

Notes on solutions of KZ equations modulo p^s and p -adic limit

$s \rightarrow \infty$

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Abstract. We consider the differential KZ equations over $\mathbb{C}$ in the case, when the hypergeometric solutions are one-dimensional hyperelliptic integrals of genus g . In this case the space of solutions of the differential KZ equations is a $2g$ -dimensional complex vector space.

We also consider the same differential equations modulo p^s , where p is an odd prime number and s is a positive integer, and over the field $\mathbb{Q}_p$ of p -adic numbers.

We describe a construction of polynomial solutions of the differential KZ equations modulo p^s . These polynomial solutions have integer coefficients and are p^s -analogs of the hyperelliptic integrals. We call them the p^s -hypergeometric solutions. We consider the space M_{p^s} of all p^s -hypergeometric solutions, which is a module over the ring of polynomial quasi-constants modulo p^s . We study basic properties of M_{p^s} , in particular its natural filtration, and the dependence of M_{p^s} on s .

We show that the p -adic limit of M_{p^s} as $s \rightarrow \infty$ gives us a g -dimensional vector space of solutions of the differential KZ equations over the field $\mathbb{Q}_p$. The solutions over $\mathbb{Q}_p$ are power series at a certain asymptotic zone of the KZ equations.

In the appendix written jointly with Steven Sperber we consider all asymptotic zones of the KZ equations in the special case $g = 1$ of elliptic integrals. It turns out that in this case the p -adic limit of M_{p^s} as $s \rightarrow \infty$ gives us a one-dimensional space of solutions over $\mathbb{Q}_p$ at every asymptotic zone. We apply Dwork's theory of the classical hypergeometric function over $\mathbb{Q}_p$ and show that our germs of solutions over $\mathbb{Q}_p$ defined at different asymptotic zones analytically continue into a single global invariant line subbundle of the associated KZ connection. Notice that the corresponding KZ connection over $\mathbb{C}$ does not have proper nontrivial invariant subbundles, and therefore our invariant line subbundle is a new feature of the KZ equations over $\mathbb{Q}_p$.

Also in the appendix we follow Dwork and describe the Frobenius transformations of solutions of the KZ equations for $g = 1$. Using these Frobenius transformations we recover the unit roots of the zeta functions of the elliptic curves defined by the affine equations $y^2 = \beta x(x - 1)(x - \alpha)$ over the finite field $\mathbb{F}_p$. Here $\alpha, \beta \in \mathbb{F}_p^\times$, $\alpha \neq 1$. Notice that the same elliptic curves considered over $\mathbb{C}$ are used to construct the complex holomorphic solutions of the KZ equations for $g = 1$.

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1. Introduction

1.1. The KZ equations were introduced in [KZ] as the differential equations satisfied by conformal blocks on sphere in the Wess-Zumino-Witten model of conformal field theory. The solutions of the KZ equations in the form of multidimensional hypergeometric integrals were constructed more than 30 years ago, see [SV1]. The KZ equations and the hypergeometric solutions are related to many subjects in algebra, representation theory, theory of integrable systems, enumerative geometry.

The polynomial solutions of the KZ equations over the finite field F_p of a prime number p of elements were constructed relatively recently in [SV2], see also [V4]-[V8], [RV1,RV2]. These solutions were called the F_p -hypergeometric solutions. The general problem is to understand relations between the hypergeometric solutions of the KZ equations over C and the F_p -hypergeometric solutions and observe how the remarkable properties of hypergeometric solutions are reflected in the properties of the F_p -hypergeometric solutions. For example, the F_p -hypergeometric solutions inherit some determinant properties of the hypergeometric solutions and some Selberg integral properties, see [V8,RV1,RV2].

This program is in the first stages, where we consider essential examples and study the corresponding F_p -hypergeometric solutions by direct methods.

In this paper we consider the differential KZ equations over C in the case, when the hypergeometric solutions are one-dimensional hyperelliptic integrals of genus g . In this case the space of solutions of the differential KZ equations is a $2g$ -dimensional complex vector space. We also consider the same differential equations modulo p^s , where p is an odd prime number and s is a positive integer, and over the field Q_p of p -adic numbers.

We give a construction of polynomial solutions of the differential KZ equations modulo p^s for positive integers s . We call such solutions the p^s -hypergeometric solutions. This construction is a straightforward modification of the construction in [SV2] of polynomial solutions modulo p .

In this paper we consider the space M_{p^s} of all p^s -hypergeometric solutions, which is a module over the ring of polynomial quasi-constants modulo p^s . We study basic properties of M_{p^s} , in particular its natural filtration, and dependence of M_{p^s} on s .

We show that the p -adic limit of M_{p^s} as $s \rightarrow \infty$ gives us a g -dimensional vector space of solutions of the differential KZ equations over the field $\mathbb{Q}_p$. The solutions over $\mathbb{Q}_p$ are power series at a certain asymptotic zone of the KZ equations. This is the main result of the paper, see Lemma 9.5 and Theorem 10.7.

1.2. In the appendix written jointly with Steven Sperber we consider all six asymptotic zones of the KZ equations in the special case $g = 1$ of elliptic integrals. It turns out that in this case the p -adic limit of M_{p^s} as $s \rightarrow \infty$ gives us a onedimensional space of solutions over $\mathbb{Q}_p$ at every asymptotic zone. We apply Dwork's theory of the classical hypergeometric function over $\mathbb{Q}_p$ and show that our germs of solutions over $\mathbb{Q}_p$ defined at different asymptotic zones analytically continue into a single global invariant line subbundle of the associated KZ connection. Notice that the corresponding KZ connection over $\mathbb{C}$ does not have proper nontrivial invariant subbundles, and therefore our invariant line subbundle is a new feature of the KZ equations over $\mathbb{Q}_p$.

Following Dwork we show that our line subbundle is spanned at any point of the base by the germs of all solutions of the KZ equations bounded in their discs of convergence. This statement gives a definition of the line subbundle independent of asymptotic zones and analytic continuation.

Also in the appendix we follow Dwork and describe the Frobenius transformations of solutions of the KZ equations for $g = 1$. Using these Frobenius transformations we recover the unit roots of the zeta functions of the elliptic curves defined by the affine equations $y^2 = \beta x(x-1)(x-\alpha)$ over the finite field $\mathbb{F}_p$. Here

$\alpha, \beta \in \mathbb{F}_p^\times, \alpha \neq 1$. Notice that the same elliptic curves considered over $\mathbb{C}$ are used to construct the complex holomorphic solutions of the KZ equations for $g = 1$.

In the end of Section A.10 we argue that the KZ equations for $g = 1$ contain more arithmetic information than the associated hypergeometric differential equation (1.2) for the hypergeometric function $I(z)$ in (1.1), studied in [Dw].

1.3. Our p -adic limit of M_{p^s} as $s \rightarrow \infty$ is similar to the p -adic limit in the following classical example, see [Ig,Ma,Cl,BV1]. Consider the elliptic integral

$$(1.1) \quad I(z) = \frac{1}{\pi} \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-z)}} = \sum_{k=0}^{\infty} \binom{-1/2}{k} z^k.$$

It satisfies the hypergeometric differential equation

$$(1.2) \quad z(1-z)I'' + (1-2z)I' - (1/4)I = 0.$$

The coefficients of the power series $I(z)$ are p -adic integers and the power series $I(z)$ converges p -adically for $|z|_p < 1$, where $|z|_p$ is the p -adic norm of $z \in \mathbb{Q}_p$. One may show that for any positive integer s the polynomial

$$(1.3) \quad I_{(p^s-1)/2}(z) = \sum_{k=0}^{(p^s-1)/2} \binom{(p^s-1)/2}{k} z^k$$

is a solution of the differential equation (1.2) modulo p^s . Thus we get a sequence $(I_{(p^s-1)/2}(z))_{s=1}^\infty$ of polynomials with integer coefficients, each of which is a solution of the differential equation (1.2) modulo p^s , and the p -adic limit of the sequence, as s tends to ∞ , is the p -adic power series solution $I(z)$ of the differential equation

(1.2).

The p^s -hypergeometric solutions of our differential KZ equations are analogs of the polynomial construction of the $I_{(p^s-1)p/2s}$ -hypergeometric solutions does not indicate the analogous p -adic limit. The difference is that

p -adic limiting solutions $I(z)$, and the analogous limiting p -adic power series solutions $I(z)$ can be discovered only after rewriting the p^s -hypergeometric solutions in a suitable asymptotic zone of the differential KZ equations.

In the simplest example of our differential KZ equations, the p -adic solution is the 3-vector

$$(1.4) \quad I(u_1, u_2) = u_1^{-3/2} \sum_{k=0}^{\infty} \binom{-1/2}{k+1} \binom{-3/2}{k} \left(\frac{k+1}{-1/2-k}, 1, \frac{-1/2}{-1/2-k} \right) u_2^k,$$

while the sequence $(I_{(p^s-3)/2}(u_1, u_2))_{s=1}^\infty$ of the p^s -hypergeometric solutions modulo p^s of the same equations is given by the formula

$$(1.5) \quad I_{(p^s-3)/2}(u_1, u_2) = u_1^{(p^s-3)/2} \sum_{k=0}^{(p^s-3)/2} \binom{(p^s-1)/2}{k+1} \binom{(p^s-3)/2}{k} \times \left(\frac{k+1}{(p^s-1)/2-k}, 1, \frac{(p^s-1)/2}{(p^s-1)/2-k} \right) u_2^k,$$

see Section 9.7.

The sum $\sum_{k=0}^{(p^s-3)/2}$ in (1.5) is the truncation of the sum $\sum_{k=0}^\infty$ in (1.4), similar to what happens in (1.1) and (1.3). A new feature appears when we compare the prefactor $u_1^{-3/2}$ and the sequence of prefactors $(u_1^{(p^s-3)/2})_{s=1}^\infty$. As $s \rightarrow \infty$ the sequence of prefactors $(u_1^{(p^s-3)/2})_{s=1}^\infty$ tends p -adically to the prefactor $u^{-3/2}$ multiplied by a Teichmüller constant on a suitable domain in $\mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p -adic integers, see Section 10.3 and Theorem 10.5.

1.4. The paper is organized as follows. In Section 2 we define our system of KZ equations. In Section 3 we describe its complex solutions as hyperelliptic integrals. In Section 4 we describe the p^s -hypergeometric solutions of our KZ equations modulo p^s

2. KZ equations

$$\Omega = \sum_i h_i \otimes h_i \in \mathfrak{g} \otimes \mathfrak{g}$$
$$\frac{\partial I}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{i,j}}{z_i z_j} I, \quad i = 1, \dots, n,$$

The object of our discussion is the following particular case.

$$(2.1) \quad \frac{\partial I}{\partial z_i} = \frac{1}{2} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} I, \quad i = 1, \dots, n, \quad I_1(z) + \dots + I_n(z) = 0$$
[illegible]

...

and all other entries are zero. This joint system of *differential and algebraic equations* will be called the *system of KZ equations* in this paper.

System (2.1) is the system of the differential KZ equations with parameter $\kappa = 2$ associated with the Lie algebra $\mathfrak{sl}_2$ and the subspace of singular vectors of weight $2g - 1$ of the tensor power $(\mathbb{C}^2)^{\otimes(2g+1)}$ of two-dimensional irreducible $\mathfrak{sl}_2$ -modules, up to a gauge transformation, see this example in [V3, Section 1.1].

We consider system (2.1) over the field $\mathbb{C}$. We also consider the same system of equations modulo p^s and over the field $\mathbb{Q}_p$ of p -adic numbers.

3. Complex solutions

Consider the *master function*

$$(3.1) \quad \Phi(x, z) = \prod_{a=1}^n (x - z_a)^{-1/2}$$

and the column n -vector of hyperelliptic integrals

$$(3.2) \quad I^{(\gamma)}(z) = (I_1(z), \dots, I_n(z)), \quad I_j = \int \frac{\Phi(x, z)}{x - z_j} dx.$$

The integrals I_j are over an element γ of the first homology group of the algebraic curve with affine equation

$$y^2 = (x - z_1) \dots (x - z_n).$$

Starting from such γ , chosen for given values $\{z_1, \dots, z_n\}$, the vector $I^{(\gamma)}(z)$ can be analytically continued as a multivalued holomorphic function of z to the complement in $\mathbb{C}^n$ of the union of the diagonal hyperplanes $z_i = z_j$, $i \neq j$.

Theorem 3.1. *The vector $I^{(\gamma)}(z)$ is a solution of system (2.1).*

Theorem 3.1 is a classical statement. Much more general algebraic and differential equations satisfied by analogous multidimensional hypergeometric integrals were considered in [SV1]. Theorem 3.1 is discussed as an example in [V3, Section 1.1].

Proof. The theorem follows from Stokes' theorem and the two identities:

$$(3.3) \quad -\frac{1}{2} \left(\frac{\Phi(x, z)}{x - z_1} + \dots + \frac{\Phi(x, z)}{x - z_n} \right) = \frac{\partial \Phi}{\partial x}(x, z),$$

$$(3.4), \quad \left(\frac{\partial}{\partial z_i} - \frac{1}{2} \sum_{j \neq i} \frac{\Omega_{i,j}}{z_i - z_j} \right) \left(\frac{\Phi(x, z)}{x - z_1}, \dots, \frac{\Phi(x, z)}{x - z_n} \right) = \frac{\partial \Psi^i}{\partial x}(x, z)$$

where $\Psi^i(x, z)$ is the column n -vector $\dots, 0, -\frac{\Phi(x, z)}{x - z_i}, 0, \dots, (0 \ 0)$ with the nonzero element at the i -th place.

Theorem 3.2 ([V1, Formula (1.3)]). *All solutions of system (2.1) have this form. Namely, the complex vector space of solutions of the form (3.2) is $n - 1$ dimensional.*

This theorem follows from the determinant formula for multidimensional hypergeometric integrals in [V1], in particular, from [V1, Formula (1.3)].

4. Solutions modulo p^s

4.1. Leading terms. For a ring R denote $R[z] = R[z_1, \dots, z_n]$. For a positive integer t denote $R[z^{p^t}] = R[z_1^{p^t}, \dots, z_n^{p^t}]$.

Consider the lexicographical ordering of monomials $z_1^{d_1} \dots z_n^{d_n}$, so we have $z_1 > \dots > z_n$ and so on. For a nonzero polynomial $f(z) = \sum_{d_1, \dots, d_n} a_{d_1, \dots, d_n} z_1^{d_1} \dots z_n^{d_n}$ let $f_i(z)$ be the nonzero summand $a_{d_1, \dots, d_n} z_1^{d_1} \dots z_n^{d_n}$ with the largest monomial $z_1^{d_1} \dots z_n^{d_n}$. We call $f_i(z)$ the *leading term* of $f(z)$, the coefficient $a_{d_1, \dots, d_n}$ – the *leading coefficient*, the monomial $z_1^{d_1} \dots z_n^{d_n}$ – the *leading monomial*.

Let s be a positive integer. An element $a \in \mathbb{Z}/p^s\mathbb{Z}$ has a unique presentation $a = a_0 + a_1 p + \dots + a_{s-1} p^{s-1}$, where $a_i \in \{0, \dots, p-1\}$. An element a is invertible if and only if $a_0 \neq 0$.

Denote $F_p = \mathbb{Z}/p\mathbb{Z}$.

Let π_s denote the homomorphisms $\mathbb{Z} \rightarrow \mathbb{Z}/p^s\mathbb{Z}$, $\mathbb{Z}[z] \rightarrow (\mathbb{Z}/p^s\mathbb{Z})[z]$, $\mathbb{Z}[z]^n \rightarrow (\mathbb{Z}/p^s\mathbb{Z})[z]^n$ and for $t < s$ let $\pi_{s,t}$ denote the homomorphisms $\mathbb{Z}/p^s\mathbb{Z} \rightarrow \mathbb{Z}/p^t\mathbb{Z}$, $(\mathbb{Z}/p^s\mathbb{Z})[z] \rightarrow (\mathbb{Z}/p^t\mathbb{Z})[z]$, $(\mathbb{Z}/p^s\mathbb{Z})[z]^n \rightarrow (\mathbb{Z}/p^t\mathbb{Z})[z]^n$.

4.2. Quasi-constants. We say that a polynomial $f(z) \in \mathbb{Z}[z]$ is a *quasi-constant modulo p^s* if $\frac{\partial f}{\partial z_i} \in p^s \mathbb{Z}[z]$ for $i = 1, \dots, n$. The quasi-constants modulo p^s form a subring of $\mathbb{Z}[z]$ denoted by $\mathbb{Z}[z]_{p^s}$. For example $(z_1 + z_2)^{p^s} \in \mathbb{Z}[z]_{p^s}$.

Lemma 4.1. *As a $\mathbb{Z}$ -module, the ring $\mathbb{Z}[z]_{p^s}$ is spanned by the monomials $p^{s-t} z_1^{d_1} \dots z_n^{d_n}$, where t is the maximal integer such that $t \leq s$ and p^t divides every $d_1, \dots, d_n$. For example, $z_1^{p^s}$ and $p^{s-1} z_1 p z_2 p^2$ are such monomials.*

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Proof. Let $f(z) = \sum_d c_d z_1^{d_1} \dots z_n^{d_n} \in \mathbb{Z}[z]_{p^s}$. We show that each summand

$c_d z_1^{d_1} \cdots z_n^{d_n}$ is a multiple of a monomial of Lemma 4.1. Indeed, let $c_{d^0} z_1^{d_1^0} \cdots z_n^{d_n^0}$ be the leading term of $f(z)$. Then all first partial derivatives of it must lie in $p^s \mathbb{Z}[z]$. Hence $c_{d^0} \in p^{s-t} \mathbb{Z}$, where t is the maximal integer such that $t \leq s$ and p^t

divides every $d_1^0, \dots, d_n^0$. Subtracting the leading term from $f(z)$ and repeating the reasoning we prove the lemma.

Lemma 4.2. *Let $f(z)$ be a quasi-constant modulo p^s and $t \in \mathbb{Z}_{\geq 0}$. Then $p^t f(z)$ is a quasi-constant modulo p^r for any $1 \leq r \leq s + t$.*

The rings of quasi-constants form a decreasing filtration, $\mathbb{Z}[z]_p \supset \mathbb{Z}[z]_{p^2} \supset \dots$.

4.3. Solutions of system (2.1) modulo p^s . We say that a column n -vector $I(z) \in \mathbb{Z}[z]^n$ of polynomials with integer coefficients is a *solution of system (2.1) modulo p^s* , if $\pi_s I(z) \in (\mathbb{Z}/p^s \mathbb{Z})[z]^n$ satisfies system (2.1).

Lemma 4.3. *Let $I(z)$ be a solution of system (2.1) modulo p^s .*

$t \in \mathbb{Z}_{\geq 0}$ (i) $\leq \leq$ Let $p^t I(z)$ is a solution of system (2.1) modulo p^r for any

$1 \leq r \leq s + t$.

(ii) Let $f(z)$ be a quasi-constant modulo p^s . Then $f(z)I(z)$ is a solution of system (2.1) modulo p^s . (iii) Let $1 \leq t < s$ and $I(z) \in p^t \mathbb{Z}[z]^n$. Let $f(z)$ be a quasi-constant modulo p^{s-t} . Then $f(z)I(z)$ is a solution of system (2.1) modulo p^s .

4.4. p^s -Hypergeometric solutions. Let M be the least positive integers such that

$$(4.1) \quad M \equiv -\frac{1}{2} \pmod{p^s}.$$

We have

$$M = \frac{p^s - 1}{2} = \frac{p - 1}{2} (1 + p + \cdots + p^{s-1}).$$

Introduce the *master polynomial*

$$(4.2) \quad \Phi_{p^s}(x, z) = \prod_{i=1}^n (x - z_i)^M \in \mathbb{Z}[x, z].$$

Let

$$(4.3) \quad P_{p^s}(x, z) = \left(\frac{\Phi_{p^s}(x, z)}{x - z_1}, \dots, \frac{\Phi_{p^s}(x, z)}{x - z_n} \right) = \sum_i P_{p^s}^i(z) x^i,$$

where $P_{p^s}(x, z)$ is a column n -vector of polynomials in $x, z_1, \dots, z_n$ and $P_{p^s}^i(z)$ are n -vectors of polynomials in $z_1, \dots, z_n$ with coefficients in $\mathbb{Z}$. For a positive integer l , denote

$$I_{p^s}^{[lp^s-1]}(z) = P_{p^s}^{lp^s-1}(z).$$

Theorem 4.4. *For any positive integer l , the vector of polynomials $I_{p^s}^{[lp^s-1]}(z) \in \mathbb{Z}[z]^n$ is a solution of system (2.1) modulo p^s .*

Proof. We have the following modifications of identities (3.3), (3.4):

$$(4.4) \quad M \left(\frac{\Phi_{p^s}(x, z)}{x - z_1} + \dots + \frac{\Phi_{p^s}(x, z)}{x - z_n} \right) = \frac{\partial \Phi_{p^s}}{\partial x}(x, z),$$

$$(4.5) \quad \left(\frac{\partial}{\partial z_i} + M \sum_{j \neq i} \frac{\Omega_{i,j}}{z_i - z_j} \right) \left(\frac{\Phi_{p^s}(x, z)}{x - z_1}, \dots, \frac{\Phi_{p^s}(x, z)}{x - z_n} \right) = \frac{\partial \Psi_{p^s}^i}{\partial x}(x, z),$$

where $\Psi_{p^s}^i(x, z)$ is the column n -vector $\dots, 0, -\frac{\Phi_{p^s}(x, z)}{x - z_i}, 0, \dots, (0 \ 0)$ with the nonzero -

element at the i -th place. The theorem follows from identities (4.4), (4.5).

Remark. In [SV2] it was explained on how to construct polynomial solutions modulo p of an arbitrary system differential KZ equations, associated with any KacMoody algebra and any tensor product of highest weight representations. The same construction gives polynomial solutions modulo p^s . The details will be provided elsewhere.

The range for the index l is defined by the inequalities $0 < lp^s - 1 \leq n \frac{p^s - 1}{2}$. Hence $l = 1, \dots, g$. The solutions $I_p^{[lp^s - 1]}(z)$, $l = 1, \dots, g$, given by this construction, will be called the p^s -hypergeometric solutions in $\mathbb{Z}[z]^n$. For $t = 1, \dots, s - 1$ and $l = 1, \dots, g$, the vector $p^{s-t} I_p^{[lp^t - 1]}(z)$ is a solution of system (2.1) modulo p^s , see Lemma 4.3. Such solutions also will be called p^s -hypergeometric solutions in $\mathbb{Z}[z]^n$.

4.5. Modules. Consider the increasing filtration

$$(4.6) \quad 0 = \mathcal{M}_{p^s}^0 \subset \mathcal{M}_{p^s}^1 \subset \dots \subset \mathcal{M}_{p^s}^{s-1} \subset \mathcal{M}_{p^s}^s = \mathcal{M}_{p^s},$$

where

$$(4.7) \quad \mathcal{M}_{p^s}^t = \left\{ \pi_s \left(\sum_{r=1}^t \sum_{l=1}^g c_{r,l}(z) p^{s-r} I_p^{[lp^r - 1]}(z) \right) \mid c_{r,l}(z) \in \mathbb{Z}[z]_{p^r} \right\},$$

$t = 1, \dots, s$. We have $\mathcal{M}_{p^s} \subset (\mathbb{Z}/p^s\mathbb{Z})[z]^n$. Every element of $\mathcal{M}_{p^s}$ is a polynomial solution of system (2.1) with coefficients in $\mathbb{Z}/p^s\mathbb{Z}$, see Lemma 4.3. The set $\mathcal{M}_{p^s}$ is a module over the ring $\mathbb{Z}[z]_{p^s}$ of quasi-constants modulo p^s , where $f(z) \in \mathbb{Z}[z]_{p^s}$ acts by multiplication by $\pi_s f(z)$. Each $\mathcal{M}_{p^s}^t$ is an $\mathbb{Z}[z]_{p^s}$ -submodule of $\mathcal{M}_{p^s}$.

Each $\mathcal{M}_{p^s}^t$ is also a module over the larger ring $\mathbb{Z}[z]_{p^t}$ of quasi-constants modulo p^t , where $f(z) \in \mathbb{Z}[z]_{p^t}$ acts by multiplication by $\pi_s f(z)$.

The elements of $\mathcal{M}_{p^s}$ will be called the p^s -hypergeometric solutions in $(\mathbb{Z}/p^s\mathbb{Z})[z]^n$.

5. Independence of modules from the choice of M

5.1. More general construction of solutions. For $i = 1, \dots, n$, let M_i be a positive integer such that

$$(5.1) \quad M_i \equiv -\frac{1}{2} \pmod{p^s}.$$

Denote $\vec{M} = (M_1, \dots, M_n)$. Consider the *master polynomial*

$$(5.2) \quad \Phi(x, z, \vec{M}) = \prod_{i=1}^n (x - z_i)^{M_i} \in \mathbb{Z}[x, z],$$

and the Taylor expansion

$$P(x, z, \vec{M}) = \left(\frac{\Phi(x, z, \vec{M})}{x - z_1}, \dots, \frac{\Phi(x, z, \vec{M})}{x - z_n} \right) = \sum_i P^i(z, \vec{M}) x^i,$$

where $P^i(z, \vec{M})$ are n -vectors of polynomials in $z_1, \dots, z_n$ with coefficients in $\mathbb{Z}$. For a positive integer l , denote

$$I^{[lp^s-1]}(z, \vec{M}) = P^{lp^s-1}(z, \vec{M}).$$

THEOREM 5.1. *For any positive integers $l, t, t \leq s$, the vector of polynomials $I^{lp^s-1}[z, M] \in \mathbb{Z}[z]^n$ is a solution of system (2.1) modulo p^t .*

Proof. The theorem follows from straightforward modifications of identities (4.4), (4.5).

5.2. More modules. Consider the increasing filtration

$$(5.3) \quad 0 = \mathcal{M}_{p^s}^0(\vec{M}) \subset \mathcal{M}_{p^s}^1(\vec{M}) \subset \dots \subset \mathcal{M}_{p^s}^{s-1}(\vec{M}) \subset \mathcal{M}_{p^s}^s(\vec{M}) = \mathcal{M}_{p^s}(\vec{M})$$

, where

$$(5.4) \quad \mathcal{M}_{p^s}^t(\vec{M}) = \left\{ \pi_s \left(\sum_{r=1}^t \sum_{l \geq 1} c_{r,l}(z) p^{s-r} I^{[lp^r-1]}(z, \vec{M}) \right) \mid c_{r,l}(z) \in \mathbb{Z}[z]_{p^r} \right\},$$

$t = 1, \dots, s$. We have $\mathcal{M}_{p^s}(\vec{M}) \subset (\mathbb{Z}/p^s\mathbb{Z})[z]^n$. Every element of $\mathcal{M}_{p^s}(\vec{M})$ is a polynomial solution of system (2.1) with coefficients in $\mathbb{Z}/p^s\mathbb{Z}$, see Lemma 4.3. The set $\mathcal{M}_{p^s}(\vec{M})$ is a module over the ring $\mathbb{Z}[z]_{p^s}$ of quasi-constants modulo p^s , where $f(z) \in \mathbb{Z}[z]_{p^s}$ acts by multiplication by $\pi_s f(z)$. Each $\mathcal{M}_{p^s}^t(\vec{M})$ is an $\mathbb{Z}[z]_{p^s}$ -submodule of $\mathcal{M}_{p^s}(\vec{M})$.

Each $\mathcal{M}_{p^s}^t(\vec{M})$ is also a module over the larger ring $\mathbb{Z}[z]_{p^t}$ of quasi-constants modulo p^t , where $f(z) \in \mathbb{Z}[z]_{p^t}$ acts by multiplication by $\pi_s f(z)$.

Theorem 5.2. *Filtration (5.3) does not depend on the choice of $\vec{M} = (M_1, \dots, M_n)$, satisfying congruences (5.1). Moreover, filtration (5.3) coincides with filtration (4.6).*

For $s = 1$ the statement is [Sliv, Theorem 3.1].

Proof. First we show that $\mathcal{M}_{p^s}(\vec{M})$ and filtration (5.3) do not depend on the choice of M . Let $M = (M_1, \dots, M_n)$, $M' = (M'_1, \dots, M'_n)$ be two vectors satisfying congruences (5.1). We say that $\vec{M}' \geq \vec{M}$ if $M'_i \geq M_i$ for all i . The vector $(\frac{p^s-1}{2}, \dots, \frac{p^s-1}{2})$ is the minimal vector with respect to this partial order. To show that $\mathcal{M}_{p^s}(\vec{M})$ and filtration (5.3) do not depend on the choice of M it is enough to show that the filtrations are the same for a vector M and for a vector

$\vec{M}' = \vec{M} + (0, \dots, 0, \frac{p^s-1}{2}, 0, \dots, 0)$, where the nonzero element stays at the j -th position for some j . Then

$$P(x, z, \vec{M}') = P(x, z, \vec{M}) \cdot (x - z_j)^{p^s} = P(x, z, \vec{M}) \sum_{a=0}^{p^s} (-1)^{p^s-a} \binom{p^s}{a} x^a z_j^{p^s-a}.$$

Recall that $P(x, z, \vec{M}) = \sum_i P^i(z, \vec{M}) x^i$, $P(x, z, \vec{M}') = \sum_i P^i(z, \vec{M}') x^i$, For any $r \leq s$ and l we have

$$(5.5) \quad P^{lp^r-1}(z, \vec{M}') = \sum_{a=0}^{p^s} (-1)^{p^s-a} \binom{p^s}{a} z_j^{p^s-a} P^{lp^r-a-1}(z, \vec{M}).$$

We are interested in this formula, since $I^{[lp^r-1]}(z, \vec{M}') = P^{lp^r-1}(z, \vec{M}')$ is a solution of system (2.1) modulo p^r .

Lemma 5.3. *Let $b, c \in \mathbb{Z}_{>0}$ be such that $bp^c \leq p^s$, $p \nmid b$. Then p^{s-c} is the maximal power of p dividing $\binom{p}{bp^c}$.*

Proof. For $a \leq p^s$ we have, $a \binom{p^s}{a} = p^s \binom{p^s-1}{a-1}$ $p \nmid \binom{p^s-1}{a-1}$ and by Lucas' theorem, [Lu].

Lemma 5.4. *Let $a \in \{0, \dots, p^s\}$, $a = bp^c$, $p \nmid b$. Consider the vector*

$$V = (-1)^{p^s-a} \binom{p^s}{a} z_j^{p^s-a} P^{lp^r-a-1}(z, \vec{M})$$

appearing in (5.5). If $lp^r \leq a$, then $V = 0$. For $lp^r > a$, we write $lp^r - a = vp^u$, where $u = \min(r, c)$. Then

$$(5.6) \quad V = d(z) p^{r-u} I^{[vp^u-1]}(z, \vec{M}),$$

where $d(z) = z_j^{p^s-a} (-1)^{p^s-a} \binom{p^s}{a} / p^{r-u}$ is a quasi-constant modulo p^u .

Proof. The lemma follows from Lemma 5.3.

Corollary 5.5. *For any $r = 1, \dots, s$, we have $\mathcal{M}_{p^s}^r(\vec{M}') \subset \mathcal{M}_{p^s}^r(\vec{M})$.* **Lemma 5.6.** *For any $r = 1, \dots, s$, we have $\mathcal{M}_{p^s}^r(\vec{M}') \supset \mathcal{M}_{p^s}^r(\vec{M})$.*

Proof. Let w be the greatest integer such that $wp^r \leq \deg_t P(t, z, \vec{M})$. Then $w' = w + p^{s-r}$ is the greatest integer such that $w'p^r \leq \deg_t P(t, z, \vec{M}')$. Comparing the coefficients in (5.5) and using Lemma 5.5, we observe that for any $l = 1, \dots, w$ we have

$$(5.7) \quad I^{[(l+p^{s-r})p^r-1]}(z, \vec{M}') = I^{[lp^r-1]}(z, \vec{M}) + \sum_{m=1}^{l-1} c_{r,m}(z) I^{[mp^r-1]}(z, \vec{M}) + \sum_{k=1}^{r-1} c_{k,m}(z) p^{r-k} I^{[mp^k-1]}(z, \vec{M}),$$

where $c_{i,j}(z) \in \mathbb{Z}[z]_{p^i}$. This triangular system of equations with respect to $I^{[lp^r-1]}(z, \vec{M})$, $l = 1, \dots, w$, can be written as

$$(5.8) \quad I^{[lp^r-1]}(z, \vec{M}) = \sum_{m=1}^{l-1} c'_{r,m}(z) I^{[mp^r-1]}(z, \vec{M}') + \sum_{k=1}^{r-1} \sum_{m=1}^{l-1} c'_{k,m}(z) p^{r-k} I^{[mp^k-1]}(z, \vec{M}'),$$

$l = 1, \dots, w$, for suitable $c'_{i,j}(z) \in \mathbb{Z}[z]_{p^i}$. Applying the previous reasoning to the

sum, we obtain

(5.9)

$$I^{[lp^r-1]}(z, \vec{M}) = \sum_{m1} c'_{r,m}(z) I^{[mp^r-1]}(z, \vec{M}') + \sum_{k=1}^{r-1} \sum_{m1} c''_{k,m}(z) p^{r-k} I^{[mp^k-1]}(z, \vec{M}'),$$

$l = 1, \dots, w$, for suitable $c''_{i,j}(z) \in \mathbb{Z}[z]_{p^i}$. This proves the lemma.

Corollary 5.7. *The module $\mathcal{M}_{p^s}(\vec{M})$ and filtration (5.3) do not depend on the choice of $M = (M_1, \dots, M_n)$, satisfying congruences (5.1).*

Lemma 5.8. *Let $\vec{M}^{\min} = (\frac{p^s-1}{2}, \dots, \frac{p^s-1}{2})$. Then $\mathcal{M}_{p^s}(\vec{M}^{\min}) = \mathcal{M}_{p^s}$.*

Proof. The proof of the lemma is a straightforward modification of the proof of

Corollary 5.5 and Lemma 5.6. Theorem 5.2 is proved.

6. Filtrations and homomorphisms

6.1. Reduction from modulo p^s to modulo p^{s-m} . If $I(z)$ is a polynomial solution of system (2.1) modulo p^s , then $I(z)$ is also a polynomial solution of system (2.1) modulo p^{s-m} for any $1 \leq m < s$. This defines a map

(6.1)

$$\begin{aligned} \mathbf{r}_{s,s-m} : \mathcal{M}_{p^s}(\vec{M}) &\rightarrow \mathcal{M}_{p^{s-m}}(\vec{M}), \\ \pi_s \left(\sum_{r=1}^s \sum_{l \geq 1} c_{r,l}(z) p^{s-r} I^{[lp^r-1]}(z, \vec{M}) \right) &\mapsto \pi_{s-m} \left(\sum_{r=m+1}^s \sum_{l \geq 1} c_{r,l}(z) p^{s-r} I^{[lp^r-1]}(z, \vec{M}) \right), \end{aligned}$$

where M is a vector with coordinates satisfying congruences (5.1). See these sums in (5.4). In the last sum we have

$p^{s-r} I^{[lp^r-1]}(z, \vec{M}) = p^{s-m-(r-m)} I^{[(lp^m)p^{r-m}-1]}(z, \vec{M})$ and a solution $I^{[lp^{r-1}]}(z, \vec{M})$ modulo p^r also can be considered as a solution $I^{[(lp^m)p^{r-m}-1]}(z, \vec{M})$ modulo p^{r-m} .

For any $r = 1, \dots, s$, the submodule $\mathcal{M}_{p^s}^r(\vec{M}) \subset \mathcal{M}_{p^s}(\vec{M})$ is mapped by $\mathbf{r}_{s,s-m}$ to the submodule $\mathcal{M}_{p^{s-m}}^{r-m}(\vec{M})$. The induced map

$$(6.2) \quad \mathbf{r}_{s,s-m}^r : \mathcal{M}_{p^s}^r(\vec{M}) \rightarrow \mathcal{M}_{p^{s-m}}^{r-m}(\vec{M})$$

is a homomorphism of $\mathbb{Z}[z]_{p^r}$ -modules. Thus the map (6.1) is a homomorphism of filtered modules decreasing the index of filtration by m .

By Theorem 5.2 we have $\mathcal{M}_{p^s}(\vec{M}) = \mathcal{M}_{p^s}$. Hence homomorphism (6.1) also can be considered as a homomorphism of filtered modules,

$$(6.3) \quad \mathbf{r}_{s,s-m} : \mathcal{M}_{p^s} \rightarrow \mathcal{M}_{p^{s-m}},$$

decreasing the index of filtration by m .

It is rather nontrivial to write a formula for this map in terms of the generators $I^{[lp^{r-1}]}(z)$ of these modules.

6.2. Multiplication by p^m . If $I(z)$ is a polynomial solution of system (2.1) modulo p^s , then for any positive integer m the polynomial $p^m I(z)$ is a polynomial solution of system (2.1) modulo p^{s+m} . In particular, multiplication by p^m defines a map

$$(6.4) \quad \mathfrak{p}_{s,s+m} : \mathcal{M}_{p^s}^t \rightarrow \mathcal{M}_{p^{s+m}}^t,$$

$$\pi_s \left(\sum_{r=1}^t \sum_{l=1}^g c_{r,l}(z) p^{s-r} I_{p^r}^{[lp^r-1]}(z) \right) \mapsto \pi_{s+m} \left(\sum_{r=1}^t \sum_{l=1}^g c_{r,l}(z) p^{s+m-r} I_{p^r}^{[lp^r-1]}(z) \right)$$

for any $t = 1, \dots, s$. See these sums in (4.7). Clearly this map is an isomorphism of filtered $\mathbb{Z}[z]_{p^t}$ -modules.

6.3. The composition of homomorphisms. For $m < s$ denote by $\mathfrak{c}_{s,m}$ the composition $\mathfrak{p}_{s-m, m} \mathfrak{r}_{s, s-m}$,

$$(6.5) \quad \mathfrak{c}_{s,m} : \mathcal{M}_{p^s} \rightarrow \mathcal{M}_{p^s}, \quad I(z) \mapsto p^m I(z).$$

For any $t = 1, \dots, s$, this map induces a homomorphism $\mathcal{M}_{p^s}^t \rightarrow \mathcal{M}_{p^s}^{t-m}$ of $\mathbb{Z}[z]_{p^t}$ -modules.

We have $\mathfrak{c}_{s,m} = (\mathfrak{c}_{s,1})^m$ for $m < s$ and $\mathfrak{c}_{s,m} = (\mathfrak{c}_{s,1})^m = 0$ for $m \geq s$.

As we know, the module $\mathcal{M}_{p^s}$ is generated by the elements $\pi_s(p^{s-r} I_{p^r}^{[lp^r-1]}(z))$, $r = 1, \dots, s$, $l = 1, \dots, g$. For $l = 1, \dots, g$, we have

$$(6.6) \quad \pi_s(I^{[lp^s-1]}(z)) \mapsto \pi_s(p I^{[lp^s-1]}(z)) = \pi_s \left(\sum_{r=1}^{s-1} \sum_{k=1}^g c_{r,k}^{l,s}(z) p^{s-r} I_{p^r}^{[kp^r-1]}(z) \right)$$

$\mathfrak{c}_{s,1}$

$$= 1 = 1$$

for suitable coefficients $c_{r,k}^{l,s}(z) \in \mathbb{Z}[z]_{p^r}$.

The set of the coefficients $(c_{r,k}^{l,s}(z))_{l,s,r,k}$ determines the homomorphisms $\mathfrak{c}_{s,m}$ for all s, m . In what follows we shall describe the coefficients $c_{s-1,k}^{l,s}(z)$ for all l, s, k , see Theorem 8.4.

6.4. Graded modules and homomorphisms. Denote

$$(6.7) \quad \text{gr} \mathcal{M}_{p^s} = \bigoplus_{t=1}^s \text{gr} \mathcal{M}_{p^s}^t, \quad \text{gr} \mathcal{M}_{p^s}^t = \mathcal{M}_{p^s}^t / \mathcal{M}_{p^s}^{t-1}.$$

Lemma 6.1. *For any $t = 1, \dots, s$, the action of $\mathbb{Z}[z]_{p^t}$ on $\mathcal{M}_{p^t}$ makes $\text{gr} \mathcal{M}_{p^s}^t$ an $\mathbb{F}_p[z^{p^t}]$ -module. Multiplication by p on $\mathcal{M}_{p^s}^t$ induces a homomorphism*

$$(6.8) \quad \text{gr} \mathfrak{c}_{s,1} : \text{gr} \mathcal{M}_{p^s}^t \rightarrow \text{gr} \mathcal{M}_{p^s}^{t-1}$$

of $\mathbb{F}_p[z^{p^t}]$ -modules.

7. Coefficients of solutions

7.1. Homogeneous polynomials. For $l = 1, \dots, g$, the solution $I^{[lp^s-1]}(z) = (I_1^{[lp^s-1]}, \dots, I_n^{[lp^s-1]})$ is a homogeneous polynomial in z of degree

$$(7.1) \quad \delta_l = (2g+1) \frac{p^s-1}{2} - lp^s = (g-l)p^s + \frac{p^s-1}{2} - g.$$

Notice that $(-1)^{\delta_l} = (-1)^{s \frac{p-1}{2} + l}$.

7.2. Formula for coefficients. Recall that $M = \frac{p^s-1}{2}$. Projection of this integer to $\mathbb{Z}/p^s\mathbb{Z}$ is invertible. Let

$$I^{[lp^s-1]}(z) = \sum_{d_1, \dots, d_n} I_{d_1, \dots, d_n}^{[lp^s-1]} z_1^{d_1} \dots z_n^{d_n}, \quad I_{d_1, \dots, d_n}^{[lp^s-1]} \in \mathbb{Z}^n.$$

Lemma 7.1 ([V8, Lemma 3.1]). *We have*

$$(7.2) \quad I_{d_1, \dots, d_n}^{[lp^s-1]} = (-1)^{\delta_l} \prod_{i=1}^n \binom{M}{d_i} \left(1 - \frac{d_1}{M}, \dots, 1 - \frac{d_n}{M}\right).$$

The sum of coordinates of this vector is divisible by p^s .

Lemma 7.2 (cf. [V8, Theorem 6.1]). *For $l = 1, \dots, g$, the leading term of the p^s -hypergeometric solution $I^{[lp^s-1]}(z)$ is*

$$(7.3) \quad I_l^{[lp^s-1]}(z) = (-1)^{\delta_l} \binom{M}{l} \left(0, \dots, 0, \frac{l}{M}, 1, \dots, 1\right) z_1^M \dots z_{2g-2l}^{M-2l} z_{2g-2l+1}^{M-l},$$

where 0 is repeated $2g - 2l$ times and 1 is repeated $2l$ times.

Proof. The lemma follows from Lemma 7.1.

Lemma 7.3. *The projections to $\mathbb{Z}/p^s\mathbb{Z}$ of the integers $\binom{M}{l}$, $\binom{M}{l} \frac{l}{M}$ are invertible.*

Proof. The invertibility of $\binom{M}{l}$ follows from Lucas' theorem, [Lu].

8. Multiplication by p and Cartier-Manin matrix

8.1. Linear independence.

Lemma 8.1. *The projections of the p^s -hypergeometric solutions $I^{[lp^s-1]}(z) \in$*

$\mathbb{Z}[z]^n$, $l = 1, \dots, g$, *to $\mathbb{F}_p[z]^n$ are linearly independent over $\mathbb{F}_p[z]$, that is, if*

$$(8.1) \quad \sum_{l=1}^g c_l(z) I^{[lp^s-1]}(z) \in p\mathbb{Z}[z]^n$$

for some $c_l(z) \in \mathbb{Z}[z]$, then all $c_l(z) \in p\mathbb{Z}[z]$.

Proof. Recall that the projection $\mathbb{Z}[z]^n \rightarrow \mathbb{F}_p[z]^n$ is denoted by π_1 . By Lemma 7.3, the leading coefficient of $\pi_1(c_l(z) I^{[lp^s-1]}(z))$ equals the product of the leading coefficient of $\pi_1(c_l(z))$ and the leading coefficient of $\pi_1(I^{[lp^s-1]}(z))$, if $\pi_1(c_l(z))$ is nonzero. In that case the leading coefficient of $\pi_1(c_l(z) I^{[lp^s-1]}(z))$ is a nonzero multiple of the nonzero vector $\pi_1((0, \dots, 0, \frac{l}{M}, 1, \dots, 1))$.

If relation (8.1) holds and some of the coefficients $c_l(z)$ have nonzero projections $\pi_1(c_l(z))$, then for several values of such indices l the sum of the corresponding leading

coefficients has to be equal to zero, which is impossible due to the fact that the vectors $\pi_1((0, \dots, 0, \frac{l}{M}, 1, \dots, 1))$ are linear independent over $\mathbb{F}_p$.

Corollary 8.2. *The projections of the p^s -hypergeometric solutions $I^{[lp^{s-1}]}(z) \in \mathbb{Z}[z]^n$, $l = 1, \dots, g$, to $\mathbb{F}_p[z]^n$ are linearly independent over $\mathbb{F}_p[z^{p^s}]$.*

Denote by

$$(8.2) \quad \text{gr}_t: \mathcal{M}_{p^s}^t \rightarrow \text{gr}\mathcal{M}_{p^s}^t$$

the natural projection. Then the elements $\text{gr}_t(\pi_s(p^{s-t} I^{[lp^{t-1}]}(z)))$, $l = 1, \dots, g$, generate the $\mathbb{F}_p[z^{p^t}]$ -module $\text{gr}\mathcal{M}_{p^s}^t$.

Corollary 8.3. *For $t = 1, \dots, s$, the $\mathbb{F}_p[z^{p^t}]$ -module $\text{gr}\mathcal{M}_{p^s}^t$ is a free module of rank g with a basis $\text{gr}_t(\pi_s(p^{s-t} I^{[lp^{t-1}]}(z)))$, $l = 1, \dots, g$.*

Denote the basis vectors of the $\mathbb{F}_p[z^{p^t}]$ -module $\text{gr}\mathcal{M}_{p^s}^t$ by

$$(8.3) \quad v_{s,t}^l := \text{gr}_t(\pi_s(p^{s-t} I^{[lp^{t-1}]}(z))), \quad l = 1, \dots, g.$$

8.2. Cartier-Manin matrices. Let

$$f(x, z) = (x - z_1) \dots (x - z_n), \quad n = 2g + 1.$$

Consider the hyperelliptic curve X defined by the affine equation

$$y^2 = (x - z_1) \dots (x - z_n).$$

Consider the space $\Omega^1(X)$ of regular 1-forms on X with basis $\frac{x^{i-1} dx}{y}$, $i = 1, \dots, g$. Define the Cartier map $C: \Omega^1(X) \rightarrow \Omega^1(X)$ as follows. We have

$$\frac{x^{i-1} dx}{y} = \frac{x^{i-1} y^{p-1} dx}{y^{p-1} y} = \frac{x^{i-1} f(x)^{(p-1)/2} dx}{y^p}.$$

Let $x^{i-1} f(x, z)^{(p-1)/2} = \sum_j c_i^j(z) x^j$. Define

$$C: \frac{x^{i-1} dx}{y} \mapsto \sum_{j=1}^g c_i^{jp-1}(z) \frac{x^{j-1} dx}{y},$$

see [AH]. The map C is identified with the $g \times g$ -matrix $(C^i_j(z))^{g_{ij}=1}$,

$$C_i^j(z) = c_{jp-1}^i(z).$$

8.3. Matrix of $\text{grc}_{s,1}$. Recall that multiplication of solutions by p defines a map

$$(8.4) \quad \text{grc}_{s,1}: \text{gr}\mathcal{M}_{p^s}^t \rightarrow \text{gr}\mathcal{M}_{p^s}^{t-1},$$

where $\text{gr} \mathcal{M}_{p^s}^t$ is a free $\mathbb{F}_p[z^{p^t}]$ -module with a basis $(v_{s,t}^l)_{l=1}^g$ and $\text{gr} \mathcal{M}_{p^s}^{t-1}$ is a free $\mathbb{F}_p[z^{p^{t-1}}]$ -module with a basis $(v_{s,t-1}^l)_{l=1}^g$, see (8.3). The map (8.4) is a homomorphism of $\mathbb{F}_p[z^{p^t}]$ -modules.

Theorem 8.4. *The matrix of $\text{grc}_{s,1}$ is the Cartier-Manin matrix $C(z^{p^{t-1}})$,*

$$(8.5) \quad \text{grc}_{s,1} : v_{s,t}^l \mapsto \sum_{m=1}^g v_{s,t-1}^m C_m^l(z^{p^{t-1}}), \quad l = 1, \dots, g.$$

Proof. The problem is to express $\text{modulop}^{s-t+2} Z[z]^n$ the element $p \cdot p^{s-t} I_{p^t}^{[lp^t-1]}(z)$ in terms of the elements. In $p^{s-t+1} I_{p^{t-1}}^{[mp^{t-1}-1]}(z)$, $m = 1, \dots, g$, other words, we need to express $I_{p^t}^{[lp^t-1]}(z)$ in terms of $I_{p^{t-1}}^{[mp^{t-1}-1]}(z)$, $m = 1, \dots, g$, modulo $pZ[z]^n$.

By definition, the vector $I_{p^t}^{[lp^t-1]}(z)$ is the coefficient of x^{lp^t-1} in the Taylor expansion of the polynomial

$$P_{p^t}(x, z) = P_{p^{t-1}}(x, z) \prod_{i=1}^n (x - z_i)^{p^{t-1}(p-1)/2},$$

while the vector $I_{p^{t-1}}^{[mp^{t-1}-1]}(z)$ is the coefficient of $x^{mp^{t-1}-1}$ in the Taylor expansion of the polynomial $P_{p^{t-1}}(x, z)$, see notations in Section 4.4. We have

$$\prod_{i=1}^n (x - z_i)^{p^{t-1}(p-1)/2} \equiv \prod_{i=1}^n (x^{p^{t-1}} - z_i^{p^{t-1}})^{(p-1)/2} \pmod{p}.$$

Hence $I_{p^t}^{[lp^t-1]}(z) \equiv \sum_{m=1}^g I_{p^{t-1}}^{[mp^{t-1}-1]}(z) C_m^l(z^{p^{t-1}}) \pmod{p}$. Theorem 8.4 is proved.

Corollary 8.5. *The matrix of $\text{grc}_{s,m} : \text{gr} \mathcal{M}_{p^s}^t \rightarrow \text{gr} \mathcal{M}_{p^s}^{t-m}$ is the product of Cartier-Manin matrices $C(z^{p^{t-1}})C(z^{p^{t-2}}) \dots C(z^{p^{t-m}})$. Moreover, this statement, applied to the map*

$$\text{grc}_{s,s-1} : \text{gr} \mathcal{M}_{p^s}^s \rightarrow \text{gr} \mathcal{M}_{p^s}^1 = \mathcal{M}_{p^s}^1,$$

can be reformulated as follows. For any $l = 1, \dots, g$, the solution $I_{p^s}^{lp^s-1}(z)$ modulo p^s of system (2.1), projected to $\mathbb{F}_p[z]^n$, equals the projection to $\mathbb{F}_p[z]^n$ of the solution

$$\sum_{m_1, \dots, m_{s-1}=1}^g I_p^{[m_1 p-1]}(z) C_{m_1}^{m_2}(z^p) \dots C_{m_{s-2}}^{m_{s-1}}(z^{p^{s-2}}) C_{m_{s-1}}^l(z^{p^{s-1}})$$

modulo p of system (2.1).

See these sums in [V5, Section 8].

9. Change of variables

9.1. Change of the variable x . Change the variable x and set $x = v + Z_n$.

Then

$$(9.1) \quad \tilde{\Phi}_{p^s}(v, z) := \Phi_{p^s}(v + z_n, z) = \left(\prod_{i=1}^{n-1} (v - (z_i - z_n)) \right)^{(p^s-1)/2} v^{(p^s-1)/2}.$$

Let

$$(9.2) \quad \begin{aligned} \tilde{P}_{p^s}(v, z) &:= P_{p^s}(v + z_n, z) \\ &= \left(\frac{\tilde{\Phi}_{p^s}(v, z)}{v - (z_1 - z_2)}, \dots, \frac{\tilde{\Phi}_{p^s}(v, z)}{v - (z_{n-1} - z_n)}, \frac{\tilde{\Phi}_{p^s}(v, z)}{v} \right) = \sum_i \tilde{P}_{p^s}^i(z) v^i, \end{aligned}$$

where $P_{p^s}^i(z)$ are n -vectors of polynomials in z with integer coefficients. For a positive integer l , denote

$$(9.3) \quad \tilde{I}_{p^s}^{[lp^s-1]}(z) := \tilde{P}_{p^s}^{lp^s-1}(z).$$

The polynomial $\tilde{I}_{p^s}^{[lp^s-1]}(z)$ is nonzero if $l = 1, \dots, g$. Notice that every polynomial $\tilde{I}_{p^s}^{[lp^s-1]}(z)$ is a function of differences $z_i - z_n$, $i = 1, \dots, n-1$.

Consider the increasing filtration

$$(9.4) \quad 0 = M^{\sim}_{0p^s} \subset M^{\sim}_{1p^s} \subset \dots \subset M^{\sim}_{sp^s-1} \subset M^{\sim}_{sp^s} = M^{\sim}_{p^s},$$

where

$$(9.5) \quad \tilde{\mathcal{M}}_{p^s} = \left\{ \pi_s \left(\sum_{r=1}^s \sum_{l=1}^g c_{r,l}(z) p^{s-r} \tilde{I}_{p^r}^{[lp^r-1]}(z) \right) \mid c_{r,l}(z) \in \mathbb{Z}[z]_{p^r} \right\}$$

$$(9.6) \quad \tilde{\mathcal{M}}_{p^s}^t = \left\{ \pi_s \left(\sum_{r=1}^t \sum_{l=1}^g c_{r,l}(z) p^{s-r} \tilde{I}_{p^r}^{[lp^r-1]}(z) \right) \mid c_{r,l}(z) \in \mathbb{Z}[z]_{p^r} \right\},$$

$$t = 1, \dots, s.$$

Theorem 9.1. *For any l , the vector of polynomials $\tilde{I}_{p^s}^{[lp^s-1]}(z) \in \mathbb{Z}[z]^n$ is a solution of system (2.1) modulo p^s . For any $t = 1, \dots, s$ we have $\tilde{\mathcal{M}}_{p^s}^t = \mathcal{M}_{p^s}^t$.*

Proof. The proof is the same as the proof of Theorem 5.2 and the proof of [V5, Lemma 5.2]. In the proof of Theorem 9.1 the following Lemma 9.2 is used instead of Lemma 5.3.

Lemma 9.2. *Let $r = 0, \dots, s-1$ and $m \nmid p$, then is $\binom{mp^r+lp^s-1}{lp^s-1}$ divisible by p^{s-r} .*

$$\text{Proof. We have } \binom{mp^r+lp^s-1}{lp^s-1} = \binom{mp^r+lp^s-1}{mp^r} = \frac{lp^s}{mp^r} \binom{mp^r+lp^s-1}{mp^r-1}.$$

9.2. Change of variables z . We introduce the new variables $u_1, \dots, u_n$ by the formulas:

$$(9.7) \quad u_1 = z_1 - z_n, \quad u_2 = \frac{z_2 - z_n}{z_1 - z_n}, \quad \dots \quad u_{n-1} = \frac{z_{n-1} - z_n}{z_{n-2} - z_n}, \quad u_n = z_1 + \dots + z_n,$$

$$\text{or } z_i - z_n = u_1 \cdots u_i, i = 1, \dots, n-1, \quad z_1 + \dots + z_n = u_n.$$

For any l ,s we denote $u = (u_1, \dots, u_{n-1})$,

$$(9.8) \quad \hat{I}_{p^s}^{[lp^s-1]}(u) := \tilde{I}_{p^s}^{[lp^s-1]}(z(u)).$$

Each $\hat{I}_{p^s}^{[lp^s-1]}(u)$ is an n -vector of polynomials in u with coefficients in $\mathbb{Z}^n$.

Each $\hat{I}_{p^{[lp_s s-1]}}(u)$ is a solution of system (2.1) modulo p^s , in which the change of variables $z = z(u)$ is performed.

9.3. Change of variables in the KZ equations. It is known that system (2.1) of the differential KZ equations has suitable asymptotic zones with appropriate local coordinates, in which the differential KZ equations have singularities only at the coordinate hyperplanes. See a definition of the asymptotic zones, for example, in [V2]. The coordinates u defined in (9.7) are local coordinates in one of the asymptotic zones. In these coordinates, system (2.1) takes the form,

$$(9.9) \quad \begin{aligned} \frac{\partial I}{\partial u_n} &= 0, & \frac{\partial I}{\partial u_i} &= \frac{1}{2} \left(\frac{\Omega_i}{u_i} + \text{Reg}_i(u) \right) I, & i &= 1, \dots, n-1 \\ I_1 + \dots + I_n &= 0, \end{aligned}$$

where $\Omega^i = \sum_{i \leq k < l \leq n} \Omega_{k,l}$ and $\text{Reg}_i(u)$ is an $n \times n$ -matrix depending on u and regular at the origin $u = 0$. The origin is a regular singular point of system (9.9) and one may expand solutions at the origin in suitable series in the variables u .

Any polynomial $\hat{I}_{p^s}^{[lp^s-1]}(u)$ is a solution of system (9.9) modulo p^s . We will expand the polynomial $\hat{I}_{p^{[lp_s-1]}}(u)$ at $u = 0$ and show that this expansion has a p -adic limit as $s \rightarrow \infty$. In that way we will construct a g -dimensional space of p -adic solutions of system (9.9), which is the same as the original system (2.1) of the differential KZ equations up to the change of variables, $z = z(u)$.

9.4. Taylor expansion of $\hat{I}_{p^s}^{[lp^s-1]}(u)$. Denote

$$(9.10) \quad \hat{\Phi}_{p^s}(v, u) := \tilde{\Phi}_{p^s}(v, z(u)) = \left(\prod_{i=1}^{n-1} \left(v - \prod_{j=1}^i u_j \right) \right)^{(p^s-1)/2} v^{(p^s-1)/2},$$

$$(9.11) \quad \hat{P}_{p^s}(v, u) := \left(\frac{\hat{\Phi}_{p^s}(v, u)}{v - u_1}, \dots, \frac{\hat{\Phi}_{p^s}(v, u)}{v - u_1 \cdots u_{n-1}}, \frac{\hat{\Phi}_{p^s}(v, u)}{v} \right) = \sum_i \hat{P}_{p^s}^i(u) v^i,$$

where $\hat{P}_{p^s}^i(u)$ are n -vectors of polynomials in u with coefficients in $\mathbb{Z}$. For a positive integer l , we have

$$(9.12) \quad \hat{P}_{p^s}^{lp^s-1}(u) = \hat{I}_{p^s}^{[lp^s-1]}(u),$$

where $\hat{I}_{p^{[lp_s s-1]}}(u)$ is defined in (9.8). For

$l = 1, \dots, g$, denote

$$(9.13) \quad u^{l,s} = (-1)^{\delta_l} (u_1 \cdots u_{n-2l})^{-l} \prod_{i=1}^{n-2l} (u_1 \cdots u_i)^{\frac{p^s-1}{2}},$$

or

$$\begin{aligned} u^{g,s} &= (-1)^{\delta_g} u_1^{\frac{p^s-1}{2}-g}, & u^{g-1,s} &= (-1)^{\delta_{g-1}} u_1^{3\frac{p^s-1}{2}-g+1} u_2^{2\frac{p^s-1}{2}-g+1} u_3^{\frac{p^s-1}{2}-g+1}, \dots \\ u^{1,s} &= (-1)^{\delta_1} u_1^{(n-2)\frac{p^s-1}{2}-1} u_2^{(n-3)\frac{p^s-1}{2}-1} \cdots u_{n-2}^{\frac{p^s-1}{2}-1}, \end{aligned}$$

see $(-1)^{\delta_i}$ in (7.1). Denote

$$(9.14) \quad C^{l,s} = \left(\frac{p^s-1}{l} \right) \left(0, \dots, 0, \frac{2l}{p^s-1}, 1, \dots, 1 \right),$$

where 0 is repeated $2g - 2l$ times and 1 is repeated $2l$ times, cf. formula (7.3).

Theorem 9.3. For $l = 1, \dots, g$, the polynomial $\hat{l}^{[lp_s-1]}(u)$ has the following form,

$$(9.15) \quad \begin{aligned} \hat{l}^{[lp_s-1]}(u) &= u_{l,s} T_{l,s}(u), \\ T^{l,s}(u) &= (T_1^{l,s}(u), \dots, T_n^{l,s}(u)), \end{aligned}$$

with coordinates $T_j^{l,s}$ defined as follows. If $j = 1, \dots, n-1$, then

$$(9.16) \quad T_j^{l,s} = u_{j+1} \cdots u_{n-2l} \sum^{l,s,j} \left(\frac{p^s-1}{2} \right) \prod_{i=1, i \neq j}^{n-1} \left(\frac{p^s-1}{2} \right) \times \prod_{i=1}^{n-2l-1} (u_{i+1} \cdots u_{n-2l})^{a_i} \prod_{i=1}^{2l-1} (u_{n-2l+1} \cdots u_{n-2l+i})^{a_{n-2l+i}},$$

where the summation is over all $a_1, \dots, a_{n-1} \in \mathbb{Z} - \frac{s}{2}$, such

that $n-2l; a_1 + \dots + a_{n-2l} = a_{n-2l+1} + \dots + a_{n-1} + l$, if $n-2l < j \leq n-1$, (9.17)

$$T_n^{l,s} = \sum^{l,s,n} \prod_{i=1}^{n-1} \left(\frac{p^s-1}{2} \right) \prod_{i=1}^{n-2l-1} (u_{i+1} \cdots u_{n-2l})^{a_i} \prod_{i=1}^{2l-1} (u_{n-2l+1} \cdots u_{n-1})^{a_{n-2l+i}},$$

where the summation is over all $a_1, \dots, a_{n-1} \in \mathbb{Z}$, $0 \leq a_i \leq \frac{p^s-1}{2}$, such that $a_1 + \dots + a_{n-2l} = l$. The constant term of $T^{l,s}(u)$ equals $C^{l,s}$.

Notice that the factor $u_{j+1} \cdots u_{n-2l}$ in (9.16) equals 1 if $j \geq n-2l$. Proof. Make the change of variables $v = wu_1 \cdots u_{n-2l}$ in (9.11),

$$P_{p^s}^\circ(w, u) := \hat{P}_{p^s}(wu_1 \cdots u_{n-2l}, u) = \sum_i \hat{P}_{p^s}^i(u) (wu_1 \cdots u_{n-2l})^i =: \sum_i P_{p^s}^{\circ,i}(u) w^i.$$

Hence

$$\hat{I}_{p^s}^{[lp^s-1]}(u) = (u_1 \dots u_{n-2l})^{-(lp^s-1)} P_{p^s}^{\circ, lp^s-1}(u) \dots$$

We transform the factors in the polynomial $P_{p^s}^{\circ, l}(w, u)$ as follows. For any positive integers k and $i \leq n - 2l$ we write

$$(9.18) \quad (wu_1 \dots u_{n-2l} - u_1 \dots u_i)^k = (u_1 \dots u_i)^k (wu_{i+1} \dots u_{n-2l} - 1)^k \\ = (u_1 \dots u_i)^k \sum_{a=0}^k (-1)^{k-a} \binom{k}{a} (wu_{i+1} \dots u_{n-2l})^a,$$

and if $i > n - 2l$, we write

$$(9.19) \quad (wu_1 \dots u_{n-2l} - u_1 \dots u_i)^k = (wu_1 \dots u_{n-2l})^k (1 - u_{n-2l+1} \dots u_i/w)^k \\ = (wu_1 \dots u_{n-2l})^k \sum_{a=0}^k (-1)^a \binom{k}{a} (u_{n-2l+1} \dots u_i/w)^a.$$

Notice that for factors in (9.11), we have $k = \frac{p^s-1}{2}$ or $k = \frac{p^s-1}{2} - 1$. This explains the binomial coefficients in (9.16) and (9.17).

We prove formula (9.16) for $j = 1$, the proof for other values of j is similar. Our goal is to calculate the first coordinate of the vector

$$(u_1 \dots u_{n-2l})^{-(lp^s-1)} P_{p^s}^{\circ, lp^s-1}(u).$$

That is we need to calculate the coefficient of w^{lp^s-1} in

$$(u_1 \dots u_{n-2l})^{-(lp^s-1)} u_1^{\frac{p^s-1}{2}-1} (wu_2 \dots u_{n-2l} - 1)^{\frac{p^s-1}{2}-1} (u_1 u_2)^{\frac{p^s-1}{2}} \dots \\ (wu_3 \dots u_{n-2l} - 1)^{\frac{p^s-1}{2}} \dots \\ \dots (u_1 \dots u_{n-2l})^{\frac{p^s-1}{2}} (w-1)^{\frac{p^s-1}{2}} (wu_1 \dots u_{n-2l})^{\frac{p^s-1}{2}} (1 - u_{n-2l+1}/w)^{\frac{p^s-1}{2}} \dots \\ \dots (wu_1 \dots u_{n-2l})^{\frac{p^s-1}{2}} (1 - u_{n-2l+1} \dots u_{n-1}/w)^{\frac{p^s-1}{2}} (wu_1 \dots u_{n-2l})^{\frac{p^s-1}{2}},$$

which is the same as the coefficient of w^{l-1} in

$$(9.20) \quad u_2 \dots u_{n-2l} u^{l,s} (wu_2 \dots u_{n-2l} - 1)^{\frac{p^s-1}{2}-1} (wu_3 \dots u_{n-2l} - 1)^{\frac{p^s-1}{2}} \dots \\ \dots (w-1)^{\frac{p^s-1}{2}} (1 - u_{n-2l+1}/w)^{\frac{p^s-1}{2}} \dots (1 - u_{n-2l+1} \dots u_{n-1}/w)^{\frac{p^s-1}{2}}.$$

Expanding the binomials we obtain formula (9.16) for $j = 1$.

The constant term of $T^s(u)$ is given by the summands in (9.16) and (9.17),

corresponding to $a_1 = \dots = a_{n-2l-1} = a_{n-2l+1} = \dots = a_{n-1} = 0$ and $j = n - 2l, \dots, n$. Theorem 9.3

is proved.

9.5. Taylor expansion of holomorphic solutions. Recall the multivalued holomorphic solutions of system (2.1) described in Section 3,

$$I^{(\gamma)}(z) = \int_{\gamma} \left(\frac{\Phi(x, z)}{x - z_1}, \dots, \frac{\Phi(x, z)}{x - z_n} \right) dx.$$

We make the same changes of variables in the integrals $I^{(\gamma)}(z)$ as we did in the previous sections. Namely, first we change the integration variable x and set $x = v + z_n$, then we make the change of variables z and set $z = z(u)$. The resulting integral is

$$\hat{I}^{(\gamma)}(u) = \int_{\gamma} \left(\frac{\hat{\Phi}(v, u)}{v - u_1}, \dots, \frac{\hat{\Phi}(v, u)}{v} \right) dv,$$

where $\hat{\Phi}(v, u) = \left(\prod_{i=1}^{n-1} \left(v - \prod_{j=1}^i u_j \right) \right)^{-1/2} v^{-1/2}$.

For $l = 1, \dots, g$, we change the integration variable v and set $v = wu_1 \dots u_{n-2l}$. Then

$$\begin{aligned} \hat{\Phi}(wu_1 \dots u_{n-2l}, u) &= e^{(l-n/2)\pi i} (u_1 \dots u_{n-2l})^{-l} \prod_{i=1}^{n-2l} (u_1 \dots u_i)^{-1/2} \\ &\times \left((1 - wu_2 \dots u_{n-2l})(1 - wu_3 \dots u_{n-2l}) \dots (1 - w) \right. \\ &\times \left. (1 - u_{n-2l+1}/w) \dots (1 - u_{n-2l+1} \dots u_{n-1}/w) \right)^{-1/2} w^{-l}. \end{aligned}$$

Choose the integration cycle $\gamma = \gamma_l$ to be the circle $|w| = 1/2$ oriented counterclockwise. We assume that all the variables $u_2, \dots, u_{n-1}$ lie inside the circle. We fix the branch of the function

$$\begin{aligned} &\left((1 - wu_2 \dots u_{n-2l})(1 - wu_3 \dots u_{n-2l}) \dots (1 - w) \right. \\ &\times \left. (1 - u_{n-2l+1}/w) \dots (1 - u_{n-2l+1} \dots u_{n-1}/w) \right)^{-1/2} \end{aligned} \quad (9.21)$$

over the circle by choosing the argument of the function in (9.21) at $w = 1/2$, $u_2 = \dots = u_{n-1} = 0$ to be 0. We multiply the circle with the chosen branch of the integrand by $\frac{e^{n\pi i/2}}{2\pi i}$. This finishes the description of γ_l . The resulting integral is

$$\begin{aligned} &\hat{I}^{(\gamma_l)}(u) \\ &= \frac{e^{n\pi i/2}}{2\pi i} \int_{|w|=1/2} \left(\frac{\hat{\Phi}(wu_1 \dots u_{n-2l}, u)}{wu_1 \dots u_{n-2l} - u_1}, \dots, \frac{\hat{\Phi}(wu_1 \dots u_{n-2l}, u)}{wu_1 \dots u_{n-2l}} \right) u_1 \dots u_{n-2l} dw. \end{aligned}$$

Denote

$$u^l := (u_1 \dots u_{n-2l})^{-l} \prod_{i=1}^{n-2l} (u_1 \dots u_i)^{-1/2}. \quad (9.22)$$

Theorem 9.4. *For $l = 1, \dots, g$, the function $\hat{I}^{(\gamma_l)}(u)$ has the following form,*

$$\hat{I}^{(\gamma_l)}(u) = u^l T^l(u), \quad T^l(u) = (T_1^l(u), \dots, T_n^l(u)), \quad (9.23)$$

with coordinates T_j^l defined as follows. If $j = 1, \dots, n-1$, then

$$\begin{aligned} (9.24) \quad T_j^l &= u_{j+1} \dots u_{n-2l} \sum^{l,j} \left(\frac{-3}{2} \right)_{a_j} \prod_{i=1, i \neq j}^{n-1} \left(\frac{-1}{2} \right)_{a_i} \\ &\times \prod_{i=1}^{n-2l-1} (u_{i+1} \dots u_{n-2l})^{a_i} \prod_{i=1}^{2l-1} (u_{n-2l+1} \dots u_{n-2l+i})^{a_{n-2l+i}}, \end{aligned}$$

where the summation $\sum^{l,j}$ is over all $a_1, \dots, a_{n-1} \in \mathbb{Z}_{\geq 0}$ such that $a_1 + \dots + a_{n-2l} =$

; and such that $a_1 + \dots + a_{n-2l} =$

$a_{n-2l+1} + \dots + a_{n-1} + l$, if $n - 2l < j \leq n - 1$

(9.25)

$$T_n^l = \sum_{i=1}^{l,n} \prod_{i=1}^{n-1} \left(\frac{-1}{a_i} \right)^{n-2l-1} \prod_{i=1}^{n-2l-1} (u_{i+1} \dots u_{n-2l})^{a_i} \prod_{i=1}^{2l-1} (u_{n-2l+1} \dots u_{n-1})^{a_{n-2l+i}},$$

where the summation $\sum_{i=1}^{l,n} + a + l$ is over all $a_1, \dots, a_{n-1} \in \mathbb{Z}_{\geq 0}$ such that $a_1 + \dots + a_{n-2l} =$

a_{n-2} The power series $l+1 \dots n-1 T_l(u)$ converges in the polydisc $\{(u_2, \dots, u_{n-1}) \in \mathbb{C}^{n-1} \mid |u_i| < 1,$

$i = 2, \dots, n-1\}$.

Proof. We prove formula (9.16) for $j = 1$, the proof for other values of j is similar.

The function $T_1^l(u)$ equals

$$\begin{aligned} & \frac{e^{n\pi i/2}}{2\pi i} \int_{|w|=1/2} u_1^{-\frac{3}{2}} (wu_2 \dots u_{n-2l} - 1)^{-\frac{3}{2}} (u_1 u_2)^{-\frac{1}{2}} (wu_3 \dots u_{n-2l} - 1)^{-\frac{1}{2}} \dots \\ & \dots (u_1 \dots u_{n-2l})^{-\frac{1}{2}} (w-1)^{-\frac{1}{2}} (wu_1 \dots u_{n-2l})^{-\frac{1}{2}} (1 - u_{n-2l+1}/w)^{-\frac{1}{2}} \dots \\ & \dots (wu_1 \dots u_{n-2l})^{-\frac{1}{2}} (1 - u_{n-2l+1} \dots u_{n-1}/w)^{-\frac{1}{2}} \\ & \times (wu_1 \dots u_{n-2l})^{-\frac{1}{2}} u_1 \dots u_{n-2l} dw \\ & = \frac{(-1)^{l-1}}{2\pi i} u_1^{-1} (u_1 \dots u_{n-2l})^{-l+1} \prod_{i=1}^{n-2l} (u_1 \dots u_i)^{-\frac{1}{2}} \\ & \times \int_{|w|=1/2} (1 - wu_2 \dots u_{n-2l})^{-\frac{3}{2}} \left((1 - wu_3 \dots u_{n-2l}) \dots (1 - w) \right. \\ & \times \left. (1 - u_{n-2l+1}/w) \dots (1 - u_{n-2l+1} \dots u_{n-1}/w) \right)^{-\frac{1}{2}} \frac{dw}{w^l}. \end{aligned}$$

Expanding the binomials we obtain formula (9.24) for $j = 1$.

The convergence property is clear.

9.6. Formal solutions over $\mathbb{Q}_p$ and truncation. For $l = 1, \dots, g$, the formal series $\hat{I}^{(\gamma)}(u)$ is a formal solution of system (2.1), in which the change of variables $z = z(u)$ is performed and which is considered over the field $\mathbb{Q}_p$ of p -adic numbers.

Lemma 9.5. *The formal series $\hat{I}^{(\gamma)}(u)$, $l = 1, \dots, g$, are linear independent over the field $\mathbb{Q}_p$.*

Proof. The proof follows from the fact that the monomials u^l , $l = 1, \dots, n$, are linear independent over $\mathbb{Q}_p$.

9.7. Example $n = 3$. In this case we have $g = 1$, $u_1 = z_1 - z_3$, $u_2 = \frac{z_2 - z_3}{z_1 - z_3}$,

(9.26)

$$\begin{aligned} \hat{I}^{(\gamma_1)}(u_1, u_2) &= u_1^{-3/2} \sum_{a=0}^{\infty} \left(\left(\frac{-\frac{3}{2}}{a} \right) \left(\frac{-\frac{1}{2}}{a} \right), \left(\frac{-\frac{1}{2}}{a+1} \right) \left(\frac{-\frac{3}{2}}{a} \right), \left(\frac{-\frac{1}{2}}{a+1} \right) \left(\frac{-\frac{1}{2}}{a} \right) \right) u_2^a \\ &= u_1^{-3/2} \sum_{a=0}^{\infty} \left(\frac{-\frac{1}{2}}{a+1} \right) \left(\frac{-\frac{3}{2}}{a} \right) \left(\frac{a+1}{-1/2-a}, 1, \frac{-1/2}{-1/2-a} \right) u_2^a \end{aligned}$$

$$\begin{aligned}
(9.27) \quad \hat{I}^{[p^s-1]}(u_1, u_2) &= \\
&= (-1)^{\frac{p^s-3}{2}} u_1^{\frac{p^s-3}{2}} \sum_{a=0}^{\frac{p^s-1}{2}} \left(\binom{\frac{p^s-3}{2}}{a} \binom{\frac{p^s-1}{2}}{a}, \binom{\frac{p^s-1}{2}}{a+1} \binom{\frac{p^s-3}{2}}{a}, \binom{\frac{p^s-1}{2}}{a+1} \binom{\frac{p^s-1}{2}}{a} \right) u_2^a \\
&= (-1)^{\frac{p^s-3}{2}} u_1^{\frac{p^s-3}{2}} \sum_{a=0}^{\frac{p^s-1}{2}} \binom{\frac{p^s-1}{2}}{a+1} \binom{\frac{p^s-3}{2}}{a} \left(\frac{a+1}{(p^s-1)/2-a}, 1, \frac{(p^s-1)/2}{(p^s-1)/2-a} \right) u_2^a
\end{aligned}$$

10. p -Adic convergence

Consider the field $\mathbb{Q}_p$ with the standard p -adic norm $|t|_p$, $t \in \mathbb{Q}_p$. In this section we consider the polynomial solutions $((-1)^{\delta_l} T^{l,s}(u))_{s=1}^{\infty}$ and the formal solutions

$\hat{I}^{(\gamma)}(u)$ as functions on $\mathbb{Q}_p^{n-1}$.

Recall that $\mathbb{Z}_p \subset \mathbb{Q}_p$ denotes the ring of p -adic integers.

10.1. Teichmüller representatives. For $t \in \mathbb{Z}_p$ there exists the unique solution $\omega(t) \in \mathbb{Z}_p$ of the equation $\omega(t)^p = \omega(t)$ that is congruent to t modulo p . The element $\omega(t)$ is called the *Teichmüller representative*. It also can be defined by $\omega(t) = \lim_{s \rightarrow \infty} t^{p^s}$. The *Teichmüller character* is the homomorphism

$$\iota: \mathbb{F}_p^\times \rightarrow \mathbb{Z}_p^\times, \quad \alpha \mapsto \omega(\alpha), \quad \alpha \in \mathbb{F}_p^\times.$$

$\alpha \in \mathbb{F}_p$, $r > 0$, define the disc

For

$$D_{\alpha,r} = \{t \in \mathbb{Z}_p \mid |t - \omega(\alpha)|_p < r\}.$$

The space $\mathbb{Z}_p$ is the disjoint union of the discs $D_{\alpha,1}$, $\alpha \in \mathbb{F}_p$. The function $\omega: \mathbb{Z}_p \rightarrow \mathbb{Z}_p$, $t \mapsto \omega(t)$, is a locally constant function equal to $\omega(\alpha)$ on the disc $D_{\alpha,1}$. For a subset $S \subset \mathbb{Z}_p$ and a function $f: S \rightarrow \mathbb{Z}_p$ define the norm

$$\|f\| = \sup_{t \in S} |f(t)|_p.$$

Lemma 10.1. *For any $\alpha \in \mathbb{F}_p$ the sequence of polynomial functions $(x^{p^s})_{s=1}^{\infty}$ uniformly converges on $D_{\alpha,1}$ to the constant function $\omega(\alpha)$.*

Proof. We use the “fundamental inequality” from [Ro, II.4.3]: if $|t|_p \leq 1$, then $|(1+t)^{p^s} - 1|_p \leq |t|_p \cdot \max(|t|_p, 1/p)^s$. Now let $t \in D_{\alpha,1}$. Then $t^{p^{-1}} = 1 + t_1$,

$$|t_1|_p \leq 1/p. \text{ We have } |t^{p^{s+1}} - t^{p^s}|_p = |t^{p^s}|_p |t^{(p-1)p^s} - 1|_p \leq |(1+t_1)^{p^s} - 1|_p \leq 1/p^{s+1}.$$

For positive integers

$$|t^{p^{s_1+s_2}} - t^{p^{s_1}}|_p = |t^{p^{s_1+s_2}} - \dots + t^{p^{s_1+1}} - t^{p^{s_1}}|_p \leq 1/p^{s_1+1} \quad s_1, s_2 \text{ and } t \in D_{\alpha,1} \text{ we have}$$

$t^{p^{s_1+s_2-1}} + t^{p^{s_1+s_2-1}} + \dots$. Hence the sequence $(x^{p^s})_{s=1}^{\infty}$ is a Cauchy sequence.

For $t \in D_{0,1}$, $|t|_p \leq 1/p$ $|t_1^p - t_2^p|_p = |(1+t)^{p^s} - 1|_p \leq 1/p^{s+1}$ we have $|t^{p^s}|_p \leq 1/p^s$. For $t_1, t_2 \in D_{\alpha,1}$, $\alpha \neq 0$, we have $t_1/t_2 = 1+t$ with $|t|_p \leq 1/p$. The lemma is proved.

For $\alpha \in \mathbb{F}_p$ consider the sequence of polynomial functions $(x^{(p^s-1)/2})_{s=1}^\infty$ on $D_{\alpha,1}$. This sequence uniformly converges to 0 on the disc $D_{0,1}$.

Let $\alpha^{(p-1)/2} = 1$. Let $\beta \in \mathbb{F}_p$ be such that $\beta^2 = \alpha$. The function $D_{\beta,1} \rightarrow D_{\alpha,1}$, $t \mapsto t^2$, is an analytic diffeomorphism. The inverse function $D_{\alpha,1} \rightarrow D_{\beta,1}$ will be denoted by $x^{1/2}$. There are two square roots $\pm x^{1/2}$. The root $x^{1/2}$ corresponds to the chosen $\beta \in \mathbb{F}_p$ and the root $-x^{1/2}$ corresponds to $-\beta \in \mathbb{F}_p$.

We change the variable x , set $x = y^2$, and lift the sequence $(x^{(p^s-1)/2})_{s=1}^\infty$ to the sequence $(y^{p^s-1})_{s=1}^\infty$ of polynomial functions on $D_{\beta,1}$.

Lemma 10.2. *The sequence of polynomial functions $(y^{p^s-1})_{s=1}^\infty$ uniformly converges on $D_{\beta,1}$ to the function $\omega(\beta)/y$. In other words, the sequence of polynomial functions $(x^{(p^s-1)/2})_{s=1}^\infty$ uniformly converges on $D_{\alpha,1}$ to the function $\omega(\beta)x^{-1/2}$.*

Proof. The lemma follows from Lemma 10.1.

Let $\alpha^{(p-1)/2} = -1$. Then $\omega(\alpha)_{(p^s-1)/2} = (-1)^{1+p+\dots+p^{s-1}} = (-1)^s$ and the sequence $(x^{(p^s-1)/2})_{s=1}^\infty$ has no limit on $D_{\alpha,1}$.

10.2. Approximation of binomial coefficients. It is known that $\binom{-\frac{1}{2}}{a}, \binom{-\frac{3}{2}}{a} \in \mathbb{Z}_p$ for $a \in \mathbb{Z}_{\geq 0}$.

Lemma 10.3. *Let $l_1, l_2 \geq 0$ be integers. Then there exists an integer $s_0 \geq 0$, such that for any integers $s \geq s_0$ and any integer a with we have*

$$(10.1) \quad \left| \binom{-\frac{1}{2} - l_1}{l_2 + a} - \binom{\frac{p^s-1}{2} - l_1}{l_2 + a} \right|_p \leq 1/p^{s-d-a}, \quad \text{where } d = l_1 + l_2 - 1/2.$$

Proof. We have

$$\begin{aligned} \binom{-\frac{1}{2} - l_1}{l_2 + a} &= \frac{1}{(-2)^{l_2+a}(l_2+a)!} \prod_{k=1}^{l_2+a} (2(l_1+k) - 1), \\ \binom{\frac{p^s-1}{2} - l_1}{l_2 + a} &= \frac{1}{(-2)^{l_2+a}(l_2+a)!} \prod_{k=1}^{l_2+a} (2(l_1+k) - 1 - p^s). \end{aligned}$$

The p -adic norm of $\binom{-\frac{1}{2}-l_1}{l_2+a}$ is ≤ 1 . The difference $\binom{-\frac{1}{2}-l_1}{l_2+a} - \binom{\frac{p^s-1}{2}-l_1}{l_2+a}$ is the sum of products $\frac{1}{(-2)^{l_2+a}(l_2+a)!} \prod_{k=1}^{l_2+a} (2(l_1+k) - 1)$ in each of which at least one of the factors $2(l_1+k)-1$ is replaced with $-p^s$. We prove that even one such replacement implies that this summand of the difference has p -

adic norm $\leq 1/p^{s+1/2-l_1-l_2-a}$. $l_1+k-1 \leq 2(l_1+l_2+a)-1$
 $-1 = bp^c, p \nmid b$

Indeed, $\frac{p-1}{2} - s$ $bp^c \leq 2l_1 - 2 + p^s$ let $2(l_1+k)$. We have $2(l_1+k) - 1$. Hence. Hence for any s , large enough, we have $c \leq s$, and the replacement of $2(l_1+k) - 1$ with $-p^s$ makes the norm of that summand $1/p^{s-c}$.

We also have $bp^c \leq 2(l_1 + l_2 + a) - 1$. Hence $c \leq p^{c-1} < \frac{p}{2} \leq \frac{bp}{2} \leq a + l_1 + l_2 - \frac{1}{2}$. This shows that each summand has the p -adic norm $1/p^{s-c} \leq 1/p^{s+1/2-l_1-l_2-a}$. The lemma is proved.

10.3. Example $n = 3$, continuation. Consider the formal power series

$$(10.2) \quad T^1(x) = \sum_{a=0}^{\infty} \left(\binom{-\frac{3}{2}}{a} \binom{-\frac{1}{2}}{a}, \binom{-\frac{1}{2}}{a+1} \binom{-\frac{3}{2}}{a}, \binom{-\frac{1}{2}}{a+1} \binom{-\frac{1}{2}}{a} \right) x^a \quad \text{in}$$

(9.26) and the sequence of polynomials

$$(10.3) \quad T^{1,s}(x) = \sum_{a=0}^{\frac{p^s-1}{2}} \left(\binom{\frac{p^s-3}{2}}{a} \binom{\frac{p^s-1}{2}}{a}, \binom{\frac{p^s-1}{2}}{a+1} \binom{\frac{p^s-3}{2}}{a}, \binom{\frac{p^s-1}{2}}{a+1} \binom{\frac{p^s-1}{2}}{a} \right) x^a$$

in (9.27) as functions on $\mathbb{Z}_p$.

Proposition 10.4. *The power series $T^1(x)$ uniformly convergence on $D_{0,1}$. The sequence of polynomial functions $(T^{1,s}(x))_{s=1}^{\infty}$ uniformly converges on $D_{0,1}$ to the function $T^1(x)$.*

Proof. The fact that the binomials $\binom{-\frac{1}{2}}{a}, \binom{-\frac{3}{2}}{a}$ are p -adic integers implies the uniform convergence of the power series $T^1(x)$ on $D_{0,1}$.

Let us write $T^1(x) = \sum_{a=0}^{\infty} T_a^1 x^a$ and $T^{1,s}(x) = \sum_{a=0}^{\frac{p^s-1}{2}} T_a^{1,s} x^a$, where $T_a^1, T_a^{1,s} \in \mathbb{Z}_p$. Then $T^1(x) - T^{1,s}(x) = \sum_{a=\frac{p^s+1}{2}}^{\infty} T_a^1 x^a + \sum_{a=0}^{\frac{p^s-1}{2}} (T_a^1 - T_a^{1,s}) x^a$. Clearly the p -adic norm of the first sum is $1/p^{\frac{p^s-1}{2}}$. By Lemma 10.3 if s is big enough, then for each summand of the second sum and $t \in$

$D_{0,1}$ we have $|T_a^{1,s} t^a|_p \leq 1/p^{s-1}$ for some d independent of s and of the summand. This proves the proposition.

Consider the formal series

$$(10.4) \quad \hat{I}^{(\gamma_1)}(u_1, u_2) = u_1^{-3/2} \sum_{a=0}^{\infty} \left(\binom{-\frac{3}{2}}{a} \binom{-\frac{1}{2}}{a}, \binom{-\frac{1}{2}}{a+1} \binom{-\frac{3}{2}}{a}, \binom{-\frac{1}{2}}{a+1} \binom{-\frac{1}{2}}{a} \right) u_2^a$$

and the sequence of polynomials

$$(10.5) \quad (-1)^{\frac{p^s-3}{2}} \hat{I}^{[p^s-1]}(u_1, u_2) = \\ = u_1^{\frac{p^s-3}{2}} \sum_{a=0}^{\frac{p^s-1}{2}} \left(\binom{\frac{p^s-3}{2}}{a} \binom{\frac{p^s-1}{2}}{a}, \binom{\frac{p^s-1}{2}}{a+1} \binom{\frac{p^s-3}{2}}{a}, \binom{\frac{p^s-1}{2}}{a+1} \binom{\frac{p^s-1}{2}}{a} \right) u_2^a$$

as functions on $D_{\alpha,1} \times D_{0,1}$, where $\alpha = \beta^2$ for some $\beta \in \mathbb{F}_p$. Then the function $u_1^{1/2} : D_{\alpha,1} \rightarrow D_{\beta,1}$ is well-defined and the series $\hat{I}^{(r_1)}(u_1, u_2)$ is a well-defined function on $D_{\alpha,1} \times D_{0,1}$.

THEOREM 10.5. *The sequence of polynomial functions $\omega(\beta) \hat{I}^{(\gamma_1)}(u_1, u_2)$ uniformly converges on $D_{\alpha,1} \times D_{0,1}$ to the function*

$$\left((-1)^{\frac{p-3}{2}} \hat{I}^{[p^s-1]}(u_1, u_2)\right)_{s=1}^{\infty}$$

Proof. The theorem follows from Lemma 10.2 and Proposition 10.4.

10.4. p -Adic convergence for arbitrary n . Given l , $1 \leq l \leq g$, consider the sequence of polynomials $(-1)^{\delta_l} \hat{I}^{[p^s-1]}(u)$ and the series $\hat{I}^{(\gamma)}(u)$. Here $u =$

2l

$(u_1 \cdots u_{n-2l})^l (-1)^{\delta_l} \hat{I}^{[p^s-1]}(u)$ We multiply the polynomials and the series by the same factor $(z_l - z_{n-1})^l$ and study the convergence of the sequence of polynomials $u_1 \cdots u_{n-2l} (z_l - z_{n-1})^l$ to the series $\hat{I}^l := (u_1 \cdots u_{n-2l})^l \hat{I}^{(\gamma)}(u)$. Introduce

new variables:

$$\begin{aligned} x_1 &= \prod_{i=1}^{n-2l} \prod_{j=1}^i u_j = \prod_{i=1}^{n-2l} (z_i - z_n), & x_2 &= u_2 \cdots u_{n-2l} = \frac{z_{n-2l} - z_n}{z_1 - z_n}, \\ x_3 &= u_3 \cdots u_{n-2l} = \frac{z_{n-2l} - z_n}{z_2 - z_n}, & \dots & \quad x_{n-2l} = u_{n-2l} = \frac{z_{n-2l} - z_n}{z_{n-2l-1} - z_n}, \\ x_{n-2l+1} &= u_{n-2l+1} = \frac{z_{n-2l+1} - z_n}{z_{n-2l} - z_n}, & x_{n-2l+2} &= u_{n-2l+1} u_{n-2l+2} = \frac{z_{n-2l+2} - z_n}{z_{n-2l} - z_n}, \\ \dots & \quad x_{n-1} = u_{n-2l+1} \cdots u_{n-1} = \frac{z_{n-1} - z_n}{z_{n-2l} - z_n}. \end{aligned}$$

—

Let $x = (x_1, \dots, x_{n-1})$. Then

$J^l(x) = x_1^{-\frac{1}{2}} Q^l(x_2, \dots, x_{n-1}) = x_1^{-\frac{1}{2}} (Q_1^l(x_2, \dots, x_{n-1}), \dots, Q_n^l(x_2, \dots, x_{n-1}))$,
with coordinates Q_j^l defined as follows. If $j = 1, \dots, n-1$, then

$$\begin{aligned} (10.6) \quad Q_j^l &= x_{j+1} \sum^{l,j} \left(\frac{-3}{2}\right)_{a_j} \prod_{i=1, i \neq j}^{n-1} \left(\frac{-1}{2}\right)_{a_i} \prod_{i=2}^{n-2l} x_i^{a_{i-1}} \prod_{i=n-2l+1}^{n-1} x_i^{a_i}, \quad j = 1, \dots, n-2l-1, \\ Q_j^l &= \sum^{l,j} \left(\frac{-3}{2}\right)_{a_j} \prod_{i=1, i \neq j}^{n-1} \left(\frac{-1}{2}\right)_{a_i} \prod_{i=2}^{n-2l} x_i^{a_{i-1}} \prod_{i=n-2l+1}^{n-1} x_i^{a_i}, \quad j = n-2l, \dots, n-1, \end{aligned}$$

where the summation $\sum^{l,j}$ is over all $a_1, \dots, a_{n-1} \in \mathbb{Z}_{\geq 0}$ such that $a_1 + \dots + a_{n-2l} =$

$a_{n-2l+1} + \dots + a_{n-1} + l - 1$, if $j \leq n-2l$; and such that $a_1 + \dots + a_{n-2l} =$
 $a_{n-2l+1} + \dots + a_{n-1} + l$, if $n-2l < j \leq n-1$;

$$(10.7) \quad Q_n^l = \sum^{l,n} \prod_{i=1}^{n-1} \binom{-\frac{1}{2}}{a_i} \prod_{i=2}^{n-2l} x_i^{a_{i-1}} \prod_{i=n-2l+1}^{n-1} x_i^{a_i},$$

where the summation is over all $a_1, \dots, a_{n-1} \in \mathbb{Z}_{\geq 0}$ such that $a_1 + \dots + a_{n-2l} = a_{n-2l+1} + \dots + a_{n-1} + l$.

We also have

$$J^{l,s}(x) = x_1^{\frac{p^s-1}{2}} Q^{l,s}(x_2, \dots, x_{n-2}) \\ = x_1^{\frac{p^s-1}{2}} (Q_1^{l,s}(x_2, \dots, x_{n-2}), \dots, Q_n^{l,s}(x_2, \dots, x_{n-2})),$$

with coordinates $Q^{ls_j}(x)$ defined as follows. If $j = 1, \dots, n-1$, then

$$(10.8) \quad Q_j^{l,s} = x_{j+1} \sum^{l,j} \binom{\frac{p^s-3}{2}}{a_j} \prod_{i=1, i \neq j}^{n-1} \binom{\frac{p^s-1}{2}}{a_i} \prod_{i=2}^{n-2l} x_i^{a_{i-1}} \prod_{i=n-2l+1}^{n-1} x_i^{a_i}, \quad j = 1, \dots, n-2l-1, \\ Q_j^{l,s} = \sum^{l,j} \binom{\frac{p^s-3}{2}}{a_j} \prod_{i=1, i \neq j}^{n-1} \binom{\frac{p^s-1}{2}}{a_i} \prod_{i=2}^{n-2l} x_i^{a_{i-1}} \prod_{i=n-2l+1}^{n-1} x_i^{a_i}, \quad j = n-2l, \dots, n-1,$$

where the summation is over all $a_1, \dots, a_{n-1} \in \mathbb{Z}$, $0 \leq a_i \leq \frac{p^s-1}{2}$, such that $a_1 + \dots + a_{n-2l} = a_{n-2l+1} + \dots + a_{n-1} + l$, if $j = n-2l$; and such that $a_1 + \dots + a_{n-2l} = a_{n-2l+1} + \dots + a_{n-1} + l$, if $n-2l < j \leq n-1$;

$$(10.9) \quad Q_n^{l,s} = \sum^{l,n} \prod_{i=1}^{n-1} \binom{\frac{p^s-1}{2}}{a_i} \prod_{i=2}^{n-2l} x_i^{a_{i-1}} \prod_{i=n-2l+1}^{n-1} x_i^{a_i},$$

where the summation is over all $a_1, \dots, a_{n-1} \in \mathbb{Z}$, $0 \leq a_i \leq \frac{p^s-1}{2}$, such that $a_1 + \dots + a_{n-2l} = a_{n-2l+1} + \dots + a_{n-1} + l$.

Proposition 10.6. *The power series $Q^l(x_2, \dots, x_{n-1})$ uniformly converges on $D_{0,1}^{n-2}$. The sequence of polynomial functions $(Q^{l,s}(x_2, \dots, x_{n-1}))_{s=1}^\infty$ uniformly converges on $D_{0,1}^{n-2}$ to the function $Q^l(x_2, \dots, x_{n-1})$.*

Proof. The fact that the binomials $\binom{-\frac{1}{2}}{a}, \binom{-\frac{3}{2}}{a}$ are p -adic integers implies the uniform convergence of the power series $Q^l(x_2, \dots, x_{n-1})$ on $D_{0,1}^{n-2}$. The proof of the uniform convergence of $(Q^{l,s}(x_2, \dots, x_{n-1}))_{s=1}^\infty$ to $Q^l(x_2, \dots, x_{n-1})$ follows from

Lemma 10.3 in the same way as the uniform convergence in the proof of Proposition 10.4.

Consider the formal series $J^l(x) = x_1^{-\frac{1}{2}} Q^l(x_2, \dots, x_{n-1})$ and the sequence of polynomials $J^{l,s}(x) = x_1^{\frac{p^s-1}{2}} Q^{l,s}(x_2, \dots, x_{n-2})$ as functions on $D_{\alpha,1} \times D_{0,1}^{n-2}$, where $\alpha = \beta^2$ for some $\beta \in \mathbb{F}_p$. Then the function $x_1^{1/2} : D_{\alpha,1} \rightarrow D_{\beta,1}$ is well-defined and the series $J^l(x)$ is a well-defined function on $D_{\alpha,1} \times D_{0,1}^{n-2}$.

Theorem 10.7. *The sequence of polynomial functions $(J^{l,s}(x))_{s=1}^{\infty}$ uniformly converges on $D_{\alpha,1} \times D_{0,1}^{n-2}$ to the function $\omega(\beta)J'(x)$.*

Proof. The theorem follows from Lemma 10.2 and Proposition 10.6.

Appendix A. The case $n = 3$ and Dwork's theory

by Steven Sperber and Alexander Varchenko

In this appendix we consider only the special case $n = 3$ of previous considerations and show how this special case is related to Dwork's theory in the classical paper [Dw].

A.1. Dwork on Legendre family.

A.1.1. Consider the Legendre family of elliptic curves $E(\lambda)$ defined by the affine equation

$$y^2 = x(x-1)(x-\lambda).$$

Let $\gamma = \gamma(\lambda)$ be a family of 1-cycles on the curves $E(\lambda)$ flat under the Gauss-Manin connection. Then the function

$$(A.1) \quad h^{(\gamma)}(\lambda) = \int_{\gamma} \frac{dx}{y} \text{ satisfies the hypergeometric}$$

differential equation

$$(A.2) \quad \lambda(1-\lambda)h'' + (1-2\lambda)h' - (1/4)h = 0.$$

All solutions of this equation are obtained in this way. One of the solutions $h^{(\delta_1)}(\lambda)$, for a suitable δ_1 , equals

$$(A.3) \quad F(\lambda) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda\right) = \sum_{k=0}^{\infty} \binom{-1/2}{k} \lambda^k,$$

where ${}_2F_1$ is the classical hypergeometric function.

A.1.2. If $h(\lambda)$ is the elliptic integral in (A.1), then its derivative is

$$(A.4) \quad h' = \frac{1}{2} \int_{\gamma} \frac{dx}{(x-\lambda)y}.$$

A.1.3. Equation (A.2) can be written as a system of first order linear differential equations for column 2-vectors $I = (h, h')$,

$$(A.5) \quad \frac{dI}{d\lambda} = B(\lambda)I, \quad B(\lambda) = \begin{pmatrix} 0 & 1 \\ \frac{1}{4\lambda(1-\lambda)} & \frac{2\lambda-1}{\lambda(1-\lambda)} \end{pmatrix}.$$

A.1.4. In [Dw] Dwork considers equation (A.2) over the field $\mathbb{Q}_p$ of p -adic numbers and studies analytic properties of the solution $F(\lambda)$. We remind these properties below.

A.1.5. Let $D \subset \mathbb{Q}_p$ be an open subset. A function $f: D \rightarrow \mathbb{Q}_p$ is called analytic at $\alpha \in D$ if f can be presented by a power series $\sum_{k=0}^{\infty} c_k(\lambda - \alpha)^k$ with nonzero radius of convergence.

A function $f: D \rightarrow \mathbb{Q}_p$ is called analytic on D if f is the uniform limit on D of a sequence of rational function regular on D . In that case f is analytic at every point of D .

For $i = 1, \dots, N$ let $f_i: D_i \rightarrow \mathbb{Q}_p$ be an analytic function on some domain D_i .

Assume that $D_i \cap D_{i+1} \neq \emptyset$ for $i = 1, \dots, N-1$, and $f_i = f_{i+1}$ on $D_i \cap D_{i+1}$, then we say that the collection of functions f_i defines an analytic element on $\bigcup_{i=1}^N D_i$. Cf. [Dw, Section 0].

A.1.6. Igusa noted in [Ig] that modulo p the polynomial (A.6)

$$g(\lambda) = \sum_{j=0}^{(p-1)/2} \binom{-1/2}{j}^2 \lambda^j$$

is the unique polynomial solution of equation (A.2) of degree less than p up to multiplication by a constant. Define

$$D_1 = \{\lambda \in \mathbb{Z}_p \mid |g(\lambda)|_p = 1\}, \quad D_2 = \{\lambda \mid \lambda^{-1} \in D_1\}, \quad D = D_1 \cup D_2.$$

Notice that D_1, D_2 are open and $D_1 \cap D_2 \neq \emptyset$. More precisely, $D_1 \cap D_2$

$$= \{\lambda \in \mathbb{Z}_p \mid |g(\lambda)|_p = 1, |\lambda|_p = 1\}.$$

A.1.7. Dwork considers the functions

$$(A.7) \quad f(\lambda) = \frac{F(\lambda)}{F(\lambda^p)}, \quad \eta(\lambda) = \frac{F'(\lambda)}{F(\lambda)},$$

defined in a neighborhood of $0 \in D_1$ as ratios of the corresponding convergent power series expansions.

Dwork proves that $f(\lambda)$ can be analytically continued to the domain D_1 . For that, he indicates a sequence of regular rational functions on D_1 , that sequence uniformly converges on D_1 , and its limit equals $F(\lambda)/F(\lambda^p)$ in a neighborhood of 0 , see [Dw, Lemma 3.4].

From that Dwork deduces that $\eta(\lambda)$ has analytic continuation to the domain D_1 in the same sense, see [Dw, Lemma 3.1].

Since $\eta(\lambda)$ is analytic on D_1 , the function $\eta(1/\lambda)$ is analytic on D_2 .

Using the properties of equation (A.2) Dwork shows that $\eta(1-\lambda) = -\eta(\lambda)$ on D_1 and shows that $\eta(\lambda) = -\eta(1/\lambda)/\lambda^2 - 1/(2\lambda)$ on $D_1 \cap D_2$. Hence the function $\eta(\lambda)$ on D_1 and the function $-\eta(1/\lambda)/\lambda^2 - 1/(2\lambda)$ on D_2 define an analytic element on D .

We will use the formulas

$$(A.8) \quad \eta(1-\lambda) = -\eta(\lambda), \quad \eta(1/\lambda) = -\lambda^2 \eta(\lambda) - \lambda/2,$$

in Section A.5.

A.1.8. For $\alpha \in \mathbb{Q}_p$ let V_α be the space of germs at α of holomorphic solutions of equation (A.2). For $\alpha \neq 0, 1$ we have $\dim V_\alpha = 2$ and for $\alpha = 0, 1$ we have $\dim V_\alpha = 1$.

For $\alpha \in D_1$ let U_α be the space of germs at α of analytic functions defined by the equation

$$(A.9) \quad \frac{du}{d\lambda} = \eta(\lambda)u.$$

By [Dw, Lemma 3.2], U_α is a subspace of V_α . We have $U_\alpha = V_\alpha$ for $\alpha = 0, 1$.

For $\alpha \in \mathfrak{D}_2$ let U'_α be the space of germs at α of analytic functions defined by the equation

$$(A.10) \quad \frac{du}{d\lambda} = (-\eta(1/\lambda)/\lambda^2 - 1/(2\lambda))u.$$

By [Dw, Lemma 3.2], U'_α is a subspace of V_α . For $\alpha \in D_1 \cap D_2$ we have $U_\alpha = U'_\alpha$.

A.1.9. By [Dw, Lemma 4.2], for $\alpha \in D$ the subspace $U_\alpha \subset V_\alpha$ also can be characterized as the subspace of germs at α of holomorphic functions bounded in their disc of convergence.

More precisely, let C_p be the metric completion of the algebraic closure $\mathbb{Q}_p^-$ of the field $\mathbb{Q}_p$. Let $\alpha \in D$. Let $u(\lambda) = \sum_{k=0}^{\infty} c_k(\lambda - \alpha)^k$ be an element of V_α .

Consider $u(\lambda)$ as a germ at $\alpha \in \mathbb{Q}_p \subset C_p$ of an analytic function on C_p . The germ $u(\lambda)$ is called bounded on its disc of convergence if $u(\lambda)$ is bounded on its disc of convergence in C_p .

Let r be the radius of convergence of $u(\lambda) = \sum_{k=0}^{\infty} c_k(\lambda - \alpha)^k$. Define $|u(\lambda)|_0 = \sup_k |c_k| r^k$. Then $u(\lambda)$ is bounded in its disc of convergence if and only if $|u(\lambda)|_0 < \infty$.

A.1.10. The function $F(\lambda)$ is a holomorphic solution of equation (A.2) on the disc $D_{0,1}$. A second solution of (A.5) is of the form $G(\lambda) = F(\lambda)\log\lambda + H(\lambda)$, where the function $H(\lambda)$ is holomorphic on $D_{0,1}$. Dwork specifies $H(\lambda)$ by [Dw, Equation (4.19)]. Then

$$(A.11) \quad \mathcal{F}(\lambda) = \begin{pmatrix} F & G \\ F' & G' \end{pmatrix}$$

is a fundamental matrix of solutions of equation (A.5).

In [Dw] Dwork introduces a 2×2 -matrix function $A(\lambda)$, and then proves the formula

$$(A.12) \quad \mathcal{A}(\lambda) = \mathcal{F}(\lambda) M \mathcal{F}(\lambda^p)^{-1}, \quad M = \begin{pmatrix} (-1)^{(p-1)/2} & b \\ 0 & (-1)^{(p-1)/2} p \end{pmatrix},$$

where b is a suitable number, see [Dw, Lemma 6.2] and formulas on page 72 in [Dw]. Dwork shows that $A(\lambda)$ extends to an analytic function on the domain

$D_3 \cup D_4$, where

$$(A.13) \quad D_3 = \{\lambda \in \mathbb{Z}_p \mid \varepsilon < |\lambda|_p \leq 1, |\lambda - 1|_p = 1\}, \quad D_4 = \{\lambda \in \mathbb{Z}_p \mid \varepsilon < |\lambda|_p < 1\},$$

D

Here ε is some explicit number, $0 < \varepsilon < 1$. See the bottom of page 62 in [Dw] and the first sentence of the proof of Theorem 6 in [Dw]. Formula (A.12) immediately implies that

$$(A.14) \quad A(\lambda)F(\lambda^p) = F(\lambda)M$$

on D_4 . The matrix $A(\lambda)$ is called the matrix of the Frobenius transformation of solutions of equation (A.5) relative to the fundamental matrix $F(\lambda)$. It follows from formula (A.14) that

$$(A.15) \quad \mathcal{A}(\lambda) \begin{pmatrix} F(\lambda^p) \\ F'(\lambda^p) \end{pmatrix} = (-1)^{(p-1)/2} \begin{pmatrix} F(\lambda) \\ F'(\lambda) \end{pmatrix} \text{ on } D_4. \text{ This can}$$

be reformulated as the relation

$$(A.16) \quad \mathcal{A}(\lambda) \begin{pmatrix} 1 \\ \eta(\lambda^p) \end{pmatrix} = (-1)^{(p-1)/2} f(\lambda) \begin{pmatrix} 1 \\ \eta(\lambda) \end{pmatrix}$$

on D_4 . By the already formulated analytic properties of $\eta(\lambda)$ and $A(\lambda)$, relation (A.16) can be analytically continued to the domain $D_1 \cap D_3$.

Equation (A.16) implies that for any $\alpha \in \mathbb{F}_p^\times - \{1\}$ such that $\omega(\alpha) \in D_1$, the vector $(1, \eta(\omega(\alpha)))$ is an eigenvector of the Frobenius matrix $A(\omega(\alpha))$ with eigenvalue $(-1)^{(p-1)/2} f(\omega(\alpha))$,

$$(A.17) \quad \mathcal{A}(\lambda) \begin{pmatrix} 1 \\ \eta(\omega(\alpha)) \end{pmatrix} = (-1)^{(p-1)/2} f(\omega(\alpha)) \begin{pmatrix} 1 \\ \eta(\omega(\alpha)) \end{pmatrix}.$$

It is known that the zeta function of the elliptic curve defined over $\mathbb{F}_p$ by the equation $y^2 = x(x-1)(x-\alpha)$ has two zeros, which are $1/((-1)^{(p-1)/2} f(\omega(\alpha)))$, $(-1)^{(p-1)/2} f(\omega(\alpha))/p$. It is also known that $|f(\omega(\alpha))|_p = 1$. The number $(-1)^{(p-1)/2} f(\omega(\alpha))$ is called the unit root. See [Dw] and also (A.53).

A.2. KZ equations. The KZ equations (2.1) for $n = 3$ is the following system of differential and algebraic equations for a column 3-vector $I = (I_1, I_2, I_3)$ depending on variables $z = (z_1, z_2, z_3)$:

$$(A.18) \quad \begin{aligned} \frac{\partial I}{\partial z_1} &= \frac{1}{2} \left(\frac{\Omega_{12}}{z_1 - z_2} + \frac{\Omega_{13}}{z_1 - z_3} \right) I, & \frac{\partial I}{\partial z_2} &= \frac{1}{2} \left(\frac{\Omega_{21}}{z_2 - z_1} + \frac{\Omega_{23}}{z_2 - z_3} \right) I, \\ \frac{\partial I}{\partial z_3} &= \frac{1}{2} \left(\frac{\Omega_{31}}{z_3 - z_1} + \frac{\Omega_{32}}{z_3 - z_2} \right) I, & 0 &= I_1 + I_2 + I_3, \end{aligned}$$

where $\Omega_{ij} = \Omega_{ji}$ and

$$\Omega_{12} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Omega_{13} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad \Omega_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

We introduce new variables

$$(A.19) \quad u_1 = z_1 - z_3, \quad u_2 = \frac{z_2 - z_3}{z_1 - z_3}, \quad u_3 = z_1 + z_2 + z_3,$$

see (9.7). Then system (A.18) takes the form

$$(A.20) \quad \begin{aligned} \frac{\partial I}{\partial u_1} &= \frac{1}{2} \frac{\Omega_{12} + \Omega_{13} + \Omega_{23}}{u_1} I, & \frac{\partial I}{\partial u_2} &= \frac{1}{2} \left(\frac{\Omega_{12}}{u_2 - 1} + \frac{\Omega_{23}}{u_2} \right) I, \\ \frac{\partial I}{\partial u_3} &= 0, & 0 &= I_1 + I_2 + I_3. \end{aligned}$$

The variables in system (A.20) are separated, cf. (9.9).

Denote $W = \{(I_1, I_2, I_3) \mid I_1 + I_2 + I_3 = 0\}$. Then

$$(A.21) \quad (\Omega_{12} + \Omega_{13} + \Omega_{23})|_W = -3\text{Id}.$$

Hence all solutions of system (A.20) have the form

$$(A.22) \quad I = u_1^{-3/2} (J_1(u_2), J_2(u_2), J_3(u_2)), \quad J_1 + J_2 + J_3 = 0,$$

where the column vector $J(u_2)$ is a solution of the differential equation

$$(A.23) \quad \frac{\partial J}{\partial u_2} = \frac{1}{2} \left(\frac{\Omega_{12}}{u_2 - 1} + \frac{\Omega_{23}}{u_2} \right) J.$$

A.3. Solutions over $\mathbb{C}$. Any solution of system (A.18) has the form

$$(A.24) \quad I^{(\gamma)}(z) = \int_{\gamma} \left(\frac{1}{x - z_1}, \frac{1}{x - z_2}, \frac{1}{x - z_3} \right) \frac{dx}{\sqrt{(x - z_1)(x - z_2)(x - z_3)}}.$$

where γ is a flat family of 1-cycles on the elliptic curves of our family of curves.

We change x and z in this integral by setting $x = (z_1 - z_3)w + z_3$ and $z = z(u)$ as in

(A.19). Then integral (A.24) takes the form

$$(A.25) \quad I^{(\gamma)}(u_1, u_2) = u_1^{-3/2} \int_{\gamma} \left(\frac{1}{w - 1}, \frac{1}{w - u_2}, \frac{1}{w} \right) \frac{dw}{\sqrt{(w - 1)(w - u_2)w}}.$$

We take $\gamma = \gamma_1$ to be the circle $|w| = 1/2$ oriented counter-clockwise. We assume that u_2 lies in this circle. We fix the branch of the integrand by choosing the argument of $(w - 1)(w - u_2)w$ at $w = 1/2$, u_2 of 0 to be $\pi/2$. We multiply the circle with the chosen branch of the integrand by $\frac{1}{2\pi}$.

This finishes the description of γ_1 . See the definition of cycles γ_l in Section 9.5. We

expand the integral $I^{(\gamma_1)}(u_1, u_2)$ as a power series in u_2 and obtain

$$(A.26) \quad \begin{aligned} I^{(\gamma_1)}(u_1, u_2) &= u_1^{-3/2} \sum_{a=0}^{\infty} \left(\left(\frac{-\frac{3}{2}}{a} \right) \left(\frac{-\frac{1}{2}}{a} \right), \left(\frac{-\frac{1}{2}}{a+1} \right) \left(\frac{-\frac{3}{2}}{a} \right), \left(\frac{-\frac{1}{2}}{a+1} \right) \left(\frac{-\frac{1}{2}}{a} \right) \right) u_2^a \\ &= u_1^{-3/2} \sum_{a=0}^{\infty} \left(\frac{-\frac{1}{2}}{a+1} \right) \left(\frac{-\frac{3}{2}}{a} \right) \left(\frac{a+1}{-1/2-a}, 1, \frac{-1/2}{-1/2-a} \right) u_2^a, \end{aligned}$$

see Theorem (9.4). Denote

$$(A.27) \quad I := I^{(\gamma_1)} = (z_1 - z_3)^{-3/2} \sum_{a=0}^{\infty} \left(\frac{-\frac{1}{2}}{a+1} \right) \left(\frac{-\frac{3}{2}}{a} \right) \left(\frac{a+1}{-1/2-a}, 1, \frac{-1/2}{-1/2-a} \right) \left(\frac{z_2 - z_3}{z_1 - z_3} \right)^a$$

$$(A.28) \quad = u_1^{-3/2} \sum_{a=0}^{\infty} \left(\frac{-\frac{1}{2}}{a+1} \right) \left(\frac{-\frac{3}{2}}{a} \right) \left(\frac{a+1}{-1/2-a}, 1, \frac{-1/2}{-1/2-a} \right) u_2^a.$$

This series is a solution of system (A.18).

Remark. Formulas (A.25) and (A.26) imply that

$$(A.29) \quad I(u_1, u_2) = u_1^{-3/2} \left({}_2F_1\left(\frac{1}{2}, \frac{3}{2}; 1; u_2\right), -\frac{1}{2} {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; u_2\right), -\frac{1}{2} {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; 2; u_2\right) \right),$$

where ${}_2F_1(a, b; c; \lambda)$ is the classical hypergeometric function.

A.4. Solutions as vectors of first derivatives. Introduce the function

$$(A.30) \quad \ell^{(\gamma)}(z) = \int_{\gamma} \frac{dx}{\sqrt{(x-z_1)(x-z_2)(x-z_3)}}.$$

Then

$$(A.31) \quad I^{(\gamma)}(z) = 2 \left(\frac{\partial \ell^{(\gamma)}}{\partial z_1}, \frac{\partial \ell^{(\gamma)}}{\partial z_2}, \frac{\partial \ell^{(\gamma)}}{\partial z_3} \right).$$

Changing the variable $x = w(z_1 - z_3) + z_3$ we write

$$(A.32) \quad \begin{aligned} \ell^{(\gamma)}(z) &= (z_1 - z_3)^{-1/2} \int_{\gamma} \frac{dw}{\sqrt{(w-1)(w - \frac{z_2-z_3}{z_1-z_3})w}} \\ &= (z_1 - z_3)^{-1/2} h^{(\gamma)}\left(\frac{z_2 - z_3}{z_1 - z_3}\right) \\ &= u_1^{-1/2} h^{(\gamma)}(u_2), \end{aligned}$$

where $h^{(\gamma)}(\lambda)$ is the elliptic integral in (A.1) and $h^{(\gamma)}(\lambda)$ is a solution of equation (A.2).

Denote $h(\lambda) := h^{(\gamma)}(\lambda)$. Then

$$(A.33) \quad \begin{aligned} I^{(\gamma)} &= (z_1 - z_3)^{-3/2} \left(-h\left(\frac{z_2 - z_3}{z_1 - z_3}\right) - 2h'\left(\frac{z_2 - z_3}{z_1 - z_3}\right) \frac{z_2 - z_3}{z_1 - z_3}, 2h'\left(\frac{z_2 - z_3}{z_1 - z_3}\right), h\left(\frac{z_2 - z_3}{z_1 - z_3}\right) \right. \\ &\quad \left. + 2h'\left(\frac{z_2 - z_3}{z_1 - z_3}\right) \frac{z_2 - z_1}{z_1 - z_3} \right) \\ (A.34) \quad &= u_1^{-3/2} (-h(u_2) - 2h'(u_2)u_2, 2h'(u_2), h(u_2) + 2h'(u_2)(u_2 - 1)). \end{aligned}$$

Formula (A.34) relates solutions of system (A.5) and solutions of system (A.20). If

(h, h') is a solution of system (A.5), then equation

$$(A.35) \quad \begin{pmatrix} I_1(u_1, u_2) \\ I_2(u_1, u_2) \\ I_3(u_1, u_2) \end{pmatrix} = u_1^{-3/2} \begin{pmatrix} -1 & -2u_2 \\ 0 & 2 \\ 1 & 2u_2 - 2 \end{pmatrix} \begin{pmatrix} h(u_2) \\ h'(u_2) \end{pmatrix}$$

gives a solution (I_1, I_2, I_3) of system (A.20). Conversely, if (I_1, I_2, I_3) is a solution of system (A.20) then formula

$$(A.36) \quad \begin{pmatrix} h(u_2) \\ h'(u_2) \end{pmatrix} = u_1^{3/2} \begin{pmatrix} -1 & -u_2 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} I_1(u_1, u_2) \\ I_2(u_1, u_2) \\ I_3(u_1, u_2) \end{pmatrix}$$

gives a solution (h, h') of system (A.5). Using

the cycle γ_1 we can evaluate

$$(A.37) \quad \begin{aligned} \ell^{(\gamma_1)} &= (z_1 - z_3)^{-1/2} F\left(\frac{z_2 - z_3}{z_1 - z_3}\right) \\ &= u_1^{-1/2} F(u_2), \end{aligned} \quad \text{where } F(\lambda) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda\right).$$

Denote

$$(A.38) \quad \ell := \ell^{(\gamma_1)}, \quad \mathcal{I} := \frac{I}{\ell} = \frac{2}{\ell} \left(\frac{\partial \ell}{\partial z_1}, \frac{\partial \ell}{\partial z_2}, \frac{\partial \ell}{\partial z_3} \right),$$

where I is defined in (A.27). Formulas (A.33) and (A.34) imply

$$(A.39) \quad \mathcal{I} = \frac{1}{z_1 - z_3} \left(-1 - 2\eta\left(\frac{z_2 - z_3}{z_1 - z_3}\right) \frac{z_2 - z_3}{z_1 - z_3}, 2\eta\left(\frac{z_2 - z_3}{z_1 - z_3}\right), 1 + 2\eta\left(\frac{z_2 - z_3}{z_1 - z_3}\right) \frac{z_2 - z_1}{z_1 - z_3} \right)$$

$$(A.40) \quad = \frac{1}{u_1} \left(-1 - 2\eta(u_2)u_2, 2\eta(u_2), 1 + 2\eta(u_2)(u_2 - 1) \right),$$

where the function $\eta(\lambda)$ is defined in (A.7).

A.5. Six coordinate systems. System (A.18) of KZ equations has 6 distinguished coordinate systems (asymptotic zones). They are labeled by permutations $\sigma = (i, j, k) \in S_3$. The coordinate system $u^\sigma = (u_1^\sigma, u_2^\sigma, u_3^\sigma)$ is defined by the formulas

$$(A.41) \quad u_1^\sigma = z_i - z_k, \quad u_2^\sigma = \frac{z_j - z_k}{z_i - z_k}, \quad u_3^\sigma = z_1 + z_2 + z_3.$$

For the identity element $\text{id} = (1, 2, 3)$ the corresponding coordinate system is defined in (A.19).

Having one of these coordinate systems we repeat the constructions of Sections A.3-A.4 and construct a scalar function $\ell^\sigma(z)$ and vector-valued functions $I^\sigma(z)$, $\mathcal{I}^\sigma(z)$, such that $\mathcal{I}^\sigma(z) = I^\sigma(z)/\ell^\sigma(z)$. For the identity element $\text{id} = (1, 2, 3)$ these functions are $\ell(z)$, $I(z)$, $\mathcal{I}(z)$ in (A.37), (A.27), (A.39). Notice that the functions

$\ell(z)$ and $I(z)$ are defined as integrals over γ_1 , and that γ_1 is defined with the help of coordinates u_1, u_2, u_3 .

For any σ the function $I^\sigma(z)$ is a power series solution of system (A.18) in the chart with coordinates u^σ , see (A.26) and (A.27). Below we list the functions I^σ :

$$(A.42) \quad \begin{aligned} \mathcal{I}^{123} &= \frac{1}{z_1 - z_3} \left(-1 - 2\eta\left(\frac{z_2 - z_3}{z_1 - z_3}\right) \frac{z_2 - z_3}{z_1 - z_3}, 2\eta\left(\frac{z_2 - z_3}{z_1 - z_3}\right), 1 + \eta\left(\frac{z_2 - z_3}{z_1 - z_3}\right) \frac{z_2 - z_1}{z_1 - z_3} \right), \\ \mathcal{I}^{321} &= \frac{1}{z_3 - z_1} \left(1 + 2\eta\left(\frac{z_2 - z_1}{z_3 - z_1}\right) \frac{z_2 - z_3}{z_3 - z_1}, 2\eta\left(\frac{z_2 - z_1}{z_3 - z_1}\right), -1 - 2\eta\left(\frac{z_2 - z_1}{z_3 - z_1}\right) \frac{z_2 - z_1}{z_3 - z_1} \right), \\ \mathcal{I}^{213} &= \frac{1}{z_2 - z_3} \left(2\eta\left(\frac{z_1 - z_3}{z_2 - z_3}\right), -1 - 2\eta\left(\frac{z_1 - z_3}{z_2 - z_3}\right) \frac{z_1 - z_3}{z_2 - z_3}, 1 + 2\eta\left(\frac{z_1 - z_3}{z_2 - z_3}\right) \frac{z_1 - z_2}{z_2 - z_3} \right), \\ \mathcal{I}^{132} &= \frac{1}{z_1 - z_2} \left(-1 - \eta\left(\frac{z_3 - z_2}{z_1 - z_2}\right) \frac{z_3 - z_2}{z_1 - z_2}, 1 + 2\eta\left(\frac{z_3 - z_2}{z_1 - z_2}\right) \frac{z_3 - z_1}{z_1 - z_2}, 2\eta\left(\frac{z_3 - z_2}{z_1 - z_2}\right) \right), \\ \mathcal{I}^{231} &= \frac{1}{z_2 - z_1} \left(1 + 2\eta\left(\frac{z_3 - z_1}{z_2 - z_1}\right) \frac{z_3 - z_2}{z_2 - z_1}, -1 - 2\eta\left(\frac{z_3 - z_1}{z_2 - z_1}\right) \frac{z_3 - z_1}{z_2 - z_1}, 2\eta\left(\frac{z_3 - z_1}{z_2 - z_1}\right) \right), \\ \mathcal{I}^{312} &= \frac{1}{z_3 - z_2} \left(2\eta\left(\frac{z_1 - z_2}{z_3 - z_2}\right), 1 + 2\eta\left(\frac{z_1 - z_2}{z_3 - z_2}\right) \frac{z_1 - z_3}{z_3 - z_2}, -1 - 2\eta\left(\frac{z_1 - z_2}{z_3 - z_2}\right) \frac{z_1 - z_2}{z_3 - z_2} \right). \end{aligned}$$

Theorem A.1. For $(i, j, k) \in S_3$ consider the three functions I^{ijk} , I^{kji} , I^{ikj} . Then I^{kji} is transformed to I^{ijk} by application of formula $\eta(1 - \lambda) = -\eta(\lambda)$ and I^{ikj} is transformed to I^{ijk} by application of formula $\eta(1/\lambda) = -\lambda^2\eta(\lambda) - \lambda/2$.

Proof. The proof is straightforward. For example we check the statement for $(i, j, k) = (1, 2, 3)$. In this case the functions $\mathcal{I}^{123}$, $\mathcal{I}^{321}$, $\mathcal{I}^{132}$ are

$$\begin{aligned} &\frac{1}{u_1} \left(-1 - 2\eta(u_2)u_2, 2\eta(u_2), 1 + 2\eta(u_2)(u_2 - 1) \right), \\ &\frac{1}{u_1} \left(-1 + 2\eta(1 - u_2)u_2, -2\eta(1 - u_2), 1 + 2\eta(1 - u_2)(1 - u_2) \right) \\ &\frac{1}{u_1 u_2} \left(2\eta\left(\frac{1}{u_2}\right), -1 - 2\eta\left(\frac{1}{u_2}\right) \frac{1}{u_2}, 1 + 2\eta\left(\frac{1}{u_2}\right) \frac{1 - u_2}{u_2} \right), \end{aligned}$$

where u_1, u_2 are defined in (A.19). Then formula $\eta(1 - u_2) = -\eta(u_2)$ transforms the second function to the first and the formula $\eta(1/u_2) = -u_2^2\eta(u_2) - u_2/2$ transforms the third function to the first.

Define

$$D^{\sim 0} = \{(z_1, z_2, z_3) \in \mathbb{Q}_p^3 \mid z_i \neq z_j \ \forall i \neq j\}.$$

For any $\sigma = (i, j, k) \in S_3$ define

$$D^{\sim 1} = \left\{ (z_1, z_2, z_3) \in \tilde{D}_0 \mid \frac{j - k}{z_i - z_k} \in \mathbb{Z}_p, \ g\left(\frac{j - k}{z_i - z_k}\right) \Big|_p = 1 \right\} \quad (1)$$

$$D^{\sim \sigma} = \left\{ (z_1, z_2, z_3) \in \tilde{D}_0 \mid \frac{j - k}{z_i - z_k} \in \mathbb{Z}_p, \ g\left(\frac{j - k}{z_i - z_k}\right) \Big|_p = 1 \right\} \quad (2)$$

$$D^{\sim 0} = \bigcup_{\sigma \in S_3} D^{\sim \sigma} = D_1, \quad (3)$$

where the function g is defined in (A.6).

For any $(i, j, k) \in S_3$ the functions I^{ik} , I^{kj} , I^{jk} define an analytic element on $D^{\sim \sigma}$, see Section A.1.7 and [Dw]. Theorem A.1 implies the following corollary.

Corollary A.2. *The functions $(I^{ijk})_{(i,j,k) \in S_3}$ define an analytic element on $D^{\sim}$.*

Remark. Dwork's formulas (A.8) present the S_3 -symmetries of the analytic element $(\eta(\lambda), -\eta(1 - \lambda), -\eta(1/\lambda)/\lambda^2 - 1/(2\lambda))$. Dwork's S_3 -symmetries reformulated as S_3 -symmetries of the analytic element $(I^{ijk})_{(i,j,k) \in S_3}$ look even more well-rounded.

A.6. Subbundle. Denote $W = \{(I_1, I_2, I_3) \in \mathbb{Q}_p^3 \mid I_1 + I_2 + I_3 = 0\}$. System (A.18) of KZ equations defines a flat connection on the trivial bundle $W \times D^{\sim 0} \rightarrow D^{\sim 0}$. The flat sections of that bundle are solutions of system (A.18) of KZ equations.

For any $\alpha \in D^{\sim}$ such that $\alpha \in D^{\sim \sigma}$ the vector $I^\sigma(\alpha)$ spans a one-dimensional subspace $U_\alpha \subset W$. That subspace does not depend on σ such that $\alpha \in D^{\sim \sigma}$. The union of these subspaces defines a one-dimensional subbundle $U \rightarrow D^{\sim}$ of the trivial bundle $W \times D^{\sim} \rightarrow D^{\sim}$.

Theorem A.3. *The subbundle $U \rightarrow D^{\sim}$ is invariant with respect to the KZ connection on $W \times D^{\sim} \rightarrow D^{\sim}$.*

Proof. For any $\sigma \in S_3$ the subbundle $U \rightarrow \tilde{D}^\sim$ is generated by the flat section I^σ near the points where $u_2^\sigma = 0$. Hence the subbundle $U \rightarrow \tilde{D}^\sim$ is generated by a flat section near any point of $\tilde{D}^\sim$, see Section A.1.8 and [Dw, Lemma 3.1].

Remark. For any $\sigma \in S_3$ the flat section I^σ generates the subbundle $U \rightarrow \tilde{D}^\sim$ near the points where $u_2^\sigma = 0$. The power series I^σ considered over $\mathbb{C}$ is the expansion of an integral over a cycle vanishing at the points where $u_2^\sigma = 0$. The analytic continuation over $\mathbb{C}$ of that integral over that vanishing cycle could not generate a one-dimensional subbundle of the trivial bundle $W \times \tilde{D}^\sim \rightarrow \tilde{D}^\sim$ since the monodromy representation of the complex KZ equations in this case is irreducible. In contrast with this fact over $\mathbb{C}$, the p -adic power series solutions I^σ , $\sigma \in S_3$, defined at different points glue together into a single line bundle $U \rightarrow \tilde{D}^\sim$. This line bundle is what Dwork calls a *p-adic cycle*. This p -adic phenomenon was stressed by Dwork in [Dw] who titled his paper *p-adic Cycles*.

Remark. The invariant subbundles of the KZ connection over $\mathbb{C}$ usually are related to some additional conformal block constructions, see [FSV1, FSV2, SV2, V7]. Apparently the subbundle $U \rightarrow \tilde{D}^\sim$ is of a different p -adic nature, cf. [V7].

A.7. Boundedness. Let $\sigma \in S_3$ and $\alpha \in \tilde{D}^\sim^\sigma$. For $w \in W$ let $I(z; w)$ be the germ at α of the solution of the KZ equations with initial condition $I(\alpha, w) = w$. By formula (A.34), the coordinates of $I(z; w)$ have the form

$$\begin{aligned} & (u_1^\sigma)^{-3/2}(-h(u_2^\sigma) - 2h'(u_2^\sigma)u_2^\sigma), \quad (u_1^\sigma)^{-3/2}2h'(u_2^\sigma) \\ & (u_1^\sigma)^{-3/2}(h(u_2^\sigma) + 2h'(u_2^\sigma)(u_2^\sigma - 1)), \end{aligned}$$

where h is the germ at the point $u_2^\sigma = u_2^\sigma(\alpha)$ of a solution of equation (A.2). We say that the germ $I(z; w)$ is bounded if each of the germs $-h(u_2^\sigma) - 2h'(u_2^\sigma)u_2^\sigma$, $2h'(u_2^\sigma)$, $h(u_2^\sigma) + 2h'(u_2^\sigma)(u_2^\sigma - 1)$ is bounded in its disc of convergence.

Theorem A.4. *The germ $I(z; w)$ is bounded if and only if $w \in \tilde{U}_\alpha$.*

Proof. Let $w \in \tilde{U}_\alpha$. Then the germ h belongs to the corresponding subspace $U_{u_2^\sigma(\alpha)}$

defined in Section A.1.8. By [Dw, Lemma 4.2] the germ h is bounded in

its disc of convergence, see Section A.1.9. Hence each of the three germs $-h(u_2^\sigma) - 2h'(u_2^\sigma)u_2^\sigma$, $2h'(u_2^\sigma)$, $h(u_2^\sigma) + 2h'(u_2^\sigma)(u_2^\sigma - 1)$ is bounded in its disc of convergence.

If $w \notin \tilde{U}_\alpha$ then $h \in U$, then $h \notin U_{u_2^\sigma(\alpha)}$. By [Dw, Lemma 4.2] the germ h is unbounded in its disc of convergence. Then at least one of the three germs $-h(u_2^\sigma) - 2h'(u_2^\sigma)u_2^\sigma$, $2h'(u_2^\sigma)$, $h(u_2^\sigma) + 2h'(u_2^\sigma)(u_2^\sigma - 1)$ is unbounded in its disc of convergence.

A.8. More domains. Denote
(A.43)

$$\begin{aligned} D^3 &= \{(z_1, z_2, z_3) \in \tilde{\mathfrak{D}}_0 \mid \frac{z_2 - z_3}{z_1 - z_3} \in \mathbb{Z}_p, \left| \frac{z_2 - z_3}{z_1 - z_3} \right|_p = 1, \left| \frac{z_2 - z_3}{z_1 - z_3} - 1 \right|_p = 1\}, \\ D^4 &= \{(z_1, z_2, z_3) \in \tilde{\mathfrak{D}}_0 \mid \frac{z_2 - z_3}{z_1 - z_3} \in \mathbb{Z}_p, \varepsilon < \left| \frac{z_2 - z_3}{z_1 - z_3} \right|_p < 1\}, \end{aligned}$$

where ε is the same number as in (A.13).

A.9. Frobenius map on solutions of KZ equations. Formula (A.14) describes the Frobenius map on solutions of equation (A.5). Solutions of equation (A.5) are identified with solutions of the KZ system (A.20) by formulas (A.35) and (A.36). That allows us to define the Frobenius map on solutions of the KZ system (A.20).

Denote

$$(A.44) \quad \mathcal{B}(u_1, u_2) \mathcal{C}(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{C}(u_1, u_2) \mathcal{B}(u_1, u_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\mathcal{B}(u_1, u_2) = u_1^{3/2} \begin{pmatrix} -1 & -u_2 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad \mathcal{C}(u_1, u_2) = u_1^{-3/2} \begin{pmatrix} -1 & -2u_2 \\ 0 & 2 \\ 1 & 2u_2 - 2 \end{pmatrix}.$$

we have. The

second matrix defines the identity operator on the space $W = \{(I_1, I_2, I_3) \in \mathbb{Q}_p^3 \mid I_1 + I_2 + I_3 = 0\}$.

Recall the matrix $F(u_2)$ defined in (A.11). By formula (A.35) the matrix

$$\tilde{F}(u_1, u_2) = \mathcal{C}(u_1, u_2) F(u_2)$$

is a fundamental matrix of solutions of system (A.20). Recall the matrices $A(\lambda)$, M in (A.12). Denote

$$(A.45) \quad \tilde{A}(u_1, u_2) = \mathcal{C}(u_1, u_2) A(u_2) \mathcal{B}((u_1)^p, (u_2)^p).$$

This is a 3×3 matrix valued function, whose values preserve the subspace $W \subset \mathbb{Q}_p^3$.

Theorem A.5. *We have*

$$(A.46) \quad \tilde{A}(u_1, u_2) \tilde{F}((u_1)^p, (u_2)^p) = \tilde{F}(u_1, u_2) M.$$

The matrix $\tilde{A}(u_1, u_2)$ extends to an analytic function on the domain $D^3 \cup D^4$.

Proof. The theorem is a corollary of formula (A.14) and Dwork's statements listed in Section A.1.10.

We call $\tilde{A}(u_1, u_2)$ the matrix of the Frobenius transformation of solutions of system (A.20) relative to the fundamental matrix $\tilde{F}(u_1, u_2)$ on the domain $\tilde{D}_3 \cup \tilde{D}_4$.

Recall the distinguished solution

(A.47) $I(u_1, u_2) = u_1^{-3/2}(-F(u_2) - 2F'(u_2)u_2, 2F'(u_2), F(u_2) + 2F'(u_2)(u_2 - 1))$ of system (A.20) defined near the points where $u_2 = 0$, see (A.34). By (A.15) we have

$$(A.48) \quad \tilde{A}(u_1, u_2)I((u_1)^p, (u_2)^p) = (-1)^{(p-1)/2}I(u_1, u_2)$$

on $\tilde{D}_4$. Recall $\ell(u_1, u_2) = u_1^{-1/2}F(u_2)$ in (A.37). Dividing both sides in (A.47) by $\ell((u_1)^p, (u_2)^p)$ we can reformulate (A.48) as

$$(A.49) \quad \tilde{A}(u_1, u_2) \mathcal{I}((u_1)^p, (u_2)^p) = (-1)^{(p-1)/2} u_1^{(p-1)/2} f(u_2) \mathcal{I}(u_1, u_2)$$

on $\tilde{D}_4$, see $I(u_1, u_2)$ in (A.40) and $f(u_2)$ in (A.7). As in Section A.1.10 we conclude with Dwork that relation (A.49) can be analytically continued to the domain

$$\tilde{D}_1^{(1,2,3)} \cap \tilde{\mathfrak{D}}_3.$$

Equation (A.49) implies that for any $\alpha \in F_p^\times - \{1\}$, $\beta \in F_p^\times$ such that $\omega(\alpha) \in D_1$, the vector $I(\omega(\beta), \omega(\alpha))$ is an eigenvector of the Frobenius matrix $\tilde{A}(\omega(\beta), \omega(\alpha))$ with eigenvalue $\omega(\beta^{(p-1)/2})(-1)^{(p-1)/2}f(\omega(\alpha))$,

$$(A.50) \quad \tilde{A}(\omega(\beta), \omega(\alpha))I(\omega(\beta), \omega(\alpha)) = \omega(\beta^{(p-1)/2})(-1)^{(p-1)/2}f(\omega(\alpha))I(\omega(\beta), \omega(\alpha)).$$

In this Section A.9 we described the matrix $\tilde{A}(u_1, u_2)$ of the Frobenius transformation of solutions of system (A.18) written in coordinates u_1, u_2, u_3 corresponding to the chart labeled by the identify permutation $(1, 2, 3) \in S_3$. In the same way we may start with the chart corresponding to any permutation $\sigma \in S_3$ and describe the matrix of the Frobenius transformation of solutions of system (A.18) written in coordinates $u_1^\sigma, u_2^\sigma, u_3^\sigma$.

A.10. Eigenvalue $\omega(\beta^{(p-1)/2})(-1)^{(p-1)/2}f(\omega(\alpha))$.

Theorem A.6. *The number $\omega(\beta^{(p-1)/2})(-1)^{(p-1)/2}f(\omega(\alpha))$ is the unit root of the elliptic curve $E(\alpha, \beta)$ defined over F_p by the affine equation*

$$(A.51) \quad w^2 = \beta v(v-1)(v-\alpha).$$

Proof. Assume that $\beta \in F_p^\times$ is a square, $\beta = \gamma^2$ for some $\gamma \in F_p$. Then on the one hand the change of the variable $\tilde{w} = w/\gamma$ makes $E(\alpha, \beta)$ isomorphic to $E(\alpha, 1)$. On the other hand $\beta^{(p-1)/2} = 1$ and $\omega(\beta^{(p-1)/2})(-1)^{(p-1)/2}f(\omega(\alpha)) =$

$$(-1)^{(p-1)/2}f(\omega(\alpha)E(\alpha, 1) \text{ by } [\mathbf{Dw}].)), \text{ where the last number is the unit root of the elliptic curve}$$

Assume that $\beta \in \mathbb{F}_p^\times$ is not a square. Denote by $N_{1,\beta}$ the number of points on $E(\alpha, \beta)$.

Then

$$(A.52) \quad N_{1,1} + N_{1,\beta} = 4 + 4 + 2(p-3) = 2p + 2.$$

Indeed the number $4+4$ corresponds to the points $(0,0)$, $(0,1)$, $(0,\alpha)$, ∞ on $E(\alpha, \beta)$ and on $E(\alpha, 1)$. The number $2(p-3)$ corresponds to $p-3$ elements of $\mathbb{F}_p - \{0, 1, \alpha\}$. Namely if $v_0 \in \mathbb{F}_p - \{0, 1, \alpha\}$, then exactly one of the two elements $\beta v_0(v_0-1)(v_0-\alpha)$, $v_0(v_0-1)(v_0-\alpha)$ is a square in $\mathbb{F}_p$ and exactly one of the two elliptic curves has two points over $v = v_0$, while the other curve does not have points over v_0 . It is known that the zeta function of the curve $E(\alpha, \beta)$ has the form

$$(A.53) \quad \exp\left(\sum_{s=1}^{\infty} \frac{N_{s,\beta}}{s} T^s\right) = \frac{(1 - R_\beta T)(1 - (p/R_\beta)T)}{(1-T)(1-pT)}.$$

Here $N_{s,\beta}$ is the number points on $E(\alpha, \beta)$ considered over the field $\mathbb{F}_{p^s}$, while the number R_β has $|R_\beta|_p = 1$ and is called the unit root, for example see [Mo]. Equation (A.53) implies that for any s we have $N_{s,\beta} = 1 + p^s - R_\beta^s - (p/R_\beta)^s$. In particular for $s = 1$ we have

$$(A.54) \quad N_{1,\beta} = 1 + p - R_\beta - p/R_\beta.$$

From (A.52) and (A.54) we obtain

$$0 = R_\beta + p/R_\beta + R_1 + p/R_1 = (R_\beta + R_1)(1 + p/R_\beta R_1).$$

Since the second factor is nonzero we conclude that $R_\beta = -R_1$. By [Dw] we have $R_1 = (-1)^{(p-1)/2} f(\omega(\alpha))$. Hence

$$R_\beta = -(-1)^{(p-1)/2} f(\omega(\alpha)) = \omega(\beta^{(p-1)/2}) (-1)^{(p-1)/2} f(\omega(\alpha)).$$

The theorem is proved.

The relation between the eigenvector $I(\omega(\beta), \omega(\alpha))$ and the elliptic curve $E(\alpha, \beta)$, indicated in Theorem A.6, can be explained as follows. Over $\mathbb{C}$ the vector I is given by integrals over cycles on elliptic curves with equation $y^2 = (x-z_1)(x-z_2)(x-z_3)$. After the change of variables

$$x = (z_1 - z_3)w + z_3, \quad u_1 = z_1 - z_3, \quad u_2 = \frac{z_2 - z_3}{z_1 - z_3}, \quad y = (z_1 - z_3)v,$$

the equation takes the form

$$(A.55) \quad v^2 = u_1(w-1)(w-u_2)w.$$

The eigenvector $I(\omega(\beta), \omega(\alpha))$ corresponds to the curve in (A.55) with $(u_1, u_2) = (\omega(\beta), \omega(\alpha))$, and $(\omega(\beta), \omega(\alpha)) \equiv (\beta, \alpha) \pmod{p}$.

It is more surprising that system (A.18) of KZ equations gives a bit more arithmetic information than the hypergeometric equation (A.5), despite the fact system (A.18) and equation (A.5) are equivalent by (A.35) and (A.36). Indeed Dwork's eigenvectors in (A.16) give unit roots of elliptic curves $E(\alpha, 1)$ while the eigenvectors in (A.50) coming from the KZ equations give unit roots of more general elliptic curves $E(\alpha, \beta)$.

A.11. Approximation of analytic element $(I^{ijk})_{(i,j,k) \in S_3}$ by rational functions. Let

s be a positive integer.

A.11.1. Let $M = (p^s - 1)/2$, $\Phi_{p^s}(x, z) = \prod_{i=1}^3 (x - z_i)^M$,

$$P_{p^s}(x, z) = \left(\frac{\Phi_{p^s}(x, z)}{x - z_1}, \frac{\Phi_{p^s}(x, z)}{x - z_2}, \frac{\Phi_{p^s}(x, z)}{x - z_3} \right) = \sum_i P_{p^s}^i(z) x^i.$$

Denote $I_{p^s}^{[p^s-1]}(z) = P_{p^s}^{p^s-1}(z)$. The functions

$$I_{p^s}^{[p^s-1]}(z), \quad p I_{p^{s-1}}^{[p^{s-1}-1]}(z), \quad \dots, \quad p^{s-2} I_{p^2}^{[p^2-1]}(z), \quad p^{s-1} I_p^{[p-1]}(z),$$

are solutions of system (A.18) modulo p^s by Theorem 4.4.

Let $\Phi_{p^s}(x, z) = \sum_i \Phi_{p^s}^i(z) x^i$. Denote

(A.56)
$$\ell_{p^s}(z) = \Phi_{p^s}^{p^s-1}(z).$$

A.11.2. For $k = 1, 2, 3$ let

$$P_{k,p^s}(v, z) := P_{p^s}(v + z_k, z) = \sum_i P_{k,p^s}^i(z) v^i.$$

Denote $I_{k,p^s}^{[p^s-1]}(z) = P_{k,p^s}^{p^s-1}(z)$. The functions

$$I_{k,p^s}^{[p^s-1]}(z), \quad p I_{k,p^{s-1}}^{[p^{s-1}-1]}(z), \quad \dots, \quad p^{s-2} I_{k,p^2}^{[p^2-1]}(z), \quad p^{s-1} I_{k,p}^{[p-1]}(z),$$

are solutions of system (A.18) modulo p^s by Theorem 9.1.

Let $\Phi_{k,p^s}(v, z) := \Phi_{p^s}(v + z_k, z) = \sum_i \Phi_{k,p^s}^i(z) v^i$. Denote

(A.57)
$$\ell_{k,p^s}^{[p^s-1]}(z) = \Phi_{k,p^s}^{p^s-1}(z).$$

A.11.3. Recall the homomorphisms $\mathbb{Z} \rightarrow \mathbb{Z}/p^s\mathbb{Z}$, $\mathbb{Z}[z] \rightarrow (\mathbb{Z}/p^s\mathbb{Z})[z]$, $\mathbb{Z}[z]^3 \rightarrow$

$(\mathbb{Z}/p^s\mathbb{Z})[z]^3$ denoted by π_s . Recall the subring $\mathbb{Z}[z]_{p^r} \subset \mathbb{Z}[z]$ of quasi-constants modulo p^r .

A.11.4. Define the filtration (A.58)

$$0 = \mathcal{M}_{p^s}^0 \subset \mathcal{M}_{p^s}^1 \subset \dots \subset \mathcal{M}_{p^s}^{s-1} \subset \mathcal{M}_{p^s}^s = \mathcal{M}_{p^s},$$

$$\mathcal{M}_{p^s}^t = \{ \pi_s \left(\sum_{r=1}^t c_r(z) p^{s-r} I_{p^r}^{[p^r-1]}(z) \right) \mid c_r(z) \in \mathbb{Z}[z]_{p^r} \}, \quad t = 1, \dots, s$$

where. Every element of $\mathcal{M}_{p^s}$ is a polynomial solution of system (A.18) with coefficients in $\mathbb{Z}/p^s\mathbb{Z}$. Define the filtration

$$0 = \mathcal{M}_{k,p^s}^0 \subset \mathcal{M}_{k,p^s}^1 \subset \dots \subset \mathcal{M}_{k,p^s}^{s-1} \subset \mathcal{M}_{k,p^s}^s = \mathcal{M}_{k,p^s},$$

$$\mathcal{M}_{k,p^s}^t = \{ \pi_s \left(\sum_{r=1}^t c_r(z) p^{s-r} I_{k,p^r}^{[p^r-1]}(z) \right) \mid c_r(z) \in \mathbb{Z}[z]_{p^r} \}, \quad t = 1, \dots, s$$

(A.59)

where.

Every element of $\mathcal{M}_{p^s}$ is a polynomial solution of system (A.18) with coefficients in $\mathbb{Z}/p^s\mathbb{Z}$ by Theorem 9.1.

By Theorem 9.1 filtrations (A.58) and (A.59) coincide, $M_{p^s}^t = M_{k,p^s}^t$ for any k, t .

A.11.5. Define the filtration (A.60)

$$0 = \mathcal{L}_{p^s}^0 \subset \mathcal{L}_{p^s}^1 \subset \dots \subset \mathcal{L}_{p^s}^{s-1} \subset \mathcal{L}_{p^s}^s = \mathcal{L}_{p^s},$$

$$\mathcal{L}_{p^s}^t = \left\{ \pi_s \left(\sum_{r=1}^t c_r(z) p^{s-r} \ell_{p^r}^{[p^r-1]}(z) \right) \mid c_r(z) \in \mathbb{Z}[z]_{p^r} \right\}, \quad t = 1, \dots, s$$

where.

Define the filtration

$$0 = \mathcal{L}_{k,p^s}^0 \subset \mathcal{L}_{k,p^s}^1 \subset \dots \subset \mathcal{L}_{k,p^s}^{s-1} \subset \mathcal{L}_{k,p^s}^s = \mathcal{L}_{k,p^s},$$

$$\mathcal{L}_{k,p^s}^t = \left\{ \pi_s \left(\sum_{r=1}^t c_r(z) p^{s-r} \ell_{k,p^r}^{[p^r-1]}(z) \right) \mid c_r(z) \in \mathbb{Z}[z]_{p^r} \right\}, \quad t = 1, \dots, s$$

(A.61)

where.

It is easy to see that filtrations (A.60) and (A.61) coincide, $\mathcal{L}_{p^s}^t = \mathcal{L}_{k,p^s}^t$ for any k, t .

A.11.6. Let $u = (u_1, u_2, u_3)$ be the coordinates in the chart corresponding to $(1, 2, 3) \in S_3$, see (A.19). Consider the functions $I_{3,p^s}^{[p^s-1]}(z) \in \mathcal{M}_{p^s}$, $\ell_{3,p^s}^{[p^s-1]}(z) \in$

$\mathcal{L}_{p^s}$. Denote

$$(A.62) \quad \hat{I}^{[p^s-1]}(u) := I_{3,p^s}^{[p^s-1]}(z(u)), \quad \hat{\ell}^{[p^s-1]}(u) := \ell_{3,p^s}^{[p^s-1]}(z(u)).$$

We have

$$(A.63) \quad (-1)^{\frac{p^s-3}{2}} \hat{I}^{[p^s-1]} = u_1^{\frac{p^s-3}{2}} \sum_{a=0}^{\frac{p^s-1}{2}} \left(\binom{\frac{p^s-3}{2}}{a} \binom{\frac{p^s-1}{2}}{a}, \binom{\frac{p^s-1}{2}}{a+1} \binom{\frac{p^s-3}{2}}{a}, \binom{\frac{p^s-1}{2}}{a+1} \binom{\frac{p^s-1}{2}}{a} \right) u_2^a$$

$$(A.64) \quad (-1)^{\frac{p^s-1}{2}} \hat{\ell}^{[p^s-1]} = u_1^{\frac{p^s-1}{2}} \sum_{a=0}^{\frac{p^s-1}{2}} \binom{\frac{p^s-1}{2}}{a}^2 u_2^a,$$

see formula (A.63) and (10.5).

Notice that $\sum_{a=0}^{\frac{p^s-1}{2}} \binom{\frac{p^s-1}{2}}{a}^2 u_2^a$ is a solution of equation (A.2) modulo p^s .

As $s \rightarrow \infty$ the sequence $((-1)^{\frac{p^s-3}{2}} \hat{I}^{[p^s-1]})_{s=1}^\infty$ of vector-valued polynomials uniformly converges to the series I in (A.28) near the points where $u_2 = 0$, see

Theorem 10.5. Similarly as $s \rightarrow \infty$ the sequence $((-1)^{\frac{p^s-1}{2}} \hat{\ell}^{[p^s-1]})_{s=1}^\infty$ of scalar polynomials ℓ uniformly converges to the series in (A.37) near the points where $u_2 = 0$.

Corollary A.7. As $s \rightarrow \infty$ the sequence $(-\hat{I}^{[p^s-1]}/\hat{\ell}^{[p^s-1]})_{s=1}^\infty$ of vectorvalued rational functions uniformly converges to the function I in (A.40) near the points where $u_2 = 0$.

A.11.7. Let $\sigma = (i, j, k) \in S_3$. Similarly to Section A.11.6 consider the coordinates u^σ and the functions $I_{k,p^s}^{[p^s-1]}(z) \in \mathcal{M}_{p^s}$, $\ell_{k,p^s}^{[p^s-1]}(z) \in \mathcal{L}_{p^s}$. Denote

$$(A.65) \quad \hat{I}^{[\sigma, p^s-1]}(u^\sigma) := I_{k,p^s}^{[p^s-1]}(z(u^\sigma)), \quad \hat{\ell}^{[\sigma, p^s-1]}(u^\sigma) := \ell_{k,p^s}^{[p^s-1]}(z(u^\sigma)).$$

Similarly to Section A.11.6 we obtain the following corollary.

Corollary A.8. As $s \rightarrow \infty$ the sequence $\left(-\hat{I}^{[\sigma, p^s-1]} / \hat{\ell}^{[\sigma, p^s-1]}\right)_{s=1}^{\infty}$ of vectorvalued rational functions uniformly converges to the function I^{σ} in (A.40) near the points where $u_2^{\sigma} = 0$.

A.11.8. This Appendix A is devoted to the relation between the analytic element $(I^{ijk})_{(i,j,k) \in S_3}$ and Dwork's theory in [Dw].

As additional information, for any $\sigma = (i, j, k) \in S_3$, Corollary A.8 indicates the sequence of polynomials $I_{k, p^s}^{[p-1]}(z) \in \mathcal{M}_{p^s}$, $\ell_{k, p^s}^{[p-1]}(z) \in \mathcal{L}_{p^s}$ whose ratio padically tends to the function I^{ijk} near the points $u_2^{\sigma} = 0$ where the function I^{ijk} is initially defined.

A.12. Further directions. In Sections 2–10 we considered system (2.1) of KZ equations with parameter $n = 2g + 1$ and constructed polynomial solutions of system (2.1) modulo p^s . We defined the module M_{p^s} of the constructed solutions and studied the limit of M_{p^s} as $s \rightarrow \infty$. Namely we considered a special coordinate system $u = u(z)$ in (9.7) associated with one of the asymptotic zones of the KZ equations and showed that in this coordinate system the limit of M_{p^s} as $s \rightarrow \infty$ produces a g -dimensional space of solutions of system (2.1) over p -adic numbers $\mathbb{Q}_p$ in the neighborhood of the point $u = 0$.

Constructions in this appendix for $g = 1$ and Dwork's theory in [Dw] suggest the following project. Consider all asymptotic zones of system (2.1), see their definition for example in [V2]. The asymptotic zones are labeled by suitable trees T . These trees are analogs of the elements $\sigma \in S_3$ in the appendix. Each asymptotic zone has a distinguished system of coordinates u^T . Probably, for every asymptotic zone the limit of M_{p^s} as $s \rightarrow \infty$ produces a g -dimensional space V_T of solutions of system (2.1) considered over $\mathbb{Q}_p$ in a neighborhood of the point $u^T = 0$. Probably the spaces V_T of p -adic solutions, defined at different places $u^T = 0$, analytically continue into a single global invariant g -dimensional vector subbundle of the associated KZ connection on the trivial vector bundle of rank $2g$. Following Dwork and Theorem A.4 we may expect that this subbundle is spanned at any point of the base by the germs of all solutions of the KZ equations bounded in their polydiscs of convergence. This subbundle would give a generalization of the line subbundle generated by the analytic element $(I^{ijk})_{(i,j,k) \in S_3}$ constructed in this appendix. Probably, that g -dimensional subbundle will determine the set of unit roots of the curves with equation $y^2 = \prod_{i=1}^n (x - z_i)$ over the field $\mathbb{F}_p$ similarly to how it is done in Sections A.9 and A.10 for the elliptic curves.

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