On Perrot's Index Cocycles

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Abstract. We shall present a simplied version of a construction due to Denis Perrot that recovers the Todd class of the complexi-ed tangent bundle of a smooth manifold from a JLO-type cyclic cocycle. The construction takes place within an algebraic framework, rather than the customary functional-analytic framework for the JLO theory. The series expansion for the exponential function is used in place of the heat kernel from the functional-analytic theory; the Dirac operator chosen is far from elliptic; and a remarkable new trace discovered by Perrot replaces the operator trace. In its full form Perrot's theory constitutes a wholly new approach to index theory. The account presented here covers most but not all of his approach.

1. Introduction

The purpose of this paper is to give an exposition of some remarkable ideas about index theory in the framework of cyclic cohomology that Denis Perrot introduced about a decade ago [Per13b]. Despite the originality of Perrot's work, his ideas seem not yet to have been studied in much detail, or at any rate not by many. Our aim is to address this circumstance by presenting a streamlined account of many of the main ideas to as wide an audience as possible.

In [Per13b], Perrot gives a more or less complete account of the Atiyah-Singer index theorem using his new approach. We shall not go that far. Our aim instead will be to present Perrot's striking construction of the Todd class within the context of cyclic cohomology

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theory. In the course of doing so we shall come into contact with many of Perrot's most fascinating discoveries.

In later works Perrot has applied his new approach to index theory to new contexts—involving groupoids [Per13a, Per16] and, in joint work with Rudy Rodsphon, foliations [PR14]. The latter work settles in almost complete generality a longstanding open problem of Alain Connes and Henri Moscovici [CM95, CM98]. These applications give ample evidence of the power of the new approach. But once again our aims will be more limited, and we shall not discuss these developments here.

Let M be a smooth, closed manifold and let ∇ be a torsion-free connection on M. The curvature of ∇ is a 2-form R on M with values in $\operatorname{End}(TM)$, and the Todd form associated to R is

$$\operatorname{Todd}(R) = \det\left(\frac{R}{\exp(R) - 1}\right).$$

This is a closed differential form on M of mixed even degree. Although its origins are in algebraic geometry, the Todd class is probably most famous for the role it plays in the Atiyah-Singer index theorem. For instance, if P is an elliptic $N \times N$ system of pseudodifferential operators on M, then

$$\operatorname{Index}(P) = \int_{S^*M} \operatorname{ch}(\sigma)\operatorname{Todd}\left(\frac{R}{2\pi i}\right),$$

where σ is the symbol of P, which is a smooth map from the cosphere bundle S^*M into invertible $N \times N$ matrices, and $\operatorname{ch}(\sigma)$ is its Chern character, which is a closed differential form on S^*M of mixed odd degree. The (normalized) Todd form is pulled back from M to S^*M.

The purpose of this paper is to present, following Perrot, a new construction of the Todd class using Dirac operators and cyclic cohomology.

Let us recall very briefly that if A is a complex associative algebra, and if

$$C^p(A) = \{\text{complex } (p+1)\text{-multilinear functionals on } A\},$$

then there are differentials

$$b\colon C^p(A) \to C^{p+1}(A) \quad \text{and} \quad B\colon C^p(A) \to C^{p-1}(A)$$

with $(b+B)^2 = 0$. A periodic cyclic cocycle is a finitely supported family of $\{\varphi_p\}_{p\geq 0}$ with $\varphi_p \in C^p(A)$ and

$$B\varphi_{p+1} + b\varphi_{p-1} = 0 \qquad \forall p \geq 0.$$

If A is the algebra of smooth functions on a smooth, closed manifold, and if C is a de Rham current in degree p, then the formula

$$\varphi_C(a^0, \dots, a^p) = \frac{1}{p!}\int_C a^0 da^1 \cdots da^p$$

defines an element $\varphi_C \in C^p(A)$ with

$$b\varphi_C = 0 \quad\text{and}\quad B\varphi_C = \varphi_{d'C},$$

where d' is the de Rham differential on currents. So if C is closed, then φ_C is a periodic cyclic cocycle (concentrated in degree p).

It follows from the above that if we pull the Todd form back from M to S^*M, as in the index theorem, and if we denote by $\mathrm{Todd}^{2q}(R)$ the degree $2q$-component, then for $p+2q = 2\dim(M) - 1$ the formula

$$\varphi_p(a^0, \dots, a^p) = \frac{1}{p!}\int_{S^*M} a^0 da^1 \cdots da^p\, \mathrm{Todd}^{2q}(R) \qquad (p \text{ odd});$$

defines a (mixed degree) periodic cyclic cocycle.

It is this cocycle that Perrot constructs in a new way. From the point of view of the formalism of cyclic theory, Perrot's construction is a variation on the famous JLO formula

$$\mathrm{STr}\left(\int_{\Delta^p} a^0 \exp(-s_0 \not{D}^2)[\not{D}, a^1] \exp(-s_1 \not{D}^2) \cdots [\not{D}, a^p] \exp(-s_p \not{D}^2)\, ds\right).$$

See [JLO88, Qui88]. What makes it noteworthy are:

(i) Perrot's definition of the Dirac operator — among other things it is not an elliptic operator;

(ii) Perrot's approach to the heat operators $\exp(-s\not{D}^2)$ — they are defined algebraically, using the series expansion for the exponential function; and

(iii) Perrot's definition of the (super)trace in the JLO formula — it is not by any means an operator trace, or even a residue trace [Wod87, Kas89], although it is related to the latter.

Our goal in this paper is to discuss each of these points in detail. But to conclude our introduction, let us indicate the parts of Perrot's work that we shall not cover in the paper.

In [Per13b], Perrot works throughout with the algebra of order zero classical pseudodifferential symbols on a smooth, closed manifold M. Our more limited goals in this paper make it possible for us to substantially simplify matters by working instead with the commutative associated graded algebra of polyhomogeneous smooth functions on the cotangent bundle. We shall be able to explain Perrot's realization of the Todd form while working exclusively in this context, thereby replicating a major part of [Per13a]. But to complete the proof of the index theorem it would be necessary to lift the entire discussion from polyhomogeneous functions to symbols.

In the context of pseudodierential symbols, Perrot actually constructs two periodic cyclic cocycles. One is eectively the cocycle that we shall construct and compute in this paper. The other is constructed in a very similar fashion, but Perrot shows that it is precisely the so-called Radul cocyle [Rad91], which produces the analytic index [Nis97, Per12]. He also proves that his two cocycles are cohomologous. So, putting everything together, the computations of the two cocycles amount to a proof of the Atiyah-Singer index theorem for elliptic pseudodierential systems.

A note on the text. Nearly all the arguments presented below are adapted from Perrot's paper [Per13b], some of them very closely. In places, for example in Section 2 where we present a simple global construction of the Dirac operator, we have been able to take advantage of our simplied context to streamline some of Perrot's constructions. In other places, for example in our treatment of the coordinate independence of Perrot's trace in Section 4, we have chosen an approach that deviates from [Per13b], typically because we felt it was illuminating to do so. But overall we owe a huge debt to [Per13b].

Acknowledgement. Beyond acknowledging our indebtedness to Perrot, it is a pleasure to thank Rudy Rodsphon for sharing his insights into Perrot's work during a number of stimulating conversations.

2. Perrot's Dirac Operator

Throughout the paper, M will always be a smooth manifold without boundary. We shall always denote by n the dimension of M. In various places (where we need to integrate over M) we shall in addition assume that M is compact. We shall be working with vector elds, dierential forms, etc, on M; all these will be assumed to be smooth.

2.1. Definition. Throughout the paper, we shall denote by $\underline{T}M$ the complement of the set of zero covectors in the cotangent bundle TM:

$$\underline{T}M = TM \text{ n } M:$$

In this section we shall work with a xed ane connection r on the tangent bundle of M (eventually we will require r to be torsion-free). Following Perrot (but with some variations, as explained in the introduction) we are going to use r to construct a Dirac operator on the total space $\underline{T}M$ that acts on sections of the pullback to $T\underline{M}$ of the exterior algebra bundle of M. But we should say at the outset that this Dirac operator will not be elliptic, and in particular it will not be a typical Dirac operator from index theory.

The construction requires some facts about horizontal vector elds on $T M$. In what follows we shall denote by : $T M$! M the standard projection mapping.

2.2. Definition. We shall identify vector elds on M with ber-wise linear smooth functions on T M via the isomorphism of $C^1(M)$-modules

$$: \text{vector elds on } M \;\; \overset{=}{!} \;\; \left\{ \begin{array}{c} \text{berwise linear smooth} \\ \text{functions on } T\ M \end{array} \right.$$

dened by $(X)(!) = !(X)$ for every 1-form ! on M. In local coordinates,

$$\frac{@}{@x^i} = {}_i;$$

where ${}_1; : : : ; {}_n$ are the usual berwise (linear) coordinate functions on T M dual to $x^1; : : : ; x^n$, dened by ${}_i() = (@=@x^i)$.

2.3. Definition. Let X be a vector eld on M. Its horizontal lift to the total space of the cotangent bundle T M (with respect to the connection r) is the vector eld X^H on T M that is -related to X and that is characterized by the further requirement that

$$X^H((Y)) = (r_X Y); \text{ for}$$

every vector eld Y on M.

In local coordinates, if we introduce the usual Christoel symbols by

$$r_{@=@x^i} \frac{@}{@x^j} = {}^k_{ij} \frac{@}{@x^k}$$

(with the usual summation convention), then

$$\left(\frac{@}{@x^i}\right)^H = \frac{@}{@x^i} + {}^k_{ij}\, {}_k \frac{@}{@_j};\tag{2.4}$$

where ${}_1; : : : ; {}_n$ are, as above, the berwise linear coordinate functions on T M dual to $x^1; : : : ; x^n$.

2.5. Definition. We introduce the following additional identication: we dene an isomorphism of $C^1(M)$-modules

$$: \text{End } T M \;\; \overset{=}{!} \;\; \left\{ \text{vertical vector elds } a_{ij}\, {}_i\, @=@_j \text{ on } T M \right\}$$

by the formula $(A)((X)) = (A(X))$. Thus in local coordinates, if A $@=@x^i = a_{ij}@=@x^j$, then

$$(A) = a_{ij}\ {}_i\ \frac{@}{@_j}:$$

2.6. Lemma. If X and Y are vector elds on M, then

$$[X^H; Y^H] \quad [X; Y]^H = \quad R(X; Y)$$

Proof. Both sides of the identity are vertical vector elds, so it suces to check that both sides agree on the functions (Z) in Deni-tion 2.2. The identity in this case is an immediate consequence of the denitions.

2.7. Definition. Form the exterior algebra bundle TM over M. Equip it with the connection induced from r, and then pull the bundle and its connection back from M to the manifold TM. For brevity, we shall write

$$S = TM$$

for the bundle, and keep the notation r for the connection.

In order to express the connection on S in local coordinates it will be convenient to introduce the following notation:

2.8. Definition. Given local coordinates $x^1; : : : ; x^n$ on an open set $U \quad M$, dene operators $\ ^i$ and $\bar{\ }_j$ acting on sections of S by

$$^i = \text{left exterior multiplication by } dx^i \text{ on } TM \quad {}_j =$$

$$\bar{\ } \text{contraction by } @=@x^j \text{ on } TM:$$

Using the above notation, along with the previously introduced Christoel symbols $\ ^k_{\ell j}$, the induced connection on TM is

$$(2.9) \qquad \begin{aligned} r_{(@=@x^i)^H} &= \frac{@}{@x^i} + \ ^k_{i\ell}\ {}_k \frac{@}{@\ \ell} \qquad {}^k_{\ell} \ {}^{\ell}\ \bar{\ }_k \\ r_{@=@_j} &= \frac{@}{@_j}: \end{aligned}$$

2.10. Definition. Given local coordinates on an open set $U \quad M$, dene operators

$$Ð_{horiz}; Ð_{vert}: \quad TU; TM \quad ! \quad TU; TM \text{ using the}$$

formulas

$$Ð_{horiz} = \ ^i r_{(@=@x^i)^H} \quad \text{and} \quad Ð_{vert} = \bar{\ }_j\ r_{@=@_j}:$$

Using the Christoffel symbols for ∇ and the standard local frame for S associated to a coordinate system, we find that

$$\mathcal{D}_{\mathrm{horiz}} = \psi^i \frac{\partial}{\partial x^i} + \Gamma^k_{ij}\psi^i\psi_k \frac{\partial}{\partial \psi_j} - \Gamma^k_{ij}\psi^i\psi^j\frac{\partial}{\partial\psi_k} \quad\text{and}\quad \mathcal{D}_{\mathrm{vert}} = \bar{\psi}^j \frac{\partial}{\partial \psi_j}.$$

2.11. Lemma. The operators $\mathcal{D}_{\mathrm{horiz}}$ and $\mathcal{D}_{\mathrm{vert}}$ are independent of the choice of local coordinate system used to define them.

Proof. This follows from the $C^\infty(M)$-linearity of the map .

2.12. Lemma. If the connection ∇ on TM is torsion-free, then

$$\mathcal{D}_{\mathrm{horiz}} = \psi^i \frac{\partial}{\partial x^i} + \Gamma^k_{ij}\psi^i\psi_k \frac{\partial}{\partial \psi_j}$$

in any local coordinate system.

Proof. This is because $\Gamma^k_{ij} = \Gamma^k_{ji}$ for a torsion-free connection, so that the third term in the defining formula

$$\mathcal{D}_{\mathrm{horiz}} = \psi^i \frac{\partial}{\partial x^i} + \Gamma^k_{ij}\psi^i\psi_k \frac{\partial}{\partial \psi_j} - \Gamma^k_{ij}\psi^i\psi^j\frac{\partial}{\partial\psi_k}$$

is zero in this case.

2.13. Lemma. If the connection ∇ on TM is torsion-free, then $\mathcal{D}^2_{\mathrm{horiz}} = \frac{1}{2}\psi^i\psi^j R(\partial/\partial x^i, \partial/\partial x^j)$ in any local coordinate system.

Proof. Fix a local coordinate system on $U \subseteq M$. Give S the local frame associated to these coordinates, and define operators

$$\partial^H_i = \frac{\partial}{\partial x^i} + \Gamma^k_{i\ell}\psi_k\frac{\partial}{\partial \psi_\ell}$$

on smooth sections of S (these are not the operators $\nabla_{(\partial/\partial x^i)^H}$, nor do they have any particularly special meaning; they are merely introduced for the sake of the computation). We now calculate from Lemma 2.12 that

$$\begin{aligned}
\mathcal{D}^2_{\mathrm{horiz}} &= \psi^i\partial^H_i\,\psi^j\partial^H_j \\
&= \psi^i\psi^j\partial^H_i\partial^H_j \\
&= \sum_{i<j}\psi^i\psi^j\left(\partial^H_i\partial^H_j - \partial^H_j\partial^H_i\right).
\end{aligned}$$

But it follows from Lemma 2.6 that

$$\partial^H_i\partial^H_j - \partial^H_j\partial^H_i = R(\partial/\partial x^i, \partial/\partial x^j);$$

and as a result

$$
\begin{aligned}
D^2_{horiz} &= \sum_{i<j} {}^{i}\;{}^{j}\; R(\partial/\partial x^i; \partial/\partial x^j) \\
&= \tfrac{1}{2}\; {}^{i}\;{}^{j}\; R(\partial/\partial x^i; \partial/\partial x^j);
\end{aligned}
$$

as required.

2.14. Definition. We shall write

$$(R) = \tfrac{1}{2}\, {}^{i}\;{}^{j}\; R(\partial/\partial x^i; \partial/\partial x^j) :$$

This is a rst-order linear dierential operator acting on section of S, independent of the choice of coordinates. More explicitly, if we write

$$R\left(\frac{\partial}{\partial x^i}; \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^{\ell}} = R_{ij\ell}^{\;\;\;k}\frac{\partial}{\partial x^k};$$

then

$$(R) = \frac{1}{2}\, {}^{i}\;{}^{j} R_{ij\ell}^{\;\;\;k}\; {}_{k}\frac{\partial}{\partial_{\ell}}:$$

In this paper, following Perrot we shall be concerned with the \Dirac operator" $D = D_{horiz} + D_{vert}$ (or actually a small variation of this; see Denition 3.18) and its square. Lemma 2.13 is obviously relevant to the computation of the square, as is the simple formula

$$D^2_{vert} = 0: \tag{2.15}$$

What remains is to compute the cross-term

$$\{D_{horiz}; D_{vert}\} = D_{horiz}D_{vert} + D_{vert}D_{horiz}:$$

2.16. Lemma. If the connection ∇ on TM is torsion-free, then

$$\{D_{horiz}; D_{vert}\} = \nabla_{\partial_i}\nabla_{(\partial_{x^i})^H}$$

(with the summation convention, as always) in any local coordinate system.

Proof. We shall use the same operators ∂^H_i introduced in the proof of Lemma 2.13. With these, we may write

$$
\begin{aligned}
\{D_{horiz}; D_{vert}\} &= {}^{i}\partial^H_i\; {}^{j}\frac{\partial}{\partial_j} + {}^{j}\frac{\partial}{\partial_j}\; {}^{i}\partial^H_i \\
&= {}^{i}\;{}_{j}\,\partial^H_i\frac{\partial}{\partial_j} + {}_{j}\;{}^{i}\frac{\partial}{\partial_j}\partial^H_i;
\end{aligned}
\tag{2.17}
$$

Next, we compute that

$$\partial^H_i\frac{\partial}{\partial_j} = \left(\frac{\partial}{\partial x^i} + {}_{i\ell}^{\;k}\,{}_k\frac{\partial}{\partial_\ell}\right)\frac{\partial}{\partial_j} = \frac{\partial}{\partial_j}\partial^H_i \quad {}_{i\ell}^{\;j}\frac{\partial}{\partial_\ell};$$

and as a result we find from (2.17) that

$$
(2.18)\qquad
\begin{aligned}
\{\not{D}_{\mathrm{horiz}}, \not{D}_{\mathrm{vert}}\} &= \frac{\partial}{\partial \xi_i}\frac{\partial}{\partial x^i} + \Gamma^k_{i\ell}\,\xi_k \frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \xi_\ell} + \Gamma^i_{i\ell}\frac{\partial}{\partial \xi_\ell} - \Gamma^j_{i\ell}\,\frac{\partial}{\partial \xi_\ell} \\
&= \frac{\partial^2}{\partial \xi_i \partial x^i} + \Gamma^k_{i\ell}\,\xi_k \frac{\partial^2}{\partial \xi_i \partial \xi_\ell} + \Gamma^i_{i\ell}\frac{\partial}{\partial \xi_\ell} - \Gamma^j_{i\ell}\,\frac{\partial}{\partial \xi_\ell}.
\end{aligned}
$$

On the other hand,

(2.19)

$$
\begin{aligned}
\nabla_{\partial_{\xi_i}} \nabla_{(\partial/\partial x^i)^H} &= \frac{\partial}{\partial \xi_i}\Big(\frac{\partial}{\partial x^i} + \Gamma^k_{i\ell}\,\xi_k \frac{\partial}{\partial \xi_\ell}\Big) - \Gamma^k_{\ell i}\ldots \\
&= \frac{\partial^2}{\partial \xi_i \partial x^i} + \Gamma^k_{i\ell}\,\xi_k \frac{\partial^2}{\partial \xi_i \partial \xi_\ell} + \Gamma^i_{i\ell}\frac{\partial}{\partial \xi_\ell} - \Gamma^k_{i\ell}\,\frac{\partial}{\partial \xi_k}.
\end{aligned}
$$

Since ∇ is torsion-free, $\Gamma^k_{i\ell} = \Gamma^k_{\ell i}$, and so we find that (2.18) and (2.19) are equal, as required.

If we write

$$\nabla^2 = \nabla_{\partial_{\xi_i}} \nabla_{(\partial_{x^i})^H}$$

(this is coordinate-independent), then the computations in this section may be summarized as follows:

$$(2.20)\qquad \not{D}^2_{\mathrm{horiz}} = \Omega(R); \qquad \not{D}^2_{\mathrm{vert}} = 0 \quad \text{and} \quad \{\not{D}_{\mathrm{horiz}}, \not{D}_{\mathrm{vert}}\} = \nabla^2 :$$

These identities will be crucial in the sequel.

2.21. Remark. Using the connection ∇ we may identify the tangent bundle of T^*M with the pullback of $TM \oplus T^*M$ to T^*M. If we equip the latter with its canonical symmetric bilinear form, which of course is nondegenerate but of indefinite signature, then S carries an irreducible representation of the associated bundle of Clifford algebras, and the operator $\not{D}_{\mathrm{horiz}} + \not{D}_{\mathrm{vert}}$ is a Dirac operator in the usual sense, although since the bilinear form is not definite, the operator is not elliptic.

3. Infinite-Order Polyhomogeneous Operators

In this section we shall describe the infinite-order linear partial differential operators (presented as formal series) with which we shall work in the rest of the paper. We shall begin by considering differential operators that act on scalar functions, but then we shall turn to differential operators acting on spinors, which will be our principal objects of interest.

3.1. **Definition.** We shall denote by E the Euler vector field on T^*M, so that

$$(Ef)(\xi) = \frac{d}{dt}\Big|_{t=0} f(e^t\xi)$$

for all smooth functions f on T^*M and all $\xi \in T^*M$. We shall say that a smooth function on T^*M is homogeneous of degree $k \in \mathbb{Z}$ if it is a k-eigenfunction for the action of E, or in other words if

$$f(e^t\xi) = e^{kt}f(\xi)$$

for all $t \in \mathbb{R}$ and all $\xi \in T^*M$. We shall say that f is polyhomogenous if it is a (finite) linear combination of smooth homogeneous functions of various integer degrees. We shall write

$$P^k(M) = \text{Smooth degree-}k\text{ homogeneous functions on } T^*M$$

and

$$P(M) = \text{Smooth polyhomogeneous functions on } T^*M \,.$$

3.2. **Definition.** A linear partial differential operator D on T^*M, acting on smooth functions on T^*M, is homogeneous of degree $k \in \mathbb{Z}$ if $\mathrm{ad}_E(D) = kD$, where E is the Euler vector field, and polyhomoge-neous if it is a (finite) linear combination of homogeneous operators of various integer degrees. We shall denote by $PDO(M)$ the algebra of polyhomogeneous linear partial differential operators on T^*M.

3.3. **Remark.** In local coordinates, the polyhomogeneous operators are linear combinations of operators of the form

$$f\,\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\beta}}{\partial \xi^{\beta}}$$

with f a polyhomogeneous smooth function on T^*M.

3.4. **Definition.** We shall denote by F the field of formal complex Laurent series $\sum_{-N}^{\infty} a_k\varepsilon^k$ (with only finite singular parts). We shall denote by $FPDO(M)$ the algebra of formal Laurent series in ε with coefficients from $PDO(M)$ (again, with only finite singular parts).

We turn now to one of the main constructions of the paper, which is that of a subspace

$$FB(M) \subseteq FPDO(M)$$

on which will be defined a crucial trace functional. The subspace is not a subalgebra, let alone an ideal, but it is for instance a bimodule under the left and right actions of $PDO(M)$ on $FPDO(M)$ (the name is supposed to suggest this, and Perrot calls his version, on which ours is very closely modelled, the bimodule of trace class operators). We shall dene the bimodule in this section and construct the trace in the next section.

The space $FB(M)$ will be dened in stages, and to begin we shall work in a xed coordinate system on an open subset U M. We shall use the following operator extensively:

3.5. Definition. Let U be an open subset of M and let $x^1; : : : ; x^n$ be coordinates on U. Given these coordinates, the associated Laplacian on T_U is

$$= \frac{@^2}{@x_i @_i}$$

(summation convention).

3.6. Remarks. The operator is not invariant under changes of coordinates (except for linear changes of coordinates). In addition, the operator is not elliptic.

Following Perrot we introduce the following increasing ltration on the algebra $PDO(U)$.

3.7. Proposition/Definition. Let U be an open subset of M and let $x^1; : : : ; x^n$ be coordinates on U. Dene an increasing ltration of the algebra $PDO(U)$, indexed by Z, and an associated notion of order on partial dierential operators on U, that we shall call bimodule order, as follows. The bimodule ltration is constructed by considering at rst all increasing algebra ltrations

$$PDO_p(U) \quad PDO_{p+1}(U) \quad PDO(U)$$

indexed by p 2 Z and with $@=@x^i$ 2 $PDO_3(U)$, $@=@_i$ 2 PDO $_1(U)$, and f 2 PDO_{2p} if f is homogeneous of degree p. Then dene the pth bimodule ltration space to be the intersection of all $PDO_p(U)$ over all these ltrations. Of course we write bimodule-order(T) p if T belongs to this intersection. We have the following identities:

$$\begin{aligned} \text{bimodule-order}(@=@x^i) &= 3 \qquad 8i \\ \text{bimodule-order}(@=@_j) &= 1 \qquad 8j \\ \text{bimodule-order}(f) &= 2p \qquad \text{if degree}(f) = p. \end{aligned}$$

Proof. A straightforward computation in local coordinates.

3.8. Remark. We shall introduce a second, quite dierent ltration on dierential operators in Section 6, and it should not be confused with the bimodule ltration dened here.

3.9. *Definition.* We shall denote by $FB(U) \subseteq FPDO(U)$ the space of all Laurent series

$$\sum_k \varepsilon^k D_k \exp(\varepsilon) \qquad (D_k \in PDO(U))$$

for which there exists $N > 0$ with

$$\text{bimodule-order}(D_k) \le k + N;$$

for all k.

3.10. *Remark.* It is not so easy to explain at the outset the reason for dening the bimodule ltration, and hence $FB(U)$, in the way that we just have. In fact there is a certain amount of exibility in how one might dene $FB(U)$, which needs to be small enough that the trace functional to be discussed in the next section is well-dened, and large enough that it is closed under some simple operations, most notably conjugation by $\exp(\varepsilon)$. The denition given seems to be the simplest one that meets these requirements.[1]

The following is another easy computation:

3.11. *Lemma.* If T is any operator in $PDO(U)$, then

$$\text{bimodule-order}\,[\;;T] \le \text{bimodule-order}(T) + 1:$$

Now suppose that $T \in FPDO(U)$ and that $s \in \mathbb{R}$. The conjugated operator

$$\mathrm{Ad}_{\exp(\varepsilon s)}(T) = \exp(\varepsilon s)T\exp(-\varepsilon s) \in FPDO(U) \text{ may}$$

be alternatively expressed as

$$\mathrm{Ad}_{\exp(\varepsilon s)}(T) = \exp(\varepsilon s\,\mathrm{ad})(T) = \sum_{k=0}^{\infty} \frac{s^k \varepsilon^k}{k!}\,\mathrm{ad}(T)^k$$

(thanks to the powers ε^k, the sum makes sense in $FPDO(U)$; to use language that will be introduced in the next section, the sum is convergent in the adic topology). Using Lemma 3.11 we nd that:

3.12. *Corollary.* If $T \in PDO(U)$, then for every $s \in \mathbb{R}$ we may write

$$\mathrm{Ad}_{\exp(\varepsilon s)}(T) = \sum_{k=0}^{\infty} \varepsilon^k T_k$$

[1] One might perhaps compare this to the situation with traceable Hilbert space operators in index theory, where one might work with smoothing operators, or trace-class operators, or operators whose singular value sequence has rapid decay, and so on.

where T_k 2 PDO(U) and order(T_k) k + order(T). In particular the subspace FB(M) FPDO(M) is closed under left and right multiplication by elements of PDO(M) in the algebra FPDO(U).

3.13. Corollary. If X 2 FB(U), then $Ad_{exp("s)}(X)$ 2 FB(U) for every s 2 R.

The bimodule ltration on PDO(U) is independent of the choice of coordinates on U; this can be checked by the usual change-of-coordinates formulas. The following result is a bit more involved:

3.14. Lemma. Let U be an open subset of M, let $x^1; : : : ; x^n$ be coordinates on U, and let and FB(U) be the associated Laplacian and bimodule. If ^{0} is the Laplacian associated to another coordinate sys-tem on U, then

$$\exp("^0)\ 2\ FB(U):$$

Proof. The Laplacian ^{0} in a new coordinate system has the form

$$^0 = + a_i \frac{@}{@_i} + b_{jk'j} \frac{@^2}{@_k @_{'}}$$

in terms of the original coordinate system, where a_i and $b_{jk'}$ are smooth functions on U, and so bimodule-order(^{0}) 0.

To proceed, within FPDO(U) there is a Duhamel-type formula

$$\exp("^0) \quad \exp(") = " \int_0^1 \exp("s^0)(^0 \quad) \exp("(1 \quad s))\, ds:$$

The integrand, viewed as a power series in ", is a polynomial function of s in each degree, with values in PDO(U). So the denition of the integral presents no problems (and the formula may be proved in the usual way).

By iterating the formula we obtain the perturbation series

$$\exp("^0) = \sum_{p=0}^{1} "^p \int_{p} \exp("s_0)R\exp("s_1)R \quad \exp("s_{p\ 1})R\exp("s_p)\, ds;$$

in FPDO(U), where R = ^{0} and where ^{p} is the standard p-simplex. Writing the above as

$$\exp("^0) = \sum_{p=0}^{1} "^p \int_{p} Ad_{exp("s_0)}(R)\, Ad_{exp("(s_0+s_1))}(R) \quad Ad_{exp("(s_0+s_1++s_{p\ 1}))}(R)\, ds \ \exp(")$$

and using Corollary 3.12 we obtain the result.

3.15. Theorem. The subspace FB(U) FPDO(U) is independent of the choice of local coordinates.

Proof. First, Lemma 3.14 shows that exp("^{0}) 2 FB(U). Sec-ond, because the bimodule order is obtained from an algebra ltration, FB(U) is in fact closed under left multiplication by series $\sum_k$ "$^k T_k$ with T_k 2 PDO(U) and bimodule-order(T_k) k + N for some N and all k. These two observations show that the FB-bimodule dened using ^{0} is included in the FB-bimodule dened using . By symmetry, the two are in fact equal.

We can now dene a global version of the bimodule FB(U):

3.16. Definition. We denote by FB(M) FPDO(M) the sub-space of all elements that restrict to elements in FB(U) whenever U is an open subset of M that supports a coordinate system.

We shall now make the minor changes needed to carry over all of the results above from operators that act on scalar functions to operators that act on sections of the bundle S. The most direct way to do proceed is to adapt the notion of bimodule order to this context:

3.17. Definition. Let U be an open subset of M and let $x^1; : : : ; x^n$ be coordinates on U. Trivialize the bundle S over U using the asso-ciated frame for the exterior algebra bundle over U, and dene an increasing ltration of the algebra PDO(U; S), indexed by Z, and an associated notion of bimodule order on partial dierential operators on U, by

bimodule-order(@=@x^i) 3 8i

bimodule-order(@=@$_j$) 1 8j

bimodule-order(f) 2p if f is a scalar function of degree p

bimodule-order(^{i}) 0 8i

bimodule-order$\overline{(}$ $_j$) 0 8j

Thus the bimodule order on PDO(U; S) does not involve the bun-dle S at all. It is coordinate-independent, and all of the results and denitions given above pass over to the context of operators on S with-out any change at all (of course one must replace homogeneous scalar functions on TU with homogeneous endomorphisms of S, noting that since S is pulled back from M, the notion of homogeneity makes sense here, as it does for any pull-back bundle, since the operation of scalar multiplication on the bers of TM lifts canonically to an action on the bundle).

To conclude this section we shall situate the Driac operator and its exponential within the bimodule $FB(M;S)$.

3.18. **Definition.** The Dirac operator on $T_\varepsilon M$, associated to a torsion-free connection on M, is the operator

$$\not D = \varepsilon \not D_{\mathrm{horiz}} + \not D_{\mathrm{vert}};$$

whose summands are described in Definition 2.10.

It follows from the formulas in (2.20) that

$$\not D^2 = \varepsilon \nabla^2 + \varepsilon^2 (R);$$

and therefore the exponentials

$$\exp(s\not D^2) = \sum_{k=0}^{\infty} \frac{s^k}{k!} \left(\varepsilon \nabla^2 + \varepsilon^2 (R)\right)^k$$

are well-defined elements of $FPDO(M;S)$.

We shall use the following two lemmas when we construct the periodic cyclic cocycle associated to the Dirac operator $\not D$ in Section 5.

3.19. **Lemma.** If $T \in PDO(U;S)$, then for every $s \in \mathbb{R}$ we may write

$$\mathrm{Ad}_{\exp(s\not D^2)}(T) = \sum_{k=0}^{\infty} \varepsilon^k T_k$$

where $T_k \in PDO(U;S)$ and $\mathrm{order}(T_k) \leq k + \mathrm{order}(T)$.

Proof. This is a variation of Corollary 3.13 and it is proved the same way, using the formula

$$\varepsilon \quad \not D^2 = \varepsilon R_1 + \varepsilon^2 R_2; \tag{3.20}$$

where R_0 and R_1 have bimodule orders 0 and 1, respectively.

3.21. **Lemma.** If $\not D$ is the Dirac operator associated to any torsion-free connection on M, then $\exp(\not D^2) \in FB(M;S)$.

Proof. The method use to prove Lemma 3.14 applies here, in view of (3.20).

4. Perrot's Trace

The most remarkable ingredient in Perrot's work is a supertrace functional defined on the bimodule $FB(M;S)$ (this space is $\mathbb{Z}/2\mathbb{Z}$-graded in the usual way, using the degree operator on differential forms). We shall construct the supertrace in this section.

As we did in the previous section, we shall begin by ignoring spinors: we shall first construct a trace functional on the space $FB(M)$ (which is not Z=2Z-graded). The rst ingredient in the denition of the trace here is the usual method of \integrating" homogeneous functions on TM of degree n, which we shall now review.

4.1. Definition. If W is any smooth manifold, then we shall denote by Dens(W) the $C^1(W)$-module of smooth densities on W (see [BGV92, pp.30-31] or [Lee03, Ch.14] for instance).

View the cotangent sphere bundle SM as the quotient of TM by the action of positive scalar multiplication on TM (generated by the Euler vector eld) and denote by

$$p\colon TM \; ! \; SM$$

the projection mapping. In addition, denote by the volume form on TM that in any local coordinate system has the form

$$= dx^1 dx^2 \, dx^n d_1 d_2 \, d_n :$$

4.2. Definition. Suppose that f 2 P ^{n}(M). The (2n 1)-form f $_E$ is basic for the action of R on TM, and we shall denote by $_f$ the unique (2n 1)-form on SM such that

$$p_f = f \; _E :$$

4.3. Definition. Throughout the paper we shall equip SM with the unique orientation such that $_{SM}$ $_f$ 0 (with $_f$ as in Deni-tion 4.2) for every nonnegative f 2 P ^{n}(M).

We obtain from the denitions above an isomorphism of $C^1(SM)$-modules

(4.4) $$P^{\,n}(M) \overset{=}{!} \text{Dens}(SM):$$

We shall use this identication throughout the rest of the paper, and for f 2 P ^{n}(M) we shall write

(4.5) $$\int_{SM} f = \text{integral of the associated density} = \int_{SM} {}_f :$$

For general f 2 P(M) we shall write

(4.6) $$\int_{SM} f = \int_{SM} f^{[\ n]};$$

where $f^{[\ n]}$ denotes the degree n component of f.

4.7. Remark. The above integral is a commutative version of the noncommutative residue on pseudodierential symbols. Indeed, up to an overall constant, the noncommutative residue of an order n pseudodierential symbol is the integral of its principal symbol function, as dened above.

The eld F of formal Laurent series, the algebra $FPDO(U)$ and its subspace $FB(U)$ all carry the usual adic topologies, in which a base for the open neighborhoods of 0 is the collection of spaces

$$\left\{ \sum_{k=\ell}^{\infty} \varepsilon^{k} D_{k} \right\}$$

for $\ell = 0; 1; 2; \dots$ (with D_k either a scalar, in the case of F, or a partial dierential operator in the other cases).

Now denote by $FB_c(U)$ the subspace of $FB(U)$ comprised of series whose coecents are partial dierential operaotrs on $\underline{T}U$ that are compactly supported in the U-direction, and dene $P_c(U)$ similarly. The main result of this section will be as follows:

4.8. **Theorem.** *Let $U \subseteq M$ be an open set and with a xed sys-tem of local coordinates. There is a unique continuous (for the adic topologies) F-linear functional*

$$\operatorname{Tr}_F : FB_c(U) \to F$$

such that

$$\operatorname{Tr}_F\big(f \exp(\varepsilon)\, g\big) = \varepsilon^{-n} \int_U fg$$

for all $f; g \in P_c(U)$. This functional has the following trace property:

$$\operatorname{Tr}_F(DX) = \operatorname{Tr}_F(XD) \qquad \forall X \in FB_c(U);\ \forall D \in PDO(U):$$

The uniquess part of the theorem is easy to prove.

Proof of uniqueness in Theorem 4.8. Since the nite sums $\varepsilon^{k} D_k \exp(\varepsilon)$ are dense in $FB_c(U)$, it suces to show that Tr_F is determined on these by the formula in the statement of the theorem. We shall show that each such nite sum is a nite Laurent polynomial in ε with coeecients of the form $f \exp(\varepsilon) g$. This certainly suces. To do so, we need only note that the formulas

$$\begin{aligned}
f \exp(\varepsilon)\, x^{j} g - f x_{j} \exp(\varepsilon)\, g
&= \varepsilon f \frac{\partial}{\partial_j} \exp(\varepsilon)\, g \\
&= \varepsilon \frac{\partial}{\partial_j} f \exp(\varepsilon)\, g - \varepsilon \frac{\partial f}{\partial_j} \exp(\varepsilon)\, g
\end{aligned}$$

and

$$\exp(\varepsilon)\,{}^{i} - {}^{i} \exp(\varepsilon) = \varepsilon \frac{\partial}{\partial x_i} f \exp(\varepsilon)\, g - \varepsilon \frac{\partial f}{\partial x_i} \exp(\varepsilon)\, g;$$

show that the span of all f exp(") g over the Laurent polynomials in " is closed under left multiplication by operators in PDO(U).

Although the proof of uniqueness suggests an approach to the construction of the trace, it is more convenient to proceed in a dierent direction. Recall that if A is a positive-denite 2n2n matrix, and if p: R^{2n} ! C is a polynomial function, then

$$\int_{R^{2n}} p(w) \exp \tfrac{1}{2} hw; Awi\, dw = \frac{(2)^{n}}{\sqrt{\det(A)}} p(\sqrt{\ } 1 @_w) \exp \tfrac{1}{2} hw; A^{1} wi \Big|_{w=0}$$

(see for instance [Hor83, Sec.7.6]). The following denition in eect extends this formula to the matrix A = $[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}]$, which is not positive-denite.

4.9. Definition. Let p: R^{2n} ! C be a polynomial function We dene

$$\Big\langle p\ @=@x; @=@ \exp "\Big\rangle = "^{n} p\Big(\sqrt{\ } 1 @=@x; \sqrt{\ } 1 @=@ \Big) \exp " ^{1} x^{i}{}_{i} \Big|_{x=0;=0}$$

$$= "^{n} p(@=@x; @=@) \exp " ^{1} x^{i}{}_{i} \Big|_{x=0;=0}$$

This value lies in C[" ^{1}; "].

4.10. Remark. A comparison of the two integral formulas shows that we have dropped an overall multiplicative constant of $(2i)^{n}$. For our purposes it is a little tidier to do so.

Perrot calls the construction in the denition above the contraction of p(@=@x; @=@) exp("). It will be convenient to follow Perrot and extend this concept of contraction to cover more cases, as follows:

4.11. Definition. If D is any operator in PDO(U), say

$$= \sum_{;} f\ @x\ @^{;} \frac{@}{} \frac{@}{} D$$

then we dene $D\exp(\varepsilon) \in P(U)[\varepsilon^{-1};\varepsilon]]$ by

$$D\exp\varepsilon = \sum_{\alpha;\beta} f_{\alpha\beta}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\beta}}{\partial \xi^{\beta}}\exp\varepsilon$$

$$= \varepsilon^{-n}\sum_{\alpha;\beta} f_{\alpha\beta}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\beta}}{\partial \xi^{\beta}}\exp\left(\varepsilon^{-1}x^{i}\xi_{i}\right)\Big|_{x=0;\xi=0}.$$

Later on we shall make use of the following alternative formula for the trace. Form the function $\exp(\varepsilon^{-1}(x^i-y^i)(\xi_i-\eta_i))$ on the direct product of two copies of T^*U, and then for an operator $D \in PDO(U)$ form the function

$$D\exp\left(\varepsilon^{-1}(x^i-y^i)(\xi_i-\eta_i)\right)$$

on the product, with D acting on the rst copy (the x- and ξ-variables) alone. If the restriction to the diagonal of this function is regarded as a function on T^*U via the projection onto the rst factor of the product, then

$$D\exp(\varepsilon) = \varepsilon^{-n} D\exp\left(\varepsilon^{-1}(x^i-y^i)(\xi_i-\eta_i)\right)\Big|_{x^i=y^i;\ \xi_i=\eta_i}. \tag{4.12}$$

We are now ready to give the formula for Perrot's trace functional.

4.13. **Definition.** Let $X \in FB(U)$. Write X as a Laurent series in ε with nite singular part,

$$X = \sum_k \varepsilon^k D_k\exp(\varepsilon);$$

as in Denition 3.9. Assuming that each D_k has compact support in the U-direction, we dene $\mathrm{Tr}_F(X)$ in the eld of complex Laurent series with nite singular parts by

$$\mathrm{Tr}_F(X) = \sum_k \varepsilon^k \int_{S^*M} D_k\exp(\varepsilon).$$

Since an innite sum is involved, we need to check that the deni- tion actually makes sense:

4.14. **Lemma.** The innite sum in Denition 4.13 converges in the ε-adic topology on the eld F of formal Laurent series with nite singular parts.

Proof. The bimodule order plays an important role here: we shall use the fact that if $X \in FB(U)$, and if

$$X = \sum_{\alpha;\beta;p;k} \varepsilon^k f_{pk\alpha\beta}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\beta}}{\partial \xi^{\beta}}\exp(\varepsilon);$$

then according to Denition 3.9, there is some N for which p

$$+ 3jj \quad jj \; k + N$$

for all , , p and k for which $f_{pk} = 0$.

Recall that

$$\underset{SM}{} f_{pk} = 0 \quad) \quad p = \quad n:$$

In addition, it follows from Denition 4.9 that D

$$\frac{@}{@x}\frac{@}{@}\exp('') \; E = 0 \quad) \quad = ;$$

while if = , then

$$D \; \frac{@}{@x}\frac{@}{@}\exp('') \; E = \text{scalar } '' \; jj \; n: \text{Of}$$

course if = , then

$$p + 3jj \quad jj = p + 2jj:$$

and so if in addition p = n, then

$$n + 2jj \; k + N$$

and therefore

$$jj \quad \frac{k}{2} + m;$$

where m = (N + n)=2. We nd that the series giving the trace in Denition 4.13 has the form

$$\mathrm{Tr}_F(X) = \sum \sum c_{k;} \; ''^{k \; jj \; n}: \; k \quad {}_{jj} \; k=2 + m$$

Since k jj n k=2 + m n under the indicated condition on , the sum is certainly convergent in the adic topology, as required: the powers of '' are bounded from below each power occurs only nitely many times in the sum.

We shall now prove that Tr_F has the trace property in Theorem 4.8. To do so, we shall follow the argument of Perrot closely. The proof obviously reduces immediately to consideration of the three cases

$$D = \frac{@_i}{@x}; \quad D = \frac{@}{@_j} \quad \text{and} \quad D = f: \tag{4.15}$$

Moreover using the continuity of the trace and its F-linearity, the proof simultaneously reduces to the case where

$$X = g \frac{@}{@x}\frac{@}{@}\exp(''):$$

In the rst two cases from (4.15) we have

$$[D; X] = \frac{\partial g_i}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial}\exp('') \qquad \text{and} \qquad [D; X] = \frac{\partial g}{\partial_j}\frac{\partial}{\partial x}\frac{\partial}{\partial}\exp(''):$$

To handle these two cases of the proof it suces to show that

$$_{SU}\ \frac{\partial g}{\partial x^i} = 0 \quad \text{and} \quad _{SU}\ \frac{\partial g}{\partial_j} = 0:$$

In the rst case we may as well assume that g has degree n (otherwise the integral is certainly zero, by denition), and then by Stokes' theorem

$$_{SU}\ \frac{\partial g}{\partial x^i} = \ _{SU}\ d_{\partial=\partial x}\, g = 0;$$

in the notation of Denition 4.2. In the second case we may as well assume that g has degree 1 n. But then the form $_{E\partial=\partial}\, g_j$ on TU is basic, so it is the pullback of a form $_g$ on SU, and from

$$d(_{E\partial=\partial_j} g) = {}_E\ ; \frac{\partial g}{\partial_j}$$

we nd that

$$_{SU}\ \frac{\partial g}{\partial_j} = \ _{SU}\ \partial g{=}\partial_j = \ _{SU}\ d_g = 0;$$

again by Stokes' theorem.

It remains to consider the case D = f in (4.15). Consider rst the cases where f is a coordinate function: $f = x^i$ or $f = {}_j$.

4.16. Lemma. For any i, and j and any and ,

$$\left\langle x^i; \frac{\partial}{\partial x}\frac{\partial}{\partial}\exp('')\right\rangle = 0 \qquad \text{and} \qquad \left\langle {}_j; \frac{\partial}{\partial x}\frac{\partial}{\partial}\exp('')\right\rangle = 0:$$

Proof. We shall present the argument for x^i (the argument for $_j$ is exactly the same) and we'll describe the case of n=1 dimensions (higher dimensions present additional notational complexities but are otherwise identical).

We calculate that

$$\frac{\partial^a}{\partial x^a}\frac{\partial^b}{\partial^b}\exp(''); x = a\frac{\partial^{a-1}}{\partial x^{a-1}}\frac{\partial^b}{\partial^b}\exp('') + ''\frac{\partial^a}{\partial x^a}\frac{\partial^{b+1}}{\partial^{b+1}}\exp('')$$

and therefore that the contraction is

$$\frac{\partial^b}{\partial^b}\left(a\frac{\partial^{a-1}}{\partial x^{a-1}} + ''\frac{\partial^a}{\partial x^a}\frac{\partial}{\partial}\right)\exp(\ ''^{\,1}x)\Big|_{x=0;=0}:$$

But

$$\left(\alpha\frac{\partial^{\alpha-1}}{\partial \xi^{\alpha-1}} + \varepsilon\frac{\partial^{\alpha}}{\partial \xi^{\alpha}}\frac{\partial}{\partial x}\right)\exp(-\varepsilon^{-1}x\xi) = (-1)^{\alpha-1}\varepsilon^{-\alpha}\alpha^{-1}x\exp(-\varepsilon^{-1}x\xi)\text{: So,}$$

thanks to the factor of x, the contraction is 0.

4.17. *Remark.* The lemma can be understood more conceptually (but at the expense of a longer argument) using the fact that the contraction is derived from an integral formula on $\mathbb{R}^{2n}$, and the fact that the integral of a partial derivative in that context is zero.

Proof of the trace property in Theorem 4.8. The calculations in the proof of Lemma 4.16 show that if p is any polynomial on $\mathbb{R}^{2n}$, possibly with coefficients that are polynomials in ε, then

$$\left[x^i, p\left(\frac{\partial}{\partial x},\frac{\partial}{\partial \xi}\right)\exp(\varepsilon\Delta)\right] = q\left(\frac{\partial}{\partial x},\frac{\partial}{\partial \xi}\right)\exp(\varepsilon\Delta)$$

for some other q. The same is true for commutators with ξ_j, and as a result the lemma in fact shows that if α and β are any multi-indices, then

$$\left\langle \mathrm{ad}_x^{\alpha}\,\mathrm{ad}_{\xi}^{\beta}\left(p\left(\frac{\partial}{\partial x},\frac{\partial}{\partial \xi}\right)\exp(\varepsilon\Delta)\right)\right\rangle = 0.$$

But if f is any polyhomogeneous function, then

$$\left[p\left(\frac{\partial}{\partial x},\frac{\partial}{\partial \xi}\right)\exp(\varepsilon\Delta), f\right] = \sum_{\alpha,\beta\neq 0}\frac{1}{\alpha!\beta!}\,\mathrm{ad}_x^{\alpha}\,\mathrm{ad}_{\xi}^{\beta}\left(p\left(\frac{\partial}{\partial x},\frac{\partial}{\partial \xi}\right)\exp(\varepsilon\Delta)\right)\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\beta} f}{\partial \xi^{\beta}}$$

with the sum convergent in the adic topology. So the contraction of the commutator is 0. We have handled the third and final case in (4.15), and therefore the proof is complete.

The second crucial fact about Tr_F is its coordinate independence:

4.18. *Theorem.* The trace functional Tr_F is independent of the choice of coordinate system used to define it.

Proof. The theorem isn't an immediate consequence of the uniqueness part of Theorem 4.8 because the Laplacian Δ is not invariant under coordinate changes. What we need to show is that if Tr_F and τ are constructed using a first coordinate system, and if Δ' is the Laplacian in a second coordinate system, then

$$\mathrm{Tr}_F(f\exp(\varepsilon\Delta)) = \int_{S^*M} f = \mathrm{Tr}_F(f\exp(\varepsilon\Delta')).$$

As we already noted in the proof of Lemma 3.14, the second Laplacian has the form

$$^{0} = \; + \; a_i @_i + b^j_k @_j @_k \frac{@}{@} \quad \frac{@^2}{}$$

in the rst coordinate system, where a_i and b^j_k are smooth functions on M. If $T = {}^{0}$ then we can write (using the contraction operation associated to the rst coordinate system)

$$f \exp("^{0}) \quad f \exp(") = \int_0^1 f \exp(s")"T \exp((1 \; s)"^{0})ds:$$

If we expand the operators $\exp(s")"T \exp((1 \; s)"^{0})$ as power series in ", then the coecient of $"^k$, which is an element of $PDO(U)$, has dierential order $k \; 1$ in the x-direction (because T includes no x-derivative) and is homogeneous of degree k in the vertical direction (since each of , ^{0} and T has degree 1). This means that if we write the same operators in the standard form

$$\exp(s")"T \exp((1 \; s)"^{0}) = \sum_{k=0}^{1} "^k D_k \exp(")$$

and then write

$$D = \sum g_{k;;} \frac{@}{@x} \frac{@}{@}^k ;$$

then

$$jj \; jj \quad) \quad g_{k;;} = 0:$$

The contraction is therefore zero and the theorem is proved.

With Theorems 4.8 and 4.18 available, we can now extend the trace beyond coordinate charts to a continuous and F-linear trace functional

$$Tr_F : FB(M) \; ! \; F$$

in the obvious way using partitions of unity. But for the purposes of this paper we only need a part of the trace we have just dened, which is scalar-valued, not F-valued:

4.19. **Definition.** Let M be a smooth, closed manifold. The Perrot trace on $FB(M)$ is

$$Tr : FB(M) \; ! \; C$$

$$Tr(X) = \text{Coecient of } "^{0} \text{ in } Tr_F(X):$$

Finally, we need to pass from a (scalar) trace on $FB(M)$ to a supertrace on $FB(M; S)$. First denote by (End(S)) the smooth polyhomogeneous sections of the endomorphism bundle of S. This is a Z=2Z-graded algebra over $P(M)$ and there is a standard $P(M)$-linear supertrace

$$\mathrm{str}\colon \ (\mathrm{End}(S)) \quad ! \quad P(M)$$

In local coordinates:

$$\mathrm{str}(\ ^{-}) = \begin{cases} 1 & jj = jj = n \\ 0 & \text{otherwise:} \end{cases}$$

Then, in any coordinate chart U M we may use the natural (coordinate-dependent) vector space isomorphism

$$FB(U; S) = \quad (\mathrm{End}(Sj_U))$$

$_{P(U)}$ $FB(U)$ to dene

$$\mathrm{STr}_F : FB(U; S) \quad ! \quad F$$
$$\mathrm{STr}_F (A$$
$$X) = \mathrm{Tr}(\mathrm{str}(A)X)$$

(strictly speaking this is dened only on elements compactly supported in the U-direction). Thanks to Theorem 4.8, the above is a bimodule supertrace for the left and right actions of the algebra

$$PDO(U; S) = \quad (\mathrm{End}(Sj_U))$$
$$_{P(U)} PDO(U):$$

Thanks to Theorems 4.8 and 4.18 it is independent of the choice of coordinates used to dene it.

We can now use partitions of unity to dene (independently of all choices) a supertrace

$$\mathrm{STr}_F : FB(M) \quad ! \quad F;$$

and nally we make the following denition:

4.20. Definition. Let M be a smooth, closed manifold. The Perrot supertrace on $FB(M; S)$ is

$$\mathrm{STr}\colon FB(M; S) \quad ! \quad C$$
$$\mathrm{STr}(X) = \text{Coecient of } "^0 \text{ in } \mathrm{STr}_F (X):$$

5. The JLO-Type Cocycle

The purpose of this section is to construct from the Dirac operator D, together with Perrot's supertrace and one more ingredient, a periodic cyclic cocycle for the algebra $C^1(SM)$.

Smooth functions on SM can be viewed as order-one homogeneous functions on TM. Seen this way, the algebra $C^1(SM)$, together with bimodule $FB(M;S)$, equipped with its supertrace, and the operator $D\!\!\!/$ form something similar to one of Connes' spectral triples [Con94, Sec.IV.2][2]: the bimodule, with its trace, plays a role similar to the role of the Hilbert space in a spectral triple, with its associated operator trace (or perhaps with a residue trace obtained from this operator trace; see for instance [Hig04] for an exposition).

With this in mind we shall attempt to adapt the well-known JLO-cocycle in periodic cyclic theory to Perrot's context. Let ^{p} be the standard p-simplex. If $s\ 2\ ^p$ and $X^0;:::;X^p\ 2\ PDO(M;S)$, then let us write

$$(5.1)\qquad X^0;:::;X^p\ _s = X^0\exp(s_0 D\!\!\!/^2)X^1\exp(s_1 D\!\!\!/^2)\ X^p\exp(s_p D^2)\!\!\!/\!:$$

The most obvious JLO cocycle that one might try to attach to $D\!\!\!/$ is given by the formula

$$(5.2)\qquad JLO^{D\!\!\!/}(a^0;:::;a^p) = \quad STr\ a^0;[D;a^1];:::;[D;a^p]\ ds:\!\!\!/\ _s$$

This is the original formula given by Jae, Lesniewski and Osterwalder [JLO88]; see [Qui88, Sec.8] or [Con94, Sec.IV.8] for expositions.

However it follows from the supertrace property that the cocycle (5.2) vanishes in odd degrees, whereas the Todd class that we are trying to understand corresponds to a cyclic cocycle that vanishes in even degrees. So the above is not the cocycle we are seeking.[3]

To address this issue we need one further ingredient, as follows. Fix a Riemannian metric on M, and hence a Euclidean structure on the bundle TM. Dene

$$q: TM\ \ !\ \ (0;1); \qquad q() = kk;$$

and then dene a derivation

$$: PDO(M;S)\ \ !\ \ PDO(M;S)$$

by the formula

$$(X) = \mathrm{ad}_{\log(q)}(X)$$

(as usual we are adapting Perrot's work here, although we are adapting it to our simplied context). Note that

$$\frac{@}{@x^i} = \frac{@q}{@x^i}\,\frac{1}{q} \quad \text{and} \quad \frac{@}{@_j} = \frac{@q}{@_j}\,\frac{1}{q};$$

[2] In this reference the older term K-cycle is used, rather than spectral triple.

[3] In fact it may be shown that (5.2) vanishes in all degrees. In the more elaborate context of Perrot's work on pseudodierential symbols, this corresponds to the vanishing of the noncommutative residue on pseudodierential projectors [Per13b, Thm.6.5].

while vanishes on other (local) generators of PDO(M; S). So does indeed map PDO(M; S) into itself, despite the fact that log(q) is not a polyhomogeneous function.

Now extends to a derivation of the algebra FPDO(M; S) by applying to each coecient in a formal series. And this extension maps the bimodule FB(M; S) to itself. This follows from the formula

$$(\exp(")) = \int^{1} \exp(s")"() \exp((1 \quad s")) \, ds \; {}^{0}{}_{1}$$
$$= \quad \mathrm{Ad}_{\exp(s")} \; "() \; \exp(") \; ds \; {}_{0}$$

together with Lemma 3.12.

5.3. Lemma. If X 2 FB(M; S), then STr((X)) = 0.

Proof. Let X be any linear partial dierential operator on TM and let u 2 R. Consider the binomial-type formula

$$\mathrm{ad}_{q^u}(X) = \sum_{k=1}^{1} \binom{u}{k} \mathrm{ad}_q^k(X) q^{u \quad k};$$

which actually involves a nite sum on the right-hand side, since if h is any smooth function on TM, then the action of ad_h is locally nilpotent on partial dierential operators. Dierentiating the formula with respect to u and then setting u = 0, we obtain

$$\mathrm{ad}_{\log(q)}(X) = \sum_{k=1}^{1} \frac{1}{k}(\quad 1)^{k \quad 1} \mathrm{ad}_q^k(X) q^{\quad k}$$

Again the right-hand side is a nite sum, and it is a combination of commutators, since $\mathrm{ad}_q^k(X)q^{\quad k} = \mathrm{ad}_q^k(Xq^{\quad k})$ for all k.

Now we turn to FPDO(M; S). The above formulas involve innite sums when X 2 FPDO(M; S), but they are convergent in the adic topology and for that reason remain valid. Restricting to FB(M; S), it follows that every element of the form (X) = $\mathrm{ad}_{\log(q)}(X)$ is a limit of sums of commutators, and since the trace is continuous the trace of (X) is zero.

5.4. Remark. It is worth noting that the trace does not extend as a trace if we adjoin log(q) to the bimodule FB(M) (so this is not the explanation for the lemma above). For instance, in case where M is a circle, if q = jj, and if s is the characteristic function of f > 0g (which is a polyhomogeneous function), then

$$\left[\frac{@}{@}; s \log(q) \exp(") \right] = s^{\quad 1} \exp(");$$

and the trace of the right-hand side is nonzero (despite it being a commutator in the enlarged bimodule).

5.5. Remark. On a related note, although the denition of re- quires a choice of Riemannian metric, dierent choices yield derivations that dier only by an inner derivation.

Using Lemma 5.3 it is not dicult to adjust the denition of the JLO cocycle so as to incorporate . The following theorem is a generalization of the well-known fact that if is any derivation on an algebra, and if is a trace with $((a)) = 0$ for all a, then the formula $'(a^0; a^1) = (a^0(a^1))$ denes a cyclic cocycle. As Rodsphon explains in [Rod15], the theorem ts very naturally into Quillen's formalism for the JLO cocycle [Qui88], and this is probably the best way of understanding it.

First a simple preliminary calculation:

5.6. Lemma. Let r be a torsion-free connection on M and let D be the associated Dirac operator, as in Denition 3.18. If p 0, s 2 $^{p+1}$ and $X_0; : : : ; X_{p+1}$ 2 PDO(M; S), then

$$X^0; : : : ; X^{p+1} \quad {}_s \; 2 \; FB(M; S);$$

to use the notation of (5.1).

Proof. This is a combination of Lemmas 3.19 and 3.21.

5.7. Theorem. The formulas

$$JLO^{D;}(a^0; : : : ; a^p) = \sum_{k=1}^{p+1} (\;\; 1)^k \quad {}_{p+1} \quad STr \quad a^0; [D; a^1]; : : :$$

$$: : : ; [D; a^{k-1}]; (D); [D; a^k]; : : : ; [D; a^p] \quad {}_s \quad ds$$

for p odd and $a^0; : : : ; a^p$ 2 $C^1(SM)$ dene a periodic cyclic cocycle for the algebra $C^1(SM)$.

5.8. Remark. The integrands in the dening formula for $JLO^{D;=}$ are polynomial functions of s 2 $^{p+1}$, so the integrals are certainly well-dened, and moreover these polynomial functions are identically zero for p suciently large (this will be made clear in the next section), so that only nitely many components of $JLO^{D;}$ are nonzero, as required in the denition of periodic cyclic cocycle. This is in contrast to the form of the usual JLO cocycle (involving the Levi-Civita Dirac operator and the operator trace), which is rather an entire cyclic cocycle [Con94, Ch.4, Sec.7].

5.9. **Remark.** In general the Quillen formalism would produce a formula with further terms, involving factors (a^j); see [Rod15, Sec.3]. But in the case at hand, $(a^j) = 0$.

We shall not prove Theorem 5.7 in this paper, rstly because it is a general result that is encompased by the Quillen formalism and is explained perfectly well by Rodsphon in [Rod15], and secondly because we shall not actually need the result: we shall calculate all the components of $JLO^{\not{D}}$, and it will be evident from these computations that they constitute a periodic cyclic cocycle.

6. Perrot Order of a Dierential Operator

The remainder of the paper will be dedicated to the computation of the JLO-type cocycle that was dened in the previous section. We shall introduce a new notion of order on dierential operators that we shall call the Perrot order (it is distinct from the bimodule order discussed earlier). The Perrot order has the property that operators of negative order have vanishing trace. Roughly speaking, upon analyzing the constituents of the cocycle, we shall nd that many parts have negative order, and so they can be removed without altering the value of the cocycle.

The Perrot order resembles somewhat the notion of order that Getzler uses to compute the index of the spinor Dirac operator [Get83] (see [Roe98] or [HY19] for surveys). Both have the property that the curvature operator[4] acquires the same order as the square of the Dirac operator, which leads to the curvature contributing to leading order computations. But the leading order is 2 for Getzler, while it is 0 for Perrot. An important consequence is that the formal exponential $\exp(\not{D}^2)$ also has Perrot order 0. Indeed all of the constituents of the JLO cocycle introduced in the previous section have Perrot order 0.

Despite its role in capturing curvature, the Perrot order does not in fact depend on the choice of connection, whereas Getzler's certainly does (which is a paradox, perhaps, but not a contradiction). But to begin with, to dene the Perrot order it is convenient to x a torsion-free connection on M (and then later we shall show that the Perrot order does not depend on this choice).

6.1. **Definition.** The Perrot order of an operator in $PDO(M; S)$ (which is an integer, possibly negative) is dened by the following:

[4] In Perrot's case, we are referring here to the operator (R) that was introduced in Denition 2.14 and appears in the formula for $\not{D}^2$. The actual curvature operators $R(X; Y)$ have order 1; see the proof of Lemma 6.6.

(i) If f is any degree p-homogeneous smooth function on T_M, then Perrot-order(f) p.

(ii) If X is any vector eld on M, with horizontal lift X^H on T M, then Perrot-order(r_{X^H}) 1.

(iii) If Y is any vertical vector eld on T M (that is, if Y commutes with functions pulled back from M), and if Y is translation-invariant on each cotangent ber, then Perrot-order(r_Y) 1.

(iv) If ! is any 1-form on M, acting on sections of S over T_M by left exterior multiplication, then Perrot-order(!) 0.

(v) If X is any vector eld on M, and if (X) denotes the action on sections of S by contraction, then Perrot-order((X)) 1.

6.2. Remark. Compare Proposition/Denition 3.7, which explains the way in which an increasing ltration on PDO(M; S) is constructed from inequalities such as those above. As with the bimodule order considered there, the inequalities in the denition above are actually equalities; this will be clear from the lemma below.

An operator has Perrot order p or less if and only if its restriction to every T_U, with U M open, has Perrot order p or less. So the Perrot order may be computed locally. The following lemma shows that moreover the Perrot order is easily determined in local coordinates. It also shows that the Perrot order is independent of the choice of ane connection r .

6.3. Lemma. Let $x_1; : : : ; x_n$ be coordinates on an open set U M and let p 2 Z. If the bundle S is trivialized over T U using the standard frame for the exterior algbra bundle associated to these coordinates, then the linear space of all operators of Perrot order p on less on T U is precisely the linear span of operators of the form

$$f \quad \frac{@ \quad @}{@x \; @}$$

with f a homogeneous smooth scalar function on T_U and with

$$\text{degree}(f) + jj \quad jj + jj \;\; p:$$

Here and are multi-indices with non-negative integer entries, while and are multi-indices with 0-1 entries.

Proof. It follows from (2.9) that

$$\frac{@}{@x^i} = r_{(@=@x^i)^H} \qquad {}^k_{i`\,k} \frac{@}{@`} + \quad {}_{i`}{}^{k} \quad {}^{`}\,{}_{k}:$$

All three terms on the right-hand side have Perrot order 1 or less, and therefore @=@x^{j} has Perrot order 1 or less. In addition

$$\frac{@}{@_{j}} = r_{@=@_{j}}:$$

It follows that all the operators in the statement of the lemma have Perrot order p or less.

In the reverse direction, if we dene $PDO_p(U; S)$ to be the linear span of the operators in the statement of the lemma, then we obtain an algebra ltration for which the associated order satises the relations in Denition 6.1. It therefore follows from the construction of the Perrot ltration and order (c.f. Propositon/Denition 3.7) that every operator of Perrot order p or less belongs to $PDO_p(U; S)$.

Let us now extend the Perrot order to the algebra $FPDO(M; S)$ and to the bimodule $FB(M; S)$ as follows: a formal series $\sum "^{k} T_k$ has Perrot order p or less if and only if each T_k has Perrot order p or less. (With this denition, there are many elements of innite Perrot order, but that will not be an issue for us.)

6.4. **Proposition.** Let M be a closed manifold. The supertrace vanishes on every element of $FB(M; S)$ of negative Perrot order.

Proof. It suces to consider negative-order generating elements

$$f \quad - \frac{@ \quad @}{@x\,@} \exp(")$$

of the bimodule $FB(U; S)$ in a coordinate neighborhood. According to the denition of the supertrace and the formula for the contraction operation in Denition 4.9, if = , then the supertrace of this element is zero. In addition, if jj = n then the supertrace is zero. If = and if jj = n, then the Perrot order is equal to degree(f) n, and if this is negative, then by the denition of the integral f in (4.6) the supertrace is zero.

Our purpose in the remainder of this section is to develop a technique to compute the supertrace of order zero operators. In the lemma below we shall use the fact that if : W ! M is any submersion with compact bers, then there is an induced morphism of $C^{1}(M)$-modules

$$: \mathrm{Dens}(W) \quad ! \quad \mathrm{Dens}(M)$$

that is characterized by the formula

$$\int_{M} () = \int_{W}$$

(in the context of oriented smooth manifolds this is the operator of integration over the fiber).

6.5. Lemma. There is a unique morphism of $C^\infty(M)$-modules

$$\mathrm{str}\colon FB(M) \to \mathrm{Dens}(M)$$

such that

$$\mathrm{STr}(X) = \int_M \mathrm{str}(X)$$

for every $X \in FB^0(M)$.

Proof. The trace was constructed (locally, at first) as the integral of a density on SM, and pushing this density forward along the projection $SM \to M$ we obtain from a partition of unity argument the existence of a map as in the statement of the lemma.

To prove uniqueness, if str and str$'$ are two morphisms of modules as in the statement of the lemma, then from the identities

$$\begin{aligned}\int_M f \cdot \mathrm{str}(X) = \int_M \mathrm{str}(f \cdot X) &= \mathrm{STr}(X)\\ &= \int_M \mathrm{str}'(f \cdot X) = \int_M f \cdot \mathrm{str}'(X)\end{aligned}$$

we conclude that $\int_M f \cdot \big(\mathrm{str}(X) - \mathrm{str}'(X)\big) = 0$ for all $f \in C^\infty(M)$, and hence $\mathrm{str}(X) - \mathrm{str}'(X) = 0$.

We shall now evaluate the density $\mathrm{str}(X) \in \mathrm{Dens}(M)$ defined above at a single point $m \in M$ under the assumption that $X \in FB(M; S)$ has order zero.

Form the associated graded algebra

$$\mathrm{gr}\,\mathrm{PDO}(M;S) = \bigoplus_p \mathrm{PDO}_p(M;S)/\mathrm{PDO}_{p-1}(M;S)$$

for the filtration by Perrot order. Notice that since smooth functions on M (viewed as operators on sections of S by pointwise multiplication) have Perrot order zero, the associated graded algebra is in fact an algebra over $C^\infty(M)$. We can therefore speak of the fiber

$$\mathrm{gr}\,\mathrm{PDO}(M;S)_m$$

at a given point $m \in M$, which is an algebra in its own right. Our first aim is to compute this fiber.

6.6. Lemma. If X and Y are vector elds on M, then the degree 1 classes of r_{X^H} and r_{Y^H} commute with one another in the associated graded algebra.

Proof. Let 2 TM and let $m = ()$. Because the connec-tion we are using on TM is a pullback, the curvature operator $R(X^H; Y^H)j$ is the curvature operator $R(X; Y)j_m$ for the original con-nection:

$$(6.7) \qquad r_{X^H} r_{Y^H} \quad r_{Y^H} r_{X^H} = r_{[X^H;Y^H]} + R(X; Y)$$

(this is an identity of operators acting on the sections of $S = TM$). If we introduce local coordinates on M and write the curvature operator on TM as

$$R \quad \frac{@}{@x^i}; \frac{@}{@x^j} \quad \frac{@}{@x^\ell} = R^k_{ij\ell} \frac{@}{@x^k};$$

as in Denition 2.14, then the curvature operator as it appears in (6.7) is

$$R \quad \frac{@}{@x^i}; \frac{@}{@x^j} \quad = \quad {}^{\ell}{}_{k} R^k_{ij\ell};$$

and in particular the operator $R(X; Y)$ in (6.7) has Perrot order 1.

In addition, according to Lemma 2.6,

$$[X^H; Y^H] = [X; Y]^H + \quad R(X; Y):$$

The covariant derivatives attached to the two terms on the right have Perrot-orders 1 and 0, respectively. So we nd from (6.7) that

$$\begin{aligned} \text{Perrot-order } & r_{X^H} r_{Y^H} \quad r_{Y^H} r_{X^H} \\ & 1 \\ & < \text{Perrot-order } r_{X^H} + \text{Perrot-order } r_{Y^H} ; \end{aligned}$$

which proves the lemma.

6.8. Remark. The lemma stands in contrast to the situation with the Getzler's order, in which the commutator covariant derivatives is a curvature operator.

Other, more easily derived, relations in the associated graded algebra are as follows:

(i) The degree 1 classes of the operators r_Y associated to vertical vector elds that are berwise translation invariant (that is, scalar combinations of the $r_{@=@_j}$) commute with one another and the classes of the r_{X^H}.

(ii) The degree 0 and 1 (respectively) classes of the exterior multiplication and contraction operators anticommute with one another and commute with all the classes of covariant derivatives already mentioned.

(iii) The degree p classes of scalar degree p functions commute with one another and with all the classes mentioned above, except that

$$[\nabla_{\partial/\partial\xi_j}][f] - [f][\nabla_{\partial/\partial\xi_j}] = [\partial f/\partial\xi_j]$$

as in $\mathrm{PDO}(M;S)$.

6.9. Definition. For $m \in M$, denote by $\mathcal{A}_m$ the tensor product of the symmetric algebra of T_mM (whose elements we shall write as constant coefficient differential operators on T^*_mM), the algebra of polyhomogeneous partial differential operators on T^*_mM and the exterior algebra $\Lambda(T_mM \oplus T^*_mM)$ (which, given local coordinates on M near m, we shall regard as generated by $dx^i \in T^*_mM$ and $d\xi_j \in T_mM$).

We shall suppress tensor product signs when describing elements of $\mathcal{A}_m$, such as for example

$$dx^i \frac{\partial}{\partial x^i} + d\xi_j \frac{\partial}{\partial \xi_j} \in \mathcal{A}_m \tag{6.10}$$

We equip $\mathcal{A}_m$ with an integer grading by assigning degree 1 to each vector in T_mM, the usual degree to each homogeneous operator on T^*_mM, degree 0 to each dx^i and degree 1 to each $d\xi_j$. For instance, the terms in (6.10) have degrees 1 and 0, respectively.

6.11. Lemma. Let ∇ be a torsion-free connection on M and let $m \in M$. There is a unique isomorphism of complex $\mathbb{Z}$-graded algebras

$$\mathrm{Symb}^\nabla_m \colon \operatorname{gr}\mathrm{PDO}(M;S)_m \xrightarrow{\cong} \mathcal{A}_m$$

such that for all local coordinates near m,

(i) $[\nabla_{(\partial/\partial x^i)^H}] \mapsto \partial/\partial x^i$ and $[\nabla_{\partial/\partial\xi_j}] \mapsto \partial/\partial\xi_j$.

(ii) If f is any polyhomogeneous function on T^*M, then $f \mapsto f|_{T^*_mM}$.

(iii) $[\varepsilon^i] \mapsto dx^i$ and $[\bar\iota_j] \mapsto d\xi_j$.

Proof. The reversed correspondences define a morphism of graded algebras in the reverse direction by the universal property inherent in the definition of $\mathcal{A}_m$ as a tensor product. It follows from the local coordinate description of the Perrot order in Lemma 6.3 that this morphism is an isomorphism.

6.12. Remark. Even though the Perrot order and therefore the associated graded algebra $\operatorname{gr}\mathrm{PDO}(M;S)$, are independent of the choice of connection used in their denitions, the homomorphism in Lemma 6.11 does depend on the choice of connection.

We shall now extend the symbol morphism Symb^r_m from polyhomogeneous dierential operators to elements of the bimodule $FB(M;S)$. To do so we shall rst look to Denition 3.9 to nd a suitable target for the symbol homomorphism.

6.13. **Definition.** Dene the bimodule order of an element in A_m by

$$\text{bimodule-order}\ \frac{\partial}{\partial x^i} = 3; \quad \text{bimodule-order}\ \frac{\partial}{\partial_j} = -1$$

$$\text{bimodule-order}\ {}^{i} = 0; \quad \text{bimodule-order}\ {}_{j} = 0$$

and $\text{bimodule-order}(f) = -2\,\text{degree}(f)$ for a homogeneous function f. Denote by FA_m the space of formal Laurent series

$$\sum_k \varepsilon^k D_k \exp(\varepsilon) \in A_m[\varepsilon^{-1};\varepsilon]]$$

with nite singular parts for each of which there exists N with

$$\text{bimodule-order}(D_k) \le k + N$$

for all k.

The space FA_m is a bimodule over A_m, and applying the symbol homomorphism termwise to a Laurent series, we obtain a morphism

$$\mathrm{Symb}^r_m \colon \operatorname{gr} FB(M;S)|_m \to FA_m$$

that is compatible with bimodule structures. We are going to factor Perrot's supertrace on order zero elements through this symbol morphism, as follows: we shall dene a supertrace morphism

$$\mathrm{str}_m \colon FA_m \to \mathrm{Dens}(M)|_m$$

and show that

$$\mathrm{str}(X)|_m = \mathrm{str}_m(\mathrm{Symb}^r_m(X))$$

for every $X \in FB(M;S)$ of Perrot order 0.

To begin the denition of str_m, suppose we are given an order $-n$ homogeneous smooth function f on $T_m M$, and let $\omega \in \wedge^n T_m M$. Think of ω as a (translation-invariant) top-degree form on $T_m M$. If E is the Euler vector eld, as usual, then the contracted form $\iota_E(f\omega)$ is basic for the action of the positive real numbers on $T_m M$ by scalar multiplication, and it is therefore the pullback of a form $\alpha_{f;\omega}$ on the sphere $S_m M$. Consider next the integral

$$\int_{S_m M} \alpha_{f;\omega}; \tag{6.14}$$

formed using the orientation on the sphere for which the integral is positive when f is positive. This orientation depends on , and so the integral is not bilinear in f and , but rather

$$\int_{S_m M} f;t = |t| \int_{S_m M} f; \qquad \forall t \in \mathbb{R}:$$

The map $\mapsto \int_{S_m M} f;$ therefore defines a density at m (see [BGV92, pp.30-31] or [Lee03, Ch.14] again), which we shall write as

$$\int_{S_m M} f \in \mathrm{Dens}(M)|_m \tag{6.15}$$

(and as before we extend this to polyhomogeneous f by selecting the degree n component). This construction has the property that if f is a smooth, degree n function on $T M$ that is compactly supported in the M-direction, then

$$\int_M \int_{S_m M} f|_{T_m M} = \int_{SM} f: \tag{6.16}$$

To define a trace[5] functional on FA_m, we use the integral above and essentially the same contraction operation that we used for Perrot's trace: namely for $D \in A_m$ we define

$$D \exp(") = "^{\,n} D \exp(\ "^{-1}(x^i \quad y^i)(\quad ^i \quad ^i)) \Big|_{x^i = y^i = 0;\ _i = _i} :$$

This is a polyhomogeneous function on $T_m M$ with values in the exterior algebra $(T_m M \ T_m M)$. Then we define

$$\mathrm{str}_{m;F}\ D \exp(") = \text{coefficient of } dx^1 \ dx^n d_1 \ d_n \text{ in } \int_{S_m M} D \exp("); \tag{6.17}$$

and then extend termwise to the Laurent series in FA_m. The result is a Laurent series in $"$ and, just as we did when we constructed Perrot's trace, we then define str_m by taking the coefficient of $"^0$.

Thanks to our direct mimicry of the constructions in Section 4, and thanks to (6.16), the following result is clear:

6.18. **Lemma.** Let $x_1; : : : ; x_n$ be local coordinates on M, defined near a point $m \in M$. If ∇ is the canonical flat connection on TM

[5] The functional is indeed a trace, although we shall not prove this fact because we shall not need it.

associated to these coordinates (in which all the Christoel symbols $_{ij}{}^{k}$ are identically zero), then

$$\mathrm{str}(X)_m = \mathrm{str}_m(\mathrm{Symb}^r_m(X))$$

for every X 2 $FB(M;S)$ of Perrot order 0.

In fact, the same result holds for any connection:

6.19. Theorem. If r is any torsion-free connection on TM, if m 2 M, and if

$$\mathrm{Symb}^r_m : FB_0(M) \;\; ! \;\; FA_m$$

is the symbol map at m dened by r and acting on Perrot order zero operators, then

$$\mathrm{str}(X)_m = \mathrm{str}_m(\mathrm{Symb}^r_m(X))$$

for every X 2 $FB^0(M)$.

Proof. Given r and m, choose local coordinates near m for which all the Christoel symbols for r vanish at m. Then dene r^0 to be the canonical at connection for these coordinates. By examining the two morphisms Symb^r_m and $\mathrm{Symb}^{r^0}_m$ in this given coordinate system, we see that these two symbol morphisms are in fact equal. So the theorem follows from the previous lemma.

6.20. Remark. Needless to say, the theorem relies heavily on the coordinate independence of Perrot's trace (and in particular the theorem is far from trivial).

7. Computation of the JLO cocycle

In this nal section we shall use Theorem 6.19 and one nal additional computation to identify the cocycle $JLO^{\eth;}$ from Section 5 with the cocycle associated with the Todd class that was described in Section 1. The starting point is the following computation:

7.1. Lemma. If r is any torsion-free connection on M, then

$$\text{Perrot-order } \underset{=}{D}_{\text{vert}} \; 0 \qquad \text{and} \qquad \text{Perrot-order } f\underset{=}{D}_{\text{horiz}}; \underset{=}{D}_{\text{vert}} g \; 0:$$

In addition if a is any smooth, degree zero function on T_M, then

$$\text{Perrot-order } [\underset{=}{D}; a] \; 0:$$

Proof. This evident from the formulas in Denition 2.10 and in Lemmas 2.13 and 2.16.

It follows that the argument of the supertrace in the formula for $\mathrm{JLO}^{D,\tau}(a^0,\dots,a^p)$ has Perrot order 0, and therefore we may use the techniques of Section 6 to compute the supertrace.

With this in mind, we now fix a point $m \in M$. As explained in the Section 6, the quantity $\mathrm{JLO}^{D,\tau}(a^0,\dots,a^p)$ is the integral of a density on M; we shall denote by

$$\mathrm{JLO}^{D,\tau}(a^0,\dots,a^p)|_m \in \mathrm{Dens}(M)|_m$$

the value of this density at m.

If we write

$$R\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k} = R_{ijk}^{\ell}\frac{\partial}{\partial x^\ell},$$

as we did earlier, and then define

$$R^k_\ell = \tfrac{1}{2} R^k_{ij\ell}\,dx^i dx^j\big|_m \in \wedge^2 T^*_m M,$$

then

$$\mathrm{Symb}^\nabla_m(\exp(s\mathbb{D}^2)) = \exp\big(-s(\varepsilon\Delta+\varepsilon^2\xi_k R^k_\ell\,\partial/\partial\xi_\ell)\big) \tag{7.2}$$

Now choose coordinates $x^1,\dots,x^n$ near m for which the Christoffel symbols Γ^k_{ij} vanish at m. Then

$$\begin{aligned}\mathrm{Symb}^\nabla_m([\mathbb{D},a]) &= \varepsilon\frac{\partial a}{\partial x^i}\,dx^i + \frac{\partial a}{\partial \xi_j}\,d\xi_j\\ \mathrm{Symb}^\nabla_m(\delta(\mathbb{D})) &= -\varepsilon q^{-1}\frac{\partial q}{\partial x^i}\,dx^i - q^{-1}\frac{\partial q}{\partial \xi_j}\,d\xi_j\end{aligned} \tag{7.3}$$

where as in Section 5, $q\colon T^*M \to (0,\infty)$ is the norm-function associated to a metric on TM.

For brevity, we shall write

$$\mathrm{Symb}^\nabla_m([\mathbb{D},a]) = d_\varepsilon a = \varepsilon d_{\mathrm{horiz}} a + d_{\mathrm{vert}} a$$

$$\mathrm{Symb}^\nabla_m(\delta(\mathbb{D})) = -q^{-1}d_\varepsilon q = -\varepsilon q^{-1} d_{\mathrm{horiz}} q - q^{-1} d_{\mathrm{vert}} q.$$

In addition, to streamline some of the formulas that follow, we shall write

$$\Delta_R = \Delta + \varepsilon\xi_k R^k_\ell\,\partial/\partial\xi_\ell = \Delta + \varepsilon\xi R\,\partial/\partial\xi.$$

Then according to the formulas (7.2) and (7.3) above and the formula in Theorem 5.7, the density $\mathrm{JLO}^{D,\tau}(a^0,\dots,a^p)|_m$ is the sum from $k=1$ to $k=p+1$ of the following integrals:

$$\begin{aligned}(-1)^{k-1}\int_{\Delta^{p+1}} \mathrm{str}_m\Big(& a^0 \exp(s_0\varepsilon\Delta_R)d_\varepsilon a^1 \exp(s_1\varepsilon\Delta_R)\\ &\dots d_\varepsilon a^{k-1}\exp(s_{k-1}\varepsilon\Delta_R)q^{-1}dq\exp(s_k\varepsilon\Delta_R)d_\varepsilon a^k\\ &\qquad\cdots\exp(s_{p-1}\varepsilon\Delta_R)d_\varepsilon a^{p-1}\exp(s_p\varepsilon\Delta_R)\Big)\,ds.\end{aligned} \tag{7.4}$$

Using the formula

$$\exp(s"_R)X\exp(\ \ s"_R) = \exp(s"\,\mathrm{ad}_R)(X)$$

we can write the argument of str_m in (7.4) as an innite linear combination (convergent in the adic topology) of terms

$$(7.5)\quad "^k X_0\,\mathrm{ad}_R^{k_0}(X_1)\,\mathrm{ad}_R^{k_1}(X_2)\cdots\mathrm{ad}_R^{k_{p\ \ 2}}(X_{p\ \ 1})\,\mathrm{ad}_R^{k_{p\ \ 1}}(X_p)\exp("_R);$$

where each X_j is one of the $d_" a^i$ or $q\ \ ^1 d_" q$.

Next, as long as any $X\ \ 2\ A_m$ includes no -derivatives

$$\mathrm{ad}_R(X) = \quad @=@x^j + "_i R_{ij}\quad [@=@_j;X];$$

and we note that in this case the result on the right-hand side also includes no -derivatives. Now we appeal to the following result, whose proof we shall defer for a moment:

7.6. **Proposition.** If an element $X\ \ 2\ A_m$ includes no -derivatives, then

$$\mathrm{str}_m\ \ "^r(@_{x^j} + "_i R_{ij})X\exp("_R)\ \ = 0$$

for all r and all j.

It follows from the lemma that all the terms (7.4) have vanishing trace, except when all k_j are zero, and hence that (7.4) is equal to

$$(\ \ 1)^{k\ \ 1}\qquad \mathrm{str}_m\ \ a^0 d_" a^1\cdots d_" a^{k\ \ 1} q\ \ ^1 d_" q\cdots d_" a^p\exp("_R)ds:\ {}_{p+1}$$

The integrand now no longer depends on s, and so the integral is simply

$$\frac{(\ \ 1)^{k\ \ 1}}{(p+1)!}\,\mathrm{str}_m\ \ a^0 d_" a^1\cdots d_" a^{k\ \ 1} q\ \ ^1 d_" q\cdots d_" a^p\exp("_R):$$

In summary, then:

$$(7.7)\quad JLO^{D_?}(a^0;\ldots;a^p)|_m$$

$$= \sum_{k=1}^{p+1}\frac{(\ \ 1)^{k\ \ 1}}{(p+1)!}\,\mathrm{str}_m\ \ a^0 d_" a^1\cdots d_" a^{k\ \ 1} q\ \ ^1 d_" q\cdots d_" a^p\exp("_R)\ {}_{p+1}$$

$$= \sum_{k=1}\frac{1}{(p+1)!}\,\mathrm{str}_m\ \ q\ \ ^1 d_" q\, a^0 d_" a^1\cdots d_" a^p\exp("_R) = \ \ p!$$

$$\frac{1}{}\mathrm{str}_m\ \ q\ \ ^1 d_" q\, a^0 d_" a^1\cdots d_" a^p\exp("_R):$$

Now we quote another result, whose proof, like that of Proposition 7.6, we shall defer for a short time:

7.8. **Proposition.** If X 2 A_m has no x- or -derivatives, then

$$X\ \exp("_R) = "\ ^n X\ \mathrm{Todd}("^2 R):$$

To continue the computation, let us now write

$$q\ ^1 d_" q\ a^0 d_" a^1\ d_" a^p\ \mathrm{Todd}("^2 R)^{top}$$
$$= \text{ top exterior form-degree part of}$$
$$q\ ^1 d_" q\ a^0 d_" a^1\ d_" a^p\ \mathrm{Todd}("^2 R):$$

The Todd class is a polynomial in ", for which the coecient of $"^k$ lies in ^{k}TM (with only even k appearing). Each of the $d_"$-dierentials lies in $"^1$T M + ^1T_mM. It therefore follows from an examination of degree in T_m M that

$$q\ ^1 d_" q\ a^0 d_" a^1\ d_" a^p\ \mathrm{Todd}("^2 R)^{top}$$
$$= "^n\ \ q\ ^1 dq\ a^0 da^1\ da^p\ \mathrm{Todd}(R)\ ^{top}:$$

As a result, it follows from (7.7) and Proposition 7.8, and the denition of str_m in (6.17) that

$$(7.9)\quad \frac{1}{p!}\ \mathrm{str}_m\ \ q\ ^1 d_" q\ a^0 d_" a^1\ d_" a^p \exp("_R)$$
$$= \text{ coecient of } dx^1\ dx^n d_1\ d_n$$
$$\text{in } \frac{1}{p!}\ _{S_m M}\ [q\ ^1 dq\ a^0 da^1\ da^p\ \mathrm{Todd}(R)]^{top}:$$

In this formula $[q\ ^1 dq\ a^0 da^1\ da^p\ \mathrm{Todd}(R)]^{top}$ is to be regarded as a polyhomogeneous, exterior algebra-valued function on T_mM and then written as a scalar function times $dx^1\ dx^n d_1\ d_n$; the integral is applied to this scalar function as in (6.15).

The right-hand side in (7.9) involves a contraction by the Euler vector eld, so let us now note that

$$_E\ \ q\ ^1 dq\ a^0 da^1\ da^p\ \mathrm{Todd}(R) = {}_E\ \ q\ ^1 dq a^0 da^1\ da^p\ \mathrm{Todd}(R)$$
$$= a^0 da^1\ da^p\ \mathrm{Todd}(R);$$

since $_E$ $q\ ^1 dq = 1$ while $_E$ $a^0 da^1\ da^p\ \mathrm{Todd}(R) = 0$. It follows that

$$\mathrm{str}_m\ \ q\ ^1 d_" q\ a^0 d_" a^1\ d_" a^p \exp("_R)$$
$$= \ [a^0 da^1\ da^p\ \mathrm{Todd}(R)]^{top}\ _m;$$

where $\pi\colon SM \to M$ and where the differential form is treated as a density using the orientation of SM given in Definition 4.3. We have arrived at our goal:

7.10. Theorem. Let M be a smooth, closed manifold, let ∇ be a torsion-free connection on TM with curvature R, and let $D\!\!\!/$ be the associated Dirac operator, as in Definition 3.18. If SM is equipped with the orientation in Definition 4.3, then

$$JLO^{D\!\!\!/}(a^0, \dots, a^p) = \frac{1}{p!}\int_{SM} a^0 da^1 \cdots da^p \,\mathrm{Todd}(R)$$

for all $p > 0$ and all $a^0, \dots, a^p \in C^\infty(SM)$.

It remains to prove Propositions 7.6 and 7.8.

Proof of Propositions 7.6 and 7.8. The following beautiful argument (along with everything else in this section) is due to Perrot [Per13b, Lem.4.4], to which we refer for full details. It is reminiscent of approaches to the Baker-Campbell-Hausdorff formula going back to Schur [Sch89].

For $t \in \mathbb{R}$ form the element

$$Y(t) = \exp(\varepsilon\theta + t\varepsilon^2 R\,\partial/\partial\theta)\exp(-\varepsilon\theta)$$

in $A_m[\varepsilon]$; it is a polynomial function of t and ε thanks to the nilpotence of the subspace

$$\wedge^2(T_mM \oplus T^*_mM) \otimes_m (T_mM \oplus T^*_mM).$$

If $X \in A_m[\varepsilon]$, then according to (4.12),

$$\begin{aligned}\sigma\big(X \exp(\varepsilon\theta + t\varepsilon^2 R\,\partial/\partial\theta)\big) &\\ = \varepsilon^{-n}\,\sigma\big(X\,Y(t)\big)\exp\big(\varepsilon^{-1}(x^i - y^i)(\xi_i - \eta_i)\big)&\Big|_{x^i=y^i;\ \xi_i=\eta_i}\end{aligned}$$

Consider now the (form-valued) function

$$H(t) = \varepsilon^{-n}\,\sigma(Y(t))\exp\big(\varepsilon^{-1}(x^i - y^i)(\xi_i - \eta_i)\big)$$

on the product of T^*T_mM with itself. A direct computation shows that it satisfies a differential equation of the type

$$\frac{dH}{dt}(t) = L(t)H(t);$$

where $L(t)$ is a polynomial in t and ε with coefficients in the algebra A_m, and that the function

$$\mathrm{Todd}(\varepsilon^2 tR)\exp\Big(\varepsilon R(x-y)\Big(\cdots\Big)\frac{1}{1} - \frac{t\varepsilon R}{\exp(-t\varepsilon^2 R)}(x-y)\Big)$$

satises the same dierential equation, with the same initial condi-tion at $t = 0$. The two solutions are necessarily equal, and the two propositions follow by setting $t = 1$ in the explicit solution, applying respectively $(\partial/\partial x^j + \varepsilon_i R_{ij})X$, as in the rst proposition, or X alone as in the second, and restricting to the diagonal to obtain an explicit formula for the contractions.

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