=\ell$ and (5.27) we get

(5.28) $$(\mu_j-L)(Uc_{j,\ell}+c_{j,\ell}-\nu_{j,\ell}\,d_j)=\mu_jc_{j,\ell},$$

which contradicts to the assumption that the Jordan chain is of maximal length.

5.3. Inverse correspondence. We recall the construction of the correspondence inverse to (5.24), cf. the construction in [**K6**]. We define it simultaneously with the construction of generic solutions to the rational RS system.

Let $\Omega(x,t,z)$ be the function in x,z, and $t = (t_1,t_2,...)$ defined by the formula

$$\Omega(x,t,z) = (1+z)^x e^{\sum_{j=1}^{\infty} t_j z^j}, \tag{5.29}$$

in which we always assume that only a finite number of the variables t_j are nonzero. The function $\Omega(x,t,z)$ in more details is considered in Section 7.3.

Let $\mu \in (\mathbb{C}^\times)^k$ with $\mu_i \neq \mu_j$ for $i \neq j$. Let $\psi(x,t,z)$ be a function of the form

$$\psi(x,t,z) = \Omega(x,t,z)\left(1 + \sum_{j=1}^{k} \frac{r_j(x,t)}{z+1-\mu_j}\right). \tag{5.30}$$

Consider the Laurent expansions

$$\psi(x,t,z) = \frac{\varphi_j^{(0)}(x,t)}{z-\mu_j+1} + \varphi_j^{(1)}(x,t) + \mathcal{O}(z-\mu_j+1), \qquad j = 1,\dots,k. \tag{5.31}$$

Lemma 5.6. If $(\mu,a) \in (\mathbb{C}^\times)^k \times \mathbb{C}^k$ with $\mu_i \neq \mu_j$ for $i \neq j$, then there is a unique function $\psi(x,t,z)$ as in (5.30), such that coefficients $\phi^{(0)}_j(x,t)$, $\phi^{(1)}_j(x,t)$ satisfy the equations

$$\phi^{(1)}_j(x,t) + a_j\phi^{(0)}_j(x,t) = 0, \qquad j = 1,\dots,k. \tag{5.32}$$

Notice that the form of the second factor in the right-hand side of (5.30) is just the simple fraction decomposition of a rational function in z with at most simple poles at the points $z = \mu_i - 1$ that equals 1 at $z = \infty$.

Proof. The lemma is proved by explicit computation of $\psi(x,t,z)$. Let $r(x,t)$ be the k-vector with coordinates $r_i(x,t)$. Taking the first coefficient of the Laurent expansion of $\psi(x,t,z)$ at $z = \mu_j - 1$ shows that equations (5.32) are equivalent to the inhomogeneous equation

$$T(x,t)r(x,t) = -e_0, \tag{5.33}$$

where e_0 is the k-vector with all coordinates equal to 1 and $T = T(x,t)$ is the $k \times k$-matrix

$$T_{ii} = a_i + \left(x\mu_i^{-1} + \sum_s st_s(\mu_i-1)^{s-1}\right), \qquad T_{ij} = \frac{1}{\mu_i-\mu_j}, \qquad i \neq j. \tag{5.34}$$

with entries

Let $\hat{T}(x,t,z)$ be the $(k+1) \times (k+1)$-matrix with the entries

$$\hat{T}_{00} = 1, \quad \hat{T}_{0,j} = \frac{1}{z+1-\mu_j}, \quad \hat{T}_{0,i} = 1, \quad \hat{T}_{ij} = T_{ij}, \quad i,j = 1,\dots k. \tag{5.35}$$

Then $\psi(x,t,z)$ equals

$$\psi(x,t,z) = \Omega(x,t,z)\,\frac{\det \hat{T}(x,t,z)}{y(x,t)}, \tag{5.36}$$

where

(5.37) $$y(x,t) = \det T(x,t).$$

The function $y(x,t)$ will also be denoted by $y(x,t|\mu,a)$. It is a polynomial in x of degree k. Let $u_i(t|\mu,a)$, $i = 1,...,k$, be its roots. Define $\gamma(t|\mu,a) = (\gamma_1(t|\mu,a),...,\gamma_k(t|\mu,a))$ by the formula

(5.38) $$\gamma_i(t|\mu,a) = \partial_{t_1} u_i(t|\mu,a).$$

Let $S \subset (\mathbb{C}^\times)^k \times \mathbb{C}^k$ be the subset of points (μ,a), such that

(a) $\mu = (\mu_1,...,\mu_k)$ has distinct coordinates;
(b) $u(0|\mu,a) = (u_1(0|\mu,a),...,u_k(0|\mu,a))$ has distinct coordinates.

Theorem 5.7. For $(\mu,a) \in S$, the map

(5.39) $$\tilde{S} \,:\, (\mu,a) \mapsto (u(0\,|\,\mu,a),\gamma(0\,|\,\mu,a))$$ is inverse to the map in (5.24).

Proof. The standard arguments based on the uniqueness of the BakerAkhiezer function prove the following statement.

Lemma 5.8. The function $\psi(x,t,z)$ given by (5.36) satisfies equation (1.9) with $y(x,t)$ defined in (5.37).

Proof. Define the function $w(x,t)$ by the formula

(5.40) $$w(x,t) = \xi_1(x,t) - \xi_1(x+1,t) - 1,$$

where $\xi_1(x,t)$ is the coefficient of the expansion of the second factor in (5.36) at $z = \infty$, i.e.

(5.41) $$\psi(x,t,z) = \Omega(x,t,z)\left(1+\sum_{s=1}^{\infty}\xi_s(x,t)z^{-s}\right).$$

Then the corresponding expansion for the function $\tilde{\psi}(x,t,z) := \partial_{t_1}\psi(x,t,z) - \psi(x+1,t,z) - w(x,t)\psi(x,t,z)$

has the form $\tilde{\psi}(x,t,z) = \Omega(x,t,z)O(z^{-1})$, i.e. the simple fraction expansion for $\tilde{\psi}$ has the

form

$$\tilde{\psi}(x,t,z) = \Omega(x,t,z)\left(\sum_{j=1}^{k} \frac{\tilde{r}_j(x,t)}{z+1-\mu_j}\right). \quad (5.42)$$

Since a_j in (5.10) is a constant, the first two coefficients of the Laurent expansion of $\tilde{\psi}$ at $\mu_j - 1$ satisfy equation (5.10), i.e. for the vector $\tilde{r}$ with coordinates $\tilde{r}_j$ the homogeneous linear equation $T\tilde{r} = 0$ holds. Hence, $\tilde{r} = 0$ and the equation $\tilde{\psi} = 0$ is proved.

It remains to show that the function $w(x,t)$, defined by (5.40), has the form (1.10) with $y(x,t)$ given by (5.37). The equation

$$\xi_1(x,t) = -\partial_{t_1} \ln y(x,t) \quad (5.43)$$

can be derived from Cramer's formulas for the coordinates r_j of the vector r and the equation

$$\xi_1(x,t) := \sum_{j=1}^{k} r_j(x,t). \quad (5.44)$$

It is more instructive to prove it directly using equation (1.9). Indeed, by definition, $\xi_1(x,t)$ is a rational function in x with poles at the zeros of $y(x,t)$. The comparison of the coefficients at $(x-u_i)^{-2}$ of the Laurent expansion of the right and left-hand sides of (1.9) at u_i gives the equation

$$\gamma_i(t) := \mathrm{res}_{x=u_i} w(x,t) = \mathrm{res}_{x=u_i} \xi_1(x,t) = \partial_{t_1} u_i(t). \quad (5.45)$$

The latter implies (5.43).

The left-hand side of (1.9) has poles only at the zeros of $y(x,t)$. Hence the right-hand side of (1.9) has no residue at $x = u_i - 1$. From (5.40) it follows that the residue of $w(x,t)$ at $x = u_i - 1$ equals $-\gamma_i(t)$ and we recover the defining condition for $\psi(x,t,z)$ in Lemma 4.3. Put $t = 0$. The theorem is proved.

5.4. Extension of the direct spectral transform. Our goal is to extend the direct spectral transform (5.24) to the whole phase space of the rational RS system.

For $(u,\gamma) \in \mathcal{P}_k$ consider the function $\psi(x,z\,|\,u,\gamma)$ defined by (5.7). The function

(5.46) $\Psi(x,z\,|\,u,\gamma) = \det L(z\,|\,u,\gamma)\psi(x,z\,|\,u,\gamma) = (z+1)^x \det \hat{L}(x,z\,|\,u,\gamma)$ has the form

$$\Psi(x,z\,|\,u,\gamma) = (1+z)^x\left(z^k + \sum_{\ell=1}^{k} \xi_\ell(x\,|\,u,\gamma) z^{k-\ell}\right). \quad (5.47)$$

It is well-defined on the whole phase space. The coefficients $\xi_\ell(x\,|\,u,\gamma)$ are rational function in u,γ holomorphic on $\mathcal{P}_k$.

Let $(\mu_i = \mu_i(u,\gamma))_{i=1}^q$ be the set of all distinct eigenvalues of $L(u,\gamma)$ with respective multiplicities $(m_i)_{i=1}^q$. We have $\sum_{i=1}^q m_i = k$ and

$$\det L(z\,|\,u,\gamma) = \prod_{i=1}^q (z-\mu_i+1)^{m_i}\,, \qquad \mu_i \neq \mu_j .$$

For a positive integer denote by $\mathbb{C}[z]_\ell$ the vector subspace of $\mathbb{C}[z]$ of polynomials of degree less than . We have dim $\mathbb{C}[z]_\ell = \ell$.

Let $\mu \in \mathbb{C}$. We will often identify $\mathbb{C}[z]_\ell$ with $\mathbb{C}[z]/\langle (z-\mu+1)^\ell\rangle$ under the isomorphism $g(z) \to g(z) + \langle (z \quad \mu+1)^\ell\rangle$. –

Theorem 5.9. Let $(u,\gamma) \in \mathcal{P}_k$. Then for $j = 1,\dots,q$, there is a unique $m_j$dimensional vector subspace $W_j(u,\gamma) \subset \mathbb{C}[z]_{2m_j}$ such that

$$\text{(5.48)}\qquad \operatorname{res}^{z=\mu_j-1} \frac{g(z)\Psi(x,z\,|\,u,\gamma)}{(z-\mu_j+1)^{2m_j}} = 0\,, \qquad \forall\, g(x) \in W_j(u,\gamma) .$$

Remark. Let $(u,\gamma) \in \mathcal{P}'_k$. Then the one-dimensional subspace $W_j(u,\gamma) \subset \mathbb{C}[z]_2$ is spanned by the polynomial $a_j(z-\mu_j+1)+1$. Then equations (5.48) take the form

$$\text{(5.49)}\qquad \operatorname{res}^{z=\mu_j-1} \frac{(a_j(z-\mu_j+1)+1)\Psi(x,z\,|\,u,\gamma)}{(z-\mu_j+1)^2} = 0, \qquad j=1,\dots,k,$$

which is the same as equations (5.10).

Proof. The coefficients of det$L(z\,|u,\gamma)$ are holomorphic functions on $\mathcal{P}_k$. Hence for any $(u',\gamma') \in \mathcal{P}'_k$ in a sufficiently small neighborhood of (u,γ) the multiple eigenvalue μ_j of $L(u,\gamma)$ splits into a set of simple eigenvalues of the matrix $L(u',\gamma')$, i.e.

$$\det L(z\,|\,u',\gamma') = \prod_{i=1}^q \prod_{s=1}^{m_j} (z-\mu_{i,s}+1)$$

where $|\mu_{j,s} - \mu_j| < \varepsilon$ for some small ε. We may assume that the ε-neighborhoods of μ_j, $j = 1,\dots,q$, do not intersect.

The set of m_j equations (5.49), corresponding to a subset of the eigenvalues $\mu_{j,s}$, can be represented in the form

$$\text{(5.50)}\qquad \oint_{c_j} \frac{g_{j,s}(z)\,\Psi(x,z\,|\,u',\gamma')}{\prod_s (z-\mu_{j,s}+1)^2}\,dz\,, \qquad s=1,\dots m_j ,$$

where c_j is the circle $|z-\mu_j+1| = \varepsilon$ and

$$\text{(5.51)}\qquad g_{j,s}(z) := (a_j(z-\mu_{j,s}+1)+1)\prod_{\ell\neq s}(z-\mu_{j,\ell}+1)^2 .$$

It is easy to see that the polynomials $g_{j,s}(z)$ are linearly independent and hence span an m_j-dimensional subspace $W_j(u',\gamma')$ of $\mathbb{C}[z]_{2m_j}$, i.e. $W_j(u',\gamma')$ can be seen as a point of the Grassmanian Gr$(m_j,2m_j)$.

The Grassmanian is compact. Therefore, for any sequence $((u^m,\gamma^m))_{m=1}^\infty \subset \mathcal{P}'_k$ converging to (u,γ) there is a subsequence of points $W_j(u^m,\gamma^m)$ of the Grassmanian converging to some point $W_j \in \mathrm{Gr}(m_j,2m_j)$. Since the integral in (5.50) is taken over a constant circle the equations (5.50) converge to (5.48).

It remains to show that W_j does not depend on the choice of a convergent sequence $((u^m,\gamma^m))^\infty_{m=1}$. Notice that if $\Psi(x,z\,|u,\gamma)$ satisfies (5.48), then

(5.52) $$\operatorname{res}_{z=\mu_j-1} \frac{g(z)\,\Psi(\ell,z\,|\,u,\gamma)}{(z-\mu_j+1)^{2m_j}} = 0\,, \qquad \ell = 0,\dots,k-1, \qquad \forall\, g(z) \in W_j .$$

The function $\Psi(\ell,z\,|\,u,\gamma)$ is a monic polynomial of degree $k+\ell$. Hence, the tuple of functions $\Psi(,z)$ defines a point $W^\perp \in \mathrm{Gr}(k,2k)$. The k-dimensional vector space

$W^\perp$ defines all spaces W_j, $j = 1,\dots,q$, uniquely. Corollary 5.10. By Theorem 5.9,

every point $(u,\gamma) \in P_k$ produces the two collections

$(\mu_i(u,\gamma))^q_{i=1}$ and $(W_i(u,\gamma) \in \mathrm{Gr}(m_i,2m_j))^q_{i=1}$. That is an extension of the map

(5.24).

Equations (5.48) imply the following lemma.

Lemma 5.11. The function $\psi(x,z\,|\,u,\gamma)$ has a pole of order $m \leqslant m_j$ at $z = \mu_j(u,\gamma)$

− 1 if and only if the corresponding subspace $W_j(u,\gamma)$ $z-\mu_j+1)^{2m_j-\ell-1}$, $\ell =$

contains $(m_j - m)$-dimensional subspace spanned by the □

polynomials (0,...,m − 1.

The following statement is used below in the proof of Theorem 7.9. Let $f(z)$ be a function holomorphic at $z = \mu_j - 1$. Multiplication by $f(z)$ defines a linear operator

. If $f_z(\mu_j - 1)$ /

(5.53) $$f_* \,:\, \mathbb{C}[z]/\langle(z-\mu_j+1)^{2m_j}\rangle \to \mathbb{C}[z]/\langle(z-\mu_j+1)^{2m_j}\rangle, \qquad g(z) \mapsto f(z)g(z).$$

Lemma 5.12

$\mathbb{C}[z]/\langle(z-\mu_j+1)^{2m_j}\rangle$ = 0, then the only m_j-dimensional subspace

$z-\mu_j+1)^\ell$, $\ell = m_j,\dots,2m_j-1$. W of

, invariant under the action of f_*, is the subspace spanned

by (

Proof. The Jordan normal form of f_* is the single Jordan block of size $2m_j$. Such an operator has a single invariant m_j-dimensional subspace. That subspace is described in the lemma.

5.5. Extension of the inverse transform. The construction of the inverse correspondence is straightforward. The spectral data is a triple (μ,m,W), where $\mu = (\mu_1,\dots,\mu_q)$ is a set of distinct nonzero complex numbers; $m = (m_1,\dots,m_q)$ a set of positive integers with $\sum_{i=1}^q m_j = k$; $W = (W_1,\dots,W_q)$ a set of spaces, where each W_j is an m_j-dimensional subspace of the space of polynomials of degree $2m_j - 1$.

Lemma 5.13. Given (μ,m,W) there is a unique function $\Psi(x,t,z)$,

$$\Psi(x,t,z) = \Omega(x,t,z)\left(z^k + \sum_{s=1}^{k} \xi_\ell(x,t) z^{k-s}\right), \tag{5.54}$$

such that equations (5.48) hold.

Proof. The proof is by explicit construction, as in its particular case of Lemma 7.3. Choose a basis $g_{j,k}(z)$ in W_j. Then equations (5.48) can be represented in the form of the inhomogeneous linear system of equations

$$M(x,t|\mu,m,W)\xi(x,t) = -e_0 \tag{5.55}$$

with some matrix M, whose entries are explicit expressions that are polynomial in x and t and linear in the coefficients of the polynomials $g_{j,k}(z)$. As before the function Ψ can be written in the same determinant form as in (5.35):

$$\Psi(x,t,z\,|\,\mu,m,W) = \frac{\det \widehat{M}(x,t,z\,|\,\mu,m,W)}{y(x,t\,|\,\mu,m,W)}, \tag{5.56}$$

where

$$y(x,t|\mu,m,W) = \det M(x,t|\mu,m,W). \tag{5.57}$$

Remark. We emphasize that unlike in the generic case considered in Section 5.3, the degree k of the polynomial $y(x,t|\mu,m,W)$ in x depends not only on the number of distinct eigenvalues μ_j and their multiplicities m_j but also on the combinatorial types of cells of Grassmannians $\mathrm{Gr}(m_j,2m_j)$, which contain the given subspaces W_j.

Denote the roots of the polynomial $y(x,t|\mu,m,W)$ by $u_i(t|\mu,m,W)$, $i = 1,\dots,k$. Define

$\gamma(t|\mu,m,W) = (\gamma_1(t|\mu,m,W),\dots,\gamma_k(t|\mu,m,W))$ by formula

(5.38).

Let $\hat{S} \subset (\mathbb{C}^\times)^q \times \prod_{j=1}^q \mathrm{Gr}(m_j,2m_j)$ be the subset of points (μ,W), such that

(a) $\mu = (\mu_1,\dots,\mu_q)$ has distinct coordinates;

(b) $u(0|\mu,W) = (u_1(0|\mu,W),\dots,u_k(0|\mu,W))$ has distinct coordinates.

Theorem 5.14. For $(\mu,W) \in \hat{S}$, the map

$$\hat{S} \,:\, (\mu,W) \mapsto (u(0\,|\,\mu,W),\gamma(0\,|\,\mu,W)) \tag{5.58}$$

is inverse to the map in Corollary 5.10.

Proof. The proof of Theorem 5.14 is similar to the proof of Theorem 5.7. The key point of the proof is the following lemma.

Lemma 5.15. Functions $\Psi(x,t,z)$ and $y(x,t)$ given by (5.56) and (5.57), respectively, satisfy equation (1.9).

The proof of the lemma is based on the uniqueness of the Baker-Akhiezer function corresponding to the data (μ,W) and almost word by word follows the proof of Lemma 5.8.

6. Solution of the rational RS hierarchy

The goal of this section is to write explicitly equations describing time dependence of the roots $(u_i(t))$ of the polynomial $y(x,t)$ corresponding to the spectral data $(\mu,W) \in \hat{S}$.

It was proved in [**KZ**] that the dependence of $(u_i(t))$ in the variable t_1 coincides with the equation of motion of the RS system. Note that in [**KZ**] this result was proved for the elliptic RS system. The dependence of $(u_i(t))$ in the variables $\bar{t} = (\bar{t}_0, \bar{t}_1, \bar{t}_2, \ldots)$, defined by formula

$$x(z+1)^{-1} + \sum_{m=1}^{\infty} m\, t_m\, z^{m-1} = \bar{t}_0\,(z+1)^{-1} + \sum_{m=1}^{\infty} m\,\bar{t}_m\,(z+1)^{m-1}, \tag{6.1}$$

was identified in [**KZ**] with the pole dynamics of the elliptic (rational) solutions of the $2D$ Toda hierarchy. In [**I**] and [**Z**] it was proved that the latter coincides with the flows defined by the higher Hamiltonians $H_k = trL^k$ of the RS system, where L is the corresponding Lax matrix.

Remark. Note that the change of variables (6.1) is well-defined only under the assumption that there are only finitely many of nonzero time variables. Nevertheless, the corresponding triangular change of the vector fields is well-defined always:

$$\partial_{\bar{t}_0} = \partial_x,\ \partial_{\bar{t}_1} = \partial_{t_1},\ \partial_{\bar{t}_2} = \partial_{t_2} + 2\partial_{t_1},\ \partial_{\bar{t}_3} = \partial_{t_3} + 3\partial_{t_2} + 3\partial_{t_1},$$

$$\ldots .$$

6.1. Hierarchies of linear equations. In this section we show that for any spectral data (μ,W) the corresponding Baker-Akhiezer function $\Psi(x,t,z)$ given by formula (5.56) satisfies a hierarchy of linear equations.

Let $T_x = e^{\partial_x}$ be the shift operator acting on functions of x, $T_x : f(x) \mapsto f(x+1)$.

Lemma 6.1. Let $\Psi(x,t,z)$ be a formal series of the form

$$\Psi = z^k \Omega(x,t,z)\left(1 + \sum_{s=1}^{\infty} \xi_s(x,t) z^{-s}\right), \tag{6.2}$$

where $k \in \mathbb{Z}$ and $\xi_s(x,t)$ are some functions of x,t. Then for each m 1 there is a unique difference operator D_m in the variable x,

$$D_m = T_x^m + \sum_{i=1}^{m} w_{i,m}(x,t)\, T_x^{m-i}, \tag{6.3}$$

such that

$$D_m\Psi(x,t,z) = z^m\Psi(x,t,z) + O(z^{k-1})\Omega(x,t,z). \tag{6.4}$$

The coefficients $w_{i,m}(x,t)$ of these operators D_m are (explicit) difference polynomials in $\xi_s(x,t)$, $s = 1,...,m-1$.

Proof. Divide both sides of (6.4) by $\Omega(x,t,z)$ and compare the leading coefficients of Laurent series. That gives a triangular system of $m-1$ linear equations for $m-1$ unknown functions $w_{i,m}(x,t)$. The system is solved recurrently.

The following theorem follows from the uniqueness of the Baker-Akhiezer function.

Theorem 6.2. Let D_m be the operator defined in Lemma 6.1 by the BakerAkhiezer function $\Psi(x,t,z\,|\mu,W)$ given by (5.56). Then

(6.5) $$(\partial_{t_m} - D_m)\,\Psi(x,t,z\,|\,\mu,W) = 0\,, \qquad m \geqslant 1.$$

Proof. The definition of D_m in Lemma 6.1 implies that the left-hand side of (6.5) has the form $\tilde{R}\,\Omega$, where $\tilde{R}$ is a polynomial in z of degree $k-1$. The function $\tilde{R}\,\Omega$ satisfies the system of equations (5.52) defining Ψ. Therefore the coefficients of $\tilde{R}$ satisfy the homogeneous linear system of equation with matrix M as in (5.55). Hence, $\tilde{R} = 0$.

Remark. Lemma 5.15 is a particular case of Theorem 6.2 for $m = 1$.

The compatibility conditions of equations (6.5) imply:

Corollary 6.3. If the Baker-Akhiezer function Ψ is given by (5.56), then the corresponding operators D_m satisfy the equations

(6.6) $[\partial_{t_i} - D_i, \partial_{t_j} - D_j] = 0$, for all i,j.

Remark. The collection of equations (6.6) is the so-called Zakharov-Shabat presentation of a part of the 2*D Toda hierarchy*. We call the collection of equations (6.6) the *positive part* of the 2*D Toda hierarchy*, see Section 8.2.

6.2. Rational RS hierarchy.

6.2.1. Let $(u,\gamma) \in \mathrm{P}_k$ be a point of the phase space. Let $L = L(u,\gamma)$ be the Lax matrix. We define recurrently a set of rational functions $\bar{w}_{1,m}(x), \bar{w}_{2,m}(x), ..., \bar{w}_{m,m}(x)$ by the formula

(6.7) $$\bar{w}_{s,m}(x) = \sum_{i=1}^{k}\left(\frac{(L^{s-1}\gamma)_i}{x-u_i} - \frac{(L^{s-1}\gamma)_i}{x-u_i+m} - \sum_{\ell=1}^{s-1}\bar{w}_{\ell,m}(x)\frac{(L^{s-1-\ell}\gamma)_i}{x-u_i+m-\ell}\right),$$

a set of matrices $H_{1,m},...,H_{m,m}$ by the formulas

(6.8) $$(H_{s,m})_{ij} = \frac{\mathrm{res}_{x=u_i}\,\bar{w}_{s,m}(x)}{u_i-u_j+m-s}\,,$$

$$(H_{m,m})_{ij} = \widetilde{H}_{m,i}\delta_{ij} + (1-\delta_{ij})\,\frac{\operatorname{res}_{x=u_i}\bar{w}_{m,m}}{u_i-u_j}, \tag{6.9}$$

where $\widetilde{H}_{m,i}$ is defined by the Laurent expansion of $\bar{w}_{m,m}(x)$ at $x = u_i$,

$$\bar{w}_{m,m}(x) = \frac{\operatorname{res}_{x=u_i}\bar{w}_{m,m}}{x-u_i} + \widetilde{H}_{m,i} + O(x-u_i), \tag{6.10}$$

and the matrix M_m by the formula

$$M_m = \sum_{s=1}^{m} H_{s,m}L^{m-s}. \tag{6.11}$$

6.2.2. Let us return to the situation of Section 5.5. Let the spectral data (μ, m, W) be given. Let $y(x,t)$ be the polynomial defined by formula (5.57) and $u(t) = (u_i(t))_{i=1}^k$ its roots. Let $\gamma(t) = (\gamma_i(t))_{i=1}^k$ with $\gamma_i(t) = \partial_{t_1}u_i(t)$. Having the pair $(u(t),\gamma(t))$ we may define all the objects of Section 6.2.1, which will depend on t.

Let $\bar{t}_m$ be the variables defined in (6.1).

Theorem 6.4. The pair $(u(t),\gamma(t))$ satisfies the equations of motion of the hierarchy of the k particle rational RS system. Namely, for all $m \geq 1$ we have

$$\partial_{\bar{t}_m}u_i = \operatorname{res}_{x=u_i}\bar{w}_{m,m}(x), \tag{6.12}$$

$$\partial_{\bar{t}_m}\gamma_i = \sum_{j=1}^{k}\left((M_m)_{ij}L_{ji} - L_{ij}(M_m)_{ji}\right). \tag{6.13}$$

Proof. The following lemma gives the Lax presentation of these flows in terms of the RS system.

Lemma 6.5. Let the linear equation

$$\left(\partial_{\bar{t}_m} - T_x^m - \sum_{s=1}^{m}\bar{w}_{s,m}(x,\bar{t})\,T_x^{m-s}\right)\psi(x,\bar{t},\bar{z}) = 0 \tag{6.14}$$

with some (a'priory unknown) coefficients $\bar{w}_{s,m}(x,\bar{t})$ has a solution of the form

$$\psi(x,\bar{t},\bar{z}) = \left(1 + \sum_{i=1}^{k_n}\frac{C_i(\bar{t},\bar{z})}{x-u_i}\right)\bar{z}^{\,x}e^{\sum_m \bar{t}\,\bar{z}^m}, \tag{6.15}$$

where $\bar{z} = z + 1$ and C is given by (4.20) with the matrix L defined in (4.18) with $\gamma_i = \gamma_i(\bar{t}_m), u_i = u_i(\bar{t}_m)$. Then equations (6.12), (6.13) hold.

Proof. The vector C with the coordinates C_i given by (4.20) solves equation (4.19), i.e.

$$(\bar{z}L^0 - L)C = \gamma, \tag{6.16}$$

where $L^0 = E$ is the identity matrix. This equation easyly implies that for any s the equation

$$(\bar{z}^sL^0 - L^s)C = \sum_{\ell=0}^{s-1}\bar{z}L^{s-\ell-1}\gamma \tag{6.17}$$

holds.

The substitution of (6.15) into (6.14) gives the equation

$$\sum_{i=1}^{k}\left(\frac{\bar z^m C_i}{x-u_i}+\frac{(\partial_{\bar t_m}u_i)C_i}{(x-u_i)^2}+\frac{\partial_{\bar t_m}C_i}{x-u_i}\right) = \sum_{i=1}^{k}\frac{\bar z^m C_i}{x-u_i+m}+\sum_{s=1}^{m}\bar z^{m-s}\bar w_{s,m}\left(1+\sum_{i=1}^{k}\frac{C_i}{x-u_i+m-s}\right). \tag{6.18}$$

Using (6.17) and then equating the coefficients at $\bar z^\ell$ for $\ell = m$ 1 2 – ,m – ,...,0 at

both sides of the equation we recurrently find that $\bar w_{s,m}(x)$ are given by formulas (6.7). The remaining part of the equations (of order $O(\bar z^{-1})$) are linear equations containing $C(\bar z)$. Equating the coefficients at $(x-u_i)^{-2}$ we get equation (6.12). Equating the coefficients at $(x-u_i)^{-1}$ we get that the vector C satisfies the equation

$$\partial_{\bar t_m}C = (M_m - L^m)C, \tag{6.19}$$

where the matrix M_m is defined in (6.11). Comparing the leading coefficients at of the expansions in $\bar z^{-1}$ of the both sides of (6.19) we get

$$(M_m - L^m)\gamma = 0. \tag{6.20}$$

From (6.16) and (6.19) it follows that

$$[\partial_{\bar t_m} - M_m, L]C = -(M_m - L^m)\gamma = 0. \tag{6.21}$$

Since equation (6.21) holds for $C = C(z)$ we have

$$\partial_{\bar t_m}L = [M_m, L]. \tag{6.22}$$

The latter is the Lax presentation of equations (6.12) and (6.13).

In the framework of the dynamical r-matrix approach the matrices M_m were obtained in [**Su**].

Now Theorem 6.4 follows from Theorem 6.2.

7. Spectral transform for N-periodic Bethe ansatz equations

7.1. Spectral data for solutions of the Bethe ansatz equations. We begin this section by identification of the spectral data corresponding to solutions of the N-periodic Bethe ansatz equations. Recall that for a given sequence of generic polynomials $(y_n(x))_{n\in\mathbb{Z}}$ whose roots satisfy the Bethe ansatz equations the solutions $(\psi_n(x,z))_{n\in\mathbb{Z}}$ of the generating linear problem constructed in Section 4 are equal to

$$\psi_n(x,z) = z^n\psi(x,z\,|\,u^{(n)},\gamma^{(n)}), \tag{7.1}$$

where $(u_i^{(n)})_{i=1}^{k_n}$ are roots of $y_n(x)$ and $(\gamma_i^{(n)})_{i=1}^{k_n}$ are defined in (4.4). Notice that $\gamma^{(n)}$ depends on the polynomials $y_n(x)$ and $y_{n+1}(x)$, only. By definition of generic polynomials, we have $(u^{(n)},\gamma^{(n)}) \in P_{k_n}$.

equations, then the matrixLemma 7.1. If $(y_n(x))_{n\in Z}L(u$represents a solution of the$(n),\gamma(n))$ has only one eigenvalue$N$-periodic Bethe ansatz$\mu = 1$ (of multi-

plicity k_n).

Proof. By Theorem 5.9 the function

$$\Psi(x,z\,|u^{(0)},\gamma^{(0)}) = \det L(u^{(0)},\gamma^{(0)})\psi_0(x,z)$$

satisfies equations (5.48) with $W_j^{(0)} := W_j(u^{(0)},\gamma^{(0)})$. From equation (4.2) it then follows that for functions $\det L(u^{(0)},\gamma^{(0)})\psi_n(x,z)$ for $n \geqslant 0$ equations (5.48) with $W_j^{(0)} := W_j(u^{(0)},\gamma^{(0)})$ hold, as well. The N-periodicity of (y_n) implies that $\psi_N = z^N\psi_0(x,z)$. Hence, $\Psi(x,z\,|u^{(0)},\gamma^{(0)})$ satisfies equation (5.48) and the equations

$$\operatorname{res}_{z=\mu_j-1} \frac{g(z)z^N\Psi(x,z\,|\,u^{(0)},\gamma^{(0)})}{(z-\mu_j+1)^{2k_j}} = 0,\ \forall g \in W_j(u^{(0)},\gamma^{(0)}) \tag{7.2}$$

Since $\Psi(x,z\,|u^{(0)},\gamma^{(0)})$ defines $W_j^{(0)}$ uniquely, equations (7.2) imply that $W_j^{(0)}$ is invariant under the action of the operator of multiplication by z^N. It follows from Lemmas 5.11 and 5.12 that $\Psi(x,z\,|u^{(0)},\gamma^{(0)})$ has zero of order m_j at $z = \mu_j - 1$ for any $\mu_j \neq 1$, or equivalently that the function $\psi(x,z\,|u^{(0)},\gamma^{(0)},z)$ is holomorphic at $z = \mu_j - 1$.

Now the reference to Lemma 5.5 finishes the proof.

Remark. In Lemma 4.5 we proved that the poles of solutions $(\psi_n(x,z))_{n\in Z}$ of the generating problem corresponding to a sequence of polynomials $(y_n(x))_{n\in Z}$ (possibly non-periodic) are n-independent away from $z = 0$. The lemma above gives a stronger statement: for periodic sequences of polynomials the solutions $(\psi_n(x,z))_{n\in Z}$ are holomorphic at $z \neq 0$.

7.2. The inverse spectral transform: construction. By Theorem 5.9 and Lemma 7.1 the functions $\psi_n(x,z)$ constructed in Section 4 are uniquely defined by a sequence of points $W^{(n)} \in \mathrm{Gr}(k_n,2k_n)$ corresponding to the only eigenvalue $\mu = 1$ of the matrix $L(u^{(n)},\gamma^{(n)})$. In this section we explicitly describe the data defining such sequences and present in a closed form the construction of the solutions of the N-periodic Bethe ansatz equations.

The parameters of the construction are nonnegative integers ν, D, and an $(N+\nu)\times(D+1)$-matrix

$$A = (a_{k,j}), \qquad k = 1,\dots,N+\nu, \qquad j = 0,\dots,D\,.$$

We say that the matrix A is *nondegenerate* if for any $n = 0,\dots,N$, the matrix $A^{(n)}$ composed of the first $n+\nu$ rows of the matrix A has rank $n+\nu$.

Two matrices A, A' are called *equivalent* if $A = GA'$ where G is an $(N+\nu)\times(N+\nu)$ nondegenerate matrix of the form

$$G = \begin{pmatrix} g & 0 \\ * & g_1 \end{pmatrix} \tag{7.3}$$

where g is a $\nu\times\nu$-matrix and g_1 is lower-triangular.

We call A *reducible* if there is a nondegenerate $\nu\times\nu$-matrix H such that

$$HA^{(0)} = \begin{pmatrix} E & 0 \\ 0 & * \end{pmatrix} \tag{7.4}$$

where E is the $\ell\times\ell$ unit matrix with $\ell \geqslant 1$. We call A *irreducible* otherwise.

7.3. Function $\Omega(x,t,z)$. Below we present some notations and properties of the function $\Omega(x,t,z)$ defined in (5.29),

$$\Omega(x,t,z) \;=\; (1+z)^x e^{\sum_{j=1}^{\infty} t_j z^j}. \tag{7.5}$$

The function $\Omega(x,t,z)$ satisfies the equation

$$(z+1)\Omega(x,t,z) = \Omega(x+1,t,z) = \partial_{t_1}\Omega(x,t,z) \tag{7.6}$$

and, more generally, the equations

$$\begin{aligned} z^\ell\,\Omega(x,t,z) &= \sum_{m=0}^{\ell}(-1)^{\ell-m}\binom{\ell}{m}\Omega(x+m,t,z) = \Delta^{(\ell)}\Omega(x,t,z) \\ \Omega(x+\ell,t,z) &= \partial_{t_1}^{\ell}\,\Omega(x,t,z) = \partial_{t_\ell}\Omega(x,t,z)\,,\ \ \ell\geqslant 1. \end{aligned} \tag{7.7}$$

Introduce the polynomials $\chi_n(x,t)$, $n\in\mathbb{Z}_{\geqslant 0}$, by using the expansion

$$\Omega(x,t,z) \;=\; \sum_{n=0}^{\infty}\chi_n(x,t)z^n, \tag{7.8}$$

where $\chi_0(x,t) = 1$,

$$\chi_n(x,t)|_{t=0} = \binom{x}{n}, \qquad \chi_n(x,t)|_{x=0,\,t'=0} = t_1^n, \tag{7.9}$$

$$\deg_x\chi_n(x,t) = \deg_{t_1}\chi_n(x,t) = n.$$

For n 0, we have

$$\chi_n(x+1,t) - \chi_n(x,t) = \partial_{t_1}\chi_n(x,t) = \chi_{n-1}(x,t), \tag{7.10}$$

where $\chi_{-1}(x,t) = 0$. More generally, we have

$$\Delta^{(\ell)}\chi_k(x,t) = \partial_{t_1}^{\ell}\chi_n(x,t) = \partial_{t_\ell}\chi_n(x,t) = \chi_{k-\ell}(x,t). \tag{7.11}$$

Let us write

$$e^{\sum_{j=1}^{\infty} t_j z^j} = \sum_{k=0}^{\infty} h_k(t) z^k,$$

where $h_0(t) = 1$. Then

$$\chi_n(x,t) = \sum_{k=0}^{n} h_{n-k}(t) \binom{x}{k}. \tag{7.12}$$

Given the spectral data $A = (a_{k,j})$, define the polynomials $f_k(x,t)$ by the formula

$$f_k(x,t) = \sum_{j=0}^{D} a_{k,j}\, \chi_j(x,t), \qquad k = 1, \ldots, N+\nu\,. \tag{7.13}$$

For $k = 1,\ldots,N + \nu$, introduce the differential operators

$$\mathcal{D}_k = \sum_{j=0}^{D} \frac{a_{k,j}}{j!} \frac{\partial^j}{\partial z^j}\,. \tag{7.14}$$

Then

$$\Big[\mathcal{D}_k \Omega(x,t,z)\Big]_{z=0} = f_k(x,t). \tag{7.15}$$

Lemma 7.2. If A is nondegenerate, then for every $n = 0,\ldots,N$,

(i) the discrete Wronskian $\widehat{W}(f_1,\ldots,f_{n+\nu})$ is nonzero;

(ii)

$$\widehat{W}(f_1,\ldots,f_{n+\nu}) = \mathrm{Wr}_{t_1}(f_1,\ldots,f_{n+\nu}), \tag{7.16}$$

where $\mathrm{Wr}_{t_1}(f_1,\ldots,f_{n+\nu}) = \det_{i,j=1}^{m}(\partial_{t_1}^{j-1} f_i)$ is the standard Wronskian with respect to the variable t_1;

(iii)

$$\deg_x \widehat{W}(f_1,\ldots,f_{n+\nu}) = \deg_{t_1} \widehat{W}(f_1,\ldots,f_{n+\nu}). \tag{7.17}$$

7.4. Baker-Akhiezer functions. For every $n = 0,\ldots,N$, consider a polynomial of degree $n + \nu$ in z of the form

$$R_n(x,t,z) = z^{n+\nu}\left(1 + \sum_{\ell=1}^{n+\nu} \xi_\ell^{(n)}(x,t)\, z^{-\ell}\right), \tag{7.18}$$

whose coefficients are some functions in x, t.

Lemma 7.3. If A is nondegenerate, then for any $n = 0,\ldots,N$, there exists a unique function $\psi_n(x,t,z)$ of the form

(7.19) $\psi_n(x,t,z) = \Omega(x,t,z)R_n(x,t,z)$, such that

$$\Big[\mathcal{D}_k \psi_n(x,t,z)\Big]_{z=0} = 0, \qquad k = 1,\ldots,n+\nu\,. \tag{7.20}$$

For fixed n,x the function $\psi_n(x,t,z)$ is a particular case of the *Baker-Akhiezer functions* introduced in [**K5**] to construct rational solutions of the KP equation.

Proof. Using equation (7.7), we rewrite equation (7.20) as

$$(7.21)\quad \Big[\mathcal{D}_k\Big(\sum_{m=0}^{n+\nu}(-1)^{n+\nu-m}\binom{n+\nu}{m}\Omega(x+m,t,z) + \sum_{\ell=1}^{n+\nu}\xi_\ell^{(n)}(x,t)\sum_{m=0}^{n+\nu-\ell}(-1)^{n+\nu-\ell-m}\binom{n+\nu-\ell}{m}\Omega(x+m,t,z)\Big)\Big]_{z=0}=0.$$

Using (7.8) and (7.13) we rewrite (7.21) as

$$(7.22)\quad \sum_{m=0}^{n+\nu}(-1)^{n+\nu-m}\binom{n+\nu}{m}f_k(x+m,t) + \sum_{\ell=1}^{n+\nu}\xi_\ell^{(n)}(x,t)\sum_{m=0}^{n+\nu-\ell}(-1)^{n+\nu-\ell-m}\binom{n+\nu-\ell}{m}f_k(x+m,t)=0.$$

The system of equations (7.22) is the systems of n+ν inhomogeneous linear equations for the coefficients $\xi_\ell^{(n)}(x,t)$,

$$(7.23)\quad \sum_{\ell=1}^{n+\nu}M_{k,\ell}^{(n)}(x,t)\,\xi_\ell^{(n)}(x,t)=F_k^{(n)}(x,t),$$

where

$$\begin{aligned} M_{k,\ell}^{(n)}(x,t) &= \sum_{m=0}^{n+\nu-\ell}(-1)^{n+\nu-\ell-m}\binom{n+\nu-\ell}{m}f_k(x+m,t)\\ &= \Delta^{n+\nu-\ell}f_k(x,t)\\ F_k^{(n)}(x,t) &= -\sum_{m=0}^{n+\nu}(-1)^{n+\nu-m}\binom{n+\nu}{m}f_k(x+m,t)\\ &= -\Delta^{n+\nu}f_k(x,t). \end{aligned}\quad (7.24)$$

Using (7.10) we may rewrite

$$(7.25)\quad \begin{aligned} M_{k,\ell}^{(n)}(x,t) &= \sum_{j=\ell-1}^{D}a_{k,j}\,\chi_{j-\ell+1}(x,t),\\ F_k^{(n)}(x,t) &= -\sum_{j=n+\nu}^{D}a_{k,j}\,\chi_{j-n-\nu}(x,t), \end{aligned}$$

cf. formula for $f_k(x,t)$ in (7.13).

Formula (7.24) implies that the determinant of the matrix $M^{(n)}(x,t)$ equals

$$(7.26)\quad y_n(x,t):=\det M^{(n)}(x,t)=\widehat{W}(f_1,\dots,f_{n+\nu}),$$

the discrete Wronskian of the polynomials $f_1(x,t),\dots,f_{n+\nu}(x,t)$ with respect to x. By Lemma 7.2 the determinant is a nonzero polynomial. Hence equations (7.20) determine uniquely a function $\psi_n(x,t,z)$. The lemma is proved.

Below we give a determinant formula for $\psi_n(x,t,z)$. Define an $(n+\nu+1)\times(n+\nu+1)$ matrix $\widehat{M}^{(n)}(x,t,z)$, whose rows and columns are labeled by indices $1,\dots,n+\nu+1$ and entries are given by the formulas:

(7.27)
$$\begin{aligned} \widehat{M}^{(n)}_{n+\nu+1,\ell} &= z^{\ell-1}, & \ell = 1,\dots,n+\nu+1 \\ \widehat{M}^{(n)}_{\ell,n+\nu+1} &= -F^{(n)}_\ell, & \ell = 1\dots,n+\nu, \\ \widehat{M}^{(n)}_{k,\ell} &= M^{(n)}_{k,\ell}, & k,\ell = 1,\dots,n+\nu. \end{aligned}$$

Using the determinant expansion of $\widehat{M}^{(n)}(x,t,z)$ by the last row we obtain

(7.28)
$$\psi_n(x,t,z) = \Omega(x,t,z)\,\frac{\det \widehat{M}^{(n)}(x,t,z)}{y_n(x,t)}.$$

Here is a useful formula for $\xi_1^{(n)}(x,t)$,

(7.29)
$$\xi_1^{(n)}(x,t) = -\frac{\Delta y_n(x,t)}{y_n(x,t)} = -\frac{\partial_{t_1} y_n(x,t)}{y_n(x,t)}.$$

Theorem 7.4. The Baker-Akhiezer functions $(\psi_m(x,t,z))^N_{m=0}$ satisfy equations (4.2) with indices $n = 0,\dots,N-1$ in which the functions $v_n(x,t)$ are given in terms of $y_n(x,t)$ and $y_{n+1}(x,t)$ by formula (4.3).

Proof. Consider the function

(7.30)
$$\tilde\psi_{n+1}(x,t,z) = \psi_n(x+1,t,z) - v_n(x,t)\psi_n(x,t,z) - \psi_{n+1}(x,t,z).$$

We need to show that $\tilde\psi_{n+1}(x,t,z)$ is the zero function.

We have

$$\begin{aligned} \tilde\psi_{n+1}(x,t,z) &= \Omega(x+1,t,z)R_n(x+1,t,z) \\ &\quad - v_n(x,t)\Omega(x,t,z)R_n(x,t,z) - \Omega(x,t,z)R_{n+1}(x,t,z) \\ &= \Omega(x,t,z)\Big((1+z)R_n(x+1,t,z) - v_n(x,t)R_n(x,t,z) - R_{n+1}(x,t,z)\Big) \\ &= \Omega(x,t,z)\,\tilde R_{n+1}(x,t,z), \end{aligned}$$

where $\tilde R_{n+1}(x,t,z)$ is a polynomial in z of degree at most $n+\nu$,

(7.31)
$$\tilde R_{n+1}(x,t,z) = \sum_{\ell=1}^{n+\nu+1} \tilde\xi_\ell^{(n+1)}(x,t)\, z^{\ell-1}.$$

Each of the three functions $\psi_n(x+1,t,z)$, $v_n(x,t)\psi_n(x,t,z)$, $\psi_{n+1}(x,t,z)$ satisfy the equations (7.20) for $k = 1,\dots,n+\nu$. Hence the function $\tilde\psi_{n+1}(x,t,z)$ satisfies equations (7.20) for $k = 1,\dots,n+\nu$.

Lemma 7.5. The function $\tilde\psi_{n+1}(x,t,z)$ satisfies equation (7.20) for $k = n+\nu+1$. Proof.

Recall the function $\det \widehat{M}^{(n)}(x,t,z)\Omega(x,t,z)$, see (7.27). Then

(7.32)
$$\mathcal{D}_{n+\nu+1}\Big[\big(\det \widehat{M}^{(n)}(x,t,z)\big)\,\Omega(x,t,z)\Big]_{z=0} = \det M^{(n+1)}(x,t) = y_{n+1}(x,t),$$

see formulas (7.7) and (7.15). Now we apply the operator $\mathcal{D}_{n+\nu+1}[\,]_{z=0}$ to both sides of equation (7.30). We have $\mathcal{D}_{n+\nu+1}\left[\psi_{n+1}(x,t,z)\right]_{z=0} = 0$ by definition of $\psi_{n+1}(x,t,z)$. Hence

$$\mathcal{D}_{n+\nu+1}\left[\tilde\psi_{n+1}(x,t,z)\right]_{z=0} = \mathcal{D}_{n+\nu+1}\left[\psi_n(x+1,t,z) - v_n(x,t)\psi_n(x,t,z)\right]_{z=0}$$
$$= \mathcal{D}_{n+\nu+1}\left[\left(\frac{\det \widehat{M}^{(n)}(x+1,t,z)}{y_n(x+1,t)} - v_n(x,t)\,\frac{\det \widehat{M}^{(n)}(x,t,z)}{y_n(x,t)}\right)\Omega(x,t,z)\right]_{z=0}$$
$$= \frac{y_{n+1}(x+1,t)}{y_n(x+1,t)} - \frac{y_n(x,t)y_{n+1}(x+1,t)}{y_n(x+1,t)y_{n+1}(x,t)}\,\frac{y_{n+1}(x,t)}{y_n(x,t)} = 0.$$

Comparing the system of equations (7.23) with $k = 1,\dots,n+\nu$ for the coefficients $(\xi_\ell^{(n)}(x,t))_{\ell=1}^{n+\nu}$ with the system of equations (7.20) with $k = 1,\dots,n+\nu+1$ for the coefficients $(\tilde\xi_\ell^{(n+1)}(x,t))_{\ell=1}^{n+\nu+1}$ we conclude that the coefficients $(\tilde\xi_\ell^{(n+1)}(x,t))_{\ell=1}^{n+\nu+1}$ satisfy the system of homogeneous equations

$$\sum_{\ell=1}^{n+\nu+1} M_{k,\ell}^{(n+1)}(x,t)\,\tilde\xi_\ell^{(n+1)}(x,t) = 0 \tag{7.33}$$

with $k = 1,\dots,n+\nu+1$. According to our previous reasonings the determinant $\det M^{(n+1)}(x,t) = y_{n+1}(x,t)$ of the matrix of this homogeneous system is a nonzero polynomial. Hence all the functions $\tilde\xi_\ell^{(n+1)}(x,t)$ are the zero functions, the function $\tilde\psi_{n+1}(x,t,z)$ is the zero function, the functions $\psi_n(x,t,z)$ with $n = 0,\dots,N-1$ satisfy equations (4.2), and the theorem is proved.

7.5. Reconstruction of A. Let A and $\tilde A$ be two nondegenerate $(N+\nu)\times(D+1)$-matrices,

$$A = (a_{k,j}), \qquad \tilde A = (\tilde a_{k,j}), \qquad k = 1,\dots,N+\nu, \qquad j = 0,\dots,D.$$

Let $(\psi_m(x,0,z))_{m=0}^N$ and $(\tilde\psi_m(x,0,z))_{m=0}^N$ be the corresponding Baker-Akhieser functions given by the above construction.

Theorem 7.6. Assume that

$$\psi_n(x,0,z) = \tilde\psi_n(x,0,z), \qquad n = 0,\dots,N. \tag{7.34}$$

Then $A = G\tilde A$ for a matrix G as in (7.3).

Proof. For any n the function $\psi_n(x,0,z)$ is given by the formulas (7.23), (7.24). Consider the linear difference operator of order $n+\nu$,

$$\Delta^{(n+\nu)} + \xi_{(1)}(x,0)\Delta^{(n+\nu-1)} + \xi_{(2)}(x,0)\Delta^{(n+\nu-2)} + \cdots + \xi_{(n+\nu)}(x,0). \tag{7.35}$$

By formulas (7.23), (7.24) the kernel of this difference operator is generated by the polynomials $f_1(x,0),\dots,f_{n+\nu}(x,0)$ given by formula (7.13).

If two matrices $A,\tilde{A}$ have the same $\psi_n(x,0,z)$ and $\tilde{\psi}_n(x,0,z)$, then the $n+\nu$ dimensional space generated by the polynomials $f_1(x,0),\dots,f_{n+\nu}(x,0)$ coincides with the space generated by the polynomials $\tilde{f}_1(x,0),\dots,\tilde{f}_{n+\nu}(x,0)$. Hence $A = G\tilde{A}$ for suitable G.

7.6. Periodicity constraint. Given spectral data $A = (a_{k,j})$, the construction of Section 7.4 gives $y_0(x,t),\dots,y_N(x,t)$ and $\psi_0(x,t,z),\dots,\psi_N(x,t,z)$. We say that these functions *extend periodically* if there exist sequences $(y_n(x,t))_{n\in Z}$ and $(\psi_n(x,t,z))_{n\in Z}$ such that

$$y_{n+N}(x,t) = y_n(x,t), \qquad \psi_{n+N}(x,t,z) = z^N\psi_n(x,t,z), \qquad n \in Z,$$

and the sequence $(\psi_n(x,t,z))_{n\in Z}$ satisfies equations (4.2) with $(v_n(x,t))_{n\in Z}$ given by (4.3) in terms of $(y_n(x,t))_{n\in Z}$.

It is clear that the periodic extension is possible if and only if

$$y_N(x,t) = y_0(x,t), \ \psi_N(x,t,z) = z^N\psi_0(x,t,z). \tag{7.36}$$

Our goal is to identify matrices A for which the periodicity equations (7.36) hold.

7.7. Construction of matrices A**.** Given ν, let W be an $(N+\nu)\times(N+\nu)$ matrix such that its upper-right $\nu\times\nu$ corner U is *nilpotent*,

$$W = \begin{pmatrix} V & U \\ * & * \end{pmatrix} \quad \text{and} \quad U^r = 0 \quad \text{for some } r < \nu. \tag{7.37}$$

Using W we construct an $(N+\nu)\times N(\nu+1)$-matrix $A = A(W)$ in three steps.

First using V and U we construct a $\nu\times N\nu$ matrix Q as follows. Let $V = (v_1,\dots,v_N)$ be column vectors of V and $Q = (q_1,\dots,q_{N\nu})$ column vectors of Q. Define $q_j = v_j$ for $j = 1,\dots,N$. Define q_j for $j > N$ recursively by the formula

$$q_{N+j} = Uq_j. \tag{7.38}$$

Define an $(N+\nu)\times N(\nu+1)$-matrix P by the formula

$$P = \begin{pmatrix} E & 0 \\ 0 & Q \end{pmatrix}, \tag{7.39}$$

where E is the $N\times N$ unit matrix. Define the matrix A by the formula

$$A = WP. \tag{7.40}$$

It is easy to see that the matrix A has the form

$$A = \begin{pmatrix} Q & 0 \\ * & * \end{pmatrix} \tag{7.41}$$

7.8. Properties of the construction.

Lemma 7.7. If a matrix $A = A(W)$ is given by the construction of Section 7.7, then the functions $y_0(x,t),...,y_N(x,t)$ and $\psi_0(x,t,z),...,\psi_N(x,t,z)$ extend periodically.

Proof. The functions $y_0(x,t)$, $\psi_0(x,t,z)$ are determined by the first ν rows of A. That gives ν equations (7.20) for $\psi_0(x,t,z)$. The functions $y_N(x,t)$, $\psi_N(x,t,z)$ are determined by the full matrix A. That gives $N + \nu$ equations (7.20) for $\psi_N(x,t,z)$. The periodicity constraint (7.36) holds if the space of linear combinations of equations defining $\psi_N(x,t,z)$ contains N equations $\partial_z^{(j)}\psi_N(x,t,z)|_{z=0} = 0$, $j = 0,...,N - 1$, and ν equations (7.20) defining $\psi_0(x,t,z)$ in which the operators $\mathcal{D}_k = \sum_j \frac{a_{k,j}}{j!}\frac{\partial}{\partial z^j}$ are replaces with the operators $\mathcal{D}_k = \sum_j \frac{a_{k,j}}{j!}\frac{\partial}{\partial z^{j+N}}$. The relations (7.39), (7.40), (7.41) mean exactly that. The lemma is proved.

Let $A = (a_{ij})$ be a nondegenerate $(N + \nu) \times (D + 1)$-matrix. Let $y_0(x,t),..., y_N(x,t)$ and $\psi_0(x,t,z),...,\psi_N(x,t,z)$ be the associated functions. Let m be a positive integer. Define the $(N +\nu)\times(D+1+m)$-matrix $\hat{A} = (\hat{a}_{ij})$ by the formula

$$\begin{aligned} \hat{a}_{ij} &= a_{ij}, \qquad j \leqslant D, \\ \hat{a}_{ij} &= a_{ij} \qquad\qquad , \qquad j > D. \end{aligned}$$

We call $\hat{A}$ the m-extension of A. Let $\hat{y}_0(x,t),...,\hat{y}_N(x,t)$ and $\hat{\psi}_0(x,t,z),..., \hat{\psi}_N(x,t,z)$ be the functions associated with $\hat{A}$. Clearly, we have

(7.42) $\hat{y}_n(x,t) = y_n(x,t)$, $\qquad \hat{\psi}_n(x,t,z) = \psi_n(x,t,z)$, $\qquad n = 0,...,N.$

Let $A = (a_{ij})$ be a nondegenerate $(N+\nu)\times(D+1)$-matrix with associated functions $y_0(x,t),...,y_N(x,t)$ and $\psi_0(x,t,z),...,\psi_N(x,t,z)$ which extend periodically Let $\hat{A}$ be the N-extension of the matrix of A. According to (7.42) the matrix $\hat{A}$ has the same associated functions $y_0(x,t),...,y_N(x,t)$ and $\psi_0(x,t,z),...,\psi_N(x,t,z)$ which extend periodically.

Lemma 7.8. Under these assumptions the matrix $\hat{A}$ is given by the construction of Section 7.7, namely, we have $\hat{A} = \hat{A}(\hat{W})$ for a suitable $\hat{W}$.

Proof. We have $\psi_N(x,0,z) = z^N\psi_0(x,0,z)$ and the function $\psi_N(x,0,z)$ is defined by the $(N + \nu) \times (D + 1 + N)$-matrix $\hat{A}$. The same function $\psi_N(x,0,z)$ is defined also by the $(N + \nu) \times (D + 1 + N)$-matrix

$$P = \begin{pmatrix} E & 0 \\ 0 & A^{(0)} \end{pmatrix}, \tag{7.43}$$

where E is the $N \times N$ unit matrix and $A^{(0)}$ is the $\nu \times (D + 1)$-matrix formed by the first ν rows of the matrix A. By Theorem 7.6 this means that $\hat{A} = \hat{W}P$ for a suitable matrix $\hat{W}$. It remains to show that the upper-right ν corner of $\hat{W}$, denoted in (7.37) by U is nilpotent. As it was already noted above from equations (7.39), (7.40) and 7.41 it follows that the columns q_j of $A(0)$ should satisfy equation (7.38). Since $A^{(0)}$ is of rank ν and $q_j = 0$ for $j > D$ we get that $U^r = 0$ for some r. If that holds for some r then $r < \nu$. From the latter it follows that the integer D used in the construction in the N periodic case is bounded by $D \leqslant N\nu$.

Theorem 7.9. If an N-periodic sequence of polynomials ($y_1^0(x), \dots, y_N^0(x))$ represents a solution of the Bethe ansatz equations (2.1), then there exists a matrix $A = A(W)$ given by the construction of Section 7.7 such that the associated polynomials $y_0(x,t),\dots,y_N(x,t)$ have the property:

(7.44) $$y_n(x,0) = y_n^0(x), \qquad n = 1,\dots,N.$$

Proof. By Lemma 7.1 the function $\psi_0(x,z)$ corresponding to a sequence polynomials $(y_n(x))_{n\in\mathbb{Z}}$ representing a periodic solutions of the Bethe ansatz equation has the form

(7.45) $$\psi_0(x,z) = \psi(x,z\,|\,u^{(0)},\gamma^{(0)}) = (z+1)^x \left(1 + \sum_{i=1}^{\nu} \xi_i^{(0)}(x) z^{-i}\right)$$

with $\xi_\nu^{(0)}$ / $= 0$. The integer $\nu \leqslant k_0$ is the order of the pole of ψ_0 at $z = 0$. By Theorem 5.9 the function $z^{m_0}\psi_0(x,z)$ satisfies (5.48) for any $g \in W_0(u^{(0)},\gamma^{(0)})$. The space $W_0(u^{(0)},\gamma^{(0)_0})$ is a k_0-dimensional subspace of polynomials of degree $2k_0 - 1$. The function $z^k\,\psi_0(x,z)$ has zero of order $k_0 - \nu$ at $z = 0$. $\sim$ Then, by Lemma 5.11 the polynomials $z^{2k_0-1-\ell}$ for $\ell = 0,\dots,k_0-\nu-$ 1 are in $W_0(u^{(0)},\gamma^{(0)})$. Hence, the space $W_0(u^{(0)},\gamma^{(0)})$ contains a ν- ν dimensional subspace $W^{(0)} \subset W_0(u^{(0)},\gamma^{(0)})$ of polynomials of degree $k_0 + \nu - 1$ such that the function

(7.46) $$z^\nu \psi(x,z\,|\,u^{(0)},\gamma^{(0)}) = (z+1)^x \left(z^\nu + \sum_{i=1} \xi_i^{(0)}(x) z^{\nu-i}\right)$$

satisfies the equations $$_{z=0} \frac{g(z) z^{k_0}\psi_0(x,z)}{z^{2k_0}} = \operatorname{res}_{z=0} \frac{g(z)(z\ \ \psi_0(x,z))}{z^{k_0+\nu}} = 0\,, \quad \forall g \in$$ ν

(7.47) res $W^{(0)}$. polynomials

Choose a basis $g_k(z), k = 1,\dots\nu$, in the space $W^{(0)}$. The coefficients $a_{k,j}$ of these

(7.48) $$g_k(z) = \sum_{j=0}^{m_0+\nu-1} a_{k,j} z^{m_0+\nu-j-1}$$

define a $\nu \times (k_0 + \nu)$-matrix $\tilde{A}^{(0)}$, which for any $D \geqslant k_0 + \nu - 1$ can be trivially extended to a $\nu \times (D + 1)$-matrix $A^{(0)}$ by setting $a_{k,j} = 0, j \; k_0 + \nu$. Then equations (7.47) coincide with equations (7.20) defining the Baker-Akhiezer function $\psi_0(x,0,z \,|A^{(0)})$, where we have included in the notation the dependence of the Baker-Akhiezer function on the defining matrix A^0, i.e.

$$z^\nu \psi_0(x,z) = \psi_0(x,0,z\,|A^{(0)}). \tag{7.49}$$

Applying recurrently equation (4.2) we get that for $n \geqslant 0$ the solution of the linear generating problem has the form

$$z^{\nu}\psi_n(x,z) = (z+1)^x \left(z^{n+\nu} + \sum_{i=1}^{n+\nu} \xi_i^{(n)}(x) z^{n+\nu-i}\right). \tag{7.50}$$

Since $z^\nu\psi_n(x,t) = z^{n+\nu}\psi(x,z\,|u^{(n)},\gamma^{(n)})$, we a'priory know that the coefficients

$\xi_i^{(n)}(x)$ are defined by a nondegenerate system of equations of the form (4.2) defined by an $(n + \nu) \times (D + 1)$ matrix $A^{(n)}$ for sufficiently large D. From (4.2) it follows that $z^\nu\psi_{n+1}(x,z)$ satisfies the system of $(n+\nu)$ linear equations defining $z^\nu\psi_n(x,z)$. Hence, $A^{(n+1)}$ can be chosen such that its first $(n + \nu)$ rows coincides with the matrix $A^{(n)}$. Then we define A in the construction of Section 5 to be equal to

$A^{(N)}$. Theorem 7.9 is proved.

7.9. Remark on difference operators. In formula (7.35) we identified, roughly speaking, the Baker-Akhiezer function $\psi_n(x,0,z)$ with the linear difference operator of order $n + \nu$, whose kernel is spanned by the polynomials $f_1(x,0), ..., f_{n+\nu}(x,0)$. From that point of view, the Baker-Akhiezer function $\psi_{n+1}(x,0)$ is identified with the linear difference operator of order $n + \nu + 1$, whose kernel is spanned by the polynomials $f_1(x,0), ..., f_{n+\nu}(x,0)$ and one new polynomial $f_{n+\nu+1}(x,0)$. The main formula of this paper, that is, formula (4.2), is the formula expressing the second of these difference operators in terms of the first one. The periodicity property of the functions $\psi_0(x,t,z)$, ..., $\psi_N(x,t,z)$ can be reformulated as a special relation between the kernels of the differential operators corresponding to $\psi_0(x,t,z)$ and $\psi_N(x,t,z)$. That property is implicitly explained in Sections 7.6 – 7.8. A version of this point of view is developed in Section 10.

7.10. Main theorem on commuting flows. By Theorem 7.9 any solution of the N-periodic Bethe ansatz equations is defined by some matrix A. Theorem 7.4 implies that the space of solutions of the N-periodic Bethe ansatz equations is invariant with respect to times t under the deformations defined by the BakerAkhiezer functions. For each n the corresponding function $\Psi_n(x,t,z)$ is a particular case of the Baker-Akhiezer function corresponding to the rational k_n-particle rational RS system. Hence, by Theorem 6.4 the dependence of roots of the corresponding polynomial $y_n(x,t)$ is described by equations of the rational RS system. Therefore we have the following theorem.

Theorem 7.10. Let $(y_n(x))_{n\in\mathbb{Z}}$ be an N-periodic sequence of polynomials of degrees (k_n) representing a solution of the N-periodic Bethe ansatz equations. Then the correspondence

(7.51) $$(y_n) \longmapsto (u^{(n)}, \gamma^{(n)}),$$

where $\gamma = (\gamma_1^{(n)}, \dots, \gamma_{k_n}^{(n)})$ is given by (4.4), is an embedding of the space of solutions of the Bethe ansatz equations into the product of N phase spaces of the k_n-particle rational RS systems, $n = 1,...,N$. The image of this map is invariant under the hierarchy the rational RS systems (6.12), (6.13) acting diagonally on the product of the phase spaces.

Consider the extension of the sequence $y = (y_n(x))_{n\in\mathbb{Z}}$ to the family $y(t) = (y_n(x,t))_{n\in\mathbb{Z}}$, defined by Theorems 7.4 and 7.9. Then the correspondence in (7.51) sends the family $y(t)$ to a solution of the rational RS hierarchy.

8. Bethe ansatz equations and integrable hierarchies

The existence of the one parameter family $\Psi(z)$ of solutions of equations (4.2) having the form (4.7) reveals the connection of the Bethe ansatz equations (4.1) with basic hierarchies of the soliton theory. We begin this section with a brief review of the hierarchy, which is referred throughout the paper as the positive part of the 2D-Toda hierarchy.

8.1. Pseudo-difference operators. We regard sequences $g = (g_n)_{n\in\mathbb{Z}}$ with $g_n \in \mathbb{C}$ as elements of the ring of functions of the discrete variable n. In particular we have addition $f + g$ and multiplication fg of sequences defined by the formulas $(f+g)_n = f_n + g_n$, $(fg)_n = f_n g_n$. Let T be the shift operator acting on sequences $g = (g_n)_{n\in\mathbb{Z}}$ by the formula, where $(Tg)_n =$ $T : f \mapsto Tg$ g_{n+1}.

The space of pseudo-difference operators is the space F of Laurent polynomials in T^{-1}, whose coefficients are functions of the variable $n \in \mathbb{Z}$, i.e.

(8.1) $$F = \sum_{s=-M}^{\infty} f_s T^{-s}, \qquad f_s = (f_{n,s}), \qquad n \in \mathbb{Z},$$

for some integer M. Recall that the coefficient f_0 in (8.1) is called the *residue* of F,

(8.2) $$\mathrm{res}_T F := f_0.$$

The ring structure on F is defined by the ring structure on the space of coefficients and the composition rule

(8.3) $T(fT^m) := (Tf)T^{m+1}$, where f is a sequence.

In what follows we will apply the pseudo-differential operators to sequences $\phi(z) = (\phi_n(z))_{n\in\mathbb{Z}}$, whose elements are formal Laurent series in z of the form

(8.4)
$$\varphi_n(z) = z^n \left(\sum_{s=-K}^{\infty} \varphi_{n,s} z^{-s} \right),$$
where K is some integer.

8.2. Positive part of the $2D$ **Toda hierarchy.** The difference analog of the KP hierarchy is defined almost verbatim to the definition in the continuous case, cf. [**SW**], [**Di**]. It leads us to the positive part of the $2D$ Toda hierarchy.

Consider the affine space of monic pseudo-difference operators of degree 1, i.e., the space of pseudo-difference operators of the form

(8.5)
$$\mathcal{L} = T + \sum_{s=0}^{\infty} w_s T^{-s}.$$

The positive part of the $2D$ Toda hierarchy has time variables $t = (t_1, t_2, \ldots)$. The flow corresponding to the time variable t_m is defined by the equation

(8.6)
$$\partial_m \mathcal{L} = [\mathcal{L}^m_+, \mathcal{L}], \qquad \partial_m := \partial_{t_m},$$

where $\mathcal{L}^i_+$ is the nonnegative part of the operator L^m, i.e. the difference operator such that $\mathcal{L}^m_- = \mathcal{L}^m - \mathcal{L}^m_+ = \mathcal{O}(T^{-1})$.

The standard arguments show that (8.6) is a well-defined system of equations on the coefficients of the operator L. For that one needs to show that the right-hand side of (8.6) is a pseudo-difference operator of degree at most zero. That follows from the equality $[\mathcal{L}^m_+, \mathcal{L}] = -[\mathcal{L}^m_-, \mathcal{L}]$ and the fact that $\mathcal{L}^m_-$ is a pseudo-difference operator of degree $\leqslant -1$.

The flows commute. The proof of the commutativity of the flows, i.e. the proof that equations (8.6) imply the equations

(8.7)
$$[\partial_m - (\mathcal{L}^m_+),\, \partial_\ell - (\mathcal{L}^\ell_+)] = 0,$$

is standard and word by word follows its continuous variant, see [**Di**].

Remark. The hierarchy of commuting flows (8.6) is a part of the $2D$ Toda hierarchy. Recall that the full $2D$ Toda hierarchy is defined on the space of pairs of pseudo-difference operators, one of which is as in (8.5) and the other is a pseudodifference operator with respect to T,

(8.8)
$$\mathcal{L}^- = \sum_{s=-1}^{\infty} w^-_s T^s.$$

The full set of time variables of the $2D$ Toda hierarchy consists of the variables $t = (t_1, t_2, \ldots)$ as above and the variables $t^- = (t^-_1, t^-_2, \ldots)$. We do not give further details, see [**TU**], since the second part of the $2D$ Toda hierarchy is not relevant for our purposes.

For any pseudo-difference operator L of the form (8.5) there is a unique formal solution $\Psi^w(z) = (\Psi^w_n(z))_{n\in\mathbb{Z}}$ of the equation

(8.9)
$$\mathrm{L}\Psi^w(z) = z\Psi^w(z)$$

of the form

(8.10) $$\Psi_n^w(z) = z^n\left(1+\sum_{s=1}^{\infty}\xi_{n,s}z^{-s}\right)$$

normalized by the condition

(8.11) $$\Psi_0^w(z)=1 \quad \Leftrightarrow \quad \xi_{0,s}=0, \quad s>0.$$

The solution $\Psi^w(z)$ is called the *wave solution*.

Let the pseudo-difference operator L depend on times, L = L(t). One can check that this pseudo-difference operator is a solution of the hierarchy equation (8.6) if and only if the following equations hold:

(8.12) $$\partial_m\Psi^w(t,z) = \mathcal{L}_+^m(t)\Psi^w(t,z) + h_m(t,z)\Psi^w(t,z),$$

where $h_m(t,z)$ is a scalar (not a sequence) Laurent series in z. The comparison of the right and left-hand sides shows that $h_m(t,z)$ has the form

(8.13) $$h_m(t,z) = z^m + O(z^{-1}).$$

From equation (8.7) it follows that

(8.14) $$\partial_m h_\ell(t,z) = \partial_\ell h_m(t,z).$$

Hence, there is a unique Laurent series $h(t,z)$ such that $\partial_m h(t,z) = h_m(t,z)$ and normalized by the condition $h(0,z) = 1$. Then equation (8.13) implies that

(8.15) $$h(t,z) = \sum_{m=1}^{\infty} t_m z^m + \mathcal{O}(z^{-1}).$$

It is easy to see that the sequence $\Psi(t,z) := \Psi^w(t,z)e^{-h(t,z)}$ satisfies the equations

(8.16) $$L(t)\Psi(t,z) = z\Psi(t,z), \qquad (\partial_m - \mathcal{L}_+^m)\Psi(t,z) = 0.$$

The elements $\Psi_n(t,z)$ of the sequence $\Psi(t,z)$ have the form

(8.17) $$\Psi_n(t,z) = z^n\left(1+\sum_{s=1}^{\infty}\chi_s(t)z^{-s}\right)e^{\sum_{m=1}^{\infty}t_m z^m}.$$

8.3. Discrete N mKdV hierarchy. Consider sequences of functions g = of(g_nT($x$is as above. The operator))$_{n\in\mathbb{Z}}$. There are two shift operators acting on them: T_x = $e^{\partial_x}$is the shift in the xTvariable, (and T_x. The actionT_xg)$_n(x)$ =

$g_n(x+1)$.

Recall the generating equation (1.17), that can be written in the form

(8.18) $$H\Psi = 0,$$

where

(8.19) $H = T - T_x + v$, $v = (v_n(x))_{n\in\mathbb{Z}}$, is a difference operator in x

and n.

The hierarchy, which we call the *discrete N mKdV hierarchy*, is the compatibility condition of the positive part of the 2*D* Toda hierarchy, defined in (8.6), with the generating equation (8.18). More precisely, the full set of equations of the discrete *N* mKdV hierarchy are equations (8.6) and equations

(8.20) $$[\partial_m - \mathcal{L}_+^m, H] = D_m H, \qquad m \geqslant 1,$$

where D_m is some difference operator in *x* and *n* depending on *m*.

Remark. The meaning of (8.20) is that the operators $\partial_m - \mathcal{L}_+^m$ and H commute on the space of solutions of equation (8.18). In the theory of integrable systems this type of representation is called an *L,A,B* triple, see [**DKN**].

By division with remainder it is easy to see that any difference operator D in *x* and *n* of degree *M* has a unique presentation

(8.21) $$\mathrm{D} = DH + D_1,$$

where D_1 is a degree *M* difference operator in *n* only, i.e. D_1 is a polynomial of degree *M* in *T* with coefficients that are sequences of functions $(g_n(x))$. Equation (8.20) says that the corresponding operator D_1 equals zero. Therefore, for any given monic difference operator *B* in *n*,

(8.22) $$B = T^M + \sum_{s=1}^{M} b_s T^{M-s},$$

the equation

(8.23) $$[\partial_m - B, H] = DH$$

with some *D* is a system of *M*+1 equations on *M*+1 unknown coefficients $b_1,\dots,b_M$ and *v*. The first *M* of them are difference equations. Unlike in the differential case, where the corresponding equations allow us to express the coefficients $b_1,\dots,b_M$ as the differential polynomials in *v* and get a well-defined system of equations for the coefficients of *H* only, in the difference case the reconstruction of $b_1,\dots,b_M$ in terms of *v* requires some additional assumptions, see more on that below.

Equations (8.6) and (8.20) is a system of equations on the coefficients of the pseudo-difference operator L and the sequence *v*. These equations can be written more explicitly following the argument identical to the one in the proof of equation (5.66) in [**K8**]. Namely, let

(8.24) $$F_m := \mathrm{res}_T \mathrm{L}^m, \qquad F_m = (F_{m,n})_{n\in\mathbb{Z}}.$$

Lemma 8.1. The system consisting of equations (8.6) and (8.20) is equivalent to the system consisting of equations (8.6) and equations

(8.25) $$\partial_m(\ln v_n(x)) = F_{m,n}(x) - F_{m,n}(x+1), \qquad m \geqslant 1.$$

8.4. Remark. Notice again that the system of equations (8.25) is not a closed system with respect to $v(x)$ since the right-hand sides are expressed in terms of the operator L.

A possible approach to eliminate L from equations (8.25) is as follows. Having an arbitrary N-periodic $v(x)$ determine a family of solutions $\psi(x,z)$ of equation the $H\psi = 0$. Then L is uniquely determined from the equation $L\psi(x,z) = z\psi(x,z)$. Put that L into (8.25) and obtain a system of equations on $v(x)$ only. Such an approach works well in similar situations but not in this one since the desired family of solutions $\psi(x,z)$ to equation $H\psi = 0$ is not unique.

Below we explain a construction of $\psi(x,z)$ from $v(x)$ and indicate why $\psi(x,z)$ is not unique. That fails this attempt to eliminate L from equations (8.25). The problem of elimination of L from (8.25) deserves further analysis.

Lemma 8.2. Let $v = (v_n(x))_{n\in Z}$ be any N-periodic sequence of functions, $v_n(x) = v_{n+N}(x)$. Then there is a formal solution $\psi = (\psi_n(x,z))_{n\in Z}$ of equation

(1.17),

(8.26) $$H\psi = 0$$

with $\psi_n(x,z)$ of the form

(8.27) $$\psi_n(x,z) = z^n(z+1)^x\left(1+\sum_{s=1}^{\infty}\xi_{n,s}(x)z^{-s}\right)$$ with periodic coefficients

(8.28) $$\xi_{n,s}(x) = \xi_{n+N,s}(x).$$

Proof. The substitution of (8.27) into (8.26) gives a system of equations for the unknown coefficients $\xi_{n,s}(x)$ in (8.27)

(8.29) $(T_x - T)\xi_{s+1} = -(v + T_x)\xi_s$, i.e.,

(8.30) $\xi_{n,s+1}(x+1)-\xi_{n+1,s+1}(x) = -v_n(x)\xi_{n,s}(x)-\xi_{n,s}(x+1)$, $s = 1,2,\ldots$.

We prove the existence of N-periodic solutions of these equation by induction. The induction starts with $\xi_0 = (\xi_{n,0})_{n\in Z}$ and $\xi_{n,0} = 1$ for all n. Suppose that $\xi_s = (\xi_{n,s})$ is known and is N-periodic. Let us apply the operator $\mathcal{T}_N := \sum_{i=0}^{N-1} T_x^{N-i-1}T^i$ to both sides of (8.29). Using the periodicity of ξ_s and v we get the equation

(8.31) $$(T_x^N - 1)\xi_{s+1} = \mathrm{T}_N(T_x\xi_s - v).$$

Invert the operator $T_x^N - 1$,

(8.32)
$$(T_x^N - 1)^{-1} := \sum_{i=1}^{\infty} T_x^{-iN}.$$

Then the N-periodic solutions of (8.29) can be recurrently defined by the formula

(8.33)
$$\xi_{s+1} = (T_x^N - 1)^{-1}\mathcal{T}_N(T_x\xi_s - v).$$

The lemma is proved.

The choice of $(T_x^N - 1)^{-1}$ is not unique. It can be replaced by

(8.34)
$$(T_x^N - 1)^{-1} := -\sum_{i=0}^{\infty} T_x^{iN}.$$

It is easy to see that for any formal solution $\psi(x,z)$ of (8.26) of the form (8.27) there is a unique pseudo-difference operator L such that

(8.35)
$$\mathrm{L}\psi(x,z) = z\psi(x,z).$$

Hence any choice of such a $\psi(x,z)$ makes the discrete N-periodic mKdV equations a well-defined system of equations for the functions $(v_n(x))_{n\in Z}$ only.

Notice that if a sequence $(v_n(x))$ is not an arbitrary N-periodic sequence of functions, but a sequence defined by formula (4.3) with $(y_n(x))$ satisfying the Bethe ansatz equations, then Theorem 4.2 gives us another way to construct the family of solutions $\psi(x,z)$ to equation $H\psi(x,z) = 0$. In that case by constructing L from (8.35) we may eliminate L from (8.25) and then solve the resulting equations on $(v_n(x))$ only.

8.5. Solutions of the discrete N mKdV hierarchy from solutions of the Bethe ansatz equations. Let $y = (y_n(x))$ be an N-periodic sequence of polynomials representing a solution of the Bethe ansatz equations. By Theorems 7.4 and 7.9 we can extend y to a family $y(t) = (y_n(x,t))$. Consider the corresponding solution of the generating problem $(\Psi_n(x,t,z))$,

(8.36)
$$\Psi_n(x,t,z) = z^n\left(1 + \sum_{s=1}^{\infty} \xi_s(x,t)z^{-s}\right)\Omega(x,t,z).$$

This solution satisfies the hierarchy of linear equations (6.5). Equations (6.4) identify the difference operators D_m in (6.5) with the operators $\mathcal{L}_+^m$. Hence we have the following theorem.

Theorem 8.3. The N-periodic sequence $(v_n(x,t))$ defined in terms of $y(t)$ by (4.3) is a solution the discrete N mKdV hierarchy.

8.6. Remark on discrete Miura opers. Denote by $L(z)$, $V(x)$ the $N\times N$ matrices

(8.37)
$$\begin{aligned} L(z) &= E_{2,1} + E_{3,2} + \cdots + E_{N,N-1} + z^{-N}E_{1,N} \\ V(x) &= v_1(x)E_{1,1} + \cdots + v_N(x)E_{N,N}, \end{aligned}$$

where $v_1(x),\dots,v_N(x)$ are some given functions of x. The first order linear difference operator

$$T - L - V \tag{8.38}$$

is called a *discrete Miura oper*, cf. [**MV3**]. Assume that $(y_n(x))_{n\in\mathbb{Z}}$ is an N-periodic sequence of polynomials representing a solution of the Bethe ansatz equations (2.1), $y_{N+n}(x) = y_n(x)$. Consider the corresponding N-periodic sequence $(v_n(x))$ defined by formula (4.3) and the N-periodic sequence of Baker-Akhieser functions $(\Psi_n(x,z))_{n\in\mathbb{Z}}$ given by Theorem 4.2, $\Psi_{N+n}(x,z) = z^N\Psi_n(x,z)$. Consider the N column vector $\Psi(x,z)$ with coordinates $\Psi_1(x,z),\dots,\Psi_N(x,z)$. Then

$$(T_x - L(z) - V(x))\Psi(x,z) = 0. \tag{8.39}$$

For example, if $N = 3$, then

$$\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\end{pmatrix}(x+1,z) = \begin{pmatrix} v_1(x) & 0 & z^{-3}\\ 1 & v_2(x) & 0\\ 0 & 1 & v_3(x)\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\end{pmatrix}(x,z).$$

Our study in this paper of periodic sequences of $(y_n(x))_{n\in\mathbb{Z}}$, $(\Psi_n(x,z))_{n\in\mathbb{Z}}$ is the study of the difference equation (8.39).

The discrete Miura opers are discrete analogs of differential Miura opers, which are the first order differential operators of the form

$$\frac{d}{dx} - \Lambda - V\,. \tag{8.40}$$

These differential operators play an important role in the theory of the N mKdV hierarchy, see for example [**DS**,**VWr**].

9. Combinatorial data

In this section we follow Section 6 in [**VWr**] and review some combinatorial data, which will be used in Section 10 to describe Baker-Akhieser functions of points of an infinite-dimensional Grassmannian.

9.1. Subsets of virtual cardinal zero. By a *partition* we mean an infinite sequence of nonnegative integers $\lambda = (\lambda_0 \geqslant \lambda_1 \geqslant \cdots)$ such that all except a finite number of the λ_i are zero. The number $|\lambda| = \sum_i \lambda_i$ will be called the *weight of* λ.

Following [**SW**], we say that a subset $S = \{s_0 < s_1 < s_2 < \dots\} \subset \mathbb{Z}$ is of *virtual cardinal zero*, if $s_j = j$ for all sufficiently large j. If n is such that $s_j = j$ for all $j > n$, then we say that S is of *depth n*.

If S is of depth n, then it is also of depth $n + 1$.

Lemma 9.1 ([**SW**]). There is a one to one correspondence between elements of S and partitions, given by $S \leftrightarrow \lambda$ where

$$\lambda_i = i - s_i.$$

For a subset $S = \{s_0 < s_1 < s_2 < ...\} \subset \mathbb{Z}$ and an integer $k \in \mathbb{Z}$ we denote by $S + k$ the subset $\{s_0 + k < s_1 + k < s_2 + k < ...\} \subset \mathbb{Z}$.

Let S be a subset of virtual cardinal zero. Let $A = \{a_1,...,a_k\} \subset \mathbb{Z}$ be a finite subset of distinct integers.

Lemma 9.2 ([**VWr**]). If $\{a_1,...,a_k\} \cap (S + k) = \emptyset$, then $\{a_1,...,a_k\} \cup (S + k)$ is a subset of virtual cardinal zero.

9.2. KdV subsets. Fix an integer $N > 1$. We say that a subset S of virtual cardinal zero is a *KdV subset* if $S + N \subset S$. For example, for any $N > 1$,

$$S^{\emptyset} = \{0,1,2,...\}$$

is a KdV subset.

Lemma 9.3 ([**VWr**]). Let S be a KdV subset. Then there exists a unique N-element subset $A = \{a_1 < \cdots < a_N\} \subset \mathbb{Z}$ such that $S = A \cup (S + N)$.

The subset A of the Lemma 9.3 will be called the *leading term of S*.

The leading term A uniquely determines the KdV subset S, since S is the union of N non-intersecting arithmetic progressions $\{a_i, a_i+N, a_i+2N,...\}$, $i = 1,...,N$. Let S be a KdV subset with leading term A. For any $a \in A$ the subset

$$S[a] = \{a + 1 - N\} \cup (S + 1) \tag{9.1}$$

is a KdV subset with leading term $A[a] = (A + 1) \cup \{a + 1 - N\} - \{a + 1\}$. The subset $S[a]$ will be called the *mutation of the KdV subset S* at $a \in A$.

Lemma 9.4 ([**VWr**]).

(i) Let S_1 be a KdV subset with leading term A. Let S_2 be a KdV subset such that $S_1 + 1 \subset S_2$. Then S_2 is the mutation of S_1 at some element $a \in A$.

(ii) Any KdV subset S can be transformed to the KdV subset $S^{\emptyset}$ by a sequence of mutations.

(iii) A subset $A = \{a_1 < \cdots < a_N\}$ is the leading term of a KdV subset if and only if equation

$$\sum_{i=1}^{N} a_i = \frac{N(N-1)}{2} \tag{9.2}$$

holds true and $a_i - a_j$ is not divisible by N for any $i \neq j$.

9.3. mKdV tuples of subsets. We say that an N-tuple $S = (S_1,\dots,S_N)$ of KdV subsets is an *mKdV tuple of subsets* if $S_i + 1 \subset S_{i+1}$ for all i, in particular,

$S_N + 1 \subset S_1$.

For example, for any N, the N-tuple

$$S^{\emptyset} = (S^{\emptyset},\dots,S^{\emptyset})$$

is an mKdV tuple of subsets.

If $S = (S_1,\dots,S_N)$ is an mKdV tuple, then $(S_i,S_{i+1},\dots,S_N, S_1,S_2,\dots,S_{i-1})$ is an mKdV tuple of subsets for any i.

Let S be a KdV subset with leading term $A = \{a_1 < \cdots < a_N\}$. Let σ be an element of the permutation group Σ_N. Define an N-tuple $S_{S,\sigma} = (S_1,\dots,S_N)$, where

(9.3) $S_i = \{\overline{a_{\sigma(1)} + iN}, \overline{a_{\sigma(2)} + iN}, \dots, \overline{a_{\sigma(i)} + iN}\} + (S + i)$, $i = 1\dots,N.$

$$S_N = A \cup (S+N) = S.$$

In particular,

Lemma 9.5 ([**VWr**]).

(i) The N-tuple $S_{S,\sigma}$ is an mKdV tuple.
(ii) Every mKdV tuple is of the form $S_{S,\sigma}$ for some KdV subset S and some element $\sigma \in \Sigma_N$.

9.4. Mutations of mKdV tuples.

Lemma 9.6 ([**VWr**]). Let $S = (S_1,\dots,S_N)$ be an mKdV tuple. Then for any $i = 1,\dots,N$, there exists a unique mKdV tuple

(9.4) $S^{(i)} = (S_1,\dots,S_{i-1},\tilde{S}_i,S_{i+1},\dots,S_N)$ which differs from S at the i-th

position only.

The mKdV tuple $S^{(i)}$ will be called the *mutation* of the mKdV tuple S at the i-th position. Denote by $w_i : S \mapsto S^{(i)}$ the mutation map.

Let $\lambda^i, \tilde{\lambda}^i$ be the partitions corresponding to the KdV subsets S_i, $\tilde{S}_i$, respectively. The mutation $w_i : S \mapsto S^{(i)}$ will be called *degree decreasing* if $|\tilde{\lambda}^i| < |\lambda^i|$.

Theorem 9.7 ([**VWr**]). Any mKdV tuple S can be transformed to the mKdV tuple $S^{\emptyset} = (S^{\emptyset},\dots,S^{\emptyset})$ by a sequence of degree decreasing mutations.

10. Tau-functions and Baker-Akhieser functions

In this section we follow Section 7 in [**VWr**] although we define the taufunctions as discrete Wronskians while in [**VWr**] the standard Wronskians are used. The tau-functions in this paper are different from the tau-functions in [**VWr**].

10.1. Remarks on the construction of Section 7.2. In Section 10 below we assign tau-functions and Baker-Akhieser functions to vector subspaces of an infinite dimensional vector space. The assignment is based on the construction of Section 7.2. We formulate two remarks on the construction.

In Section 7.4 starting from an $(\nu + N) \times (D + 1)$-matrix

$$A = \{a_{k,j}\}, \qquad k = 1,\dots,N + \nu, \qquad j = 0,\dots,D,$$

we constructed the functions $y_n(x,t)$, $\psi_n(x,t,z)$ for $n = 0,\dots,N$.

Choose n, 0 n N. Consider the $(n + \nu) \times D$-matrix $A^{(n)}$ formed by the first $n + \nu$ rows of the matrix A. Then the functions $y_n(x,t)$ and $\psi_n(x,t,z)$ are determined by formulas (7.26) and (7.28) in terms of the matrix $A^{(n)}$ only.

Let B be a nondegenerate $(n + \nu) \times (n + \nu)$-matrix. Let $y_{n,B}(x,t)$ and $\psi_{n,B}(x,t,z)$ be the functions determined by formulas (7.26) and (7.28), respectively, in which the entries of the matrix $A^{(n)}$ are replaced with the corresponding entires of the matrix $BA^{(n)}$. Then $y_{n,B}(x,t) = y_n(x,t)$ and $\psi_{n,B}(x,t,z) = \psi_n(x,t,z)$. That is, the functions $y_n(x,t)$ and $\psi_n(x,t,z)$ are determined by the $(n + \nu)$-dimensional vector space spanned by the first $n+\nu$ rows of the matrix A and do not depend on the choice of a basis in that space.

Consider the new $(\nu + N + 1) \times (D + 2)$-matrix

$$\tilde{A} = \{\tilde{a}_{k,j}\}, \qquad k = 0,\dots,N + \nu, \qquad j = 0,\dots,D + 1,$$

defined by the formulas

$$\begin{aligned} \tilde{a}_{0,j} &= \delta_{0,j}, & j &= 0,\dots,D + 1, \\ \tilde{a}_{k,0} &= 0, & k &= 1,\dots,N + \nu, \\ \tilde{a}_{k,j} &= a_{k,j-1}, & j &= 1,\dots,D + 1. \end{aligned} \tag{10.1}$$

Apply the construction of Section 7.4 to the matrix $\tilde{A}$ and construct the functions $\tilde{y}_n(x,t)$ and $\tilde{\psi}_n(x,t,z)$ for $n = 0,\dots,N$.

Lemma 10.1. We have

$$\tilde{y}_n(x,t) = y_n(x,t), \tag{10.2}$$

$$\tilde{\psi}_n(x,t,z) = z\,\psi_n(x,t,z), \qquad n = 0,\dots,N.$$

Lemma 10.1 says that the functions $y_n(x,t)$ and $\psi_n(x,t,z)$, determined by the $(n + \nu)$-dimensional vector space spanned by the first $n + \nu$ rows of the matrix A, do not change up to multiplication of $\psi_n(x,t,z)$ by z, if the $(n + \nu)$-dimensional vector space is extended to the $(n + \nu + 1)$-dimensional vector space by formulas

(10. 1).

10.2. Grassmannian $\mathrm{Gr}_0(H)$. For a Laurent polynomial $v = \sum_i v_i z^i$, the number ord$v = \min\{i : v_i \neq 0\}$ will be called the *order* of *v*.

Following [**SW**], let H be the Hilbert space $L^2(S^1)$ with orthonormal basis {span of$z_j\}_{j\in\mathbb{Z}}$. Let$\{z_j\}_{j<}H_{0+}$. We have the orthogonal decompositionbe the closure of the span of $\{z^j\}_{j\geqslant 0}$ and $H_-H = H_+$the closure of the$\oplus$ H_-.

We consider the set of all closed subspaces $W \subset H$ such that

10.3. $z^qH_+ \subset W \subset z^{-q}H_+$

for some $q > 0$. Such subspaces can be identified with subspaces W/z^qH_+ of $z^{-q}H_+/z^qH_+$. We say that W is of *virtual dimension zero* if $\dim W/z^qH_+ = q$. Denote by $\mathrm{Gr}_0(H)$ the set of all subspaces of virtual dimension zero.

Any $W \in \mathrm{Gr}_0(H)$ has a basis$\{v_j\}_{j\geqslant 0}$ consisting of Laurent polynomials. We may assume that the numbers $s_j = \mathrm{ord}\, v_j$ form a strictly increasing sequence $S_W = \{s_0 < s_1 < s_2 < \dots\}$. The assignment$W \mapsto S_W$ is well-defined. The subset S_W will be called the *order subset* of W. The order subset S_W is of virtual cardinal zero.

For $W \in \mathrm{Gr}_0(H)$, a basis$\{v_j = \sum_{i\geqslant s_j} v_{j,i}z^i\}_{j\geqslant 0}$ of W is called *special of depth n*, if it consists of Laurent polynomials such that $v_j = z^j$ for $j > n$ and $v_{j,i} = 0$ if $i > n$ and $j \leqslant n$.

If$\{v_j\}_{j\geqslant 0}$ is a basis of depth n, then it is also a basis of depth $n + 1$.

10.3. Points in $\mathrm{Gr}_0(H)$ **and finite-dimensional spaces of polynomials in** x,t.
Let $S = \{s_0 < s_1 < \dots\}$ be a set of virtual cardinal zero of depth n. For $W \in \mathrm{Gr}_0(H)$ with order subset$S$ let $\{v_j = \sum_{i\geqslant s_j} v_{j,i}z^i\}_{j\geqslant 0}$ be a special basis of depth n.

Introduce the $n + 1$-dimensional complex vector space $V_{W,n}$ of polynomials in x,t as the space spanned by the polynomials $f_{j,n}(x,t), j = 0,\dots,n$, where

$$f_{j,n}(x,t) = \sum_{i=0}^{n-s_j} v_{j,n-i}\,\chi_i(x,t)\,, \qquad j = 0,\dots,n\,. \tag{10.4}$$

We have $\deg_x f_j(x,t) = n - s_j$.

It is clear that the space $V_{W,n}$ does not depend on the choice of a special basis of W with depth n.

The same basis of depth n is also a basis of depth n+1. Then the space $V_{W,n+1}$ is spanned by the polynomials

$$f_{j,n+1}(x,t) = \sum_{i=0}^{n-s_j} v_{j,n-i}\,\chi_{i+1}(x,t)\,, \qquad j=0,\dots,n,$$

(10.5) $$f_{n+1,n+1}(x,t) = \chi_0(x,t)\,.$$

Therefore, the n + 2-dimensional space $V_{W,n+1}$ consists of all linear combinations $g(x,t)$ of polynomials $\chi_i(x,t)$ such that $\Delta g(x,t) \in V_{W,n}$.

The space $V_{W,n+2}$ is related to the space $V_{W,n+1}$ in a similar way, and so on. Thus, to a space $W \in \mathrm{Gr}_0(H)$ we assigned a sequence of spaces $V_{W,n}$, $V_{W,n+1}$, ...related by formulas (10.4) and (10.5).

The construction in the opposite direction goes as follows. Let S be a set of virtual cardinal zero. Let n be such that $s_j = j$ for all $j > n$. Let V be an n + 1dimensional complex vector space spanned by linear combinations of polynomials $\chi_i(x,t)$, such that V has a basis $(f_j(x,t))_{j=0}^n$ with $\deg_x f_j(x,t) = n - s_j$. To this vector space V with such a basis

(10.6) $$f_{j,n}(x,t) = \sum_{i=0}^{n-s_j} v_{j,n-i}\,\chi_i(x,t)\,, \qquad j=0,\dots,n,$$

we assign $W_V \in \mathrm{Gr}_0(H)$ with special basis $\{v_j\}_{j\geqslant 0}$ of depth n, where

(10.7) $$v_j = \sum_i v_{j,i} z^i, \qquad \text{for} \qquad j = 0,\dots,n,$$

and $v_j = z^j$ for all $j > n$. The set S is the order subset of W_V. We also have $V_{W_V,n} = V$.

For $W \in \mathrm{Gr}_0(H)$ with order subset $S = \{s_0 < s_1 < ...\}$ of depth n, we have $W = W_{V_{W,n}}$.

10.4. Tau and Baker-Akhieser functions. Let $W \in \mathrm{Gr}_0(H)$ have order subset $S = \{s_0 < s_1 < ...\}$ of depth n. Let $\{v_j = \sum_{i\geqslant s_j} v_{j,i} z^i\}_{j\geqslant 0}$ be a special basis of W of depth n. Consider the polynomials $(f_j(x,t))_{j=0}^n$ defined in (10.4). Define the *tau-function* of W by the formula

(10.8) $$\tau_W(x,t) = \widehat{W}(f_0(x,t),\dots,f_n(x,t)),$$

cf. [**SW**]. The tau-function is independent of the choice of n up to multiplication by a nonzero number, see Lemma 10.1.

Let the order subset $S = \{s_0 < s_1 < ...\}$ corresponds to a partition λ. Then

(10.9) $$\tau_W(x,t) = a x^{|\lambda|} + (\text{low order terms in } x),$$

where a is a nonzero number independent of x,t.

Define the *Baker-Akhieser function* of W by the formula

$$\psi_W^{(n)}(x,t,z) = \Omega(x,t,z)\,\frac{\det \widehat{M}_W^{(n)}(x,t,z)}{\tau_W(x,t)}, \tag{10.10}$$

where the matrix $\widehat{M}_W^{(n)}(x,t,z)$ is defined as follows.

First we define an $(n+1)\times(n+1)$-matrix $M_W^{(n)}(x,t)$ by the formula

$$M_{W,k,\ell}^{(n)}(x,t) = \Delta^{(\ell)} f_k(x,t), \qquad k,\ell = 0,\dots,n, \tag{10.11}$$

cf. (7.24). Define an $(n+2)\times(n+2)$-matrix $\widehat{M}_W^{(n)}(x,t,z)$, whose rows and columns are labeled by indices $0,\dots,n+1$ and entries are given by the formulas:

$$\begin{aligned} \widehat{M}_{W,k,\ell}^{(n)} &= M_{W,k,\ell}^{(n)}, && k,\ell = 0,\dots,n,\\ \widehat{M}_{W,n+1,\ell}^{(n)} &= z^{\ell}, && \ell = 0,\dots,n+1\\ \widehat{M}_{W,\ell,n+1}^{(n)} &= \Delta^{(n+1)} f_k(x,t), && \ell = 0\dots,n, \end{aligned} \tag{10.12}$$

cf. formula (7.27).

Lemma 10.2.

(i) Let $\{v_j = \sum_{i\geqslant s_j} v_{j,i} z^i\}_{j\geqslant 0}$ be a special basis of W of depth n. Then the Baker-Akhieser function $\psi_W^{(n)}(x,t,z)$ does not depend on the choice of the special basis.

(ii) If another number n is chosen such that $s_j = j$ for all $j > n'$, then

$$\psi_W^{(n')}(x,t,z) = z^{n'-n}\,\psi_W^{(n)}(x,t,z). \tag{10.13}$$

Proof. The lemma follows from Lemma 10.1.

10.5. mKdV tuples of subspaces. Fix an integer $N > 1$. We say that a subspace $W \in \mathrm{Gr}_0(H)$ is a *KdV subspace* if $z^N W \subset W$.

For example, for any N the subspace H_+ is a KdV subspace.

Lemma 10.3 ([**VWr**]). Let W be a KdV subspace with order subset S. Then S is a KdV subset.

We say that an N-tuple $W = (W_1,\dots,W_N)$ of KdV subspaces is an *mKdV tuple of subspaces* if $zW_i \subset W_{i+1}$ for all i, in particular, $zW_N \subset W_1$. Denote by Gr_{mKdV} the set of all mKdV tuples of subspaces.

For example, for any N the tuple $W^{\emptyset} = (H_+,\dots,H_+)$ is an mKdV tuple.

If $W = (W_1,\dots,W_N) \in \mathrm{Gr}_{mKdV}$, then $(W_i, W_{i+1},\dots,W_N, W_1, W_2,\dots,W_{i-1}) \in \mathrm{Gr}_{mKdV}$ for any i.

Let $W = (W_1,\dots,W_N) \in \mathrm{Gr}_{mKdV}$. Let S_i be the order subset of W_i and $S = (S_1,\dots,S_N)$. Then S is an mKdV tuple of subsets.

Let W be a KdV subspace with order subset S. Let $A = \{a_1 < \cdots < a_N\}$ be the leading term of S. Let $v = (v_1,\ldots,v_N)$ be a tuple of elements of W such that $\operatorname{ord} v_i = a_i$ for all i. Let $\sigma \in \Sigma_N$. Define an N-tuple $W_{W,v,\sigma} = (W_1,\ldots,W_N)$ of subspaces by the formula

$$W_i = \langle z^{i-N} v_{\sigma(1)}, z^{i-N} v_{\sigma(2)}, \ldots, z^{i-N} v_{\sigma(i)} \rangle + z^i W, \tag{10.14}$$

in particular, $W_N = z^N W + span\langle v_1, \ldots, v_N \rangle = W$.

Theorem 10.4 ([**VWr**]). The N-tuple $W_{W,v,\sigma}$ is an mKdV tuple of subspaces. Moreover, every mKdV tuple of subspaces is of the form $W_{W,v,\sigma}$ for suitable W,v,σ.

Here is another description of mKdV tuples of subspaces.

Theorem 10.5 ([**VWr**]). Let W be a KdV subspace. Let $z^N W = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{N-1} \subset V_N = W$ be a complete flag of vector subspaces such that $\dim V_i/V_{i-1} = 1$ for all i. Set

$$W_i = z^{i-N} V_i, \qquad i = 1,\ldots,N. \tag{10.15}$$

Then $W = (W_1,\ldots,W_{N-1},W_N = W)$ is an mKdV tuple of subspaces. Moreover, every mKdV tuple of subspaces is of this form.

Let W be a KdV subspace. It follows from Theorem 10.5 that the set of mKdV tuples of subspaces with prescribed last term $W_N = W$ is identified with the set of complete flags in the N-dimensional complex vector space $W/z^N W$.

10.6. Relations between Baker-Akhieser functions. Let $(W_1,\ldots,W_N) \in \mathrm{Gr}_{mKdV}$. Let $(\tau_{W_1}(x,t),\ldots,\tau_{W_N}(x,t))$ and $(\psi_{W_1}(x,t,z),\ldots,\psi_{W_N}(x,t,z))$ be the corresponding tau and Baker-Akhieser functions.

Recall that each Baker-Akhieser function $\tau_{W_i}(x,t)$ is defined up to multiplication by a monomial z^m, see Lemma 10.2. A Baker-Akhieser function with a choice of this factor will be called a *graded Baker-Akhieser function* of W_i.

Theorem 10.6. There exist graded Baker-Akhieser functions $\psi_{W_1}(x,t,z), \ldots, \psi_{W_N}(x,t,z)$ such that

$$\begin{aligned} \psi_{W_{i-1}}(x,t,z) &= \psi_{W_i}(x+1,t,z) \\ &- \frac{\tau_{W_i}(x,t)\, y_{W_{i-1}}(x+1,t)}{y_{W_i}(x+1,t)\, y_{W_{i-1}}(x,t)}\, \psi_{W_i}(x,t,z) \end{aligned}, \tag{10.16}$$

for $i = 2,\ldots,N$, and

$$\begin{aligned} z^N \psi_{W_N}(x,t,z) &= \psi_{W_1}(x+1,t,z) \\ &- \frac{\tau_{W_1}(x,t)\, y_{W_N}(x+1,t)}{y_{W_1}(x+1,t)\, y_{W_N}(x,t)}\, \psi_{W_1}(x,t,z) \end{aligned}. \tag{10.17}$$

Denote $y_n(x,t) := \tau_{W_{N-n+1}}(x,t)$, $n = 1,\dots,N$, and extend this sequence by periodicity, $y_{N+n}(x,t) = y_n(x,t)$ for all values of $n \in \mathbb{Z}$. Denote $\psi_n(x,t,z) := \psi_{W_{N-n+1}}(x,t,z)$, $n = 1,\dots,N$, and extend this sequence by periodicity, $\psi_{N+n}(x,t,z) = z^N\psi_n(x,t,z)$ for all values of $n \in \mathbb{Z}$. Introduce the sequence $(v_n(x,t))_{n\in\mathbb{Z}}$ by formula

$$v_n(x,t) = \frac{y_n(x,t)\,y_{n+1}(x+1,t)}{y_n(x+1,t)\,y_{n+1}(x,t)}, \tag{10.18}$$

see (4.3).

Corollary 10.7. For any fixed t, the functions $(v_n(x,t))_{n\in\mathbb{Z}}$ and $(\psi_n(x,t,z))_{n\in\mathbb{Z}}$ satisfy relations (4.2).

Proof of Theorem 10.6. Since the tuple $(W_2, W_3,\dots,W_N, W_1)$ is also an mKdV tuple, it is enough to prove (10.16) for $i = N$ only.

By Theorem 10.4 the pair W_{N-1}, W_N has the following form. Let $S = \{s_0 < s_1 < \dots\}$ be the order subset of W_N. Let $A = \{a_1 < \cdots < a_N\}$ be the leading term of S. Choose one element $a \in A$.

Let S be of depth n. Let $\{v_j = \sum_{i\geqslant s_j} v_{j,i}z^i\}_{j\geqslant 0}$ be a special basis of W of depth n. Let $w = \sum_i w_i z^i$ be the element of the basis with ord$w = a$. Then W_{N-1} is the space with basis $\{z^{1-N}w\}\cup\{zv_j\}_{j\geqslant 0}$. This basis of W_{N-1} is a basis of depth $n + 1$.

The tau and Baker-Akhieser functions of W_N are defined in terms of the basis $\{v_j\}_{j\geqslant 0}$ of depth n by polynomials $f_{j,n}(x,t)$, $j = 0,\dots,n$, see formula (10.4).

The tau and Baker-Akhieser functions of W_{N-1} are defined in terms of its basis $\{z^{1-N}w\}\cup\{zv_j\}_{j\geqslant 0}$ of depth $n+1$ by the same polynomials $f_{j,n}(x,t)$, $j = 0,\dots,n$, and one additional polynomial $f_{n+1}(x,t)$ corresponding to the basis element $z^{1-N}w$,

$$f_{n+1}(x,t) = \sum_i w_{N+n-i}\chi_i(x,t).$$

Now the functions $\tau_{W_{N-1}}, \tau_{W_{N-1}}$, $\psi_{W_{N-1}}, \psi_{W_{N-1}}$ satisfy (10.16) for $i = N$ by Theorem 7.4.

10.7. Generation of new mKdV tuples of subspaces. Let $W = (W_1,\dots, \dots W_N) \in \mathrm{Gr}_{mKdV}$. By Theorem 10.5, the tuple W is determined by a flag

$$z^N W_N = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{N-1} \subset W_N.$$

The quotient V_2/V_0 is two-dimensional. Any line $\tilde V_1/V_0$ in V_2/V_0 determines a an mKdV

tupleflag $z^N W_N = V_0$ $W \subset \tilde V_1 \subset V_2 \subset \cdots \subset V_{N-1} \subset W_N$, which in its turn determines $\tilde{}$ =

$z^{1-N}\tilde V_1$. Thus we get

$_{(1)} = (\tilde W_1, W_2, ..., W_N)$ with W_1

a family of mKdV tuples of subspaces parameterized by points of the projective line $P(V_2/V_0)$. The new tuples are parametrized by points of the affine line A = $P(V_2/V_0) - \{V_1/V_0\}$. We get a map $X^{(1)} : A \to \mathrm{Gr}_{mKdV}$ which sends $a \in A$ to the corresponding mKdV tuple $W^{(1)}(a) = (\tilde W_1(a), W_2, ..., W_N)$. This map will be called the *generation of mKdV tuples from the tuple W in the first direction*.

Similarly, for any $i = 2,...,N$, we construct a map $X^{(i)} : A \to \mathrm{Gr}_{mKdV}$, where

tupleA = $PW(V_{i+1}/V_{i-1}) - \{V_i/\tilde V_{i-}(a_1\}),...,W$which sends$_N$). This map will be called the$a$ $\in$ A to the corresponding mKdV*generation*

$_{(i)}(a) = (W_1,...,W_i$ *of mKdV tuples of subspaces from the tuple W*

in the i-th direction.

We say that the generation in the i-th direction is *degree increasing* if for any $a \in$ A, we have $\deg_x \tau_{W^{(i)}(a)}(x,t) > \deg_x \tau_W(x,t)$.

The tau-function $\tau_{\tilde W_i(a)}$ depends on a linearly in the following sense. Let $\{v_i\}_{i\geqslant 1}$ be a basis of V_{i-1}. Let $v_0 \in V_i$ be such that $\{v_i\}_{i\geqslant 0}$ is a basis of V_i. Let $\tilde v_0 \in V_{i+1}$ be such that $\{\tilde v_0, v_0, v_1, v_2, ...\}$ is a basis of V_{i+1}. Then the points of A =

Presponds to the line generated by the subspace$(V_{i+1}/V_{i-1}) - \{V_i/V_{i-1}\}$ are parametrized

by complex numbers$\tilde V_i(c)$ with basis $\{\tilde v c_0$. A number$+cv_0, v_1, v_2 c$...cor-}.

This c is an affine coordinate on A. Calculating the tau-function of the subspace $\tilde W_i(c)$ = $z^{i-N}\tilde V_i(c)$ with respect to the basis $\{z^{i-N}(\tilde v_0 + cv_0), z^{i-N}v_1, z^{i-N}v_2 ...\}$ we get the formula

(10.19) $$\tau_{\tilde W_i(c)} = \tau_{\tilde W_i(0)} + c\tau_{W_i}.$$

Theorem 10.8. For the generation in the i-th direction, the tau-functions of the subspaces $\tilde W_i(c), W_i, W_{i-1}, W_{i+1}$ satisfy the equation

(10.20) $$\widehat{W}(\tau_{W_i}(x,t), \tau_{\tilde W_i(c)}(x,t)) = \text{const}\, \tau_{W_{i-1}}(x,t)\tau_{W_{i+1}}(x+1,t),$$

where const is a number independent of x,t.

Proof. The proof of this theorem is word by word the same as the proof of Theorems 6.10 and 7.10 in [**VWr**], see also the proof of Theorem 10.6. Define an infinite N-periodic sequence of polynomials $(y_n(x,t))_{n\in\mathbb{Z}}$ by the formula

$$y_n(x,t) := \tau_{W_{-n}}(x,t). \tag{10.21}$$

Corollary 10.9. For any mKdV tuple $W = (W_1,...,W_N)$ and any fixed t, the sequence $(y_n(x,t))_{n\in\mathbb{Z}}$ of polynomials in x is fertile.

Remark. Theorem 10.8 says that the generation of mKdV tuples in the i-th direction from the tuple W corresponds to the generation of tuples of polynomials in the i-th direction from the tuple $(\tau_{W_1}(x,t),...,\tau_{W_N}(x,t))$, where the latter generation procedure is described in Section 3.2. In other words, the two generation procedure and the functor, which assigns to a point of $\mathrm{Gr}_0(H)$ its tau-function, commute.

10.8. Transitivity of the generation procedure.

Theorem 10.10 ([**VWr**]). Any mKdV tuple $W \in \mathrm{Gr}_{mKdV}$ can be obtained from the mKdV tuple $W^{\emptyset} = (H_+,...,H_+)$ by a sequence of degree increasing generations.

Combining this theorem and Theorem 3.4 we obtain the following corollary.

Corollary 10.11. If a tuple $(y_1(x),...,y_N(x))$ represents a solution of the Bethe ansatz equations (2.1), then there exists an mKdV tuple of subspaces $(W_1,...,...W_N)$ such that

$$(y_1(x),...,y_N(x)) = (\tau_{W_1}(x,0),...,\tau_{W_N}(x,0)). \tag{10.22}$$

In particular, the tuple $(y_1(x),...,y_N(x))$ can included into the family $(\tau_{W_1}(x,t), ..., \tau_{W_N}(x,t))$ of tuples depending on t, and then extended to the sequences of functions $(v_n(x,t))_{n\in\mathbb{Z}}$ and $(\psi_n(x,t,z))_{n\in\mathbb{Z}}$, as explained in Corollary 10.7, and those sequences $(v_n(x,t))_{n\in\mathbb{Z}}$ and $(\psi_n(x,t,z))_{n\in\mathbb{Z}}$ give a solution of the generating linear problem equation (4.2) depending on t as stated in Corollary 10.7.

10.9. Commuting flows on $\mathrm{Gr}_0(H)$**.** For a subspace $W \in \mathrm{Gr}_0(H)$, the subspace

$$W(t) := e^{\sum_{i=1}^{\infty} t_i z^i} W \tag{10.23}$$

is a well-defined subspace in $\mathrm{Gr}_0(H)$. Given W, the space $W(t)$ depends only on finitely many of $t_1,t_2,...$. This construction gives us a family of commuting flows on $\mathrm{Gr}_0(H)$ with times $t_1,t_2,...$. We will call them the *discrete mKdV flows*.

The discrete mKdV flows on $\mathrm{Gr}_0(H)$ induce a family of commuting flows on the space of N-tuples $(\tau_{W_1}(x,0),...,y_{W_N}(x,0))$, representing solutions of the Bethe ansatz equations (2.1). The construction goes as follows.

Let $(W_1,...,W_N) \in \mathrm{Gr}_0(H)$. Let $(\tau_{W_1}(x,t),...,y_{W_N}(x,t))$ be the collection of tau-functions assigned to $(W_1,...,W_N)$ in Section 10.4. The collection of polynomials $(\tau_{W_1}(x,0)$, ...,

$\tau_{W_N}(x,0)$) in x will be called the tuple of *reduced tau-functions* of $(W_1,...,W_N)$. When the tuple $(W_1,...,W_N)$ becomes dependent on t we obtain a family of tuples of reduced tau-functions $(\tau_{W_1(t)}(x,0),...,\tau_{W_N(t)}(x,0))$. Thus we obtain a family of commuting flows on the space of tuples of reduced tau-functions, which will also be called the *discrete mKdV flows*.

Lemma 10.12. For any $(W_1,...,W_N) \in \mathrm{Gr}_0(H)$ we have

(10.24) $$(\tau_{W_1(t)}(x,0),...,\tau_{W_N(t)}(x,0)) = (\tau_{W_1}(x,t),...,\tau_{W_N}(x,t)).$$

11. Appendix

After this article had been finished, the authors decided, for the sake of completeness, to revisit the results of the work [**VWr**] and present them in a new form, analogous to Theorem 7.10.

Recall, that in [**VWr**] a family of commuting flows acting on the space of

constructed and identified with the flows of the N mKdV integrable hierarchy.

solutions of the Bethe ansatz equations (1.1) for the affine Lie algebra sl_N was

In terms of the theory of finite dimensional integrable systems of particles the corresponding result is as follows.

Theorem 11.1. Let $(y_n(x))_{n\in\mathbb{Z}}$ be an N-periodic sequence of polynomials of degrees (k_n) representing a solution of the Bethe ansatz equations (1.1) for the affine Lie algebra sl_N. Then the correspondence

(11.1) $(y_n) \longmapsto (u^{(n)}, p^{(n)})$, where

$$p_i^{(n)} := \sum_{j\neq i}^{k_n} \frac{1}{u_i^{(n)} - u_j^{(n)}} - \sum_{\ell=1}^{k_{n+1}} \frac{1}{u_i^{(n)} - u_\ell^{(n+1)}}, \qquad i = 1, \ldots, k_n$$

,

is an embedding of the space of solutions of the Bethe ansatz equations into the product of N phase spaces of the k_n-particle CM systems, $n = 1,...,N$.

The image of this map is invariant under the hierarchy of the CM systems, acting diagonally on the product of the phase spaces.

The proof of the theorem goes along the same lines as the proof of Theorem 7.10. Its starting point is the following linear generating problem (compare it with Theorem 4.2).

Test

Theorem 11.2. The equations

(11.2) $\partial_x\psi_n(x) = \psi_{n+1}(x) + v_n(x)\psi_n(x)$, $n \in \mathbb{Z}$, with the potential

$$v_n(x) = \partial_x \ln \left(\frac{y_n(x)}{y_{n-1}(x)} \right) \tag{11.3}$$

have a meromorphic in x solution $\psi_n(x)$, with simple poles at zeros at $y_{n-1}(x)$, if and only if equations (1.1) hold. Moreover, if (1.1) hold, then there exists a family of solutions $\Psi_n(x,z)$, $z \in \mathbb{C}$, of (11.2) of the form

$$\Psi_n(x,z) = z^n e^{zx} \left(1 + \sum_{i=1}^{k_{n-1}} \xi_s^{(n)}(x) z^{-i} \right), \qquad n \in \mathbb{Z} \tag{11.4}$$

where $\xi_s^{(n)}(x)$ are rational functions in x such that all the functions $y_{n-1}(x)\xi_s^{(n)}(x)$ are holomorphic in x.

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