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THE NAKAYAMA FUNCTOR AND ITS COMPLETION FOR GORENSTEIN ALGEBRAS

by Srikanth B. Iyengar & Henning Krause

To Bill Crawley-Boevey on his 60th birthday.

Abstract. — Duality properties are studied for a Gorenstein algebra that is finite and projective over its center. Using the homotopy category of injective modules, it is proved that there is a local duality theorem for the subcategory of acyclic complexes of such an algebra, akin to the local duality theorems of Grothendieck and Serre in the context of commutative algebra and algebraic geometry. A key ingredient is the Nakayama functor on the bounded derived category of a Gorenstein algebra and its extension to the full homotopy category of injective modules.

Résumé (*Le foncteur de Nakayama et sa complétion pour les algèbres de Gorenstein*). — Des propriétés de dualité sont étudiées pour une algèbre de Gorenstein finie et projective sur son centre. En utilisant la catégorie homotopique des modules injectifs, il est démontré qu'il existe un théorème de dualité locale pour la sous-catégorie des objets acycliques d'une telle algèbre, semblable aux théorèmes de dualité locale de Grothendieck et Serre dans le cadre de l'algèbre commutative et de la géométrie algébrique. Un ingrédient clé est le foncteur de Nakayama sur la catégorie dérivée bornée d'une algèbre de Gorenstein, et son extension à toute la catégorie homotopique des modules injectifs.

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1. Introduction

This work is a contribution to the representation theory of Gorenstein algebras, both commutative and noncommutative, with a focus on duality phenomena. The notion of a Gorenstein variety was introduced by Grothendieck [26, 25, 29, 30] and grew out of his reinterpretation and extension of Serre duality [43] for projective varieties. A local version of his duality is that over a Cohen–Macaulay local algebra R of dimension d, with maximal ideal $\mathfrak{m}$, and for complexes F, G with F perfect, there are natural isomorphisms

$$\operatorname{Hom}_R(\operatorname{Ext}^i_R(F,G), I(\mathfrak{m})) \cong \operatorname{Ext}^{d-i}_R(G, R\Gamma_{\mathfrak{m}}(\omega_R \otimes^{\mathrm{L}}_R F)),$$

where ω_R is a dualizing module, and $I(\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$. The functor $R\Gamma_{\mathfrak{m}}$ represents local cohomology at $\mathfrak{m}$. Serre duality concerns the case where R is the local ring at the vertex of the affine cone of a projective variety. The ring R (equivalently, the variety it represents) is said to be Gorenstein if, in addition, the R-module ω_R is projective. Serre observed that this property is characterized by R having a finite self-injective dimension. This result appears in the work of Bass [4], who gave numerous other characterizations of Gorenstein rings.

Iwanaga [31] launched the study of Noetherian rings, not necessarily commutative, having finite self-injective dimension on both sides. Now known as Iwanaga–Gorenstein rings, these form an integral part of the representation theory of algebras. In that domain, the principal objects of interest are maximal Cohen–Macaulay modules and the associated stable category. Auslander [1] and Buchweitz [13] have proved duality theorems for the stable category of a Gorenstein algebra with *isolated* singularities. The driving force behind our work was to understand what duality phenomena can be observed for general Gorenstein algebras. Theorem 1.2 below is what we found, following Grothendieck's footsteps.

We set the stage to present that result and begin with a crucial definition.

Definition 1.1. — Let R be a commutative Noetherian ring. An R-algebra A is called *Gorenstein* if

(1) the R-module A is finitely generated and projective, and
(2) for each $\mathfrak{p}$ in $\operatorname{Spec} R$ with $A_{\mathfrak{p}} \neq 0$ the ring $A_{\mathfrak{p}}$ has finite injective dimension as a module over itself, on the left and on the right.

A Gorenstein R-algebra A itself need not be Iwanaga–Gorenstein. Indeed, for A commutative and Gorenstein, the injective dimension of A is finite precisely when its Krull dimension is finite, and there exist rings locally of finite injective dimension but of infinite Krull dimension. There are precedents to the study of Gorenstein algebras, starting with [4] and more recently in the work of Goto and Nishida [24]. Our work differs from theirs in its focus on

duality. We refer to [22] for a discussion of examples and natural constructions preserving the Gorenstein property.

Let A be a Gorenstein R-algebra and $\omega_{A/R} := \operatorname{Hom}_R(A, R)$ the dualizing *bimodule*. Unlike in the commutative case, $\omega_{A/R}$ does not need to be projective (neither on the left nor on the right), and the bimodule structure can be complicated. Nevertheless, it is a tilting object in $\mathbf{D}(\operatorname{Mod} A)$, the derived category of A-modules, inducing a triangle equivalence

$$\operatorname{RHom}_A(\omega_{A/R}, -)\colon \mathbf{D}(\operatorname{Mod} A) \xrightarrow{\sim} \mathbf{D}(\operatorname{Mod} A)\,;$$

see Section 4. The representation theory of a Gorenstein algebra A is governed by its maximal Cohen–Macaulay modules, namely, finitely generated A-modules M with $\operatorname{Ext}^i_A(M, A) = 0$ for $i \geq 1$. For our purposes, their infinitely generated counterparts are also important. Thus, we consider Gorenstein projective A-modules (abbreviated to G-projective), which are by definition A-modules occurring as syzygies in acyclic complexes of projective A-modules [13, 19]. The G-projective modules form a Frobenius exact category, and so the corresponding stable category, is triangulated. Its inclusion into the usual stable module category has a right adjoint, the Gorenstein approximation functor, $\mathrm{GP}(-)$. The functor

$$S := \mathrm{GP}(\omega_{A/R} \otimes_A -)\colon \underline{\mathrm{GProj}}A \longrightarrow \underline{\mathrm{GProj}}A$$

is an equivalence of triangulated categories and plays the role of a Serre functor on the subcategory of finitely generated G-projectives. This is spelled out in the result below. Here, the $\widehat{\operatorname{Ext}}^i_A(-,-)$ are the Tate cohomology modules, which compute morphisms in $\underline{\mathrm{GProj}}A$.

THEOREM 1.2. — *Let A be a Gorenstein R-algebra and let M, N be G-projective A-modules with M finitely generated. For each $\mathfrak{p} \in \operatorname{Spec} R$, there is a natural isomorphism*

$$\operatorname{Hom}_R(\widehat{\operatorname{Ext}}^i_A(M, N), I(\mathfrak{p})) \cong \widehat{\operatorname{Ext}}^{d(\mathfrak{p})-i}_A(N, \varGamma_{\mathfrak{p}} S(M))\,,$$

where $d(\mathfrak{p}) = \dim(R_{\mathfrak{p}}) - 1$.

This is the duality theorem we seek; it is proved in Section 9. It is new even for commutative rings. The parallel to Grothendieck's duality theorem is clear.

In the following, we explain the strategy for proving this theorem and some essential ingredients. The functor $\varGamma_{\mathfrak{p}}$ is analogous to the local cohomology functor encountered above. It is constructed in Section 7 following the recipe in [7], using the natural R-action on $\underline{\mathrm{GProj}}A$. Even if N is finitely generated, $\varGamma_{\mathfrak{p}}(N)$ need not be, which is one reason we have to work with infinitely generated modules in the first place. If R is local with maximal ideal $\mathfrak{p}$, and A has isolated singularities, $\varGamma_{\mathfrak{p}}$ is the identity, and the duality statement above is precisely the one discovered by Auslander and Buchweitz.

For a Gorenstein algebra, the stable category of G-projective modules is equivalent to $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$, the homotopy category of acyclic complexes of injective A-modules. This connection is explained in Section 6 and builds on the results from [33, 35]. In fact, much of the work that goes into proving Theorem 1.2 deals with $\mathbf{K}(\operatorname{Inj} A)$, the full homotopy category of injective A-modules; see Section 2. A key ingredient in all this is the Nakayama functor on the category of A-modules:

$$\mathbf{N}\colon \operatorname{Mod} A \longrightarrow \operatorname{Mod} A \quad \text{where} \quad \mathbf{N}(M) = \operatorname{Hom}_A(\omega_{A/R}, M).$$

As noted above, its derived functor induces an equivalence on $\mathbf{D}(\operatorname{Mod} A)$. Following [35] we extend the Nakayama functor to all of $\mathbf{K}(\operatorname{Inj} A)$, which one may think of as a triangulated analogue of the ind-completion of $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$. This *completion* of the Nakayama functor is also an equivalence:

$$\widehat{\mathbf{N}}_{A/R}\colon \mathbf{K}(\operatorname{Inj} A) \xrightarrow{\sim} \mathbf{K}(\operatorname{Inj} A)\,.$$

This is proved in Section 5, where we establish also that it restricts to an equivalence on $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$. The induced equivalence on the stable category of G-projective modules is precisely the functor S in the statement of Theorem 1.2; see Section 6 where the singularity category of A, in the sense of Buchweitz [13] and Orlov [42] also appears. To make this identification, we need to extend results of Auslander and Buchweitz concerning G-approximations; this is dealt with in Appendix A.

Our debt to Grothendieck is evident. It ought to be clear by now that the work of Auslander and Buchweitz also provides much inspiration for this paper. Whatever new insight we bring is through the systematic use of the homotopy category of injective modules and methods from abstract homotopy theory, especially the Brown representability theorem. To that end we need the structure theory of injectives over finite R-algebras from Gabriel's thesis [20]. Gabriel also introduced the Nakayama functor in representation theory of Artin algebra in his exposition of Auslander–Reiten duality; it is the categorical analogue of the Nakayama automorphism that permutes the isomorphism classes of simple modules over a self-injective algebra [21]. Moreover, it was Gabriel who pointed out the parallel between derived equivalences induced by tilting modules and the duality of Grothendieck and Roos [34].

2. Homotopy category of injectives

In this section, we describe certain functors on homotopy categories attached to Noetherian rings. Our basic references for this material are [32, 35].

Throughout, A will be a ring that is Noetherian on both sides; that is to say, A is Noetherian as a left and as a right A-module. In what follows, A-modules will mean left A-modules, and A^{op}-modules are identified with right A-modules. We write $\operatorname{Mod} A$ for the (abelian) category of A-modules and $\operatorname{mod} A$ for its full

subcategory consisting of finitely generated modules. Also, $\operatorname{Inj} A$ and $\operatorname{Proj} A$ are the full subcategories of $\operatorname{Mod} A$ consisting of injective and projective modules, respectively.

For any additive category $\mathcal{A} \subseteq \operatorname{Mod} A$, like the ones in the last paragraph, $\mathbf{K}(\mathcal{A})$ will denote the associated homotopy category, with its natural structure as a triangulated category. Morphisms in this category are denoted $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(-,-)$. An object X in $\mathbf{K}(\mathcal{A})$ is *acyclic* if $H^*(X) = 0$, and the full subcategory of acyclic objects in $\mathbf{K}(\mathcal{A})$ is denoted $\mathbf{K}_{\mathrm{ac}}(\mathcal{A})$. A complex $X \in \mathbf{K}(\mathcal{A})$ is said to be *bounded above* if $X^i = 0$ for $i \gg 0$, and *bounded below* if $X^i = 0$ for $i \ll 0$.

In the sequel our focus in mostly on $\mathbf{K}(\operatorname{Inj} A)$, the homotopy category of injective modules, and its various subcategories; the analogous categories of projectives play a more subsidiary role. From work in [33, 35, 41], we know that the triangulated categories $\mathbf{K}(\operatorname{Inj} A)$ and $\mathbf{K}(\operatorname{Proj} A)$ are compactly generated since the ring A is Noetherian on both sides; the compact objects in these categories are described further below. Let $\mathbf{D}(\operatorname{Mod} A)$ denote the (full) derived category of A-modules and $\mathbf{q}\colon \mathbf{K}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ the localization functor; its kernel is $\mathbf{K}_{\mathrm{ac}}(\operatorname{Mod} A)$. We write $\mathbf{q}$ also for its restriction to the homotopy categories of injectives and projectives. These functors have adjoints:

$$\mathbf{K}(\operatorname{Inj} A) \underset{\mathbf{i}}{\overset{\mathbf{q}}{\rightleftarrows}} \mathbf{D}(\operatorname{Mod} A) \qquad \text{and} \qquad \mathbf{K}(\operatorname{Proj} A) \underset{\mathbf{q}}{\overset{\mathbf{p}}{\rightleftarrows}} \mathbf{D}(\operatorname{Mod} A)\,.$$

Our convention is to write the left adjoint above the corresponding right one. In what follows, it is convenient to conflate $\mathbf{i}$ and $\mathbf{p}$ with $\mathbf{i}\circ\mathbf{q}$ and $\mathbf{p}\circ\mathbf{q}$, respectively. The images of $\mathbf{i}$ and $\mathbf{p}$ are the K-injectives and K-projectives, respectively. Recall that an object X in $\mathbf{K}(\operatorname{Inj} A)$ is *K-injective* if $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X) = 0$ for any acyclic complex W in $\mathbf{K}(\operatorname{Mod} A)$. We write $\mathbf{K}_{\mathrm{inj}}(A)$ for the full subcategory of $\mathbf{K}(\operatorname{Inj} A)$ consisting of K-injective complexes. The subcategory $\mathbf{K}_{\mathrm{proj}}(A) \subseteq \mathbf{K}(\operatorname{Proj} A)$ of K-projective complexes is defined similarly.

Compact objects. — Since A is Noetherian $\operatorname{Inj} A$ is closed under arbitrary direct sums, and hence so is the subcategory $\mathbf{K}(\operatorname{Inj} A)$ of $\mathbf{K}(\operatorname{Mod} A)$. As in any triangulated category with arbitrary direct sums, an object X in $\mathbf{K}(\operatorname{Inj} A)$ is *compact* if $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X, -)$ commutes with direct sums. The compact objects in $\mathbf{K}(\operatorname{Inj} A)$ form a thick subcategory, denoted $\mathbf{K}^{\mathrm{c}}(\operatorname{Inj} A)$. The adjoint pair $(\mathbf{q}, \mathbf{i})$ above restricts to an equivalence of triangulated categories

$$\mathbf{K}^{\mathrm{c}}(\operatorname{Inj} A) \underset{\mathbf{i}}{\overset{\mathbf{q}}{\rightleftarrows}} \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)\,,$$

where $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ denotes the bounded derived category of $\operatorname{mod} A$; see [35, Proposition 2.3] for a proof of this assertion. The corresponding identification

of the compact objects in $\mathbf{K}(\operatorname{Proj} A)$ is a bit more involved and is due to Jørgensen [33, Theorem 3.2]. The assignment $M \mapsto \operatorname{Hom}_{A^{\mathrm{op}}}(\mathbf{p}M, A)$ induces an equivalence

$$\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{K}^{\mathrm{c}}(\operatorname{Proj} A)\,.$$

See also [32], where these two equivalences are related. The formula below for computing morphisms from compacts in $\mathbf{K}(\operatorname{Inj} A)$ is useful in the sequel.

LEMMA 2.1. — *For $C, X \in \mathbf{K}(\operatorname{Inj} A)$ with C compact, there is a natural isomorphism*

$$\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C, X) \cong H^0(\operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A X)\,.$$

Proof. — Since C is compact its K-projective resolution $\mathbf{p}C$ is homotopy equivalent to a complex that is bounded above and consists of finitely generated projective A-modules. For each integer n, let $X(n)$ be the subcomplex $X^{\geqslant -n}$ of X. Since $X(n)$ is K-injective, the quasi-isomorphism $\mathbf{p}C \to C$ induces the one on the left

$$\operatorname{Hom}_A(C, X(n)) \xrightarrow{\sim} \operatorname{Hom}_A(\mathbf{p}C, X(n)) \xleftarrow{\sim} \operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A X(n)\,.$$

The one on the right is the standard one and holds because of the aforementioned properties of $\mathbf{p}C$ and the fact that $X(n)$ is bounded below. One thus gets a canonical isomorphism

$$\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C, X(n)) \xrightarrow{\sim} H^0(\operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A X(n))\,.$$

It is compatible with the inclusions $X(n) \subseteq X(n+1)$, so induces the isomorphism in the bottom row of the following diagram.

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C, \operatorname{hocolim}_{n\geqslant 0} X(n)) & \xrightarrow{\sim} & H^0(\operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A \operatorname{hocolim}_{n\geqslant 0} X(n)) \\ \downarrow\wr & & \downarrow\wr \\ \operatorname{colim}_{n\geqslant 0} \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C, X(n)) & \xrightarrow{\sim} & \operatorname{colim}_{n\geqslant 0} H^0(\operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A X(n))\,. \end{array}$$

The isomorphism on the left holds by the compactness of C, while the one on the right holds because $H^0(-)$ commutes with homotopy colimits. It remains to note that $\operatorname{hocolim}_{n\geqslant 0} X(n) = X$ in $\mathbf{K}(\operatorname{Inj} A)$. □

A recollement. — The functors $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A) \xrightarrow{\mathrm{incl}} \mathbf{K}(\operatorname{Inj} A) \xrightarrow{\mathbf{q}} \mathbf{D}(\operatorname{Mod} A)$ induce a recollement of triangulated categories

$$\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A) \underset{\mathbf{r}}{\overset{\mathbf{s}}{\overleftarrow{\overleftarrow{}}}} \mathbf{K}(\operatorname{Inj} A) \underset{\mathbf{i}}{\overset{\mathbf{j}}{\overleftarrow{\overleftarrow{}}}} \mathbf{D}(\operatorname{Mod} A)\,. \tag{1}$$

The functor $\mathbf{i}$ is the one discussed above; it embeds $\mathbf{D}(\operatorname{Mod} A)$ as the homotopy category of K-injective complexes. The functor $\mathbf{r}$ thus has a simple description: there is an exact triangle

$$\mathbf{r}X \longrightarrow X \longrightarrow \mathbf{i}X \longrightarrow, \tag{2}$$

where the morphism $X \to \mathbf{i}X$ is the canonical one. Indeed, $\mathbf{r}X$ is evidently acyclic, and if W is in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$, the induced map $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, \mathbf{r}X) \to \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X)$ is an isomorphism, for one has $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, \mathbf{i}X) = 0$.

The functor $\mathbf{j}\colon \mathbf{D}(\operatorname{Mod} A) \to \mathbf{K}(\operatorname{Inj} A)$ is fully faithful. The image of $\mathbf{j}$ equals the kernel of $\mathbf{s}$ and identifies with $\operatorname{Loc}(\mathbf{i}A)$, the localizing subcategory of $\mathbf{K}(\operatorname{Inj} A)$ generated by the injective resolution of A; see [35, Theorem 4.2]. One may think of $\mathbf{j}$ as the injective version of taking projective resolutions; see Lemma 2.5. To justify this claim takes preparation.

LEMMA 2.2. — *Restricted to the subcategory* $\operatorname{Loc}(\mathbf{i}A)$ *of* $\mathbf{K}(\operatorname{Inj} A)$ *there is a natural isomorphism of functors* $\mathbf{r} \xrightarrow{\sim} \Sigma^{-1}\mathbf{si}$.

Proof. — Consider anew the exact triangle (2), but for X in $\operatorname{Loc}(\mathbf{i}A)$:

$$\mathbf{r}X \longrightarrow X \longrightarrow \mathbf{i}X \longrightarrow \Sigma\mathbf{r}X\,.$$

Apply $\mathbf{s}$ and remember that its kernel is $\operatorname{Loc}(\mathbf{i}A)$. □

Projective algebras. — In the remainder of this section, we assume that the ring A (which hitherto has been Noetherian on both sides) is also projective, as a module, over some central subring R. For the moment, the only role R plays is to allow for constructions of bimodule resolutions with good properties. Set $A^{\mathrm{ev}} := A \otimes_R A^{\mathrm{op}}$, the enveloping algebra of the R-algebra A, and set

$$E := \mathbf{i}_{A^{\mathrm{ev}}} A\,.$$

This is an injective resolution of A as a (left) module over A^{ev}. Since E is a complex of A-bimodules, for any complex X of A-modules, the right action of A on E induces a left A-action on $\operatorname{Hom}_A(E, X)$. The structure map $A \to E$ of bimodules induces a morphism of A-complexes

$$\operatorname{Hom}_A(E, X) \longrightarrow \operatorname{Hom}_A(A, X) \cong X \quad \text{for } X \in \mathbf{K}(\operatorname{Mod} A). \tag{3}$$

The computation below will be used often:

LEMMA 2.3. — *The morphism in* (3) *is a quasi-isomorphism for* $X \in \mathbf{K}(\operatorname{Inj} A)$.

Proof. — By considering the mapping cone of $A \to E$, the desired statement reduces to: For any complex $W \in \mathbf{K}(\operatorname{Mod} A)$ that is acyclic and satisfies $W^i = 0$ for $i \ll 0$, one has $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X) = 0$. Without loss of generality we can assume $W^i = 0$ for $i < 0$. Then one gets the first equality below

$$\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X) = \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X^{\geqslant -1}) = 0,$$

and the second one holds because $X^{\geqslant -1}$ is K-injective. □

Since A is projective as an R-module, A^{ev} is projective as an A-module both on the left and on the right. The latter condition implies, by adjunction, that as a complex of left A-modules E consists of injectives. In particular, for any projective A-module P, the A-complex $E \otimes_A P$ consists of injective modules. Thus, one has an exact functor

$$E \otimes_A - \colon \mathbf{K}(\operatorname{Proj} A) \longrightarrow \mathbf{K}(\operatorname{Inj} A)\,.$$

For each X in $\mathbf{K}(\operatorname{Inj} A)$, one has isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(E \otimes_A \mathbf{p}X, X) &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{p}X, \operatorname{Hom}_A(E, X)) \\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{p}X, X)\,.\end{aligned}$$

The second isomorphism is a consequence of Lemma 2.3 and the K-projectivity of $\mathbf{p}X$. Thus, corresponding to the morphism $\mathbf{p}X \to X$, there is natural morphism

$$\pi(X) \colon E \otimes_A \mathbf{p}X \longrightarrow X \tag{4}$$

of complexes of A-modules.

Lemma 2.4. — *The morphism $\pi(X)$ in* (4) *is a quasi-isomorphism for each X.*

Proof. — Let $\eta \colon A \to E$ and $\varepsilon \colon \mathbf{p}X \to X$ denote the structure maps. These fit in the commutative diagram

$$\begin{array}{ccc} A \otimes_A \mathbf{p}X & \xrightarrow{\sim} & \mathbf{p}X \\ {\scriptstyle \eta \otimes_A \mathbf{p}X}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ E \otimes_A \mathbf{p}X & \xrightarrow{\pi(X)} & X\,. \end{array}$$

The map $\eta \otimes_A \mathbf{p}X$ is a quasi-isomorphism as η is one and $\mathbf{p}X$ is K-projective. Thus, $\pi(X)$ is a quasi-isomorphism. □

The stabilization functor. — The functor $\mathbf{s} \colon \mathbf{K}(\operatorname{Inj} A) \to \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ from (1) admits the following description in terms of its kernel, which uses the natural transformation $\pi \colon E \otimes_A \mathbf{p}(-) \to \mathrm{id}$ of functors on $\mathbf{K}(\operatorname{Inj} A)$ from (4).

Lemma 2.5. — *Each object X in $\mathbf{K}(\operatorname{Inj} A)$ fits into an exact triangle*

$$E \otimes_A \mathbf{p}X \xrightarrow{\pi(X)} X \longrightarrow \mathbf{s}X \longrightarrow,$$

and this yields a natural isomorphism $E \otimes_A \mathbf{p}X \xrightarrow{\sim} \mathbf{j}X$.

Proof. — Since $\pi(X)$ is a quasi-isomorphism, by Lemma 2.4, the complex $\mathbf{s}X$ is acyclic. In $\mathbf{K}(\operatorname{Proj} A)$, the complex $\mathbf{p}X$ is in $\operatorname{Loc}(A)$, and hence in $\mathbf{K}(\operatorname{Inj} A)$, the complex $E \otimes_A \mathbf{p}X$ is in $\operatorname{Loc}(E)$. It remains to observe that if $W \in \mathbf{K}(\operatorname{Inj} A)$ is acyclic, then $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(E, W) = 0$ by Lemma 2.3. □

3. The Nakayama functor and its completion

The Nakayama functor is a standard tool in representation theory of Artin algebras. For instance, the functor interchanges projective and injective modules, thereby providing an efficient method to compute the Auslander–Reiten translate of a finitely generated module [21]. In this section, we discuss the extension of the Nakayama functor from modules to the homotopy category of injectives.

Throughout the rest of this work, we say that a ring A is a *finite R-algebra* if

(1) R is a commutative Noetherian ring;
(2) A is an R-algebra, that is to say, there is a map of rings $R \to A$ whose image is in the center of A;
(3) A is finitely generated as an R-module.

These conditions imply that A is a Noetherian ring, finitely generated as a module over its center, which is thus also Noetherian. Hence, A is a finite algebra over its center. When A is a finite R-algebra, so is the opposite ring A^{op}.

Let A be a finite R-algebra. Following Buchweitz [13, §7.6], which in turn is inspired by the terminology in commutative algebra, we call the A-bimodule

$$\omega_{A/R} := \operatorname{Hom}_R(A, R)$$

the *dualizing bimodule* of the R-algebra A. It is finitely generated as an A-module, on either side. Extending the terminology from the context of finite dimensional algebras over fields we call

$$\mathbf{N}_{A/R} := \operatorname{Hom}_A(\omega_{A/R}, -)\colon \operatorname{Mod} A \longrightarrow \operatorname{Mod} A \tag{5}$$

the *Nakayama functor* of the R-algebra A. Sometimes, this name is used for the functor $\omega_{A/R} \otimes_A -$, which is left adjoint to $\mathbf{N}_{A/R}$, but in this work, the one above plays a more central role, hence our choice of nomenclature. When the algebra in question is clear, we drop the "A/R" from subscripts, to write ω and $\mathbf{N}$. In our applications, A will be projective as an R-module. Then the left adjoint of $\mathbf{N}_{A/R}$ is a Nakayama functor relative to the restriction $\operatorname{Mod} A \to \operatorname{Mod} R$ in the sense of Kvamme [37].

The Nakayama functor can be extended to $\mathbf{D}(\operatorname{Mod} A)$, yielding the *derived Nakayama functor*

$$\mathrm{R}\operatorname{Hom}_A(\omega, -)\colon \mathbf{D}(\operatorname{Mod} A) \longrightarrow \mathbf{D}(\operatorname{Mod} A)\,.$$

This functor and its left adjoint has been considered by several authors; see [27]. Here, we study the extension to $\mathbf{K}(\operatorname{Inj} A)$, following [35, §6].

The Nakayama functor is evidently additive and, therefore, admits an extension to $\mathbf{K}(\operatorname{Inj} A)$ as follows. Extend $\mathbf{N}$ to $\mathbf{K}(\operatorname{Mod} A)$, by applying it term-wise; denote this functor also $\mathbf{N}$. Brown representability yields a left adjoint to the

inclusion $\mathbf{K}(\operatorname{Inj} A) \hookrightarrow \mathbf{K}(\operatorname{Mod} A)$, say λ. Set

$$\widehat{\mathbf{N}}_{A/R} \colon \mathbf{K}(\operatorname{Inj} A) \longrightarrow \mathbf{K}(\operatorname{Inj} A) \tag{6}$$

to be the composite of functors

$$\mathbf{K}(\operatorname{Inj} A) \hookrightarrow \mathbf{K}(\operatorname{Mod} A) \xrightarrow{\mathbf{N}} \mathbf{K}(\operatorname{Mod} A) \xrightarrow{\lambda} \mathbf{K}(\operatorname{Inj} A)\,.$$

Our notation is motivated by the fact that $\mathbf{K}(\operatorname{Inj} A)$ can be viewed as a completion of $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, as is explained in [35, §2]. The next result is another reason for this choice. Here, $\mathbf{K}^{+}(\operatorname{Inj} A)$ denotes the full subcategory of $\mathbf{K}(\operatorname{Inj} A)$ consisting of complexes W that are bounded below. Note that $\mathbf{K}^{+}(\operatorname{Inj} A) \xrightarrow{\sim} \mathbf{D}^{+}(\operatorname{Mod} A)$.

LEMMA 3.1. — *On the subcategory $\mathbf{K}^{+}(\operatorname{Inj} A)$, there is an isomorphism of functors*

$$\widehat{\mathbf{N}}_{A/R} \xrightarrow{\sim} \mathbf{i} \operatorname{Hom}_A(\omega_{A/R}, -)\,.$$

making the following diagram commutative:

$$\begin{array}{ccccc}
\operatorname{Mod} A & \xrightarrowtail{\text{incl}} & \mathbf{D}^{+}(\operatorname{Mod} A) & \xrightarrowtail{\mathbf{i}} & \mathbf{K}(\operatorname{Inj} A) \\
\downarrow{\scriptstyle \mathbf{N}} & & \downarrow{\scriptstyle \operatorname{RHom}_A(\omega,-)} & & \downarrow{\scriptstyle \widehat{\mathbf{N}}} \\
\operatorname{Mod} A & \xrightarrowtail{\text{incl}} & \mathbf{D}(\operatorname{Mod} A) & \xrightarrowtail{\mathbf{i}} & \mathbf{K}(\operatorname{Inj} A)\,.
\end{array}$$

The functor $\widehat{\mathbf{N}}_{A/R} \colon \mathbf{K}(\operatorname{Inj} A) \to \mathbf{K}(\operatorname{Inj} A)$ preserves arbitrary direct sums and on compact objects $\widehat{\mathbf{N}}$ identifies with the functor

$$\operatorname{RHom}_A(\omega_{A/R}, -) \colon \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A) \longrightarrow \mathbf{D}(\operatorname{Mod} A)\,.$$

In general, the above square on the right will not be commutative, if one replaces $\mathbf{D}^{+}(\operatorname{Mod} A)$ by $\mathbf{D}(\operatorname{Mod} A)$; compare Theorem 5.1. We examine these functors in greater detail in the next section.

Proof. — Fix $X \in \mathbf{K}^{+}(\operatorname{Mod} A)$. The key observation is the following.

CLAIM. — *$\lambda X \xrightarrow{\sim} \mathbf{i}X$, the K-injective resolution of X.*

Indeed, since X is bounded below one can assume that so is $\mathbf{i}X$, and hence also the mapping cone, say Z, of the morphism $X \to \mathbf{i}X$. Since Z is also acyclic, arguing as in the proof of Lemma 2.3 one gets that $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(Z, Y) = 0$, for any $Y \in \mathbf{K}(\operatorname{Inj} A)$. Thus, the morphism $X \to \mathbf{i}X$ induces an isomorphism

$$\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{i}X, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X, Y)\,,$$

and this justifies the claim.

When X is bounded below, so is $\operatorname{Hom}_A(\omega, X)$. Thus, the claim above yields

$$\widehat{\mathbf{N}}(X) = \lambda \operatorname{Hom}_A(\omega, X) \cong \mathbf{i} \operatorname{Hom}_A(\omega, X)\,.$$

Now fix $X \in \mathbf{D}^+(\operatorname{Mod} A)$. Again, one can assume $\mathbf{i}X$ is also bounded below, and, therefore,

$$\widehat{\mathbf{N}}(\mathbf{i}X) = \lambda \operatorname{Hom}_A(\omega, \mathbf{i}X) \cong \mathbf{i} \operatorname{Hom}_A(\omega, \mathbf{i}X) = \mathbf{i} \operatorname{RHom}_A(\omega, X)\,.$$

This yields the commutativity of the right-hand square.

For the second part of the lemma, it remains to note that the functor $\mathbf{N}$ preserves direct sums, as the A-module ω is finitely generated, and λ preserves direct sums, as it is a left adjoint. □

4. Gorenstein algebras and their derived categories

In this section, we introduce Gorenstein algebras and characterize them in terms of the derived Nakayama functor. This generalizes a well-known fact for Artin algebras. In that case, the algebra is Gorenstein if and only if the dualizing module is a tilting module, so that the derived Nakayama functor is an equivalence.

Commutative Gorenstein rings. — A commutative Noetherian ring R is *Gorenstein* if for each prime (equivalently, maximal) ideal $\mathfrak{p}$, the local ring $R_\mathfrak{p}$ has finite injective dimension as a module over itself [4]. When the Krull dimension of R is finite, this condition is equivalent to R itself having finite injective dimension; see [4, Theorem, §1] for details.

Gorenstein algebras. — We say that a ring A is a *Gorenstein* R-algebra if

(1) A is a finite R-algebra;
(2) A is projective as an R-module;
(3) $A_\mathfrak{p}$ is Iwanaga–Gorenstein for each $\mathfrak{p} \in \operatorname{Spec} R$ with $A_\mathfrak{p} \neq 0$.

Condition (3) means $A_\mathfrak{p}$ has finite injective dimension as a module over itself, on the left and on the right; then the injective dimensions coincide; see [45, Lemma A].

The following lemma provides a comparison between A and R with respect to the Gorenstein property.

LEMMA 4.1. — *Let A be a Gorenstein R-algebra and $\mathfrak{p} \in \operatorname{Spec} R$. Then the ring $R_\mathfrak{p}$ is Gorenstein whenever $A_\mathfrak{p} \neq 0$.*

Proof. — As the R-module A is projective so is the $R_\mathfrak{p}$-module $A_\mathfrak{p}$, and hence for each finitely generated $R_\mathfrak{p}$-module M one has the isomorphism below

$$\operatorname{Ext}^i_{R_\mathfrak{p}}(M, R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} A_\mathfrak{p} \cong \operatorname{Ext}^i_{A_\mathfrak{p}}(M \otimes_{R_\mathfrak{p}} A_\mathfrak{p}, A_\mathfrak{p}) = 0 \qquad \text{for } i \gg 0.$$

The equality on the right holds because the injective dimension of $A_\mathfrak{p}$ is finite. We deduce from the computation above that $\operatorname{Ext}^i_{R_\mathfrak{p}}(M, R_\mathfrak{p}) = 0$ for $i \gg 0$, since $A_\mathfrak{p} \neq 0$. Hence, $R_\mathfrak{p}$ is Gorenstein; see [12, Proposition 3.1.14]. □

Let A be a finite R-algebra that is projective as an R-module. Then R admits a decomposition $R' \times R''$ such that $A_{\mathfrak{p}} \neq 0$ for all $\mathfrak{p} \in \operatorname{Spec} R'$, and A is finitely generated over R'. Thus, one may assume that A is faithful as an R-module, and then the Gorenstein property for A implies that R is Gorenstein; see [22] for details.

The preceding result also has a converse, but this plays no role in the sequel, so we discuss this at the end of this section; see Theorem 4.6. The Gorenstein condition is reflected also in the dualizing bimodule of the R-algebra A. To discuss this, we recall some aspects of perfect complexes over finite algebras.

Let A be a finite R-algebra and M a complex of A-modules. Recall that M is *perfect* if it is isomorphic in $\mathbf{D}(\operatorname{Mod} A)$ to a bounded complex of finitely generated projective A-modules; equivalently, M is compact, as an object in the triangulated category $\mathbf{D}(\operatorname{Mod} A)$; equivalently, M is in $\operatorname{Thick}(A)$; see [40, Theorem 2.2].

The following criterion for detecting perfect complexes will be handy.

LEMMA 4.2. — *Let A be a finite R-algebra. For $M \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, the following conditions are equivalent.*

(1) *M is perfect in $\mathbf{D}(\operatorname{Mod} A)$;*
(2) *$M_{\mathfrak{m}}$ is perfect in $\mathbf{D}(\operatorname{Mod} A_{\mathfrak{m}})$ for each maximal ideal $\mathfrak{m}$ in R;*
(3) *$\operatorname{Tor}^A_i(L, M) = 0$ for each $L \in \operatorname{mod} A^{\mathrm{op}}$ and $i \gg 0$.*

Proof. — The equivalence of (1) and (2) is due to Bass [5, Proposition III.6.6]. Evidently, (1) implies (3), and the reverse implication can be verified by an argument akin to that for [2, Theorem A.1.2]. □

REMARK 4.3. — We say that a complex M of A-bimodules is *perfect on both sides*, if it is perfect both in $\mathbf{D}(\operatorname{Mod} A)$ and in $\mathbf{D}(\operatorname{Mod} A^{\mathrm{op}})$; said otherwise, the restriction of M along either map $A \to A^{\mathrm{ev}} \leftarrow A^{\mathrm{op}}$ is perfect, in the corresponding category.

We note also that when M is a complex of A-bimodules, $\operatorname{RHom}_A(M, A)$ has a left A-action induced by the right A-action on M, and a right action induced by the right A-action on A. In our context A is a projective R-module, so one can realize $\operatorname{RHom}_A(M, A)$ as a complex of bimodules, namely, the complex $\operatorname{Hom}_A(M, \mathbf{i}_{A^{\mathrm{ev}}} A)$.

LEMMA 4.4. — *Let A be a finite R-algebra and M a complex of A-bimodules that is perfect on both sides. The following statements hold:*

(1) *There exists a quasi-isomorphism $P \to M$ of A-bimodules where P is bounded, consisting of finitely generated A-bimodules that are projective on both sides.*
(2) *When A is a Gorenstein R-algebra, $\operatorname{RHom}_A(M, A)$ is perfect on both sides.*

Proof. — (1) The hypothesis on M implies that the A^{ev}-module $H^*(M)$ is finitely generated. There thus exists a projective A^{ev}-resolution, say $Q \to M$ with each Q_i finitely generated and 0 for $i \ll 0$. Fix an integer

$$i \geq \max\{\operatorname{proj\,dim}_A M, \operatorname{proj\,dim}_{A^{\mathrm{op}}} M\}\,.$$

The morphism $Q \to M$ factors through the quotient complex

$$P := 0 \longrightarrow \operatorname{Coker}(d^Q_{i+1}) \longrightarrow Q_i \longrightarrow Q_{i-1} \longrightarrow \cdots$$

Since A-modules Q_i are projective on both sides, it follows by the choice of i that so is the A-module $\operatorname{Coker}(d^Q_{i+1})$. Thus, P is the complex we seek.

(2) That $\operatorname{RHom}_A(M, A)$ is perfect on the right is clear; for example, it is equivalent to $\operatorname{Hom}_A(P, A)$ with P as above; this does not involve the Gorenstein property.

As for the perfection on the left, by Lemma 4.2 it suffices to check the perfection locally on $\operatorname{Spec} R$. Thus, we can assume that the injective dimension of A is finite. For any finitely generated A^{op}-module L, one has a natural isomorphism

$$L \otimes^{\mathrm{L}}_A \operatorname{RHom}_A(M, A) \xrightarrow{\sim} \operatorname{RHom}_A(\operatorname{RHom}_{A^{\mathrm{op}}}(L, M), A)\,.$$

Since M is perfect over A^{op}, and A has finite injective dimension (on the right), so does M, and, hence, $H^*(\operatorname{RHom}_{A^{\mathrm{op}}}(L, M))$ is bounded. Then the finiteness of the injective dimension of A on the left implies that

$$H^*(\operatorname{RHom}_A(\operatorname{RHom}_{A^{\mathrm{op}}}(L, M), A))$$

is bounded. It thus follows from the quasi-isomorphism above that

$$\operatorname{Tor}^A_i(L, \operatorname{RHom}_A(M, A)) = 0 \qquad \text{for } |i| \gg 0.$$

This implies $\operatorname{RHom}_A(M, A)$ is perfect on the left; see Lemma 4.2. □

An equivalence of categories. — Let A be a Gorenstein R-algebra, $\omega_{A/R}$ its dualizing module, and $\mathbf{N}_{A/R}$ the Nakayama functor; see (5). As for finite dimensional algebras [28] the derived functor of the Nakayama functor is an auto-equivalence of the bounded derived category. In other words, $\omega_{A/R}$ is a tilting complex for A.

THEOREM 4.5. — *Let A be a Gorenstein R-algebra. The A-bimodule $\omega_{A/R}$ is perfect on both sides and induces adjoint equivalences of triangulated categories*

$$\mathbf{D}(\operatorname{Mod} A) \underset{\operatorname{RHom}_A(\omega_{A/R}, -)}{\overset{\omega_{A/R} \otimes^{\mathrm{L}}_A -}{\rightleftarrows}} \mathbf{D}(\operatorname{Mod} A)\,.$$

Moreover, these restrict to adjoint equivalences on $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$.

Proof. — The argument becomes a bit more transparent once we consider the ring $E := \operatorname{End}_A(\omega)$, and its natural left action on ω that is compatible with the left A-module structure. We first verify the following properties of ω:

(1) The natural maps $A \to E^{\mathrm{op}}$ and $A \to \operatorname{End}_E(\omega)$ of rings are isomorphisms.
(2) $\operatorname{Ext}^i_A(\omega,\omega) = 0 = \operatorname{Ext}^i_E(\omega,\omega)$ for $i \geq 1$.
(3) ω is compact both in $\mathbf{D}(\operatorname{Mod} A)$ and in $\mathbf{D}(\operatorname{Mod} E)$.

The first map in (1) is

$$A \longrightarrow \operatorname{End}_A(\omega)^{\mathrm{op}} \quad \text{where } a \mapsto (w \mapsto wa)\,.$$

A routine computation reveals that this is, indeed, a map of rings. Its bijectivity follows from the computation:

$$\begin{aligned}\operatorname{RHom}_A(\omega,\omega) &\cong \operatorname{RHom}_R(\operatorname{Hom}_R(A,R),R)\\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(A,R),R)\\ &\cong A\,,\end{aligned}$$

where the first isomorphism is an adjunction, and the others hold because the R-module A is finite and projective. The computation above also establishes that $\operatorname{Ext}^i_A(\omega,\omega) = 0$ for $i \geq 1$. This justifies the first parts of the (1) and (2). Given that $A \xrightarrow{\sim} E^{\mathrm{op}}$, applying the already established part of the result to A^{op} completes the argument for (1) and (2).

It remains to verify (3), and again, given that $E \cong A^{\mathrm{op}}$ as rings, it suffices to check that ω is perfect in $\mathbf{D}(\operatorname{Mod} A)$. Since the A-module ω is finitely generated it suffices to prove that it has finite projective dimension as an A-module. By Lemma 4.2 it suffices to verify that the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ has finite projective dimension for each $\mathfrak{p} \in \operatorname{Spec} R$. Since

$$\operatorname{Hom}_{R_{\mathfrak{p}}}(A_{\mathfrak{p}}, R_{\mathfrak{p}}) \cong \operatorname{Hom}_R(A,R)_{\mathfrak{p}}$$

as $A_{\mathfrak{p}}$-bimodules, and $A_{\mathfrak{p}}$ is a Gorenstein $R_{\mathfrak{p}}$-algebra, replacing R and A by their localizations at $\mathfrak{p}$ we can assume that $(R,\mathfrak{m},k)$ is a local ring and A is a Gorenstein R-algebra of finite injective dimension; the desired conclusion is that the projective dimension of $\operatorname{Hom}_R(A,R)$ is finite. At this point, one can invoke [13, Proposition 7.6.3(ii)] to complete the proof. The proof of *op. cit.* uses the theory of Cohen–Macaulay approximations. Here is a direct argument:

Since R is Gorenstein, by Lemma 4.1, and local, it has finite injective dimension; choose a finite injective resolution $R \to \mathbf{i}R$. Choose also a finite injective resolution $A \to \mathbf{i}A$. Then $\operatorname{Hom}_R(\mathbf{i}A, \mathbf{i}R)$ is a bounded complex of flat A-modules, quasi-isomorphic to $\operatorname{Hom}_R(A,R)$; thus the A-module $\operatorname{Hom}_R(A,R)$ has finite flat dimension. Since it is also finitely generated, it follows that its projective dimension is finite; see Lemma 4.2.

This completes the proofs of assertions (1)–(3).

Next we verify the stated equivalence of (the full derived) categories. This is a standard argument, given the properties of ω. Here is a sketch. To begin with, given the isomorphism $A \cong E^{\mathrm{op}}$ of rings, the stated adjunction can be factored as

$$\mathbf{D}(\operatorname{Mod} A) \underset{-\otimes^{\mathrm{L}}_E \omega}{\overset{\operatorname{RHom}_A(\omega,-)}{\rightleftarrows}} \mathbf{D}(\operatorname{Mod} E^{\mathrm{op}}) \xrightarrow{\ \sim\ } \mathbf{D}(\operatorname{Mod} A)\,.$$

It thus suffices to verify that the adjoint pair on the left are quasi-inverses to each other, that is to say that their counit and unit of the adjunction are isomorphisms. The counit is the evaluation map

$$\varepsilon(M)\colon \operatorname{RHom}_A(\omega, M) \otimes^{\mathrm{L}}_E \omega \longrightarrow M \quad \text{for } M \text{ in } \mathbf{D}(\operatorname{Mod} A).$$

The map above is an isomorphism, for it factors as the composition of isomorphisms

$$\begin{aligned} \operatorname{RHom}_A(\omega, M) \otimes^{\mathrm{L}}_E \omega &\xrightarrow{\ \sim\ } \operatorname{RHom}_A(\operatorname{RHom}_E(\omega,\omega), M) \\ &\xrightarrow{\ \sim\ } \operatorname{RHom}_A(A, M) \\ &\xrightarrow{\ \sim\ } M\,, \end{aligned}$$

where the first map is standard and is a quasi-isomorphism because ω is compact in $\mathbf{D}(\operatorname{Mod} E)$, by (3) above, and the second map is induced by the natural map $A \to \operatorname{RHom}_E(\omega,\omega)$ that is a quasi-isomorphism because of properties (1) and (2). Similarly, the unit map

$$N \longrightarrow \operatorname{RHom}_A(\omega, N \otimes^{\mathrm{L}}_E \omega)$$

is a quasi-isomorphism, for all N in $\mathbf{D}(\operatorname{Mod} A)$, for it factors as the composition

$$N \xrightarrow{\ \sim\ } N \otimes_E \operatorname{RHom}_A(\omega,\omega) \xrightarrow{\ \sim\ } \operatorname{RHom}_A(\omega, N \otimes_E \omega)\,,$$

where the first map is induced by the isomorphism $E \xrightarrow{\ \sim\ } \operatorname{RHom}_A(\omega,\omega)$, and the second one is standard and is an isomorphism because ω is perfect in $\mathbf{D}(\operatorname{Mod} A)$.

This completes the proof that the stated adjoint pair of functors induce an equivalence on $\mathbf{D}(\operatorname{Mod} A)$. It remains to note that for each M in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, the A-complex $\operatorname{RHom}_A(\omega, M)$ and $\omega \otimes^{\mathrm{L}}_A M$ are in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ as well, because ω is compact on both sides. Thus, they restrict to adjoint equivalences on $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$. □

We can now offer converses to Lemma 4.1; see Goto [23] for a similar statement in commutative algebra. Regarding condition (3), it is noteworthy that the injective dimension of A need not be finite; so there need not be a global bound (independent of M) on the degree i beyond which $\operatorname{Ext}^i_A(M, A)$ is zero. Indeed, there exist even commutative Gorenstein rings R that exhibit this phenomenon; see [39, A1].

THEOREM 4.6. — *Let R be a commutative Noetherian Gorenstein ring, and A a finite, projective, R-algebra. The following conditions are equivalent.*

(1) *The R-algebra A is Gorenstein.*
(2) *The A-bimodule $\omega_{A/R}$ is perfect on both sides.*
(3) *For each $M \in \operatorname{mod} A$ and $N \in \operatorname{mod} A^{\mathrm{op}}$, we have $\operatorname{Ext}^i_A(M, A) = 0$ for $i \gg 0$ and $\operatorname{Ext}^i_{A^{\mathrm{op}}}(N, A) = 0$ for $i \gg 0$.*
(4) *The functors $\operatorname{RHom}_A(-, A)$ and $\operatorname{RHom}_{A^{\mathrm{op}}}(-, A)$ induce triangle equivalences*

$$\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)^{\mathrm{op}} \underset{\operatorname{RHom}_{A^{\mathrm{op}}}(-,A)}{\overset{\operatorname{RHom}_A(-,A)}{\underset{\sim}{\rightleftarrows}}} \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{op}})\,.$$

Proof. — The proof that (1)⇒(2) is contained in Lemma 4.1 and Theorem 4.5.

(2)⇒(1) The hypotheses are local with respect to primes in $\operatorname{Spec} R$, as is the conclusion, by definition. We may thus assume that R is local and, hence, of finite injective dimension. Then, since A is a projective R-module, it follows from adjunction that the A-module $\omega = \operatorname{Hom}_R(A, R)$ has finite injective dimension on both sides. For the same reason, one gets that the following natural map is a quasi-isomorphism

$$A \longrightarrow \operatorname{RHom}_{A^{\mathrm{op}}}(\omega, \omega)\,;$$

see the proof of Theorem 4.5. As ω is perfect on the right, it is in $\operatorname{Thick}(A)$ in $\mathbf{D}(\operatorname{Mod} A^{\mathrm{op}})$, and the quasi-isomorphism above implies that A is in $\operatorname{Thick}(\omega)$ in $\mathbf{D}(\operatorname{Mod} A)$. In particular, since the injective dimension of ω as a left A-module is finite, so is that of A. Similarly, we deduce that the injective dimension of A is finite also on the right.

(1)⇒(3) Suppose A is a Gorenstein R-algebra and fix an M in $\operatorname{mod} A$. Since A is in $\operatorname{Thick}(\omega)$ in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, it suffices to verify that $\operatorname{Ext}^i_A(M, \omega)$ for $i \gg 0$. Adjunction yields

$$\operatorname{Ext}^i_A(M, \omega) = \operatorname{Ext}^i_A(M, \operatorname{Hom}_R(A, R)) \cong \operatorname{Ext}^i_R(M, R)\,.$$

As R is Gorenstein, by Lemma 4.1, the problem reduces to the commutative case, where the result is due to Goto [23, Theorem 1]. The same argument gives the result for N in $\operatorname{mod} A^{\mathrm{op}}$.

(3)⇒(1) For each prime $\mathfrak{p}$ in $\operatorname{Spec} R$ and M in $\operatorname{mod} A$, we have an isomorphism

$$\operatorname{Ext}^i_A(M, A)_{\mathfrak{p}} \cong \operatorname{Ext}^i_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, A_{\mathfrak{p}}) \qquad (i \geq 0)\,.$$

If this vanishes for each M and $i \gg 0$, then $A_{\mathfrak{p}}$ has finite injective dimension as a left $A_{\mathfrak{p}}$-module. Analogously, $A_{\mathfrak{p}}$ has finite injective dimension as a right $A_{\mathfrak{p}}$-module. Thus, A is Gorenstein.

(1)⇒(4) For each $M \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, the A^{op}-complex $\operatorname{RHom}_A(M, A)$ belongs to $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{op}})$, by the already verified implication (1)⇒(3), so it remains to

verify that the natural biduality morphism

$$M \longrightarrow \mathrm{RHom}_A(\mathrm{RHom}_A(M, A), A)$$

is an isomorphism. Since $\mathrm{RHom}_A(M, A)$ is in $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}\, A^{\mathrm{op}})$ this can be checked locally on $\mathrm{Spec}\, R$, where it holds for the injective dimension of A is locally finite. The same argument gives the result for N in $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}\, A^{\mathrm{op}})$.

(4)⇒(3) Clear. □

Remark 4.7. — The argument in the proof of Theorem 4.6 raises the question: When A is a Gorenstein R-algebra, is $\omega_{A/R}$ generated by A in $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}\, A^{\mathrm{ev}})$, that is to say, is it in $\mathrm{Thick}_{A^{\mathrm{ev}}}(A)$? By standard arguments, this question is equivalent to: Is

$$\mathrm{RHom}_R(A \otimes^{\mathrm{L}}_{A^{\mathrm{ev}}} A, R) \xrightarrow{\simeq} \mathrm{RHom}_{A^{\mathrm{ev}}}(A, \omega_{A/R})$$

perfect as a dg module over $\mathcal{E} := \mathrm{RHom}_{A^{\mathrm{ev}}}(A, A)$, the (derived) Hochschild cohomology algebra? When this condition holds, it would follow from the isomorphism above that if $\mathrm{HH}^i(A/R) = 0$ for $i \gg 0$, then also $\mathrm{HH}_i(A/R) = 0$ for $i \gg 0$.

This turns out not to be the case when A is finite dimensional and self-injective over a field: Let k be a field, $q \in k$ an element that is nonzero and not a root of unity, and set

$$\Lambda := \frac{k\langle x, y\rangle}{(x^2, xy + qyx, y^2)}\,.$$

Then Buchweitz, Madsen, Green, and Solberg prove that $\mathrm{rank}_k \mathrm{HH}^*(A/k) = 5$, whereas $\mathrm{HH}_i(A/k)$ is nonzero for each $i \geq 0$ [14].

On the other hand, the question has, trivially, a positive answer when A is a symmetric R-algebra, that is to say, when $\omega_{A/R} \cong A$ as an A-bimodule. So this begs the question: If $\omega_{A/R}$ is in $\mathrm{Thick}_{A^{\mathrm{ev}}}(A)$, is then A a symmetric R-algebra?

5. Gorenstein algebras and their homotopy categories

Let A be a Gorenstein R-algebra. We study in this case the properties of the Nakayama functor for the homotopy category of injectives $\mathbf{K}(\mathrm{Inj}\, A)$.

The Nakayama functor. — As explained in Section 3, the Nakayama functor admits a canonical extension to a functor $\widehat{\mathbf{N}}_{A/R}\colon \mathbf{K}(\mathrm{Inj}\, A) \to \mathbf{K}(\mathrm{Inj}\, A)$. The following result discusses the compatibility of this functor with the recollement for $\mathbf{K}(\mathrm{Inj}\, A)$ introduced in (1) and the equivalence on $\mathbf{D}(\mathrm{Mod}\, A)$ in Theorem 4.5.

THEOREM 5.1. — *Let A be a Gorenstein R-algebra. The functor $\widehat{\mathbf{N}}_{A/R}\colon \mathbf{K}(\operatorname{Inj} A) \to \mathbf{K}(\operatorname{Inj} A)$ is a triangle equivalence making the following square commutative:*

$$\begin{array}{ccc} \mathbf{D}(\operatorname{Mod} A) & \xrightarrow{\mathbf{i}} & \mathbf{K}(\operatorname{Inj} A) \\ \downarrow{\scriptstyle \operatorname{RHom}_A(\omega,-)} & & \downarrow{\scriptstyle \widehat{\mathbf{N}}} \\ \mathbf{D}(\operatorname{Mod} A) & \xrightarrow{\mathbf{i}} & \mathbf{K}(\operatorname{Inj} A)\,. \end{array}$$

Moreover, $\widehat{\mathbf{N}}_{A/R}$ restricts to an equivalence $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A) \xrightarrow{\sim} \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$.

The key step in the proof of the result is a "concrete" description of $\widehat{\mathbf{N}}$; see Lemma 5.2 below. To that end note that Lemma 4.4 applies to the dualizing bimodule $\omega_{A/R}$; fix a complex P provided by that result and set $\widehat{\omega}_{A/R} := P$. Thus,

$$\widehat{\omega}_{A/R} \longrightarrow \omega_{A/R}$$

is a finite resolution of $\omega_{A/R}$ by finitely generated A-bimodules that are projective on either side. This implies, in particular, that when X is a complex of injective A-modules, so is $\operatorname{Hom}_A(\widehat{\omega}_{A/R}, X)$; this follows from the standard Hom-tensor adjunction and requires only that $\widehat{\omega}_{A/R}$ consists of modules projective on the right. One thus has the induced exact functor

$$\operatorname{Hom}_A(\widehat{\omega}_{A/R}, -)\colon \mathbf{K}(\operatorname{Inj} A) \to \mathbf{K}(\operatorname{Inj} A)\,.$$

Here is the vouched for description of the completion of the Nakayama functor.

LEMMA 5.2. — *The quasi-isomorphism $\widehat{\omega}_{A/R} \to \omega_{A/R}$ induces an isomorphism*

$$\widehat{\mathbf{N}}_{A/R} \xrightarrow{\sim} \operatorname{Hom}_A(\widehat{\omega}_{A/R}, -)$$

of functors on $\mathbf{K}(\operatorname{Inj} A)$.

Proof. — For $X \in \mathbf{K}(\operatorname{Inj} A)$, the morphism $\widehat{\omega} \to \omega$ induces the morphism

$$\operatorname{Hom}_A(\omega, X) \longrightarrow \operatorname{Hom}_A(\widehat{\omega}, X)$$

of complexes of A-modules. Since $\operatorname{Hom}_A(\widehat{\omega}, X)$ consists of injective modules, one gets an induced morphism

$$\widehat{\mathbf{N}}(X) = \lambda \operatorname{Hom}_A(\omega, X) \longrightarrow \operatorname{Hom}_A(\widehat{\omega}, X)\,.$$

This is the natural transformation in question. The functors $\widehat{\mathbf{N}}$ and $\operatorname{Hom}_A(\widehat{\omega}, -)$ preserve arbitrary direct sums, the former by Lemma 3.1 and the latter because $\widehat{\omega}$ is a bounded complex of finitely generated modules, by choice. Thus, it suffices to verify that the morphism above is an isomorphism when X is compact in $\mathbf{K}(\operatorname{Inj} A)$, that is to say, when it is of the form $\mathbf{i}M$, for some $M \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$. In this case, the morphism in question is the composite

$$\widehat{\mathbf{N}}(\mathbf{i}M) \xrightarrow{\sim} \mathbf{i} \operatorname{Hom}_A(\omega, \mathbf{i}M) \to \operatorname{Hom}_A(\widehat{\omega}, \mathbf{i}M)\,,$$

where the isomorphism is taken from Lemma 3.1. The map above is a quasi-isomorphism and its source and target are K-injective; the former by construction and the latter because $\widehat{\omega}$ is a bounded complex of projectives. It remains to observe that a quasi-isomorphism between K-injectives is an isomorphism in $\mathbf{K}(\operatorname{Inj} A)$. □

Proof of Theorem 5.1. — Given Lemma 5.2, a standard dévissage argument shows that $\widehat{\mathbf{N}}$ is a triangle equivalence: the functor preserves arbitrary direct sums and identifies with $\operatorname{RHom}_A(\omega, -)$ when restricted to compacts, by Lemma 3.1. It remains to note that $\operatorname{RHom}_A(\omega, -)$ is an equivalence on $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, by Theorem 4.5.

For the commutativity of the square, fix a complex $X \in \mathbf{D}(\operatorname{Mod} A)$. We have already seen in Lemma 3.1 that

$$\widehat{\mathbf{N}}(\mathbf{i}X) \xrightarrow{\sim} \mathbf{i}\operatorname{RHom}_A(\omega, X)\,,$$

when X is bounded below. An arbitrary complex in $\mathbf{D}(\operatorname{Mod} A)$ is quasi-isomorphic to a homotopy limit of complexes that are bounded below. Thus, it remains to observe that both functors preserve homotopy limits.

It remains to verify that $\widehat{\mathbf{N}}$ restricts to an equivalence between acyclic complexes; equivalently that a complex $X \in \mathbf{K}(\operatorname{Inj} A)$ is acyclic if and only if $\widehat{\mathbf{N}}(X)$ is acyclic.

Since $\widehat{\omega}$ is perfect on the left, $\widehat{\mathbf{N}}$ preserves acyclic complexes. On the other hand, since $\operatorname{RHom}_A(\omega, A)$ is in $\operatorname{Thick}(A)$ in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ by Lemma 4.4, it follows that $\widehat{\mathbf{N}}(\mathbf{i}A)$ is in $\operatorname{Thick}(\mathbf{i}A)$. Using the isomorphism

$$H^n(X) \cong \operatorname{Hom}_{\mathbf{K}}(\mathbf{i}A, \Sigma^n X) \cong \operatorname{Hom}_{\mathbf{K}}(\widehat{\mathbf{N}}(\mathbf{i}A), \Sigma^n \widehat{\mathbf{N}}(X))$$

it follows that when $\widehat{\mathbf{N}}(X)$ is acyclic so is X. □

Remark 5.3. — One may turn $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ into a dg category such that $\mathbf{K}(\operatorname{Inj} A)$ identifies with its derived category; see [35, Appendix A]. Then $\widehat{\mathbf{N}}_{A/R}$ identifies with the lift of the Nakayama functor $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$.

Remark 5.4. — If X is a complex of projective A-modules, then so is the A-complex $\widehat{\omega}_{A/R} \otimes_A X$; this is because $\widehat{\omega}$ consists of modules projective on the left. Thus, one gets an exact functor

$$\widehat{\omega}_{A/R} \otimes_A -\colon \mathbf{K}(\operatorname{Proj} A) \longrightarrow \mathbf{K}(\operatorname{Proj} A)\,.$$

Arguing as in the proof of Theorem 5.1 one can verify that this is also an equivalence of categories.

Since the Nakayama functor $\widehat{\mathbf{N}}_{A/R}$ is an equivalence, it has a quasi-inverse. This is described below.

A quasi-inverse. — Set $V := \operatorname{Hom}_A(\widehat{\omega}_{A/R}, A)$; this is a bounded complex of A-bimodules where the left action is through the right A-module structure on $\widehat{\omega}_{A/R}$ and the right action is through the right A-module structure of A.

PROPOSITION 5.5. — *The assignment $X \mapsto \operatorname{Hom}_A(V, X)$ induces an exact functor*

$$\operatorname{Hom}_A(V, -)\colon \mathbf{K}(\operatorname{Inj} A) \longrightarrow \mathbf{K}(\operatorname{Inj} A)\,.$$

This functor is a quasi-inverse of $\widehat{\mathbf{N}}_{A/R}$, and so an equivalence of categories.

Proof. — The complex $\widehat{\omega}$ consists of modules projective on the left, and the right A-action on $V = \operatorname{Hom}_A(\widehat{\omega}, A)$ is through A, so V consists of modules that are projective on the right. Given this it is easy to verify that $\operatorname{Hom}_A(V, -)$ maps complexes of injectives to complexes of injectives and so induces an exact functor on $\mathbf{K}(\operatorname{Inj} A)$. For $X \in \mathbf{K}(\operatorname{Inj} A)$, the natural morphism of complexes

$$V \otimes_A X = \operatorname{Hom}_A(\widehat{\omega}, A) \otimes_A X \longrightarrow \operatorname{Hom}_A(\widehat{\omega}, X)$$

is an isomorphism because the complex $\widehat{\omega}$ is a bounded complex of modules projective on the left. This justifies the second isomorphism below:

$$\begin{aligned}\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X, \operatorname{Hom}_A(V, \operatorname{Hom}_A(\widehat{\omega}, X))) &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(V \otimes_A X, \operatorname{Hom}_A(\widehat{\omega}, X)) \\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\operatorname{Hom}_A(\widehat{\omega}, X), \operatorname{Hom}_A(\widehat{\omega}, X))\,.\end{aligned}$$

The first one is adjunction. Thus, the identity on $\operatorname{Hom}_A(\widehat{\omega}, X)$ induces a morphism

$$\eta(X)\colon X \longrightarrow \operatorname{Hom}_A(V, \operatorname{Hom}_A(\widehat{\omega}, X))\,,$$

which is natural in X. As functors of X, both the source and the target of η are exact and preserves direct sums; thus, to verify that $\eta(X)$ is an isomorphism for each X it suffices to verify that this is so for compact objects in $\mathbf{K}(\operatorname{Inj} A)$, that is to say, for the induced natural transformation on $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$. This is the map

$$M \mapsto \operatorname{RHom}_A(\operatorname{RHom}_A(\omega, A), \operatorname{RHom}_A(\omega, M))\,.$$

Since ω and $\operatorname{RHom}_A(\omega, A)$ are perfect as complexes of left A-modules, by Theorem 4.5 and Lemma 4.4, respectively, the map above can be obtained by applying $(-) \otimes^{\mathrm{L}}_A M$ to the natural homothety morphism

$$A \longrightarrow \operatorname{RHom}_A(\operatorname{RHom}_A(\omega, A), \operatorname{RHom}_A(\omega, A))\,.$$

Observe this a morphism in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{ev}})$. It remains to note that the map above is a quasi-isomorphism by, for example, Theorem 4.5. □

Acyclicity versus total acyclicity. — Set $E := \mathbf{i}_{A^{\mathrm{ev}}}A$, the injective resolution of A as an A-bimodule, and consider adjoint functors

$$\mathbf{K}(\operatorname{Proj} A) \underset{\mathbf{f}}{\overset{\longrightarrow}{\hookleftarrow}} \mathbf{K}(\operatorname{Flat} A) \underset{\operatorname{Hom}_A(E,-)}{\overset{E\otimes_A -}{\longleftarrow}} \mathbf{K}(\operatorname{Inj} A)\,,$$

where $\mathbf{f}$ is the right adjoint to the inclusion. It exists because $\mathbf{K}(\operatorname{Proj} A)$ is a compactly generated triangulated category, and its inclusion in $\mathbf{K}(\operatorname{Flat} A)$ is compatible with coproducts; see [32, Proposition 2.4]. One thus gets an adjoint pair

$$\mathbf{K}(\operatorname{Proj} A) \underset{\mathbf{h}}{\overset{\mathbf{t}}{\longleftarrow}} \mathbf{K}(\operatorname{Inj} A)\,,$$

where $\mathbf{t} := E \otimes_A -$ and $\mathbf{h} := \mathbf{f} \circ \operatorname{Hom}_A(E, -)$.

Let $\mathcal{A}$ be an additive category. A complex $X \in \mathbf{K}(\mathcal{A})$ is called *totally acyclic* if $\operatorname{Hom}(W, X)$ and $\operatorname{Hom}(X, W)$ are acyclic complexes of abelian groups for all $W \in \mathcal{A}$. We denote by $\mathbf{K}_{\mathrm{tac}}(\mathcal{A})$ the full subcategory of totally acyclic complexes.

THEOREM 5.6. — *Let A be a Gorenstein R-algebra. The adjoint functors $(\mathbf{t}, \mathbf{h})$ above are equivalences of categories, and they restrict to equivalences*

$$\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A) \underset{\mathbf{h}}{\overset{\mathbf{t}}{\underset{\sim}{\longleftarrow}}} \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)\,.$$

Moreover, there are equalities

$$\mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} A) = \mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A) \quad \textit{and} \quad \mathbf{K}_{\mathrm{tac}}(\operatorname{Inj} A) = \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)\,.$$

Proof. — It is clear that the functor $\mathbf{t}$ preserves direct sums. It also preserves compact objects, as we now explain. We may assume that a compact object in $\mathbf{K}(\operatorname{Proj} A)$ is of the form $\operatorname{Hom}_A(\mathbf{p}M, A)$ for some $M \in \operatorname{mod} A^{\mathrm{op}}$. This yields a complex

$$E \otimes_A \operatorname{Hom}_{A^{\mathrm{op}}}(\mathbf{p}M, A) \cong \operatorname{Hom}_{A^{\mathrm{op}}}(\mathbf{p}M, E)\,,$$

which is compact in $\mathbf{K}(\operatorname{Inj} A)$ because it is bounded below with

$$H^i \operatorname{Hom}_{A^{\mathrm{op}}}(\mathbf{p}M, E) \cong \operatorname{Ext}^i_{A^{\mathrm{op}}}(M, A) = 0$$

, for $i \gg 0$, by Theorem 4.6. In fact, the functor $\mathbf{t}$ restricted to compacts identifies with

$$\operatorname{RHom}_{A^{\mathrm{op}}}(-, A)\colon \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{op}}) \longrightarrow \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)^{\mathrm{op}}\,,$$

and this is an equivalence, again by Theorem 4.6. Thus, $\mathbf{t}$ is an equivalence of categories. Moreover, since $\mathbf{h}$ is its adjoint, the latter is the quasi-inverse to $\mathbf{t}$.

For $X \in \mathbf{K}(\operatorname{Proj} A)$, the equivalence of categories and Lemma 2.3 yield

$$H^n(X) = \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(A, \Sigma^n X) \cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(E, \Sigma^n \mathbf{t}X) = H^n(\mathbf{t}X)\,,$$

for each integer n. Thus, X is in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ if and only $\mathbf{t}X$ is in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$. Therefore, $(\mathbf{t}, \mathbf{h})$ induce an equivalence on the subcategory of acyclic complexes.

The key to verifying the remaining assertions is the following.

Claim. — $\operatorname{Inj} A \subset \operatorname{Loc}(E)$, *in* $\mathbf{K}(\operatorname{Inj} A)$.

Indeed, given the already established equivalence, it suffices to verify that $\mathbf{h}I$ is in $\operatorname{Loc}(A)$ for any injective A-module I, since $\mathbf{h}$ identifies E with A. As E is a complex of injective modules that are bounded below, $\operatorname{Hom}_A(E, I)$ is a complex of flat modules that are bounded above, and it is quasi-isomorphic to I, by Lemma 2.3. Therefore, $\mathbf{h}I = \mathbf{f}\operatorname{Hom}_A(E, I)$ is a projective resolution of I; see [32, Theorem 2.7(2)]. Thus, $\mathbf{h}I$ is in $\operatorname{Loc}(A)$, as desired.

Fix $Y \in \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$. Then $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(E, \Sigma^n Y) = 0$ for each integer n, so the claim yields $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(I, \Sigma^n Y) = 0$, for $I \in \operatorname{Inj} A$ and integers n, that is to say, Y is totally acyclic. Thus, any acyclic complex of injective modules is totally acyclic.

Fix an acyclic complex X in $\mathbf{K}(\operatorname{Proj} A)$. We want to verify that X is totally acyclic, that is to say, $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X, -) = 0$ on $\operatorname{Add} A$. Since $\mathbf{t}$ is an equivalence of categories, it suffices to verify that $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{t}X, -) = 0$ on $\operatorname{Add} \mathbf{t}A$, that is to say, on $\operatorname{Add} E$. However, $\mathbf{t}X$ is also acyclic, by the already established part of the result, and any complex in $\operatorname{Add} E$ is bounded below, and hence K-injective. This implies the desired result. □

6. Gorenstein projective modules

Let A be a Gorenstein R-algebra. An A-module M is *Gorenstein projective* (abbreviated to G-projective) if M is a syzygy in a totally acyclic complex of projective modules, that is, $M \cong \operatorname{Coker}(d_X^{-1})$, for some X in $\mathbf{K}_{\mathrm{tac}}(\operatorname{Proj} A)$. Given Theorem 5.6, one can "totally" drop from the definition. We write $\operatorname{GProj} A$ for the full subcategory of $\operatorname{Mod} A$ consisting of G-projectives, and $\operatorname{Gproj} A$ for $\operatorname{GProj} A \cap \operatorname{mod} A$.

Starting from Theorem 5.6, and also the results below, one can develop the theory of G-projective modules along the lines in [13], but we shall be content with recording a few observations needed to prove the duality theorems in Section 9. All these are well known when A is Iwanaga–Gorenstein.

Lemma 6.1. — *Let M be a G-projective A-module. The following statements hold.*

(1) *$M_{\mathfrak{p}}$ is G-projective as an $A_{\mathfrak{p}}$-module for $\mathfrak{p} \in \operatorname{Spec} R$.*
(2) *$\operatorname{Tor}_i^A(\omega_{A/R}, M) = 0 = \operatorname{Ext}^i_A(\omega_{A/R}, M)$ for $i \geq 1$.*

Proof. — Evidently, the localization of an acyclic complex is acyclic, so (1) follows.

(2) Since an A-module is zero if it is zero locally on $\operatorname{Spec} R$, given (1) and the finite generation of ω, we can reduce the verification of (2) to the case when R is local and so assume that the injective dimension of A is finite. Let I be the injective hull of the residue field of R and set $J := \operatorname{Hom}_R(\omega, I)$.

Claim. — *The A-module J is a faithful injective and has finite projective dimension.*

Indeed, as I is a faithful injective R-module, it follows by adjunction that the A-module J is faithful and injective. Since R is a Gorenstein local ring it has finite injective dimension, so I has finite projective dimension; that is to say, I is in $\operatorname{Thick}(\operatorname{Add} R)$ in $\mathbf{D}(\operatorname{Mod} R)$. Since ω is a finite projective R-module $\operatorname{Hom}_R(\omega, -)$ is an exact functor on $\mathbf{D}(\operatorname{Mod} A)$, so we deduce that J is in $\operatorname{Thick}(\operatorname{Add} \operatorname{Hom}_R(\omega, R))$ in $\mathbf{D}(\operatorname{Mod} A)$. Finally, observe that $A \cong \operatorname{Hom}_R(\omega, R)$ as A-modules.

The claim and the hypothesis that M is G-projective justify the equality below:

$$\operatorname{Hom}_R(\operatorname{Tor}^A_i(\omega, M), I) \cong \operatorname{Ext}^i_A(M, J) = 0 \quad \text{for } i \geq 1;$$

see also (7). The isomorphism is a standard adjunction. Since I is a faithful injective, it follows that $\operatorname{Tor}^A_i(\omega, M) = 0$ as desired.

A similar argument settles the claim about the vanishing of Ext-modules. □

When M is G-projective and $X \in \mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ is as above, the truncation $X_{\geqslant 0}$ is a projective resolution of M, and the total acyclicity of X implies

$$\operatorname{Ext}^i_A(M, P) = 0 \quad \text{for each projective module } P \text{ and } i \geq 1. \tag{7}$$

Here is a partial converse.

Lemma 6.2. — *A finitely generated A-module M satisfying $\operatorname{Ext}^i_A(M, A) = 0$, for $i \geq 1$, is G-projective. Moreover, such a module is a syzygy in an acyclic complex of finitely generated projective A-modules.*

Proof. — It suffices to verify that $\operatorname{Ext}^i_{A^{\mathrm{op}}}(M^*, A) = 0$, for $i \geq 1$, and that the biduality map $M \to M^{**}$ is bijective; given these, it is straightforward to construct an acyclic complex with M as a syzygy. What is more, using resolutions of M and M^* by finitely generated projective modules, one can get an acyclic complex consisting of finitely generated projective modules. Since M is finitely generated, and A is a finite R-algebra, both the conditions in question can be checked locally on $\operatorname{Spec} R$. We may thus assume that A is Iwananga–Gorenstein, in which case, the desired result is contained in [13, Lemma 4.2.2(iii)]. □

With exact structure inherited from $\operatorname{Mod} A$, the category $\operatorname{GProj} A$ is Frobenius, with projective objects $\operatorname{Proj} A$. Thus, the associated stable category,

$\underline{\mathrm{GProj}}A$, is triangulated. It is also compactly generated, with compact objects $\underline{\mathrm{Gproj}}A$; see, for example, [10, Proposition 2.10]. By the very definition, G-projectives are related to acyclic complexes of projectives. To clarify this connection, we recall from [32, §7.6] that there is an adjoint pair

$$\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A) \underset{\mathbf{a}}{\overset{\longrightarrow}{\hookleftarrow}} \mathbf{K}(\operatorname{Proj} A)\,,$$

where the left adjoint is the inclusion. The next result is well known and can be readily proved by adapting the argument for [13, Theorem 4.4.1].

PROPOSITION 6.3. — *The composition of functors* $\mathbf{a}\circ\mathbf{p}\colon \operatorname{Mod} A \to \mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ *induces a triangle equivalence*

$$\mathbf{ap}\colon \underline{\mathrm{GProj}}A \xrightarrow{\sim} \mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A)\,,$$

with the quasi-inverse defined by the assignment $X \mapsto \operatorname{Coker}(d_X^{-1})$. □

The singularity category. — Let $\mathbf{D}_{\mathrm{sg}}(A)$ be the *singularity category* of A introduced by Buchweitz [13] as the *stable derived category*. It is $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ modulo the perfect complexes. Any perfect complex is in the kernel of the functor

$$\mathbf{si}\colon \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)^{\mathrm{c}}\,,$$

where the functors $\mathbf{s}$ and $\mathbf{i}$ are from (1). Hence, there is an induced exact functor

$$\mathbf{D}_{\mathrm{sg}}(A) \to \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)^{\mathrm{c}}\,,$$

which we also denote $\mathbf{si}$. On the other hand, the embedding $\operatorname{Gproj} A \hookrightarrow \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ induces an exact functor

$$\mathbf{g}\colon \underline{\mathrm{Gproj}}A \longrightarrow \mathbf{D}_{\mathrm{sg}}(A)\,.$$

The result below was proved by Buchweitz [13, Theorem 4.4.1] when A is an Iwanaga–Gorenstein ring.

THEOREM 6.4. — *Let A be a Gorenstein R-algebra. The functors* $\mathbf{g}$ *and* $\mathbf{si}$ *are equivalences, up to direct summands, of triangulated categories:*

$$\underline{\mathrm{Gproj}}(A) \underset{\mathbf{g}}{\xrightarrow{\sim}} \mathbf{D}_{\mathrm{sg}}(A) \underset{\mathbf{si}}{\xrightarrow{\sim}} \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)^{\mathrm{c}}\,.$$

Proof. — The assertion about $\mathbf{si}$ is by [35, Corollary 5.4].

Let M, N be finitely generated G-projective A-modules. As noted in (7), one has $\operatorname{Ext}_A^i(M, A) = 0$ for $i \geq 1$. Arguing as in the proof of [42, Proposition 1.21] one gets that $\mathbf{g}$ induces a bijection:

$$\underline{\operatorname{Hom}}_A(M, N) \xrightarrow{\cong} \operatorname{Hom}_{\mathbf{D}_{\mathrm{sg}}}(\mathbf{g}M, \mathbf{g}N)\,.$$

Thus, $\mathbf{g}$ is fully faithful. It remains to prove that it is essentially surjective.

Fix X in $\mathbf{D}_{\mathrm{sg}}(A)$; we can assume that X is a bounded-above complex of finitely generated projective A-modules. Suppose $H^i(X) = 0$, for all $i < n$. Truncating at n yields a morphism $X \to \sigma_{\leq n}X$, which is an isomorphism in $\mathbf{D}_{\mathrm{sg}}(A)$ since its cone is perfect. Thus, X is isomorphic to a suspension of $M := \operatorname{Coker}(d_X^{n-1})$ in $\mathbf{D}_{\mathrm{sg}}(A)$. Since $\operatorname{Ext}_A^i(M, A)$, for $i \gg 0$ by Theorem 4.6, some syzygy of M is G-projective by Lemma 6.2, and we conclude that $\mathbf{g}$ is essentially surjective. □

A standard dévissage argument yields the following consequence.

COROLLARY 6.5. — *The composition of functor* $\mathbf{s} \circ \mathbf{i}\colon \operatorname{Mod} A \to \mathbf{K}(\operatorname{Inj} A)$ *induces a triangle equivalence*

$$\mathbf{si}\colon \underline{\mathrm{GProj}}A \xrightarrow{\sim} \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)\,.$$

Proof. — The triangulated categories $\underline{\mathrm{GProj}}A$ and $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj})$ are both compactly generated, and the functor $\mathbf{si}$ preserves coproducts. For the compact generation of $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj})$, see [35, Corollary 5.4], and $\mathbf{si}$ preserves coproducts since $\mathbf{s}$ is a left adjoint. Thus, the assertion follows from the fact that $\mathbf{si}$ is a triangle equivalence when restricted to the subcategories of compact objects; see Theorem 6.4. □

The Nakayama functor. — Via the equivalences of categories established above the auto-equivalence of $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ given by Nakayama functor induces an auto-equivalence on $\underline{\mathrm{GProj}}A$ and on the singularity category. This is made explicit in the next two results. The functor $\mathrm{GP}(-)$ that appears in the statements is the G-projective approximation whose existence is established in Theorem A.1. When A is a Gorenstein R-algebra, it follows from Theorem 4.5 that functor $\omega_{A/R}$ takes perfect complexes to perfect complexes, and hence induces a functor on the quotient $\mathbf{D}_{\mathrm{sg}}(A)$; we also denote that functor $\omega_{A/R} \otimes^{\mathrm{L}}_A (-)$.

PROPOSITION 6.6. — *Let A be a Gorenstein R-algebra. One has the following diagram of equivalences of categories*

$$\begin{array}{ccccc} \underline{\mathrm{Gproj}}(A) & \xrightarrow[\mathbf{g}]{\sim} & \mathbf{D}_{\mathrm{sg}}(A) & \xrightarrow[\mathbf{si}]{\sim} & \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)^{\mathrm{c}} \\ {\scriptstyle \mathrm{GP}(\omega_{A/R}\otimes_A(-))}\downarrow{\scriptstyle\sim} & & {\scriptstyle \omega_{A/R}\otimes^{\mathrm{L}}_A(-)}\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\downarrow{\scriptstyle \widehat{\mathbf{N}}^{-1}_{A/R}} \\ \underline{\mathrm{Gproj}}(A) & \xrightarrow[\mathbf{g}]{\sim} & \mathbf{D}_{\mathrm{sg}}(A) & \xrightarrow[\mathbf{si}]{\sim} & \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)^{\mathrm{c}} \end{array}\,,$$

where the squares commute up to an isomorphism of functors.

Proof. — The equivalences in the rows are from Theorem 6.4. We already know that $\widehat{\mathbf{N}}$ is an equivalence, so one has only to verify the commutativity of the diagram.

The commutativity of the square on the left is tantamount to: For each G-projective A-module M there is a natural isomorphism between $\omega \otimes^{\mathrm{L}}_A M$ and

$\mathrm{GP}(\omega \otimes_A M)$, viewed as objects in $\mathbf{D}_{\mathrm{sg}}(A)$. As noted in Lemma 6.2, finitely generated G-projective modules are syzygies in acyclic complexes of finitely generated projective modules. Thus, the proof of Theorem A.1 yields an exact sequence of A-modules

$$0 \longrightarrow P \longrightarrow \mathrm{GP}(\omega \otimes_A M) \longrightarrow \omega \otimes_A M \longrightarrow 0\,,$$

with $\mathrm{GP}(\omega \otimes_A M)$ a G-projective and P a finitely generated projective. This gives the isomorphism on the left

$$\mathrm{GP}(\omega \otimes_A M) \xrightarrow{\sim} \omega \otimes_A M \xleftarrow{\sim} \omega \otimes^{\mathrm{L}}_A M$$

in $\mathbf{D}_{\mathrm{sg}}(A)$. The one on the right is by Lemma 6.1(2), for the latter is tantamount to the statement that the natural morphism of complexes $\omega \otimes^{\mathrm{L}}_A M \to (\omega \otimes_A M)$ is an isomorphism in $\mathbf{D}(\operatorname{Mod} A)$, and so also in $\mathbf{D}_{\mathrm{sg}}A$.

For $X \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, from Lemma 3.1 and Theorem 4.5 one gets isomorphisms

$$\widehat{\mathbf{N}}\mathbf{i}(\omega \otimes^{\mathrm{L}}_A X) \cong \widehat{\mathbf{N}}(\omega \otimes^{\mathrm{L}}_A \mathbf{i}X) \cong \mathrm{RHom}_A(\omega, \omega \otimes^{\mathrm{L}}_A \mathbf{i}X) \cong \mathbf{i}X\,.$$

Applying $\mathbf{s}$ to the composition and observing that $\widehat{\mathbf{N}}$ commutes with $\mathbf{s}$ by Theorem 5.1, yields the commutativity of the square on the right. □

The commutativity of the outer square in Proposition 6.6 lifts to the corresponding "big" categories.

Proposition 6.7. — *The functor* $\mathrm{GP}(\omega_{A/R} \otimes_A -)\colon \underline{\mathrm{GProj}}A \to \underline{\mathrm{GProj}}A$ *is an equivalence of triangulated categories, with quasi-inverse* $\mathrm{GP}\operatorname{Hom}_A(\omega_{A/R}, -)$. *Moreover, the diagram below commutes up to an isomorphism of functors:*

$$\begin{array}{ccc} \underline{\mathrm{GProj}}A & \xrightarrow[\sim]{\mathbf{si}} & \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A) \\ {\scriptstyle \mathrm{GP}_A(\omega_{A/R}\otimes_A -)}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle \widehat{\mathbf{N}}^{-1}_{A/R}} \\ \underline{\mathrm{GProj}}A & \xrightarrow[\sim]{\mathbf{si}} & \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)\,. \end{array}$$

Proof. — The crucial observation is that the categories involved are compactly generated, and all the functors involved commute with direct sums. Thus, the desired result is a consequence of Proposition 6.6. □

7. Localization and torsion functors

As before, let A be a finite R-algebra. In what follows, we apply the theory of local cohomology and localization from [7], with respect to the action of the ring R on the homotopy category of injective A-modules. To that end we recall some results concerning the structure of injective A-modules discovered by Gabriel [20]; it extends the (by now well-known) theory for commutative rings.

To begin with, by the *spectrum* of A we mean the collection of two-sided prime ideals of A, denoted $\operatorname{Spec} A$. Since the map $\eta\colon R \to A$ is central and finite, the induced map on spectra

$$\operatorname{Spec} A \longrightarrow \operatorname{Spec} R \quad \text{where } \mathfrak{q} \mapsto \mathfrak{q} \cap R \text{ for } \mathfrak{q} \in \operatorname{Spec} A,$$

is surjective onto $\operatorname{Spec} \eta(R)$, which is a closed subset of $\operatorname{Spec} R$. Moreover, the fibers of the map are *discrete*: if $\mathfrak{q}' \subseteq \mathfrak{q}$ are elements of $\operatorname{Spec} A$ such that $\mathfrak{q}' \cap R = \mathfrak{q} \cap R$, then $\mathfrak{q}' = \mathfrak{q}$; see [20, Proposition V.11].

Torsion. — For each $\mathfrak{p}$ in $\operatorname{Spec} R$, there is a natural A-module structure of $M_{\mathfrak{p}}$ for which the canonical map $M \to M_{\mathfrak{p}}$ is A-linear.

A subset $V \subseteq \operatorname{Spec} R$ is *specialization closed* when it has the following property: If $\mathfrak{p} \subseteq \mathfrak{p}'$ are prime ideals in R and $\mathfrak{p}$ is in V, then $\mathfrak{p}'$ is in V; equivalently, that V contains the closure (in the Zariski) topology of its points. The following specialization closed subsets play a central role: Given an ideal $\mathfrak{a} \subset R$, the subset

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq \mathfrak{a}\}$$

of $\operatorname{Spec} R$, and given a prime $\mathfrak{p}$ in $\operatorname{Spec} R$, the subset

$$Z(\mathfrak{p}) := \{\mathfrak{p}' \in \operatorname{Spec} R \mid \mathfrak{p}' \not\subseteq \mathfrak{p}\}\,.$$

Observe that $Z(\mathfrak{p})$ equals $\operatorname{Spec} R \setminus \operatorname{Spec} R_{\mathfrak{p}}$.

Give a specialization closed subset V of $\operatorname{Spec} R$, the *V-torsion* submodule of an A-module M is defined by

$$\Gamma_V M := \operatorname{Ker}(M \longrightarrow \prod_{\mathfrak{p} \notin V} M_{\mathfrak{p}})\,.$$

The assignment $M \mapsto \Gamma_V M$ is an additive, left-exact, functor on $\operatorname{Mod} A$. The module M is called *V-torsion* if $\Gamma_V M = M$.

It is easy to verify that when $V := V(r)$ for an element $r \in R$, one has

$$\Gamma_{V(r)} M = \operatorname{Ker}(M \longrightarrow M_r)\,,$$

where M_r is the localization of M at the multiplicatively closed subset $\{r^n\}_{n \geqslant 0}$, and that when $V := Z(\mathfrak{p})$, for some $\mathfrak{p} \in \operatorname{Spec} R$, one gets

$$\Gamma_{Z(\mathfrak{p})} M = \operatorname{Ker}(M \longrightarrow M_{\mathfrak{p}})\,.$$

Injective modules. — Since A is Noetherian, $\operatorname{Inj} A$, the full subcategory of $\operatorname{Mod} A$ consisting of injective modules, is closed under arbitrary direct sums. For a $\mathfrak{q}$ in $\operatorname{Spec} A$ the injective hull of the A-module $A/\mathfrak{q}$ decomposes into a finite direct sum of copies of an indecomposable injective module, which we denote by $I(\mathfrak{q})$. Since A is a finite R-algebra, the assignment $\mathfrak{q} \mapsto I(\mathfrak{q})$ is a bijection between $\operatorname{Spec} A$ and the isomorphism classes of indecomposable injective A-modules, by [20, V.4]. Thus, each injective A-module is a direct sum of copies of $I(\mathfrak{q})$, as $\mathfrak{q}$ varies over $\operatorname{Spec} A$.

LEMMA 7.1. — *Let $V \subseteq \operatorname{Spec} R$ be specialization closed. For each injective A-module I, the module $\Gamma_V I$ is a direct summand of I. Thus, for $\mathfrak{q}$ in $\operatorname{Spec} A$, one has*

$$\Gamma_V I(\mathfrak{q}) = \begin{cases} I(\mathfrak{q}) & \text{when } \mathfrak{q} \cap R \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — The functor Γ_V provides a right adjoint for the inclusion of the localizing subcategory of A-modules that are V-torsion. This functor preserves injectivity, since the localizing subcategory is stable under taking injective envelopes, by [20, Proposition V.12]. Thus, $\Gamma_V I$ is a direct summand of I for every injective A-module I. In particular, we have $\Gamma_V I = I$, or $\Gamma_V I = 0$ when I is indecomposable. □

Since Γ_V is an additive functor, it induces a functor on the category of A-complexes. For each complex X of injective A-modules set

$$L_V X := \operatorname{Coker}(\Gamma_V X \longrightarrow X)\,.$$

Thus, one gets an exact sequence of A-complexes

$$0 \longrightarrow \Gamma_V X \longrightarrow X \longrightarrow L_V X \longrightarrow 0\,.$$

By Lemma 7.1 the subcomplex $\Gamma_V X$ consists of injective A-modules so the sequence above is degree-wise split exact, and hence induces in $\mathbf{K}(\operatorname{Inj} A)$ an exact triangle

$$\Gamma_V X \longrightarrow X \longrightarrow L_V X \longrightarrow \Sigma \Gamma_V X\,. \tag{8}$$

The functor L_V has an explicit description in a couple of cases.

EXAMPLE 7.2. — Suppose $V := V(r)$, for some $r \in R$. Then the map $X \to X_r$ is surjective, by Lemma 7.1, so there is an exact sequence

$$0 \longrightarrow \Gamma_{V(r)} X \longrightarrow X \longrightarrow X_r \longrightarrow 0$$

of A-complexes, and hence $L_{V(r)} X = X_r$. By the same token, when $V := Z(\mathfrak{p})$ for some prime $\mathfrak{p}$ in $\operatorname{Spec} R$, one gets an exact sequence

$$0 \longrightarrow \Gamma_{Z(\mathfrak{p})} X \longrightarrow X \longrightarrow X_{\mathfrak{p}} \longrightarrow 0$$

of A-complexes, so that $L_{Z(\mathfrak{p})} X = X_{\mathfrak{p}}$.

Localization and local cohomology. — The ring R acts on $\mathbf{K}(\operatorname{Mod} A)$ and hence on its subcategories discussed above, in the sense of [7]. We focus on $\mathcal{T} = \mathbf{K}(\operatorname{Inj} A)$.

For any localizing subcategory $\mathcal{C} \subseteq \mathcal{T}$ and object $X \in \mathcal{T}$, we call an exact triangle

$$\Gamma X \longrightarrow X \longrightarrow LX \longrightarrow \Sigma \Gamma X$$

a *localization triangle* provided that $\Gamma X \in \mathcal{C}$ and $LX \in \mathcal{C}^{\perp}$, where $\mathcal{C}^{\perp} \subseteq \mathcal{T}$ denotes the colocalizing subcategory consisting of objects Y such that $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$, for all $X \in \mathcal{C}$. If such a triangle exists for all objects $X \in \mathcal{T}$, then Γ yields a right adjoint for the inclusion $\mathcal{C} \hookrightarrow \mathcal{T}$, and L yields a left adjoint for the inclusion $\mathcal{C}^{\perp} \hookrightarrow \mathcal{T}$.

Given a specialization closed subset $V \subseteq \operatorname{Spec} R$, an object X in $\mathcal{T}$ is *V-torsion* provided that $\mathrm{Hom}_{\mathcal{T}}(C, X)$ is a V-torsion A-module for each compact $C \in \mathcal{T}$.

LEMMA 7.3. — *For a specialization closed subset $V \subseteq \operatorname{Spec} R$, the triangle* (8) *is the localization triangle associated to the localizing subcategory of V-torsion objects in $\mathbf{K}(\operatorname{Inj} A)$.*

Proof. — Fix $X \in \mathbf{K}(\operatorname{Inj} A)$. Then $\Gamma_V X$ is V-torsion by construction. Moreover, for every injective A-module I, it is easy to verify that

$$\mathrm{Hom}(\Gamma_V I, I/\Gamma_V I) = 0\,.$$

Thus, $\mathrm{Hom}_{\mathbf{K}(\mathrm{A})}(X', L_V X) = 0$, for all V-torsion $X' \in \mathbf{K}(\operatorname{Inj} A)$. □

LEMMA 7.4. — *For any $\mathfrak{p}$ in $\operatorname{Spec} R$ and X in $\mathbf{K}(\operatorname{Inj} A)$, we have $L_{Z(\mathfrak{p})}X \cong X_{\mathfrak{p}}$.*

Proof. — This follows from Example 7.2. □

For an object X in $\mathbf{K}(\operatorname{Inj} A)$, we write $\operatorname{Loc}(X)$ for the smallest localizing subcategory of $\mathbf{K}(\operatorname{Inj} A)$ that contains X.

LEMMA 7.5. — *Let $V \subseteq \operatorname{Spec} R$ be specialization closed. For any X in $\mathbf{K}(\operatorname{Inj} A)$, the A-complexes $\Gamma_V X$ and $L_V X$ are in $\operatorname{Loc}(X)$.*

Proof. — This follows from the local-to-global principle discussed in [8]. More specifically, one combines [8, Theorem 3.1] with [44, Theorem 6.9]. □

LEMMA 7.6. — *Let $V \subseteq \operatorname{Spec} R$ be specialization closed. If an A-complex X of injective A-modules is acyclic, then so are the complexes $\Gamma_V X$ and $L_V X$.*

Proof. — The subcategory $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ of $\mathbf{K}(\operatorname{Inj} A)$ is localizing, hence when X is acyclic, so are the complexes in $\operatorname{Loc}(X)$. It remains to recall Lemma 7.5. □

For an object X in $\mathbf{K}(\operatorname{Inj} A)$ and $\mathfrak{p} \in \operatorname{Spec} R$ the *local cohomology* at $\mathfrak{p}$ is

$$\Gamma_{\mathfrak{p}} X := \Gamma_{V(\mathfrak{p})}(X_{\mathfrak{p}})\,.$$

The following observation will be useful.

LEMMA 7.7. — *For any X in $\mathbf{K}(\operatorname{Inj} A)$, the complex $\Gamma_{\mathfrak{p}} X$ is a subquotient of X. In particular, if $X^i = 0$ for some $i \in \mathbb{Z}$, then $(\Gamma_{\mathfrak{p}} X)^i = 0$ as well.* □

REMARK 7.8. — The triangulated category $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is compactly generated and R-linear, so has its own localization functors for a specialization closed subset V of $\operatorname{Spec} R$. It follows from Lemma 7.6 that these are just restrictions of the corresponding functors on $\mathbf{K}(\operatorname{Inj} A)$.

The triangulated category $\mathbf{D}(\operatorname{Mod} A)$ is also compactly generated and R-linear. However, the embedding $\mathbf{i}\colon \mathbf{D}(\operatorname{Mod} A) \to \mathbf{K}(\operatorname{Inj} A)$ is not compatible with the localization functors; in other words, for a K-injective complex X, the complex $\Gamma_V X$ need not be K-injective; see [16]. On the other hand, it is easy to verify that these functors are compatible with the restriction functor $\mathbf{D}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} R)$.

REMARK 7.9. — Fix a $\mathfrak{p}$ in $\operatorname{Spec} R$ and consider the diagram of exact functors.

$$\begin{array}{ccc} \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A) & \underset{\text{incl}}{\overset{\mathbf{s}}{\rightleftarrows}} & \mathbf{K}(\operatorname{Inj} A) \\ {\scriptstyle (-)_{\mathfrak{p}}}\downarrow \uparrow{\scriptstyle \text{res}} & & {\scriptstyle (-)_{\mathfrak{p}}}\downarrow \uparrow{\scriptstyle \text{res}} \\ \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A_{\mathfrak{p}}) & \underset{\text{incl}}{\overset{\mathbf{s}_{\mathfrak{p}}}{\rightleftarrows}} & \mathbf{K}(\operatorname{Inj} A_{\mathfrak{p}}) \end{array}$$

It is clear that the two compositions of right adjoints, from the bottom left to the top right, coincide. It follows that the composition of the corresponding left adjoint functors are isomorphic: $(\mathbf{s}X)_{\mathfrak{p}} \cong \mathbf{s}_{\mathfrak{p}}(X_{\mathfrak{p}})$ for X in $\mathbf{K}(\operatorname{Inj} A)$.

Support. — Let $\mathcal{T}$ be $\mathbf{K}(\operatorname{Inj} A)$ or $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$. Specializing the definition from [7] to our context, we introduce the *support* of an object X in $\mathcal{T}$ to be the subset

$$\operatorname{supp}_R X := \{\mathfrak{p} \in \operatorname{Spec} R \mid \Gamma_{\mathfrak{p}} X \neq 0\}\,.$$

It follows from Remark 7.8 that the support an object in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is the same as its support when we view it as an object in $\mathbf{K}(\operatorname{Inj} A)$.

The *support* of $\mathcal{T}$ is the subset of $\operatorname{Spec} R$ defined by

$$\operatorname{supp}_R \mathcal{T} := \bigcup_{X \in \mathcal{T}^{\mathrm{c}}} \operatorname{supp}_R X\,.$$

Here are some alternative characterizations of support for acyclic complexes.

PROPOSITION 7.10. — *Let A be a finite R-algebra, fix $X \in \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ and $\mathfrak{p}$ in $\operatorname{Spec} R$. The following conditions are equivalent:*

(1) *The prime $\mathfrak{p}$ is not in $\operatorname{supp}_R X$.*
(2) *The complex $\Gamma_{\mathfrak{p}} X$ is contractible.*
(3) *The A-module $\Gamma_{\mathfrak{p}}(\Omega^i(X))$ is injective for each (equivalently, some) integer i.*

Proof. — An acyclic complex of injective modules is zero in $\mathbf{K}(\operatorname{Inj} A)$ if and only if it is contractible, if and only if each, equivalently, one of its syzygy

modules is injective. From this, we get that (1)⇔(2) and also that these conditions are equivalent to $\Omega^i(\Gamma_{\mathfrak{p}}X)$ injective for each, equivalently, some, i. It remains to note that since the functor $\Gamma_{\mathfrak{p}}$ is left-exact and preserves acyclicity of complexes in $\mathbf{K}(\operatorname{Inj} A)$, one gets

$$\Omega^i(\Gamma_{\mathfrak{p}}X) \cong \Gamma_{\mathfrak{p}}\Omega^i(X) \quad \text{for each integer } i.$$

This completes the proof. □

The following observation concerning generators for $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is well known.

LEMMA 7.11. — *The compact objects in* $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ *are direct summands of objects of the form* $\mathbf{s}C$*, where* C *is a compact object in* $\mathbf{K}(\operatorname{Inj} A)$*.*

Proof. — The functor $\mathbf{s}$ is left adjoint to the inclusion $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A) \subset \mathbf{K}(\operatorname{Inj} A)$, so it is essentially surjective; it also preserves compactness for the inclusion preserves direct sums. It follows that up to direct summands all compact objects of $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ are in the image of $\mathbf{s}$; see [40, Theorem 2.1]. □

A Noetherian ring A is *regular* if each $M \in \operatorname{mod} A$ has finite projective dimension; equivalently, each M in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ is perfect. We say that A is *singular* to mean that it is not regular. When A is a finite R-algebra its *regular locus* will mean the collection of primes $\mathfrak{p} \in \operatorname{Spec} R$ such that $A_{\mathfrak{p}}$ is regular. Its complement in $\operatorname{Spec} R$ is the *singular locus.*

COROLLARY 7.12. — *The singular locus of* A *equals* $\operatorname{supp}_R \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$*.*

Proof. — By Lemma 7.11 the support of $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is the union of the supports of $\mathbf{s}(\mathbf{i}M)$, for $M \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$. For any $\mathfrak{p} \in \operatorname{Spec} R$, one has isomorphisms

$$\mathbf{s}(\mathbf{i}M)_{\mathfrak{p}} \cong \mathbf{s}_{\mathfrak{p}}((\mathbf{i}M)_{\mathfrak{p}}) \cong \mathbf{s}_{\mathfrak{p}}(\mathbf{i}(M_{\mathfrak{p}}))$$

in $\mathbf{K}(\operatorname{Inj} A_{\mathfrak{p}})$, where the first one is by Remark 7.9, and the second one is standard. Thus, $\mathbf{s}(\mathbf{i}M)_{\mathfrak{p}} \cong 0$ if and only if $M_{\mathfrak{p}}$ is perfect in $\mathbf{D}(\operatorname{Mod} A_{\mathfrak{p}})$. Consequently, if $\mathfrak{p}$ is in the regular locus of A, then $\mathbf{s}(\mathbf{i}M)_{\mathfrak{p}} = 0$, for each M in $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$, and hence $\mathfrak{p}$ is not in the support of $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$.

Conversely, if $A_{\mathfrak{p}}$ is not regular, then there exists an $M \in \operatorname{mod} A$ such that $M_{\mathfrak{p}}$ is not perfect; one can choose M to be $V(\mathfrak{p})$-torsion. Then $\Gamma_{\mathfrak{p}}\mathbf{s}(\mathbf{i}M) \cong \mathbf{s}(\mathbf{i}M)_{\mathfrak{p}}$ is nonzero, so $\mathfrak{p}$ is in the support of $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$. □

8. Matlis duality and Gorenstein categories

This section is about avatars of Matlis duality in various homotopy categories we have been dealing with. To set the stage for the discussion, it helps to consider a general, compactly generated, triangulated category $\mathcal{T}$ with the

action of a commutative Noetherian ring R, in the sense of [7]. Fix an injective R-module I. For each compact object C in $\mathcal{T}$, the functor

$$X \longmapsto \operatorname{Hom}_R(\operatorname{Hom}_{\mathcal{T}}(C, X), I)\,,$$

from $\mathcal{T}$ to $\operatorname{Mod} R$, is homological and takes coproducts to products. The Brown representability theorem implies that it is representable: There is an object, say $T_I(C)$, in $\mathcal{T}$ and an isomorphism of functors

$$\operatorname{Hom}_R(\operatorname{Hom}_{\mathcal{T}}(C, -), I) \cong \operatorname{Hom}_{\mathcal{T}}(-, T_I(C))\,.$$

In this way, the assignment $C \times I \mapsto T_I(C)$ yields a functor

$$T\colon \mathcal{T}^{\mathrm{c}} \times \operatorname{Inj} R \longrightarrow \mathcal{T}\,.$$

Borrowing terminology from [18] we call the functor $T_I(-)$ the *Matlis lift* of I to $\mathcal{T}$. In what follows, for $\mathfrak{p}$ in $\operatorname{Spec} R$, we write $T_{\mathfrak{p}}(-)$ for $T_{I(\mathfrak{p})}(-)$, where $I(\mathfrak{p})$ is the injective hull of the R-module $R/\mathfrak{p}$.

Now, let A be a finite R-algebra as before. The description of the Matlis lifts of injective R-modules to the R-linear category $\mathbf{D}(\operatorname{Mod} A)$ is straightforward.

Proposition 8.1. — *The Matlis lift to* $\mathbf{D}(\operatorname{Mod} A)$ *of an injective* R*-module* I *is given by the functor* $C \mapsto \operatorname{RHom}_R(A, I) \otimes^{\mathrm{L}}_A C$.

Proof. — Given objects $X \in \mathbf{D}(\operatorname{Mod} A)$ and a finitely generated projective A-module P, there are natural isomorphisms

$$\begin{aligned}\operatorname{Hom}_R(\operatorname{Hom}_A(P, X), I) &\cong \operatorname{Hom}_R(X, I) \otimes_A P \\ &\cong \operatorname{Hom}_A(X, \operatorname{Hom}_R(A, I)) \otimes_A P \\ &\cong \operatorname{Hom}_A(X, \operatorname{Hom}_R(A, I) \otimes_A P)\,.\end{aligned}$$

It remains to observe that any compact object in $\mathbf{D}(\operatorname{Mod} A)$ is isomorphic to a bounded complex of finitely generated projective A-modules. □

The Matlis lifts of injective R-modules to the R-linear category $\mathbf{K}(\operatorname{Inj} A)$ is described in the next result, which is modeled on [36, Theorem 3.4]; the proof we give is somewhat different.

Theorem 8.2. — *Let* A *be a finite* R*-algebra. The Matlis lift to* $\mathbf{K}(\operatorname{Inj} A)$ *of an injective* R*-module* I *is given by*

$$C \longmapsto \operatorname{Hom}_R(A, I) \otimes_A \mathbf{p}C\,.$$

Proof. — Fix objects C, X in $\mathbf{K}(\operatorname{Inj} A)$ with C compact. The key input is Lemma 2.1 that yields the first isomorphism below

$$\begin{aligned}\operatorname{Hom}_R(\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C,X),I) &\cong \operatorname{Hom}_R(H^0(\operatorname{Hom}_A(\mathbf{p}C,A)\otimes_A X),I)\\ &\cong H^0(\operatorname{Hom}_R(\operatorname{Hom}_A(\mathbf{p}C,A)\otimes_A X,I))\\ &\cong H^0(\operatorname{Hom}_A(X,\operatorname{Hom}_R(\operatorname{Hom}_A(\mathbf{p}C,A),I))\\ &\cong H^0(\operatorname{Hom}_A(X,\operatorname{Hom}_R(A,I)\otimes_A \mathbf{p}C))\\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X,\operatorname{Hom}_R(A,I)\otimes_A \mathbf{p}C)\,.\end{aligned}$$

The second one holds because I is injective. The rest are standard. □

The next result describes Matlis lifts to $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$, using the functors from (1). In Lemma 7.11 we described the compact objects in that category.

COROLLARY 8.3. — *For a compact object in* $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ *of the form* $\mathbf{s}C$*, given by a compact object* C *in* $\mathbf{K}(\operatorname{Inj} A)$*, the Matlis lift of an injective* R*-module* I *is the complex*

$$T_I(\mathbf{s}C) \cong \mathbf{r}(T_I C) \cong \mathbf{r}(\operatorname{Hom}_R(A,I)\otimes_A \mathbf{p}C)\,.$$

Proof. — For any acyclic complex X of injective R-modules, one has

$$\begin{aligned}\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X,\mathbf{r}(T_I(C)) &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X,T_I(C))\\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C,X),I)\\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{s}C,X),I)\\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X,T_I(\mathbf{s}C))\,.\end{aligned}$$

This justifies the first isomorphism. For the second one, see Theorem 8.2. □

REMARK 8.4. — There is a notion of purity for compactly generated triangulated categories, analogous to the classical concept of purity for module categories; see Crawley-Boevey's survey [17]. It follows from the construction that any Matlis lift is a pure-injective object. In particular, we obtain from a Matlis lift a pure-injective module when an acyclic complex is identified with an A-module.

Gorenstein categories. — Let $\mathcal{T}$ be an R-linear category. Following [9] we say that $\mathcal{T}$ is *Gorenstein* if there is an R-linear triangle equivalence

$$F\colon \mathcal{T}^{\mathrm{c}} \xrightarrow{\sim} \mathcal{T}^{\mathrm{c}}$$

such that for each $\mathfrak{p}$ in $\operatorname{supp}_R \mathcal{T}$, there is an integer $d(\mathfrak{p})$ and a natural isomorphism

$$\Gamma_{\mathfrak{p}} \circ F \cong \Sigma^{-d(\mathfrak{p})} \circ T_{\mathfrak{p}}$$

of functors $\mathcal{T}^{\mathrm{c}} \to \mathcal{T}$. The functor F plays the role of a *global Serre functor* because it induces a Serre functor, in the sense of Bondal and Kapranov [11], on the subcategory of compacts objects in $\mathcal{T}_{\mathfrak{p}}$, the $\mathfrak{p}$-local $\mathfrak{p}$-torsion objects in $\mathcal{T}$, for $\mathfrak{p}$ in $\operatorname{Spec} R$. More precisely, localizing with respect to $\mathfrak{p}$ yields a functor $F_{\mathfrak{p}} \colon \mathcal{T}^{\mathrm{c}}_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{T}^{\mathrm{c}}_{\mathfrak{p}}$ and a natural isomorphism

$$\operatorname{Hom}_R(\operatorname{Hom}_{\mathcal{T}}(X,Y), I(\mathfrak{p})) \cong \operatorname{Hom}_{\mathcal{T}}(Y, \Sigma^{d(\mathfrak{p})} F_{\mathfrak{p}} X)\,,$$

for objects $X, Y \in \mathcal{T}_{\mathfrak{p}}$ such that X is compact and $\operatorname{supp}_R X = \{\mathfrak{p}\}$. This is explained in [9, §7]. In what follows we focus on the following special case.

Proposition 8.5. — *Let $\mathcal{T}$ be a compactly generated R-linear category that is Gorenstein, with global Serre functor F. Fix a maximal ideal $\mathfrak{m}$ in R. For any $X \in \mathcal{T}^{\mathrm{c}}$ and $Y \in \mathcal{T}$ with $\operatorname{supp}_R X = \{\mathfrak{m}\}$, there is a natural isomorphism*

$$\operatorname{Hom}_R(\operatorname{Hom}_{\mathcal{T}}(X,Y), I(\mathfrak{m})) \cong \operatorname{Hom}_{\mathcal{T}}(Y, \Sigma^{d(\mathfrak{m})} F X)\,.$$

In particular, if $\operatorname{supp}_R \mathcal{T} = \{\mathfrak{m}\}$, then $\Sigma^{d(\mathfrak{m})} F$ is a Serre functor on $\mathcal{T}^{\mathrm{c}}$.

Proof. — Since $\mathfrak{m}$ is maximal, any object supported on $\mathfrak{m}$ is already $\mathfrak{m}$-local. Thus, the desired isomorphism is a special case of [9, Proposition 7.3]. □

Gorenstein rings. — Let R be a commutative Gorenstein ring. For $\mathfrak{p}$ in $\operatorname{Spec} R$, set $h(\mathfrak{p}) = \dim R_{\mathfrak{p}}$; this is the height of $\mathfrak{p}$. The Gorenstein property for R is equivalent to the condition that the minimal injective resolution I of R satisfies

$$I^n = \bigoplus_{h(\mathfrak{p})=n} I(\mathfrak{p}) \quad \text{for each } n.$$

This translates to the condition that in $\mathbf{K}(\operatorname{Inj} R)$, there are isomorphisms

$$\Gamma_{\mathfrak{p}}(\mathbf{i}R) \cong \Sigma^{-h(\mathfrak{p})} I(\mathfrak{p}) \quad \text{for each } \mathfrak{p} \in \operatorname{Spec} R. \tag{9}$$

This result is due to Grothendieck, cf. [12, Proposition 3.5.4].

Proposition 8.6. — *Let A be a finite R-algebra that is projective as an R-module. The following conditions are equivalent:*

(1) *The R-algebra A is Gorenstein.*
(2) *The R-linear category $\mathbf{D}(\operatorname{Mod} A)$ is Gorenstein.*

When they hold the global Serre functor is $\omega_{A/R} \otimes^{\mathrm{L}}_A -$, and $d(\mathfrak{p}) = \dim R_{\mathfrak{p}}$.

Proof. — (1)⇒(2) As the R-algebra A is Gorenstein, the functor $F := \omega \otimes^{\mathrm{L}}_A -$ is an equivalence on $\mathbf{D}(\operatorname{Mod} A)$ and hence restricts to an equivalence $\mathbf{D}(\operatorname{Mod} A)^{\mathrm{c}}$ the subcategory of perfect complexes; see Theorem 4.5. With $d(\mathfrak{p})$ as in the

statement, for any perfect complex C, from Proposition 8.1 one gets the equality below

$$\begin{aligned} T_{\mathfrak{p}}(C) &= \mathrm{RHom}_R(A, I(\mathfrak{p})) \otimes^{\mathrm{L}}_A C \\ &\cong I(\mathfrak{p}) \otimes^{\mathrm{L}}_R \mathrm{Hom}_R(A, R) \otimes^{\mathrm{L}}_A C \\ &\cong \Sigma^{d(\mathfrak{p})} \Gamma_{\mathfrak{p}}(\mathfrak{i}R) \otimes^{\mathrm{L}}_R FC \\ &\cong \Sigma^{d(\mathfrak{p})} \Gamma_{\mathfrak{p}}(\mathfrak{i}R \otimes^{\mathrm{L}}_R FC) \\ &\cong \Sigma^{d(\mathfrak{p})} \Gamma_{\mathfrak{p}} FC\,. \end{aligned}$$

The third isomorphism is from (9), and the rest are standard. Thus, $\mathbf{D}(\mathrm{Mod}\, A)$ is Gorenstein, with the prescribed global Serre functor and shift $d(\mathfrak{p})$.

(2)⇒(1) It suffices to verify that the injective dimension of $A_{\mathfrak{m}}$ is finite for any maximal ideal $\mathfrak{m}$ in R. For this, it suffices to verify that $M \in \mathrm{mod}(A/\mathfrak{m}A)$ satisfy

$$\mathrm{Ext}^i_A(M, A) = 0 \quad \text{for } i \gg 0.$$

For then an argument along the lines of the proof of [2, Proposition A.1.5] yields that $A_{\mathfrak{m}}$ has finite injective dimension over itself.

Let $F\colon \mathbf{D}^{\mathrm{b}}(\mathrm{mod}\, A)^{\mathrm{c}} \to \mathbf{D}^{\mathrm{b}}(\mathrm{mod}\, A)^{\mathrm{c}}$ be a global Serre functor and F^{-1} its quasi-inverse. Since M is $\mathfrak{m}$-torsion from Proposition 8.5 we get the isomorphism below

$$\mathrm{Hom}_{\mathbf{D}(A)}(M, \Sigma^i A) \cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathbf{D}(A)}(F^{-1}A, \Sigma^{d(\mathfrak{m})-i} M), I(\mathfrak{m}))\,.$$

It remains to note that since $F^{-1}A$ is perfect one has

$$\mathrm{Hom}_{\mathbf{D}(A)}(F^{-1}A, \Sigma^j(-)) = 0 \quad \text{on } \mathrm{Mod}\, A,$$

for all $|j| \gg 0$. This implies the desired result. □

Here is the analogue of the preceding result dealing with homotopy categories.

Proposition 8.7. — *Let A be a finite R-algebra that is projective as an R-module. The R-linear category $\mathbf{K}(\mathrm{Inj}\, A)$ is Gorenstein if and only if A is regular.*

Proof. — When A is regular, the canonical functor $\mathbf{K}(\mathrm{Inj}\, A) \to \mathbf{D}(\mathrm{Mod}\, A)$ is an equivalence and $\mathbf{D}(\mathrm{Mod}\, A)$ is Gorenstein, by Proposition 8.6. As to the converse, it suffices check that $A_{\mathfrak{m}}$ is regular for each maximal ideal $\mathfrak{m}$ in R.

Arguing as in the proof of (2)⇒(1) in Proposition 8.6 one deduces that for each $M \in \mathrm{mod}(A/\mathfrak{m}A)$ and $N \in \mathrm{mod}\, A$, one has

$$\mathrm{Ext}^i_A(M, N) = 0 \quad \text{for } i \gg 0.$$

This implies that $A_{\mathfrak{m}}$ is regular. □

The preceding results concern the Gorenstein property for the derived category and the homotopy category of injectives for two of the three categories that appear in the recollement (1). That of the last one is dealt with in the next section.

9. Grothendieck duality for $\mathbf{K}_{\mathbf{ac}}(\mathbf{Inj}\, A)$

This section is dedicated to the proof of the following result. As explained in the Introduction, this has been the guiding light for the results presented in this work.

THEOREM 9.1. — *Let A be a Gorenstein R-algebra. For each compact object X in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ and $\mathfrak{p}$ in the singular locus of A, there is a natural isomorphism*

$$\Gamma_{\mathfrak{p}} X \cong \Sigma^{-d(\mathfrak{p})} T_{\mathfrak{p}}(\widehat{\mathbf{N}}_{A/R} X)\,,$$

where $d(\mathfrak{p}) = \dim(R_{\mathfrak{p}}) - 1$. In particular, the R-linear category $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is Gorenstein, with the global Serre functor the quasi-inverse of $\widehat{\mathbf{N}}_{A/R}$.

The proof is given further below. Theorem 1.2 from the Introduction is an immediate consequence.

COROLLARY 9.2. — *Let A be a Gorenstein R-algebra and let M, N be G-projective A-modules with M finitely generated. For each $\mathfrak{p} \in \operatorname{Spec} R$, there is a natural isomorphism*

$$\operatorname{Hom}_R(\widehat{\operatorname{Ext}}^i_A(M,N), I(\mathfrak{p})) \cong \widehat{\operatorname{Ext}}^{d(\mathfrak{p})-i}_A(N, \Gamma_{\mathfrak{p}} S(M)).$$

Proof. — The assertion is a direct translation of Theorem 9.1, given the equivalence $\underline{\mathrm{GProj}}A \xrightarrow{\simeq} \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ from Proposition 6.7. □

We continue with a consequence concerning duality for the category of compact objects. The statements are simpler, and perhaps more striking, when specialized to the case of local isolated singularities, and that is what we do.

Isolated singularities. — Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring and A a finite projective R-algebra. We say that A has an *isolated singularity* if its singular locus is $\{\mathfrak{m}\}$; that is to say, if the ring $A_{\mathfrak{p}}$ is regular for each non-maximal ideal $\mathfrak{p}$ in $\operatorname{Spec} R$; see the discussion around Corollary 7.12.

COROLLARY 9.3. — *Let R be a commutative Noetherian local ring of Krull dimension d. If A is a Gorenstein R-algebra with an isolated singularity, then the assignment*

$$X \mapsto \Sigma^{d-1} \widehat{\mathbf{N}}^{-1}(X)$$

is a Serre functor on the R-linear category $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)^{\mathrm{c}}$.

Proof. — Since A has an isolated singularity, the R-linear category $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is supported at $\mathfrak{m}$, the maximal ideal of R; see Corollary 7.12. Thus, Theorem 9.1 and Proposition 8.5 yield the desired result. □

Given the equivalences in Theorem 6.4 one can recast the duality statement above in terms of the singularity category and the stable category of Gorenstein projective modules. Here, too, we are following Buchweitz's footsteps [13], except that he does not require A to be projective over a central subalgebra; on the other hand, he considers only rings of finite injective dimension. We can get away with local finiteness of injective dimension, thanks to Theorem 4.6.

COROLLARY 9.4. — *For R and A as in Corollary 9.3, the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ has Serre duality, with Serre functor $\Sigma^{d-1}\omega_{A/R}\otimes^{\mathrm{L}}_{A}(-)$.*

Proof. — This is a direct translation of Corollary 9.3, made using Theorem 6.6. □

Moreover, here is Corollary 9.3 transported to the world of G-projective modules.

COROLLARY 9.5. — *For R and A as in Corollary 9.3, the functor*

$$M \mapsto \Omega^{1-d}\operatorname{GP}(\omega_{A/R}\otimes_A M)$$

is a Serre functor on the triangulated category $\underline{\mathrm{Gproj}}A$. □

REMARK 9.6. — Set $S := \Omega^{1-d}\operatorname{GP}(\omega\otimes_A(-))$; the Serre functor on $\underline{\mathrm{Gproj}}(A)$. Theorem 9.5 translates to the statement that there is an R-linear *trace* map

$$\underline{\operatorname{Hom}}_A(M, SM) \xrightarrow{\tau} I(\mathfrak{m})$$

such that the bilinear pairing

$$\underline{\operatorname{Hom}}_A(N, SM)\times\underline{\operatorname{Hom}}_A(M,N)\xrightarrow{-\circ-}\underline{\operatorname{Hom}}_A(M,SM))\xrightarrow{\tau} I(\mathfrak{m})\,,$$

where $-\circ-$ is the obvious composition, is non-degenerate. Murfet [38] describes the trace map in the case when $A = R$, that is to say, in the case of commutative rings; this involves the theory of residues and differentials forms. It would be interesting to extend his work to the present context.

We now prepare for the proof of Theorem 9.1.

LEMMA 9.7. — *Let A be a finite R-algebra. For each X in $\operatorname{Loc}(\mathbf{i}A)$ for which $\mathbf{i}X$ is in $\mathbf{K}^{+}(\operatorname{Inj} A)$, the isomorphism (2.2) induces isomorphisms*

$$\Sigma^{-1}\Gamma_{\mathfrak{p}}\mathbf{s}(\mathbf{i}X)\xrightarrow{\sim}\mathbf{r}\Gamma_{\mathfrak{p}}X \quad \textit{for each } \mathfrak{p}\in\operatorname{Spec} R.$$

Proof. — Since X is in $\operatorname{Loc}(\mathbf{i}A)$, from (2) and Lemma 2.2, we get an exact triangle

$$\Sigma^{-1}\mathbf{s}(\mathbf{i}X) \longrightarrow X \longrightarrow \mathbf{i}X \longrightarrow$$

Applying $\Gamma_{\mathfrak{p}}$ to this yields the exact triangle

$$\Sigma^{-1}\Gamma_{\mathfrak{p}}(\mathbf{s}(\mathbf{i}X)) \longrightarrow \Gamma_{\mathfrak{p}}X \longrightarrow \Gamma_{\mathfrak{p}}(\mathbf{i}X) \longrightarrow .$$

Since $\mathbf{s}(\mathbf{i}X)$ is acyclic so is the complex $\Gamma_{\mathfrak{p}}(\Sigma^{-1}\mathbf{s}(\mathbf{i}X))$, by Lemma 7.6. Hence, $\mathbf{r}(-)$ is (isomorphic to) the identity on this complex. On the other hand, since $\mathbf{i}X$ is bounded below, so is $\Gamma_{\mathfrak{p}}(\mathbf{i}X)$, by Lemma 7.7, and hence $\mathbf{r}(-)$ vanishes on this complex. Keeping these observations in mind and applying the functor $\mathbf{r}$ to the exact triangle above yields the stated isomorphism. □

Proof of Theorem 9.1. — It suffices to establish the result for objects of the form $\mathbf{s}(C)$, where $C \in \mathbf{K}(\operatorname{Inj} A)$ is a compact object; we can assume that C is bounded below. Set $D := \operatorname{Hom}_R(A, \mathbf{i}R)$. We shall be interested in the complex of injective A-modules

$$X := D \otimes_A \mathbf{p}(\widehat{\mathbf{N}}C)\,.$$

We claim that this complex satisfies the hypotheses of Lemma 9.7.

Claim. — *X is in* $\operatorname{Loc}(\mathbf{i}A)$ *and* $\mathbf{i}X \xrightarrow{\sim} C$ *and, in particular, it is bounded below.*

Indeed, the complex D consists of A-bimodules that are injective on either side, and the map $R \to \mathbf{i}R$ induces a quasi-isomorphism

$$\omega = \operatorname{Hom}_R(A, R) \longrightarrow \operatorname{Hom}_R(A, \mathbf{i}R) = D$$

of A-bimodules. Thus, D is an injective resolution of ω on both sides. It follows that in $\mathbf{D}(\operatorname{Mod} A)$ there are natural isomorphisms

$$D \otimes_A \mathbf{p}(\widehat{\mathbf{N}}C) \cong \omega \otimes^{\mathrm{L}}_A \operatorname{RHom}_A(\omega, C) \cong C\,,$$

where the second one is by Theorem 4.5. Therefore, in $\mathbf{K}(\operatorname{Inj} A)$, one gets that

$$\mathbf{i}X = \mathbf{i}(D \otimes_A \mathbf{p}(\widehat{\mathbf{N}}C)) \xrightarrow{\sim} C.$$

As to the first part of the claim, $\mathbf{p}(\widehat{\mathbf{N}}C)$ is in $\operatorname{Loc}(A) \subseteq \mathbf{K}(\operatorname{Proj} A)$, hence X is in $\operatorname{Loc}(D)$ in $\mathbf{K}(\operatorname{Inj} A)$. However, D is an injective resolution of ω, and the latter is perfect, as an object of $\mathbf{D}(\operatorname{Mod} A)$, so D is in $\operatorname{Thick}(\mathbf{i}A)$. It follows that X is in $\operatorname{Loc}(\mathbf{i}A)$, as claimed.

From the claim and Lemma 9.7, we deduce that

$$\Sigma^{-1}\Gamma_{\mathfrak{p}}\mathbf{s}(\mathbf{i}X) \cong \mathbf{r}\Gamma_{\mathfrak{p}}X \quad \text{for each } \mathfrak{p} \in \operatorname{Spec} R.$$

This justifies the penultimate isomorphism below, where h stands for $\dim R_{\mathfrak{p}}$:

$$\begin{aligned} T_{\mathfrak{p}}(\widehat{\mathbf{N}}(\mathbf{s}C)) &\cong T_{\mathfrak{p}}(\mathbf{s}(\widehat{\mathbf{N}}C)) \\ &\cong \mathbf{r}(\operatorname{Hom}_R(A, I(\mathfrak{p})) \otimes_A \mathbf{p}(\widehat{\mathbf{N}}C))) \\ &\cong \mathbf{r}(\operatorname{Hom}_R(A, \Sigma^h \Gamma_{\mathfrak{p}} \mathbf{i}R) \otimes_A \mathbf{p}(\widehat{\mathbf{N}}C))) \\ &\cong \Sigma^h \mathbf{r}\Gamma_{\mathfrak{p}}(\operatorname{Hom}_R(A, \mathbf{i}R) \otimes_A \mathbf{p}(\widehat{\mathbf{N}}C)) \\ &= \Sigma^h \mathbf{r}\Gamma_{\mathfrak{p}}(X) \\ &\cong \Sigma^{h-1} \Gamma_{\mathfrak{p}} \mathbf{s}(\mathbf{i}X) \\ &\cong \Sigma^{h-1} \Gamma_{\mathfrak{p}} \mathbf{s}(C)\,. \end{aligned}$$

The first isomorphism is by Theorem 5.1; the second is by Corollary 8.3; the third is from (9), which applies as R is Gorenstein, by Lemma 4.1. The last isomorphism is again by the claim above. This finishes the proof. □

In contrast with Proposition 8.6 and Proposition 8.7, we do not know if the Gorenstein property of $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ characterizes Gorenstein algebras; except when A is commutative.

THEOREM 9.8. — *Let R be a commutative Noetherian ring. The R-linear category $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R)$ is Gorenstein if and only if the ring R is Gorenstein.*

Proof. — The reverse implication is contained in Theorem 9.1.

Suppose $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} R)$ is Gorenstein as an R-linear category, with global Serre functor F. Let $\mathfrak{m}$ be a maximal ideal of R, and $k := R/\mathfrak{m}$ its residue field. The object $\mathbf{s}\mathbf{i}k$ in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is compact and $\mathfrak{m}$-torsion so Proposition 8.5 yields

$$\begin{aligned} \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{s}\mathbf{i}k, \mathbf{s}\mathbf{i}k)^{\vee\vee} &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{s}\mathbf{i}k, \Sigma^{d(\mathfrak{m})} F(\mathbf{s}\mathbf{i}k))^{\vee} \\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\Sigma^{d(\mathfrak{m})} F(\mathbf{s}\mathbf{i}k), \Sigma^{d(\mathfrak{m})} F(\mathbf{s}\mathbf{i}k)) \\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{s}\mathbf{i}k, \mathbf{s}\mathbf{i}k)\,. \end{aligned}$$

Thus, one gets an isomorphism of Tate cohomology modules

$$\widehat{\operatorname{Ext}}^0_R(k,k) \cong \widehat{\operatorname{Ext}}^0_R(k,k)^{\vee\vee} \quad \text{for each } i \in \mathbb{N}.$$

These modules are annihilated by $\mathfrak{m}$, and so are k-vector spaces. The isomorphism above implies that each of them has finite rank over k. It remains to recall the result of Avramov and Veliche [3, Theorem 6.4] that the finiteness of the rank of $\widehat{\operatorname{Ext}}^i_R(k,k)$ for *some* i already implies that $R_{\mathfrak{m}}$ is Gorenstein. □

The proof of the preceding result does not go through for non-commutative rings, for there exist finite dimensional algebras A over a field k that are not Gorenstein, and yet $\widehat{\operatorname{Ext}}^i_A(M,N)$ is finite dimensional over k for each i, and finite dimensional A-modules M, N; see, for example, [15, Example 4.3, (1), (2)].

Appendix A. Gorenstein approximations

Let $\mathcal{A}$ be an additive category. Recall that a complex $X \in \mathbf{K}(\mathcal{A})$ is called *totally acyclic* if the complexes of abelian groups $\operatorname{Hom}(W, X)$ and $\operatorname{Hom}(X, W)$ are acyclic for all $W \in \mathcal{A}$. When $\mathcal{A}$ is abelian, and $\mathcal{C} \subseteq \mathcal{A}$ is a class of objects, we set

$$
\begin{aligned}
{}^{\perp}\mathcal{C} &= \{X \in \mathcal{A} \mid \operatorname{Ext}^n(X, Y) = 0 \text{ for all } Y \in \mathcal{C},\ n > 0\} \\
\mathcal{C}^{\perp} &= \{Y \in \mathcal{A} \mid \operatorname{Ext}^n(X, Y) = 0 \text{ for all } X \in \mathcal{C},\ n > 0\}.
\end{aligned}
$$

A pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories of $\mathcal{A}$ is a (hereditary and complete) *cotorsion pair* for $\mathcal{A}$ if

$$
\mathcal{X}^{\perp} = \mathcal{Y} \qquad \text{and} \qquad \mathcal{X} = {}^{\perp}\mathcal{Y}\,,
$$

and every object $M \in \mathcal{A}$ fits into exact *approximation sequences*

$$
0 \longrightarrow Y_M \longrightarrow X_M \longrightarrow M \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow M \longrightarrow Y^M \longrightarrow X^M \longrightarrow 0\,,
$$

with $X_M, X^M \in \mathcal{X}$ and $Y_M, Y^M \in \mathcal{Y}$.

Gorenstein algebras. — Fix a ring A. Recall that an A-module is *G-projective*, if it is of the form

$$
C^0(X) := \operatorname{Coker}(X^{-1} \xrightarrow{d^{-1}} X^0)\,,
$$

for a totally acyclic $X \in \mathbf{K}(\operatorname{Proj} A)$. The *G-injective* modules are those of the form

$$
Z^0(X) := \operatorname{Ker}(X^0 \xrightarrow{d^0} X^1)\,,
$$

for some totally acyclic $X \in \mathbf{K}(\operatorname{Inj} A)$. We write $\operatorname{GProj} A$ for the full subcategory of all G-projective modules and $\operatorname{GInj} A$ for the full subcategory of all G-injective modules. The theorem below provides Gorenstein approximations for all modules over a Gorenstein algebra.

Let $\operatorname{Fin} A$ be the full subcategory of A-modules having finite projective and finite injective dimension. When A is a finite R-algebra, we consider the category

$$
\operatorname{Fin}(A/R) := \{M \in \operatorname{Mod} A \mid M_{\mathfrak{p}} \in \operatorname{Fin}(A_{\mathfrak{p}}) \text{ for all } \mathfrak{p} \in \operatorname{Spec} R\}.
$$

Observe that when the R-algebra A is Gorenstein, $\operatorname{Fin}(A_{\mathfrak{p}}/R)$ is the category of $A_{\mathfrak{p}}$-modules of finite projective—equivalently, finite injective—dimension. One of the consequences of the result below is that, at least for Gorenstein algebras, $\operatorname{Fin}(A/R)$ is independent of the ring R.

THEOREM A.1. — *Let A be a Gorenstein R-algebra. Then there are equalities*

$$
(\operatorname{GProj} A)^{\perp} = \operatorname{Fin}(A/R) = {}^{\perp}(\operatorname{GInj} A)\,.
$$

Also, $(\operatorname{GProj} A, \operatorname{Fin}(A/R))$ *and* $(\operatorname{Fin}(A/R), \operatorname{GInj} A)$ *are cotorsion pairs for* $\operatorname{Mod} A$.

The map $X_M \to M$ for $\mathcal{X} = \operatorname{GProj} A$ is called the *G-projective approximation*; we set $\operatorname{GP}(M) = X_M$. This module is unique up to morphisms that factor through a projective module. Analogously, the map $M \to Y^M$ for $\mathcal{Y} = \operatorname{GInj} A$ is called *G-injective approximation*, and we set $\operatorname{GI}(M) = Y^M$; it is unique up to morphisms that factor through an injective module.

Proof. — First, observe that any acyclic complex of projective or injective A-modules is totally acyclic by Theorem 5.6. This means that G-projective and G-injective modules are obtained from acyclic complexes.

We begin with the construction of G-injective approximations, using the recollement (1) as follows. Set $\mathcal{Y} = \operatorname{GInj} A$ and $\mathcal{X} = {}^{\perp}\mathcal{Y}$. Fix an A-module M. Then an injective resolution $\mathbf{i}M$ fits into an exact triangle

$$\mathbf{jq}(\mathbf{i}M) \longrightarrow \mathbf{i}M \longrightarrow \mathbf{s}(\mathbf{i}M) \longrightarrow$$

given by an exact sequence of complexes

$$0 \longrightarrow \mathbf{i}M \longrightarrow \mathbf{s}(\mathbf{i}M) \longrightarrow \Sigma(\mathbf{jq}(\mathbf{i}M)) \longrightarrow 0\,,$$

which is split-exact in each degree. Thus $Z^0(-)$ gives an exact sequence

$$0 \longrightarrow M \longrightarrow Y^M \longrightarrow X^M \longrightarrow 0\,,$$

with $X^M \in \mathcal{X}$ and $Y^M \in \mathcal{Y}$. The other sequence $0 \to Y_M \to X_M \to M \to 0$ is obtained by rotating this triangle. This justifies the claim that $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair; see [35, Theorem 7.12] for details.

It remains to identify $\mathcal{X}$, the left orthogonal to $\operatorname{GInj} A$. A standard argument yields the equality $\mathcal{X} = \operatorname{Fin} A$ when A is Iwanaga–Gorenstein. For a Gorenstein algebra A, the equality $\mathcal{X} = \operatorname{Fin}(A/R)$ follows once we can show that for each $\mathfrak{p} \in \operatorname{Spec} R$, the $\mathfrak{p}$-localization of an approximation sequence for $M \in \operatorname{Mod} A$ yields an approximation sequence for $M_{\mathfrak{p}}$ in $\operatorname{Mod} A_{\mathfrak{p}}$. It follows from the discussion in Remark 7.9 that for any A-module M, one has isomorphisms

$$(\mathbf{s}(\mathbf{i}M))_{\mathfrak{p}} \cong \mathbf{s}_{\mathfrak{p}}((\mathbf{i}M)_{\mathfrak{p}}) \cong \mathbf{s}_{\mathfrak{p}}(\mathbf{i}_{\mathfrak{p}}M_{\mathfrak{p}})\,.$$

This implies $(X^M)_{\mathfrak{p}} \cong X^{M_{\mathfrak{p}}}$ and $(X_M)_{\mathfrak{p}} \cong X_{M_{\mathfrak{p}}}$. Thus, both modules have finite projective and finite injective dimension. We conclude that $\mathcal{X} = \operatorname{Fin}(A/R)$.

Next, we consider G-projective approximations using the analogue of the recollement (1) for $\mathbf{K}(\operatorname{Proj} A)$. The proof that $(\operatorname{GProj} A, (\operatorname{GProj} A)^{\perp})$ is a cotorsion pair is similar to that for $\operatorname{GInj} A$, for it uses the right adjoint of the inclusion $\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A) \hookrightarrow \mathbf{K}(\operatorname{Proj} A)$; we omit the details. The equality $(\operatorname{GProj} A)^{\perp} = \operatorname{Fin}(A/R)$ can be verified as follows. Recall from Theorem 5.6 that there is an adjoint pair of triangle equivalences

$$\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A) \underset{\mathbf{h}}{\overset{E\otimes_A -}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)\quad.$$

Consider the exact triangle

$$E \otimes_A \mathbf{p}M \longrightarrow \mathbf{i}M \longrightarrow \mathbf{s}(\mathbf{i}M) \longrightarrow$$

from Lemma 2.5, which we used for constructing a G-injective approximation of M. Applying the equivalence $\mathbf{h}$ and rotating yields an exact triangle

$$\Sigma^{-1}\mathbf{hs}(\mathbf{i}M) \longrightarrow \mathbf{p}M \longrightarrow \mathbf{h}(\mathbf{i}M) \longrightarrow$$

which provides us with the G-projective approximation of M. We claim that for each $\mathfrak{p} \in \operatorname{Spec} R$, the $\mathfrak{p}$-localization of this triangle yields the Gorenstein-projective approximation of $M_\mathfrak{p}$. To this end consider the following diagram of exact functors.

$$\begin{array}{ccc} \mathbf{K}(\operatorname{Proj} A) & \underset{\mathbf{h}}{\overset{E\otimes_A -}{\rightleftarrows}} & \mathbf{K}(\operatorname{Inj} A) \\ {\scriptstyle (-)_\mathfrak{p}}\downarrow \uparrow{\scriptstyle \mathrm{res}} & & {\scriptstyle (-)_\mathfrak{p}}\downarrow \uparrow{\scriptstyle \mathrm{res}} \\ \mathbf{K}(\operatorname{Proj} A_\mathfrak{p}) & \underset{\mathbf{h}_\mathfrak{p}}{\overset{E_\mathfrak{p}\otimes_{A_\mathfrak{p}} -}{\rightleftarrows}} & \mathbf{K}(\operatorname{Inj} A_\mathfrak{p}) \end{array}$$

It is easily checked that for each A-module M, one has isomorphisms

$$(\mathbf{hs}(\mathbf{i}M))_\mathfrak{p} \cong \mathbf{h}_\mathfrak{p}((\mathbf{s}(\mathbf{i}M))_\mathfrak{p}) \cong \mathbf{h}_\mathfrak{p}\mathbf{s}_\mathfrak{p}(\mathbf{i}_\mathfrak{p}M_\mathfrak{p})\,.$$

This implies that $(Y^M)_\mathfrak{p} \cong Y^{M_\mathfrak{p}}$ and $(Y_M)_\mathfrak{p} \cong Y_{M_\mathfrak{p}}$. Thus, both modules have finite projective and finite injective dimension, so $(\operatorname{GProj} A)^\perp = \operatorname{Fin}(A/R)$. □

Remark A.2. — The above theorem shows that Gorenstein algebras are virtually Gorenstein in the sense of Beligiannis and Reiten [6], which means that the classes $(\operatorname{GProj} A)^\perp$ and ${}^\perp(\operatorname{GInj} A)$ coincide.

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