What can Partial Differential Equations tell us about life?

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Abstract

Partial differential equations were developed in the 18th century to model physical systems. Its inception has led to the continued development of a beautiful mathematical theory with an ever increasing range of applications. In 1890 Poincare observed that its encompassing framework can allow us see similarities in a wide range of physical applications. We now know that the similarities extend far beyond physical applications to other fields such as chemistry, biology, ecology and even sociology. In this article we provide a brief history of the applications of partial differential equations and showcase some recent works with applications in ecology and sociology.

1 Introduction

Our world is constantly changing in very complex ways and it is human nature to look for the factors driving these changes. From trying to understand planetary motion, dating back to 3000 BCE, to the Covid-19 pandemic that we are still living with, researchers have been working to gain insight into the fundamental mechanisms leading to change. The invention of Calculus in the late 17th century was an important development, as it opened up a world of possibilities to provide mathematical frameworks for numerous dynamic complex systems.

Reflecting on our daily lives we see that physical space, x, and time, t, are two key independent variables which affect how things change. Moreover, it is often a challenge to determine a physical quantity, such as heat or population density as functions of x and t. Instead, it is easier to postulate relations between these physical quantities and their derivatives. From these relations we then hope to be able to then determine the physical quantity in question. This is how partial differential equations enter the picture. A partial differential equation (PDE), which is a relation between an unknown function $u : \mathbb{R}^n \to \mathbb{R}^m$ and its derivatives provides a perfect mathematical framework to understand changing systems. A simple example has the form:

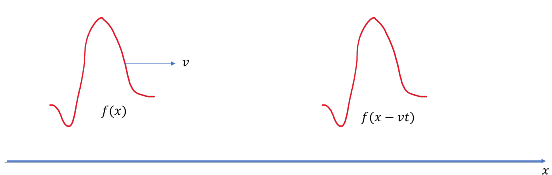
$$\mathcal{F}\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}\right) = 0,$$

where $u : \mathbb{R}^2 \to \mathbb{R}$. These equations can be seen as mechanistic models, that is, they use theory to predict the real-world. These type of equations were developed in the 18th century to model physical phenomena, such as heat and wave phenomena, by mathematicians such as Euler, d'Alembert, Lagrange, and Laplace [7]. This mathematical framework has since been used in other areas of physics and engineering. In fact, in 1890, Poincare remarked on the wide range of applications of this framework [28], emphasizing that various physical problems "had an air of similarity" when observed from the point-of-view of partial differential equations. In those days, the applications were still restricted to physical phenomena, such as electrodynamics, magnetism, fluids, optics, and heat. Moving forward we will use the notations $\partial_t u, u_t$ and $\frac{\partial u}{\partial t}$ to denote the partial derivatives interchangeably and will consider positive time $t > 0$.

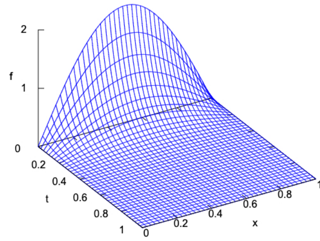
In 1759, d'Alembert introduced one of the first equations, the so-called *wave equation,* to model the vibration of a string, such as one in a musical instrument. Imagine you have a taught string from location $x = 0$ to $x = L$ and you pluck the string at $t = 0$. How do the dynamics play out? To begin to answer this, let $u(x, t)$ denote the displacement of the string at position x and time t away from its equilibrium position (a straight horizontal line). The dynamics can be modeled using Newton's second law of motion, $F = ma$ (the force, F, equals mass, m, times acceleration, a) applied to an infinitesimal length of the string. Assuming the mass is one, so $m = 1$, the right hand side of the equation is simply $a = \frac{\partial^2 u}{\partial t^2}$. Moreover, a careful analysis of the tension on the infinitesimal length of the string gives that the force is proportional to $\frac{\partial^2 u}{\partial x^2}$. The equation has the form:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad x \in \mathbb{R}, \quad t > 0. \tag{1}$$

To determine the unknown u we must impose an initial position of the string at time $t = 0$, denoted by $u(x, 0)$, this is the so-called *initial condition* and also keep in mind that $u(0, t) = u(L, t) = 0$ for all $t > 0$, which are the so-called *boundary conditions.* As one would expect, the wave equations has wave solutions, which are moving solutions at have a fixed speed and when it moves is does not change its shape. Figure 1(a) illustrates a cartoon solution of the wave equation. You can see the solution at time $t = 0$ is $f(x)$ and the form is traveling with velocity v. At time t the form has not changes, simply moved.



(a) Wave equation solution.



(b) Heat equation solution (1).

Figure 1: (a) A cartoon illustration of a solution to the wave equation (1) (b) A solution to the heat equation with boundary conditions $f(0, t) = f(1) = 0$.

(1) Credit: By Quartl - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=14318045.

The *heat equation*, another classical partial differential equation, was introduced by Fourier in his memoir "Théorie analytique de la chaleur" (1810-1822) and has the form:

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \quad x \in \mathbb{R}, \quad t > 0, \tag{2}$$

where $f(x, t)$ is the temperature at location x and time t. Imagine a rod of length L positioned between location $x = 0$ and $x = L$. Since heat moves down gradients of the temperature (from high temperature areas to low temperature) the rate of change of temperature at location x and time t is equal to the diffusion of u. Mathematically, the term $\frac{\partial f}{\partial t}$ represents the rate of change of the temperature with respect to time and $\frac{\partial^2 f}{\partial x^2}$ models diffusion, which will be discussed in the next section. These solutions spread as time increases. An example solution is illustrated in Figure 1(b): What we observe is the solution to $f_t - f_{xx} = 0$ on $[0, 1] \times [0, \infty)$ with initial condition $f(x, 0) = 2\sin(\pi x)$ and zero boundary conditions $f(0,t) = f(1,t) = 0$. Observe that the solution diffuses to zero as time increases. Note that these examples are one-dimensional. Higher dimensions may be relevant and will be discussed later.

As the decades passed the applications of PDEs widened from chemistry, in the middle of the 20th century, to the social sciences in recent decades. This work aims to provide a brief history of the use of PDEs in the fields of chemistry, where reaction-diffusion equations where introduced, biology, ecology, and the social sciences. Moreover, we will showcase a few recent advances in the areas of ecology and sociology. Note also that the development of the theory of PDEs has had a significant impact outside of applications and, in fact, it has helped develop many areas of pure mathematics. We will not discuss the connection between numerous pure mathematics fields and PDEs and instead refer the reader to Brezis and Browder's beautiful summary given in [7].

1.1 Reaction-Diffusion systems in chemistry

A subclass of PDEs with application in chemistry are Reaction-Diffusion (RD) equations which model the dynamical process of particles reacting (chemical reactions) and spreading. This field arose from the desire to understand pattern formation, which one can see all around us and at all scales. Some examples are galaxies, snowflakes, and animal coat patterns [4]. Alan Turing set out to understand the origins of pattern formation in animal coats from the point of view of chemistry. An example of such patterns is illustrated in Figure 2. Figure 2(a) illustrates the coat of a giraffe and Figure 2(b) illustrates the coat of a zebra. In his paper "Chemical basis of morphogenesis" published in 1952 [35], Turing introduced hypothetical chemical reactions and mathematically framed them in a RD system. These reactions could break the symmetry of an initially homogeneous mixture and create pattern formation when there was diffusion at play. Diffusion here is the process by which the morphogens, a substance that governs the pattern of tissue development, spread though the tissue. Reaction is the process that creates and destroys morphogens [4].

(a) Pattern formation in the coat of a giraffe.(1)

(b) Pattern formation in the coat of a zebra.(2).

Figure 2: Illustration of patterns formed in the coats of two different animals.

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To understand these processes better, let us consider individual particles that are moving around randomly, following a so-called *Brownian motion* [36]. Figure 3(b) illustrates a sample path of a particle following Brownian motion. Note that seeming randomness of the particle path. When there are a very large number of particles it becomes most cost-effective to keep track of the dynamics of the density of particles, which we can call u, instead of keeping track of each individual particle. It so happens that the time evolution of the density satisfies the heat equation, written out in three dimensional space here:

$$\frac{\partial u}{\partial t}(x,y,z,t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad (x,y,z) \in \mathbb{R}^3 \ t > 0.$$

In short notation we write

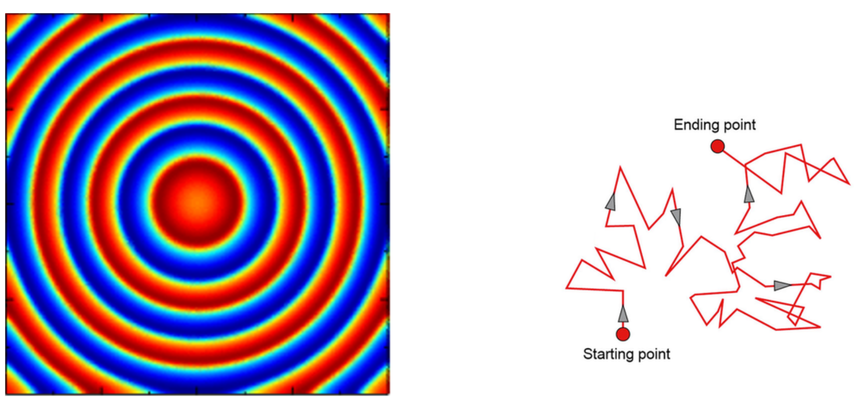
$$\frac{\partial u}{\partial t} = \Delta u,$$

where $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$, in three-dimensions, is the so-called *Laplace operator* and models diffusion, *e.g.* spread.

If we have different types of particles which are moving about randomly and colliding, they may react and form different substances. Let us denote our two densities of the morphogens by u and v, the reaction can be incorporated into our evolution systems though two production/consumption functions f and g:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(u,v), \\ \frac{\partial v}{\partial t} = \Delta v + g(u,v). \end{cases} \tag{3}$$

The FitzHugh-Nagumo system, is a special case of (3), and models the spike generation of an axon. Here we have that $f(u,v) = u - \frac{u^3}{3} - v + I_{current}$ and $g(u,v) = a(u + b - cv)$ with $a, b, c, I_{current}$ fixed constants. This is an example of an excitable system, where if the impulse, given by $I_{current}$, is sufficiently large the variables u and v embark on a journey before settling back to their relaxed states. It is this journey that allows for the existence of spiky solutions, which represent traveling signals along an axon. The solution, u, to first equation in two-dimensions and for a fixed time t is illustrated in Figure 3(a). This exemplifies the patterns formed by solutions to a reaction-diffusion equation. This is the solution at a fixed time snapshot and one can observe the pattern formation formed on the x-y plane.



(a) Pattern formation in a reaction-diffusion equation.[1]

(b) Sample Brownian motion particle path[2].

Figure 3: (a) An illustration of a solution to a reaction-diffusion equation in two-dimensions (b) A sample path of a particle in Brownian motion.

(1) Credit: Reaction diffusion target by Dr. H. U. Bödeker under GNU Free Documentation License

(2) Credit: Brownian Motion Diagram by NivedRajeev 2019 under Creative Commons Attribution-Share Alike 4.0 International.

1.2 Reaction-Advection-Diffusion equations in ecology and biology

The use of Reaction-Advection-Diffusion (RAD) equations to study movement in ecology was popularized after Skellam's 1951 paper where he related the random motion of animals with the heat equation [34]. By and large, in a context without competition, we can model population dynamics with the following general equation:

$$\begin{cases} u_t(x,t) = \mathcal{M}[u] + f(x,t,u), \quad x \in \mathbb{R}^n, \ t > 0, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}^n, \end{cases} \tag{4}$$

where $u(x,t)$ represents the population density at location x and time t and $u_0(x)$ is the initial distribution of the population. Note that $x \in \mathbb{R}^n$ is now a vector in n-dimensional space. For example, in spatial ecology the relevant spaces are $n = 2$ and $n = 3$. As in earlier models presented $u_t(x,t)$ represents the rate at which a population is changing with respect to time at location x and time t. This rate depends on the movement of species to and from location x plus the growth death that occurs at that particular location. We denote the movement by a general operator $\mathcal{M}$ and the growth pattern is denoted by f.

A classical movement strategy which is assumed for many species is that of dispersal, or spreading of a population. In this case, the appropriate movement operator is the Laplace operator, $\mathcal{M}[u] = \mu\Delta u = \mu\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}u$, stated here with diffusivity coefficient μ. Specifically, the process of dispersal and growth/death on the whole space has been traditionally modeled by the now classical reaction-diffusion equation:

$$\begin{cases} u_t = \mu\Delta u + f(u,x), \ x \in \mathbb{R}^n, \ t > 0, \\ u(x,0) = u_0(x); \end{cases} \tag{5}$$

see for example [34, 9] and references within.

In the classical reaction-diffusion model for population dynamics, species does not take the environment or population density into account when moving. However, it is known that some animals do make use the environment to inform their movement patterns. One movement strategy that incorporates environmental cues is *taxis*, which is the movement of an organism in response to a stimulus such as a chemical, light, or a general environmental signal. For example, in ecology the environmental signal can include the density of food resources or the density of predators. Classically, this has been incorporated as an *advection term*, the transfer of matter by a velocity field, $\vec{v}$, mathematically this is represented by $\nabla \cdot (u\vec{v})$. In this case, organisms determine the velocity field using an environmental signal, $\vec{v} = \nabla A(x)$, where A represents the environmental signal. This is the taxis movement shown in Table 1. Sometimes animals use non-local information (information about their surrounding neighborhood), this can be incorporated using a so-called *kernel* function, $\mathcal{K}$. The term $\mathcal{K} * u$ models a velocity field that helps a population use non-local information to aggregate. One can observe the non-local nature of the velocity field from the definition of a convolution:

$$\mathcal{K} * u = \int_{\mathbb{R}^n} \mathcal{K}(x-y)u(y)\, dy.$$

You can think of a convolution as a non-local average with function $\mathcal{K} : \mathbb{R}^n \to \mathbb{R}$ taken as a weight. The velocity then is given by $\vec{v} = \nabla\mathcal{K} * u$ and directs the population towards areas with a high concentration of the population. Table 1 displays a number of typical movement strategies and their mathematical representations.

Table 1: A sample of movement strategies and their mathematical representations

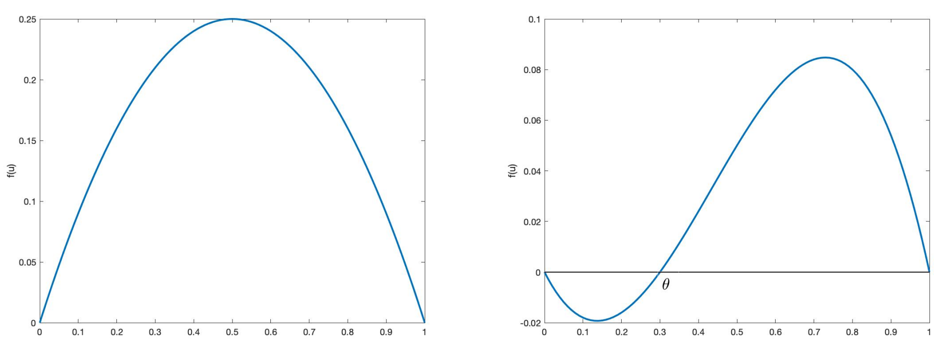
Movement Strategy	**Operator $\mathcal{M}$**
Dispersal	$\mu\Delta u$
Taxis along the signal A	$-\nabla\cdot(u\nabla A(x))$
Non-local aggregation with kernel $\mathcal{K}$	$-\nabla\cdot(u\nabla\mathcal{K}*u)$
Dispersal & taxis movement	$d\Delta u-\nabla\cdot(u\nabla A(x))$

Let us now shift to discussing the typical growth functions seen in the literature. When the population is subject to logistic growth, that is, when a population's per capita growth rate gets smaller as population size approaches a the carrying capacity, K, which is imposed by the environment, this equation us known as the Fisher-KPP equation:

$$u_t = \Delta u + u(K-u). \tag{6}$$

Note that in equation (6) $u(K-u)$ represents the product of u with $K-u$. Equation (6) was introduced by Fisher to model the spread of advantageous genetic traits in a population [14]. That same year Kolmogorov, Petrovsky, and Piskunov published a first mathematical analysis of the equation [21]. Since then it has been the subject of much interest and research to this day.

Not all populations are subject to logistic growth. In 1930 Warder Clyde Allee, a zoologist and ecologist working in the Marine Biological Laboratory at the University of Chicago, experimentally showed that survival rate of goldfish was positively correlated with population density. In his work [1], Allee concluded that aggregation and cooperation are beneficial for the survival of species. The effect of decreased individual fitness at low population densities as dubbed the Allee effect [22, 1, 24]. For population which are subject to the Allee effect the prototypical reaction term is given by $f(u) = u(1-u)(u-\theta)$, where θ is the Allee threshold and satisfies $0<\theta<1$. Note that at the microscopic level, individuals are simply moving randomly here. Figures 4(a) and 4(b) illustrate prototypical logistic and Allee growth patterns with carrying capacity one. In other words these figures illustrate two different functions f.



(a) Monostable production pattern

(b) Bistable production pattern

Figure 4: Two typical production/consumption pattern that appear in RD equations.

1.3 Social applications

Mathematical sociology aims to take sociological theories and express them in mathematical terms. Having a mathematical framework allows one to run simulations and perform mathematical analysis that can help us gain insight into very complex phenomena. The use of differential equations as mathematical frameworks to study the dynamics of social processes goes back as far as [10] and [6] who advocated for the use of differential equations to study dynamical systems. However, the subject has only recently taken off in the beginning of the 21^{st} century and remains an area full of opportunities.

The use of PDEs to understand socio-economic systems has has many significant advances during the last few of decade [8]. A prominent example, dates back to 1973, is the use of stochastic (modeling randomness) differential equations to model pricing in finance and insurance. Ideas from statistical mechanics, which studies the interactions of particles, have also been carried over to the social sciences to model the evolution of wealth, opinion formation and herding – see [8] and references within.

PDEs have also found applications in urban crime. While crime is ubiquitous it tends to aggregate spatially and create spatio-temporal patterns. In [32] the authors introduced a system of reaction-advection-diffusion equations to understand the spatio-temporal dynamics of urban crime. We will discuss the sociological assumptions being modeled in this equation in section 3.1. This has led to much research including extension of the model in [20] and [27] and interesting insight into hotspot policing [31].

2 Ecological applications

The study of movement of organisms is a key subject of spatial ecology, which investigates the plethora of spatial patterns in nature and their ecological consequences. It is known that the movement of organisms is key to their survival. In fact, animals move to forage for food, run away from predators, as well as expand and establish their territory. As our environment continues to changes more and more drastically, two questions become very relevant. First, what are the leading factors informing the movement of animals? Second, what are optimal movement strategies? Insight into these two key issues is vital in wildlife management. In fact, as stated by Andrew Allen and Navinder Singh, Professors in the Department of Wildlife, Fish and Environmental Studies, Swedish University of Agricultural Sciences in [2] "A common challenge in species conservation and management is how to incorporate species movements into management objectives. There often is a lack of knowledge of where, when, and why species move."

In the following two subsections we discuss a couple of recent work related to the issues discussed above. We first focus on how one can use reaction-diffusion-equations and animal location time series to gain gain knowledge about the factors that lead to animal movement. We use meerkats as a case study. Following that, we discuss how one can analyze movement strategies that can help species survive.

2.1 Key factors leading to movement

Partial differential equations can be used to study how species live and move in their environment by utilizing the movement factors hypothesized by field experts. This framework has been used to describe phenomena such as foraging [16, 17], aggregation [18], and home ranges [23, 25]. In [19] and [26], mechanistic home range models incorporated diffusion and attraction to a localizing center, such as a den site, to generate stable territory patterns. In some situations, it is hypothesized that animals, such as Meerkats, use their memory to inform movement [3]. In particular, this means that they are able to use non-local information to inform their movement strategies.

We model the dynamics of movement for social species that live in competing groups. We consider a non-local PDE model that incorporates the different, and sometimes competing, factors of movement. These factors include: an intra-species (within group) long-range attraction and short-range repulsion (an overcrowding effect); an inter-species (between group) repulsion; the use of an environmental signal. The model reads as follows:

$$\partial_t u_i(x,t) = \eta \Delta u_i^2(x,t) - \nabla \cdot \left[u \nabla \left(\mathcal{K} * u_i - \sum_{j=1, j\neq i}^{N} \mathcal{K} * u_j + A(x,t) \right) \right] \tag{7}$$

for $x \in \Omega \subset \mathbb{R}^d, t > 0$, where u_i represents different competing groups, with $i = 1, 2, ..., N$. In the above system the symbol $*$ represents a non-local average, specifically,

$$\mathcal{K} * u(x) = \int K(x-y)u(y)\, dy.$$

The constant η in (7) represents the intra-group dispersal rate, the convolution term represents intra-group aggregation, and inter-group repulsion is governed by the function $\mathcal{K}$. Note that the long-range aggregation term moves the group u_i with a nonlocal velocity $-\nabla\mathcal{K} * u_i$, which helps maintain the group coherent. Moreover, the long-range, inter-group repulsion term moves the population u_i away from other groups via the velocity field $\sum_{j=1,j\neq i}^{N} \nabla\mathcal{K} * u_j$ and serves as a segregation term.

Physical interpretation	**Mathematical Formulation**
Rate of change of u with respect to time	$\partial_t u_i(x,t)$
Intra-species short-range repulsion	$\eta\Delta u_i^2(x,t)$
Intra-species long-range attraction	$-\nabla \cdot (u_i \nabla\mathcal{K} * u_i)$
Inter-species segregation	$u_i \sum_{j=1,j\neq i}^{N} \nabla\mathcal{K} * u_j$
Taxis up a signal	$-\nabla \cdot (u_i \nabla A)$

Figure 5 illustrates relocation data from meerkats belonging to different competing groups. The locations where these meerkats where observed helps map the territory. One can observe that although they do not have a home center there is coherence to their territories.

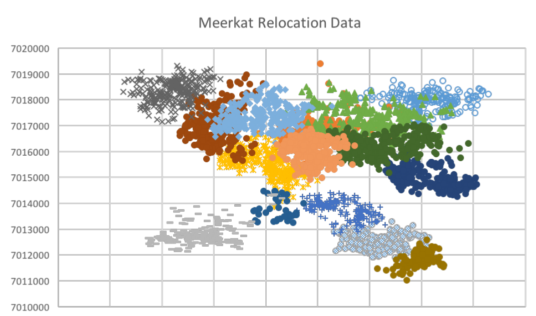
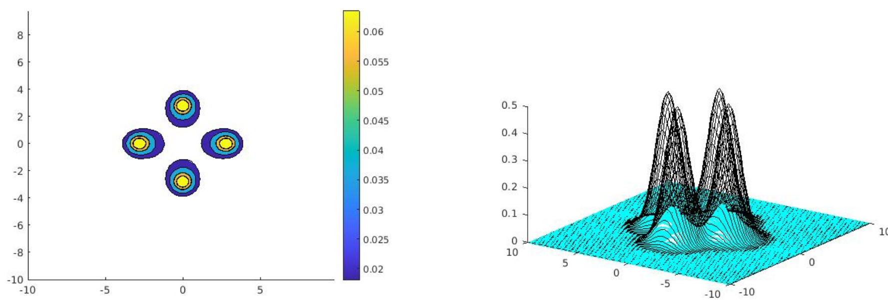


Figure 5: Relocation data for meerkat groups, each represented by a different color.

Note that it is basically impossible to find a formula for the solution to systems such as the one given in (7). Our goal then becomes to understand as much about the solutions without having an exact formula. For example, one can still try to understand the long-term behavior of the population depending on the initial distributions, properties of $\mathcal{K}$ and A, and the other parameters. Another thing we can try to do is to solve the solutions numerically. The integral term in the system becomes problematic and one has to work hard to find ways to solve the equation in an efficient way. We provide some illustrations of solutions to system (7), for various scenarios. Here we use a so-called *spectral method*, which moves our functions from physical space to frequency space, solves the equations there, and then translates them back to physical space. Figure 6 illustrates the long-time solution ($t \to \infty$) of the system (7) with four competing groups. For illustration purposes we use a signal A that represents an environment which is most beneficial at the center of the territory and the benefit decays radially as one moves away from the center. We observe that the groups segregate and form their territories around the optimal spots.



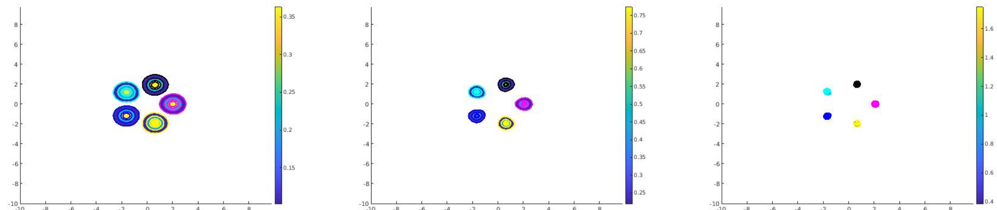
(a) Contours of the final distribution

(b) Final distribution of the groups

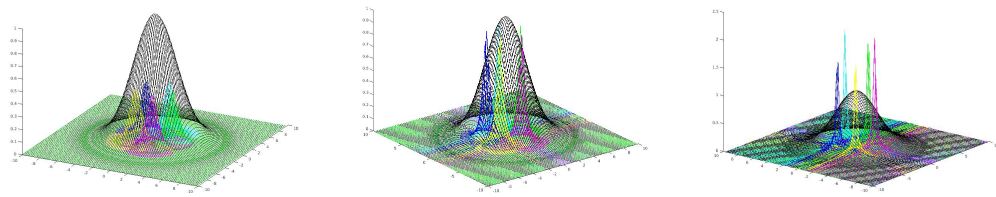
Figure 6: Final distribution of four groups interacting with a Gaussian environment.

Figures 7 and 8 provide illustrations of solutions to the model where we have removed some of the movement factors. This will illuminate the role that some of the terms play in the dynamics of the solution to our model. The solutions illustrated in Figure 7 do not incorporate any overcrowding effect (that is the short-range inter-species repulsion). In this case, the parameter is $\eta = 0$. One can observe a few time steps in the evolution of the population densities. Note that the population densities are blowing-up (concentrating into a single point). The initial population densities are Gaussian like and as time increases we begin to see that the densities become pointy and the magnitude increases. This provides evidence that short-range repulsion is an important factor in the movement strategy of meerkats. The population densities illustrated in Figure 8 are the solutions to the model without the inter-species repulsion (no segregation term). In this case, we observe that all population aggregate around the optimal environmental regions. Again, this confirm the significance of the segregation term in order for the solution of our model to match the dynamics observed in the data.

Finally, we want to see what happens if we incorporate a real life environmental signal using information about sand-type and density. It is hypothesized that they prefer clay sand over ferrous. Figure 9(a) illustrates the sand type, blue corresponding to clay sand and yellow corresponding to ferrous sand. Figure 9(b) illustrates the final solution of our system using the environment. You can see that the solutions do a reasonable job matching the data.

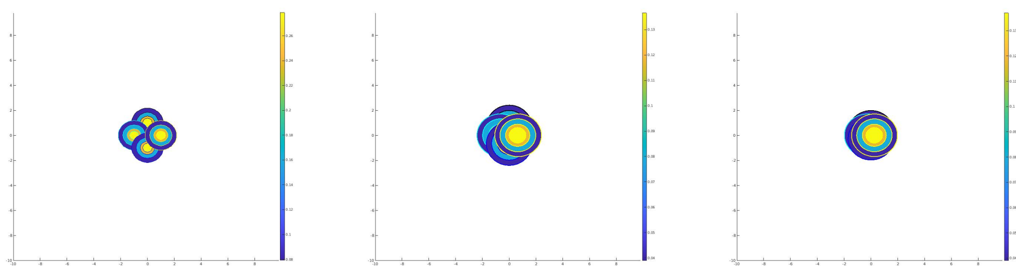


(a) $u_i(\mathbf{x}, 0)$ (b) $u_i(\mathbf{x}, t_1)$, $t_1 > 0$ (c) $u_i(\mathbf{x}, t_2)$, $t_2 > t_1$



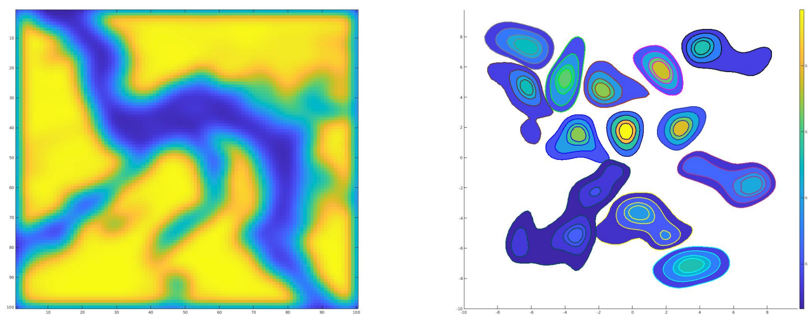
(d) $u_i(\mathbf{x}, 0)$ (e) $u_i(\mathbf{x}, t_1)$, $t_1 > 0$ (f) $u_i(\mathbf{x}, t_2)$, $t_2 > t_1$

Figure 7: Dynamics of solutions without overcrowding effect and $A(\mathbf{x})$ in black.



(a) $u_i(\mathbf{x}, 0)$ (b) $u_i(\mathbf{x}, t_1)$, $t_1 > 0$ (c) $u_i(\mathbf{x}, t_2)$, $t_2 > t_1$

Figure 8: Dynamics of the solution with no segregation function and $A(\mathbf{x})$ from Figure 7.



(a) Sand data

(b) Long-time solution

Figure 9: Sand environmental function and the corresponding equilibrium solution (long term solutions) when using the sand as an environmental function.

2.2 Optimal movement strategies

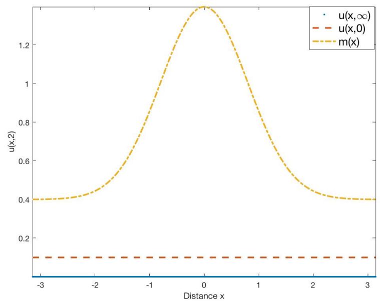
Populations tend to use heterogeneities in the environment and positive influence of the presence of congeners (similar animals) to aggregate and take advantage of social structures [30]. Thus, on the one hand, species develop movement strategies using environmental cues in order to aggregate and potentially increase their fitness. On the other hand, dispersal, the spreading of organisms from on location to another, has also been found to help populations establish themselves. For example, dispersal can help populations control their size or expand their territory. While dispersal can be beneficial to species, in some situations it has been found to be detrimental. For example, high dispersal rates can prevent the adaptation of a population to a new environment, which can hinder the survival of small populations [29].

To further understand these movement strategies we consider the evolution of a population density subject to the dynamics modeled by:

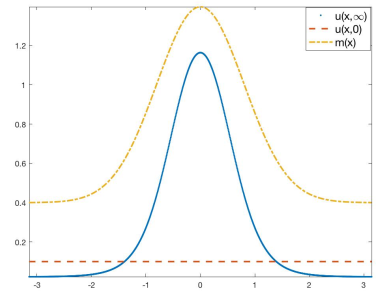
$$\begin{cases} u_t = \mu\Delta u - \chi\nabla\cdot(u\nabla m) + f(u),\ x\in[-L,L],\ t>0, \\ (\mu\nabla u - \chi u\nabla m)\cdot n = 0, \quad x\in\{-L,L\},\ t>0, \\ u(x,0) = u_0(x),\ x\in[-L,L]. \end{cases} \tag{8}$$

where $f(u) = u(1-u)(u-.3)$. The movement strategy modeled in (8) is a combination of dispersal (strength measured by μ) and movement up gradients of a spatially varying environmental signal m.

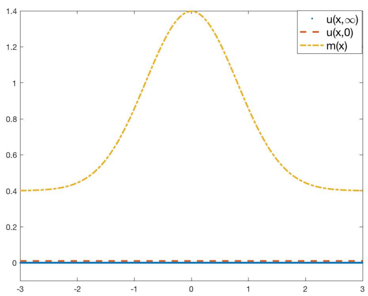
We observe that if the initial population is below the Allee threshold (here $\theta = .3$) and the "speed", χ, at which the population moves using the environment is too small then in the long term the population will become extinct in the long term. This is illustrated in Figures 10(a) and 10(c). In all cases illustrated in Figure 10 the initial population is constant. In Figure 10(a) the initial population is 0.1 everywhere (dashed red line) and $\chi = 2$; thus, the population does not aggregate sufficiently fast and eventually becomes extinct (solid blue line). On the other hand when χ is sufficiently large, this leads to the population aggregating, and ultimately persisting. Figure 10(b) illustrates this scenario with the same initial data as in Figure 10(a) but with $\chi = 3$. One can observe that the population persists and aggregates where the signal is highest (solid blue line). The smaller the initial population the faster the population must aggregate as is illustrated in Figures 10(c) and 10(d). Here the initial population is 0.01 everywhere (dashed red line) and the former figure illustrates the final density of the population (solid blue line) when $\chi = 50$ when the population becomes extinct, the latter illustrates the case when $\chi = 150$ and here the population persists.



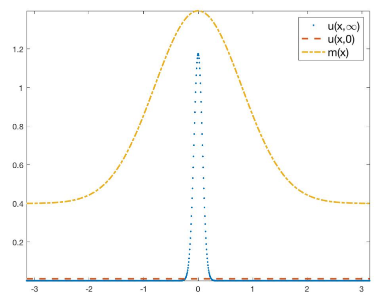
(a) Initial data $u_0(x) \equiv .1$ with $\chi = 2$



(b) Initial data $u_0(x) \equiv .1$ with $\chi = 3$



(c) Initial data $u_0(x) \equiv .01$ with $\chi = 50$



(d) Initial data $u_0(x) \equiv .01$ with $\chi = 150$

Figure 10: Illustration of numerical solutions to (8) with constant initial data, u_0, directed movement parameter χ, and growth-pattern $f(u) = u(1-u)(u-.2)$. The signal $A = m$.[1]

The key take away is that populations that are subject to an Allee effect can actually survive, even if they are very small initially, provided they aggregate sufficiently fast. Interestingly, it is also the case that when a population is too greedy (moving up gradients of the resources too quickly) they may be outcompeted by another population employing a movement strategy that is balanced between aggregation and dispersal.

3 Applications in social complex systems

While it is clear that we cannot use mechanistic models to understand individual behavior, it is also evident that when individuals act as a collective, spatio-temporal dynamics may emerge. This can be observed in data for urban crime, protesting activity, and the Covid-19 pandemic dynamics. If we forget about understanding individual behavior but rather focus on understanding the macroscopic patterns that emerge, we can make headway using partial differential equations. In the next two sections we discuss the use of PDEs in urban crime and social outburst such as protests.

3.1 Urban crime

A relatively recent application of partial differential equations which as garnered much interest is in the understanding of urban crime. While crime can happen everywhere, real-life data shows that certain regions have a disproportionate level of crime. These are the so-called *crime hotspots*. In 2008 a team at UCLA developed a PDE model for urban crime based on the sociological theories of repeat-and-near repeat victimization effects [33] and routine activity theory [13] in [32]. The former is the effect that crime in a location leads to an increased probability of a second crime occurring at the same, or nearby, locations. The latter is the theory that the most important factor in a crime transpiring is opportunity. It is important to remark that mathematical modelers only provide mathematical framework to the theories developed by the experts. Thus, the results of this model only hold true if the theories they model are true. We should point out that there was a recent article, [15], criticizing the sociological assumptions in [32]. Nevertheless, this model is our starting point:

$$\begin{cases} A_t = \mu A_{xx} - A + A\rho + \alpha, & x \in (0,L), t > 0, \\ \rho_t = (d\rho_x - 2\rho \ln(A)_x)_x - A\rho + \beta - u\rho, & x \in (0,L), t > 0, \\ (A,\rho,u)(x,0) = (A_0,\rho_0,u_0)(x) \geq 0, \not\equiv 0, & x \in (0,L), \\ A_x(x,t) = \rho_x(x,t) = u_x(x,t) = 0, & x = 0, L, t > 0. \end{cases} \tag{9}$$

Here μ, d, α, and β are constants. We discuss the model in one-dimension for simplicity. The *criminal density* is denoted by ρ and A represents a field that measures the propensity towards crime. This field is changing spatially and temporally and incorporates the repeat-near-repeat victimization effect. Notice the similarities of system (9) and equation (8). In particular, the movement strategy of criminals has similarities to the movement strategy modeled in (8). Indeed, criminals have a strategy composed of dispersal and taxis up gradients of a signal $2\ln(A)$, which means that their speed is dependent on A, slowing down as A increases. The assumption that crime leads to more crime is evident in the term $A\rho$ seen in the first equations, which models the expected number of crimes.

Later Jones and collaborators in [20] and Pitcher in [27] added the dynamics of police movement. The generalized model reads as follows:

$$\begin{cases} A_t = \mu A_{xx} - A + A\rho + \alpha, & x \in (0,L), t > 0, \\ \rho_t = (d\rho_x - 2\rho A_x/A)_x - A\rho + \beta - u\rho, & x \in (0,L), t > 0, \\ u_t = (D_u u_x - \chi u A_x/A)_x, & x \in (0,L), t > 0, \\ (A,\rho,u)(x,0) = (A_0,\rho_0,u_0)(x) \geq 0, \not\equiv 0, & x \in (0,L), \\ A_x(x,t) = \rho_x(x,t) = u_x(x,t) = 0, & x = 0, L, t > 0, \end{cases} \tag{10}$$

where we have added the dynamics of the police density, u, who are simply following a movement strategy. In a recent work [12] we set out to understand the possible long-term behavior of solutions to system (9). We found that the possibilities are rich and can be nicely summarized using illustrations. You may find the mathematics behind these pictures in a joint work with Wang and Zhang in [12]. Note that the dynamic of the solutions are determined by the parameters of the model. In a certain parameters the solutions do form spatial crime hotspots as seen in Figure 3.1. The top row illustrates a time snapshot of the solutions (A, ρ, u). The bottom row illustrates the density of the solution using a colormap (red means high and blue low) with space on the horizontal axis and time on the vertical axis.

In another parameter regime the solutions are time periodic, that is, we observe *temporal crime waves*. These solutions are illustrated in Figure 12 and are better observed in the first row of figures. Indeed, on the horizontal axis you have time and the vertical axis illustrates the value of the solutions for the specific locations of $0, \frac{\pi}{4}, \frac{\pi}{2}$, and $\frac{3\pi}{4}$. Observe that it takes a while for the time-periodic solutions to form, but by time $t = 150$ the temporal crime waves are well-formed.

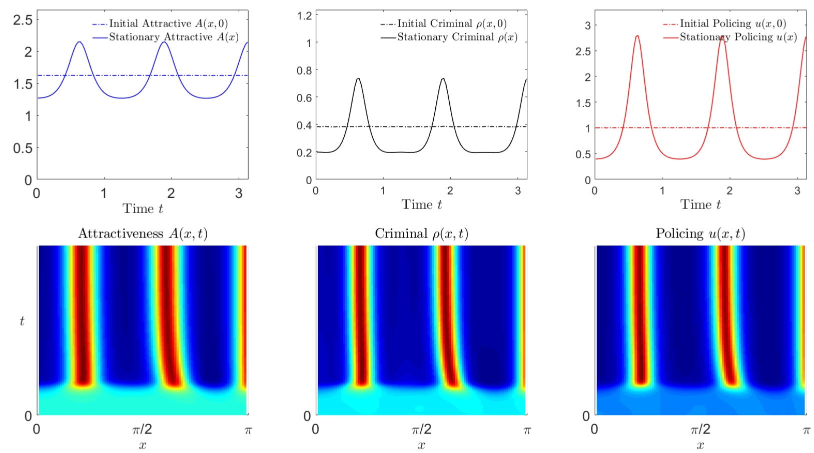


Figure 11: Formation of crime hotspots.[1]

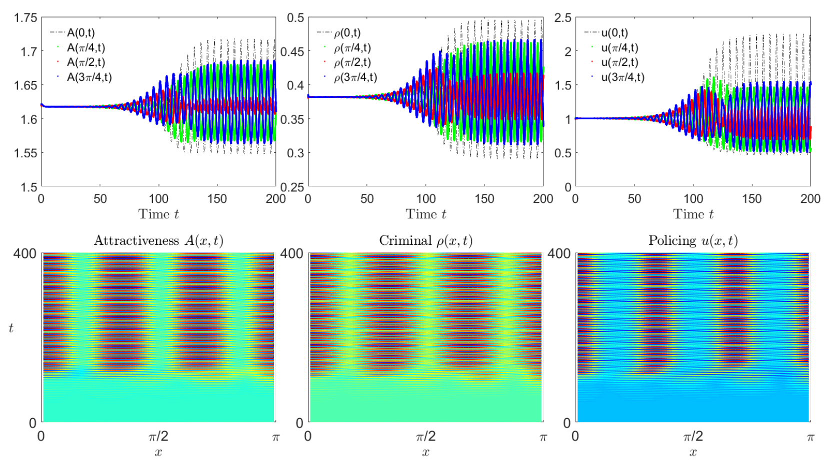


Figure 12: Formation of time-periodic hotspots.[1]

A final parameter regime leads to chaotic solutions. In this situation the crime levels, in all locations, vary in a seemingly random manner. Moreover, if the system is fed two distinct initial conditions which are infinitesimally close (as close as you would like them to be) the corresponding solutions diverge. This observation is quite important as it allows us to conclude that this model cannot be used with data, for the reason that given the error in the data one cannot trust the results in data fitting processes.

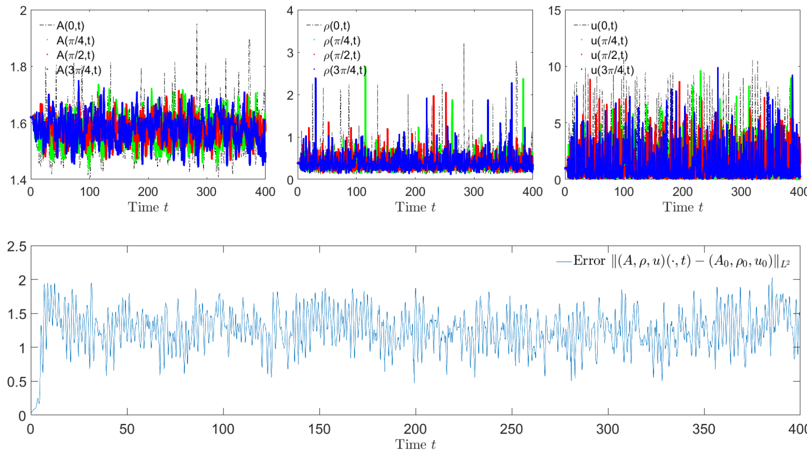


Figure 13: Chaotic dynamics.[1]

(1) Credit: Originally published in [12]

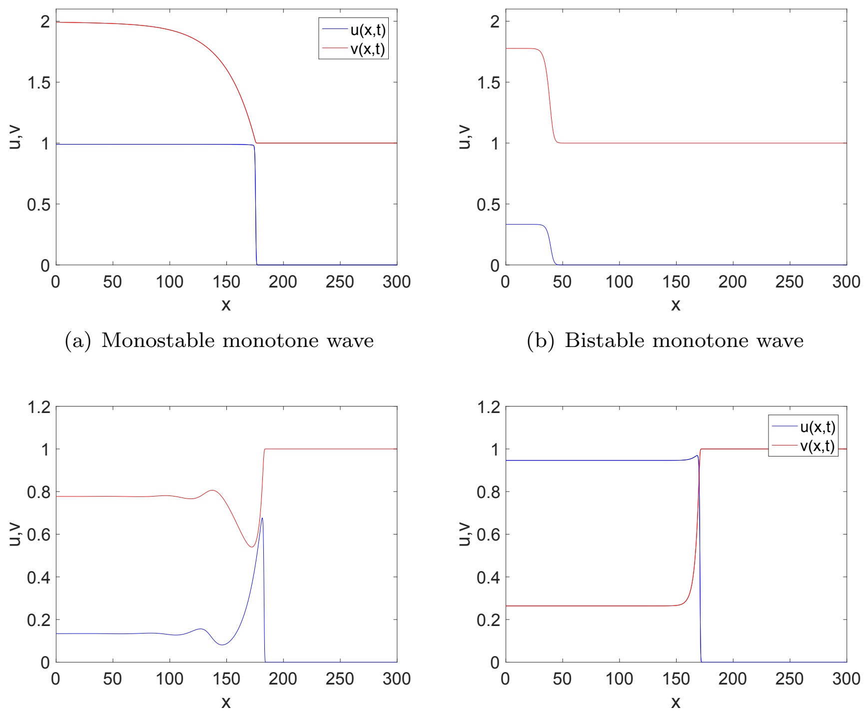
3.2 Traveling waves of social outbursts of activity

Civil unrest, protests, and rioting are tools that populations use to express objection or dissent towards an idea or action, typically political. These outbursts of social activity have been ubiquitous in time and space and, in many cases, have changed the course of history. From the religious protest in the early sixteenth century to the Euromaidan in 2014, the activity amplifies in time and sometime spread in geographical space. Moreover, there is an underlying "tension" that helps drive the activity.

In [5], in collaboration with Berestycky and Nadal, we introduce a reaction-diffusion model for the dynamics of rioting activity (or unrest) and social tension, motivated by the 2005 French riots. The model assumes a self-excitation effect on the level of unrest that turns on when the social tension is above a certain value. It further assumes spatial contagion is local and modeled by the classical diffusion operator. The levels of unrest and social tension at location x and time t are represented by $u(x,t)$ and $v(x,t)$, respectively, and they satisfy the system:

$$\begin{cases} u_t &= d_1\Delta u + r(v)G(u) - \omega u, \\ v_t &= d_2\Delta v + 1 - h(u)v, \end{cases} \tag{11}$$

satisfied for $t > 0$ and $x \in \mathbb{R}^n$ and with non-negative initial data. Some robust features observed in these social outbursts are the temporal up-and-down dynamics and, in cases like the 2005 French riots, the spatial spread of the activity. These features have been observed in the data and can be expressed mathematically as the existence of traveling wave solutions. Our model (11) does indeed allow for the existence of traveling wave solutions, which vary in qualitative behavior. Figure 14 illustrates the different type of traveling waves (which represents moving activity). Depending on the parameters used in the model, we can see waves that are either pulled by the front of the activity, Figure 14(a), pushed by the activity behind, Figure 14(b), a traveling pulse of activity, Figure 14(c), or a non-monotone wave, Figure 14(d). What these waves represent are high levels of activity invading zero or low-levels of activity. For example, Figure 14(c) illustrates a solution whose activity is concentrated in a small region, called the *front*, in this illustration the high level of activity is occurring at approximately the locations $x \in [150, 200]$. Once this front passes a certain region, it leaves behind a small number of activity.



(a) Monostable monotone wave

(b) Bistable monotone wave

(c) Oscillatory wave

(d) Non-monotone wave

Figure 14: Zoology of traveling wave solutions admitted by system (11).

4 Discussion

Partial differential equations provide a mathematical framework to understand changing systems, which are ubiquitous. If one is willing to give up following the dynamics of a single particle, animal, or individuals, this framework is mathematically much more efficient. Of course, there is no such thing as a free lunch and we pay the price of having approximated our system with an infinite number of particles (which is clearly not the case in reality). We thus have to do additional work to make sure that our "approximated" system is in fact telling us things that are relevant to our real-world system. Since its inception, this framework has been used to shed light into many complex systems from fluid dynamics to opinion formation. A fun application that we did not discussed was that of animation. Two recent hits where mathematics played a big role were Frozen and Moana. In fact, there is a lot of physics, mathematical models of the physical laws, and computer science behind making things like snow and water look realistic. Specifically, the dynamics of snow and water are modeled by famous partial differential equations, the so-called *Navier-Stokes equation* and its relatives. The macroscopic nature of these equations also make computations quite efficient. The reason behind *why* these models work so well is still a mystery. Somehow, the language of nature and life is written in Calculus. In a conversation between the influential physicist Richard Feynman and Herman Wouk, Feynman asked Wouk if he know calculus. Wouk confessed that he did not, to which Feynman replied

"You had better learn it...It is the language that God speaks." We hope that next time you hear the radio, observe nature, or experience awe, you can think of ways to try to provide a mathematical framework to what you observe or hear. Surely, there is a mathematical model behind what you are observing!

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