# MATHEMATICAL PROBLEM POSING: TASK VARIABLES, PROCESSES, AND PRODUCTS

Jinfa Cai, University of Delaware
Boris Koichu, Weizmann Institute of Science
Benjamin Rott, University of Cologne
Rina Zazkis, Simon Fraser University
Chunlian Jiang, University of Macau

*Mathematical problem posing (MPP) has been at the forefront of discussion for the past few decades, and a wide range of problem-posing topics have been studied. However, problem posing is still not a widespread activity in mathematics classrooms, and there is not yet a general problem-posing analogue to well-established frameworks for problem solving. This paper presents the state of the art on the effort to understand the cognitive and affective processes of problem posing as well as task variables of problem posing at the individual, group, and classroom levels. We end this paper by proposing a number of research questions for future studies related to task variables and processes of problem posing.*

## POSING A PROBLEM ABOUT PROBLEM POSING – PROMPT DESIGN

To open the floor for a discussion about problem posing, we invite readers to engage with a problem-posing activity. Consider the initial Situation A and several related prompts for problem posing below. How would the different prompts impact your problem posing based on Situation A?

Situation A: ABC is an equilateral triangle. D, E, and F are midpoints of the sides of ΔABC. Show that the area of ΔDEF is ¼ the area of ΔABC.

Prompt 1A: Based on the above problem, use the “what if not” strategy to pose two mathematical problems.

Prompt 2A: Based on the above problem, use the “what if not” strategy to pose as many mathematical problems as you can.



2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.). Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1, pp. 119-145). PME.

Prompt 3A: Based on the above problem, use the “what if not” strategy to pose two “easy” mathematical problems and two “difficult” mathematical problems, where the relative difficulty takes into account the levels of students.

Five of us independently responded to the question (How would the different prompts impact your problem posing based on Situation A?). A clear difference between the prompts is in the request for the number of posed problems: two in Prompt 1A, two easy and two difficult in Prompt 3A, and “as many as you can” in Prompt 2A. Further, the addition of “relative difficulty” and “levels of students” in Prompt 3A is appropriate for a problem-posing activity with teachers and can be omitted in work with students. However, the reference to difficulty may entice problem posers to consider a greater variety of problems and attend to what can make a problem easy or difficult. Moreover, problem posers’ interpretation of “difficulty” can be a fruitful venue for investigation.

Common to Prompts 1A, 2A, and 3A is the reference to the “what if not” strategy. As such, the expected variations in problem posing can attend to any of the problem attributes:

V1: What if ABC is a not-equilateral (right angle, isosceles, scalene) triangle? What then is the ratio of the areas of ΔABC and ΔDEF?
V2: What if D, E, and F are not midpoints but divide the sides in some common ratio. What then is the ratio of the areas of ΔABC and ΔDEF?
V3: What if the ratio of the areas of ΔABC and ΔDEF is a given R. How then should we place points D, E, and F on the sides of ABC to obtain the given ratio of the triangle areas?
V4: What if we are not considering ΔABC and ΔDEF? What other triangles are determined in Situation A? What is the relationship between their areas?
V5: What if the starting figure is not a triangle but a quadrilateral (or a special quadrilateral, like a square) and the “inner” quadrilateral is constructed by connecting mid points (or points placed on the sides of that quadrilateral) using a given ratio. What then is the relationship between the starting areas and the inner quadrilaterals? What if it is not a quadrilateral but any polygon?
V6: What if we aren’t looking for areas? Can you determine any relationship between the attributes of ΔABC and ΔDEF?

Situation A mentions the relationship between areas. As such, five of the six examples above explicitly mention areas of triangles. But a particular focus can be on the prompt rather than on the situation. Consider Situation B and several related prompts below.

Situation B: D, E, and F are midpoints of the sides of equilateral triangle ΔABC.

Prompt 1B: Consider the (ratio of) areas of ΔABC and ΔDEF. Use the "what if not" strategy to pose two mathematical problems.
Prompt 2B: Consider the (ratio of) areas of ΔABC and ΔDEF. Use the "what if not" strategy to pose as many mathematical problems as you can.
Prompt 3B: Consider the (ratio of) areas of ΔABC and ΔDEF. Use the "what if not" strategy to pose two "easy" mathematical problems and two "difficult" mathematical problems, where the relative difficulty takes into account the levels of students.

The focus on areas appears in the theme itself in the case of Situation A and in the prompts in the case of Situation B. This is the main difference between the two situations so far. The problem-posing variations V1 to V6 responding to prompts 1B, 2B, and 3B are not expected to be different from those resulting from Prompts 1A, 2A, and 3A. However, Situation B is more open and can be followed up with more open-ended prompts:

Prompt 4B: Based on the described Situation B, pose two mathematical problems related to ratios of measures of the attributes in the problem.
Prompt 5B: Based on the described Situation B, pose two mathematical problems related to ratios of measures (e.g., area, lengths, perimeter) of the attributes (e.g., segments, areas) in the problem.
Prompt 6B: What can you say about the described Situation B? Formulate this as questions about the different attributes and the relationships among them.

Prompts 4B and 5B both specify the number of problems as well as the focus on ratios of measures of the attributes. However, Prompt 5B explicitly suggests what measures and what attributes are to be considered. We consider Prompt 6B to be very open in terms of attributes in the focus and the number of problems to be considered. The choice to use a more open or a more specific prompt can depend on the population of problem posers and on their previous experience. Furthermore, the last three prompts (4B, 5B, and 6B) do not mention the "what if not" or any other particular strategy. Although the "what if not" strategy is a good tool for starting a problem-posing activity, other formulations can open the task for creative adventures.
For example, Prompt 6B can be modified to appeal to the affective domain of problem posing.

Prompt 7B: What can you say about the described Situation B? Formulate this as questions about the different attributes and the relationships among them that for YOU would be interesting to answer.

Prompt 7B can be used with either teachers or students. Here are several examples of what was "prompted" by Prompt 7B for us.

V7: A turtle walks along the sides of an outer ΔABC and the inner ΔDEF, beginning at point A and finishing at the same point. Can it walk so that every segment would be walked only once? If yes, suggest as many as possible trails for the turtle. If not, why not?

V8: ΔABC is an equilateral triangle. D, E, and F are points of the sides ofΔABC that divide the sides in the same ratio. That is, $AD:DB = BE:EC = CF:FA = x:y$. What should the ratio $x:y$ be so that ΔADF, ΔBDE, and ΔCEF would become: (1) an acute angle; (2) a right angle; and (3) obtuse?

V9: ΔABC is an equilateral triangle. D, E, and F are points of the sides of ΔABC that divide the sides in ratios X, Y, and Z. Suppose $AD:DB = X$; $BE:EC = Y$; and $CF:FA = Z$. Is there a relationship between the ratios X, Y, and Z and the ratio of the areas of ΔABC and ΔDEF (where $X = Y = Z$ it is a variation of V8)?

V10: ΔABC is an equilateral triangle. D, E, and F are midpoints of the sides of triangle ABC.

1. Show that the area of ΔDEF is ¼ the area of ΔABC and the perimeter of DEF is 1/3 the perimeter of ΔABC.
2. Consider the following process: The middle triangle DEF is removed, midpoints of the sides of three remaining triangles (ΔAFE, ΔFBD, and ΔEDC) are drawn, and each of these three triangles is split into four triangles as has been done for the initial triangle ABC. Then, again, the middle triangle in each of the three triangles is removed. What would be the area and the perimeter of the figure resulting from all the remaining triangles?
3. Imagine that the above process is repeated many times. Approximate the area and the perimeter of the figure consisting of all the remaining triangles after 100 iterations.
4. What would be the area and the perimeter when the number of iterations approaches infinity?

V11: What transformation(s) can map ΔABC to ΔDEF?
V12: Reverse construction: Given ΔDEF, which is the "inner" triangle? Construct ΔABC such that points D, E, and F are midpoints of AB, BC, and CA. Easy: Start with equilateral ΔDEF. Harder: Start with scalene ΔDEF. Very hard: Construct ΔABC such that points D, E, and F divide the sides of ΔABC in the given ratio.

We invite readers to examine the suggested prompts and consider which ones, if any, they will choose when working with students or teachers in their respective environments. What considerations determine your preference? What task variables are featured? Further, will the choice of a prompt be different if it is intended to be used for research data collection? What additional or different considerations will determine your choice? We also invite readers to engage in prompt design, considering Situation B as a prelude to the forthcoming discussion of processes and variables of problem posing at individual, group, and classroom levels. In the following sections, we discuss problem-posing research with regard to processes and task variables.

## PROBLEM-POSING PROCESSES: PROGRESS

Mathematical problem posing (MPP) has been at the forefront of discussion for the past few decades (Brown & Walter, 1983; Cai, 1998; Ellerton, 1986; English, 1998; Kilpatrick, 1987; Silver, 1994; Silver & Cai, 1996). Recent years have seen increased research activity in the domain of problem posing as reflected in journal special issues (Cai & Hwang, 2020; Cai & Leikin, 2020; Singer, Ellerton, & Cai, 2013), books (e.g., Felmer, Pehkonen, & Kilpatrick, 2016; Singer, Ellerton, & Cai, 2015), and conferences (e.g., ICME-14: TSG 17). This increased research on problem posing has also been reflected in the wide range of problem-posing topics studied (see Cai, Hwang, Jiang, & Silber, 2015, and Singer et al., 2013, for examples of such topics) and review papers (e.g., Baumanns & Rott, 2021; Cai & Leikin, 2020; Cai et al., 2015).

One of the important topics studied is the *processes* of problem posing as experienced by students and teachers. Although we know that students and teachers are capable of posing mathematical problems, we have a considerably less fine-grained understanding of how they go about posing those mathematical problems in any given situation. Some researchers have identified general strategies students may use to pose problems (e.g., Brown & Walter, 1983; Cai & Cifarelli, 2005; Christou, Mousoulides, Pittalis, & Pitta-Pantazi, 2005; Cifarelli & Cai, 2005; English, 1998; Koichu, 2020; Koichu & Kontorovich, 2013; Pittalis, Christou, Mousoulides, & Pitta-Pantazi, 2004; Rott, Specht, & Knipping, 2021; Silver & Cai, 1996). Others have explored some of the variables that may influence students' problem posing (e.g., Kontorovich, Koichu, Leikin, & Berman, 2012; Leung & Silver, 1997; Silber & Cai, 2017). Still others have explored the affective processes of mathematical problem posing (e.g., Schindler & Bakker, 2020).

However, there is not yet a general problem-posing analogue to well-established frameworks for problem solving such as Pólya's (1945) four phases of problem solving, Garofalo and Lester's (1985) cognitive-metacognitive processes of problem solving, and Schoenfeld's (1985) problem-solving attributes. More research is needed to develop a broadly applicable understanding of the fundamental processes and strategies of mathematical problem posing. For now, we remain in the beginning stages of understanding the cognitive and affective processes of problem posing, and this is one of the reasons for which this activity is implemented in mathematics instruction in a rather cursory way (Cai & Hwang, 2020; Cai & Leikin, 2020).

Even though the products of problem posing (i.e., new problems) are important as they constitute the heart of mathematical activities, problem-posing processes are equally important because it is in the processes that problem posers come up with ideas for new problems, evaluate those ideas, and develop or reject them (Baumanns, in press).

**Earlier attempts at understanding problem-posing processes**

In several earlier studies (e.g., Cai & Hwang, 2002; English, 1998; Silver & Cai, 1996), researchers have tried to use students' posed problems as a base for examining problem-posing processes. For example, Cai and Hwang (2002) used pattern situations to examine students' problem posing and problem solving. They observed that the sequence of pattern-based problems posed by students appeared to reflect a common sequence of thought when solving pattern problems (gathering data, analyzing the data for trends, making predictions). Silver and Cai (1996) found that students tend to pose related and parallel problems when they were asked to pose three problems. They observed a clear tendency of students to pose later problems by varying a single element in earlier problems, which is known as the "what if not" strategy (Brown & Walter, 1983) referred to in several of the prompts considered in the previous section.

Earlier studies have also tried to identify problem-posing strategies as a way to understand problem-posing processes. There are consistent findings about the use of the "what if not" strategy in problem posing (Cai & Ciffarelli, 2005; Cifarelli & Cai, 2005; Lavy & Bershadsky, 2003; Song, Yim, Shin, & Lee, 2007). For example, Lavy and Bershadsky (2003) identified two stages to pose problems. In the first stage, all the attributes included in the statement of the original problem are listed. In the second stage, each of the listed attributes is negated by asking “what if not attribute k?” and alternatives are proposed. Each of the alternatives could yield a new problem.

**Phases of the problem-posing process**

For problem solving, several models of the problem-solving process have been developed, initiated by reflections on their processes by mathematicians, most notably Poincaré (1908) and Pólya (1945). Later, researchers from mathematics education picked up this topic; important representatives of such research are Mason, Burton, and Stacey (1982), Fernandez, Hadaway, and Wilson (1994), and Schoenfeld (1984; see Rott et al., 2021, for an overview). For problem posing, on the other hand, as stated above, there is no well-known and generally accepted phase model (cf. Cai et al., 2015, p. 14). Some researchers argue that both problem solving and problem posing are strongly related and that there might be no need for a specific problem-posing-process model; however, we argue that cognitive processes in both kinds of processes are different enough to warrant individual models (cf. Baumanns & Rott, 2022; Pelczer & Gamboa, 2009).

Before going into detail regarding research on problem-posing models, we ponder the question of why such models are important. Process models can be used for normative and descriptive purposes (Rott et al., 2021). On the one hand, *normative* models sketch a (more or less) ideal process, stripped of unnecessary detours, that can be used in teaching and instruction. For example, Pólya’s four-step problem-solving model is a rather simple model that in its sequence of steps does not account for errors, being stuck, or realizing that the problem formulation needs to be read again. However, it was never intended to map real processes in their “non-smooth” nature but to instruct problem solvers in what steps to do and how to become a better problem solver or poser, respectively. On the other hand, *descriptive* models are designed to account for non-ideal sequences of steps in processes. Such models are used by researchers (or educators) to interpret processes they have observed, make sense of their observations, look for patterns, compare processes by experts and novices, and so on. Reviews of the literature reveal that for problem solving, mostly normative and only very few descriptive models have been developed (Rott et al., 2021) and, for problem posing, only a handful of models has been developed at all (Baumanns & Rott, 2022). In their review, Baumanns and Rott (2022) identified three models of the problem-posing process and added their own—all of which are descriptive phase models. These five models will now be described briefly.

The first model identified by Baumanns and Rott is that of Cruz (2006), who described the process of problem posing in teaching-learning situations and, thus, included educational needs and goals (see Figure 1). After setting a *goal*, a teacher *formulates a problem* and *tries to solve it*, which might fail or lead to regressions. After the problem has been solved, the problem is reflected upon, possibly *improved* to meet the goals, and then selected or rejected. This is a normative model of the problem-posing process intended to guide teachers; actually, it is based on a professional development program for teachers.

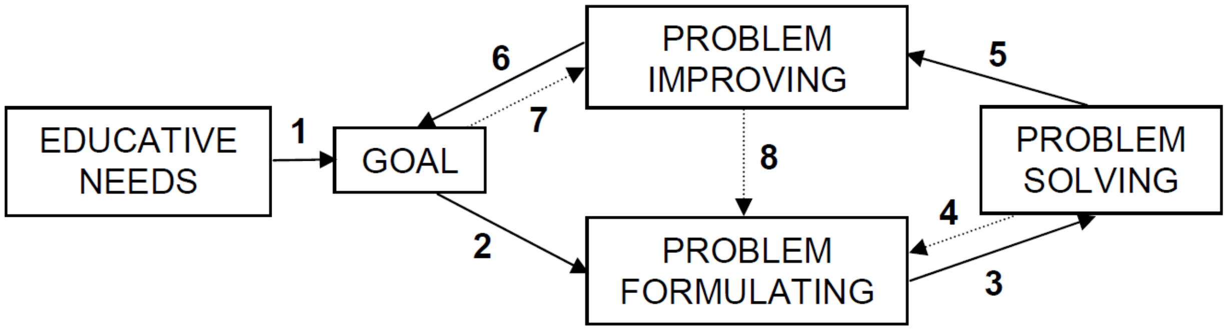


*Figure 1. Problem-posing phase model by Cruz (2006)*

The second model, based on an analysis of problem-posing processes, is that by Pelczer and Gamboa (2009), who developed a descriptive phase model with five phases, namely *setup*, *transformation*, *formulation*, *evaluation*, and *final assessment*. The setup phase is the starting point, including a reflection about the context of a given situation and the required knowledge. In the transformation phase, the given situation is analyzed and possible modifications are reflected upon and then executed. During the formulation phase, problem formulations and possible alterations are explored. In the next phase, the posed problem is evaluated to see whether it satisfies the initial conditions. In the final phase, much like Pólya's looking-back phase, the whole process is reflected upon.

Koichu and Kontorovich (2013) also developed a descriptive model. Based on two activities by prospective mathematics teachers called "success stories," they identified four phases of problem posing. The first phase is called *warming-up*, in which spontaneous ideas and typical problems regarding a given situation are posed. The next phase is called *searching for an interesting mathematical phenomenon*, in which the initially posed problems are critically considered and modified. In the next phase, problem posers are *hiding the problem-posing process in the problem formulation*, which was a behavior that had not been observed before (Koichu & Kontorovich, 2013, p. 82). In the final *reviewing* phase, the posed problems are evaluated and possibly tested with peers.

Zhang et al. (2022) described the problem-posing process as comprised of the following three major steps: (a) understanding the task (i.e., the context of the problem-posing task); (b) constructing the problem involving selecting and determining which elements to be used and recognizing the relationships among them to construct a new problem space; and (c) expressing the problem which involves organizing the language to express the problem space obtained in the previous stage.

Baumanns and Rott (2022) then developed their own descriptive phase model, the development of which was based on the problem-posing processes of 64 preservice mathematics teachers (see Figure 2). After an initial *situation analysis*, the model allows for differentiation between activities of *variation*, in which a given problem is altered, and *generation*, in which a new problem is generated—a differentiation that had been proposed by Silver (1994) but that had not been made in an operationalized way with empirical data. The duplication of Figure 2 aims to denote that after one problem has been posed, the process can be repeated for posing the second problem, third problem, and so on.

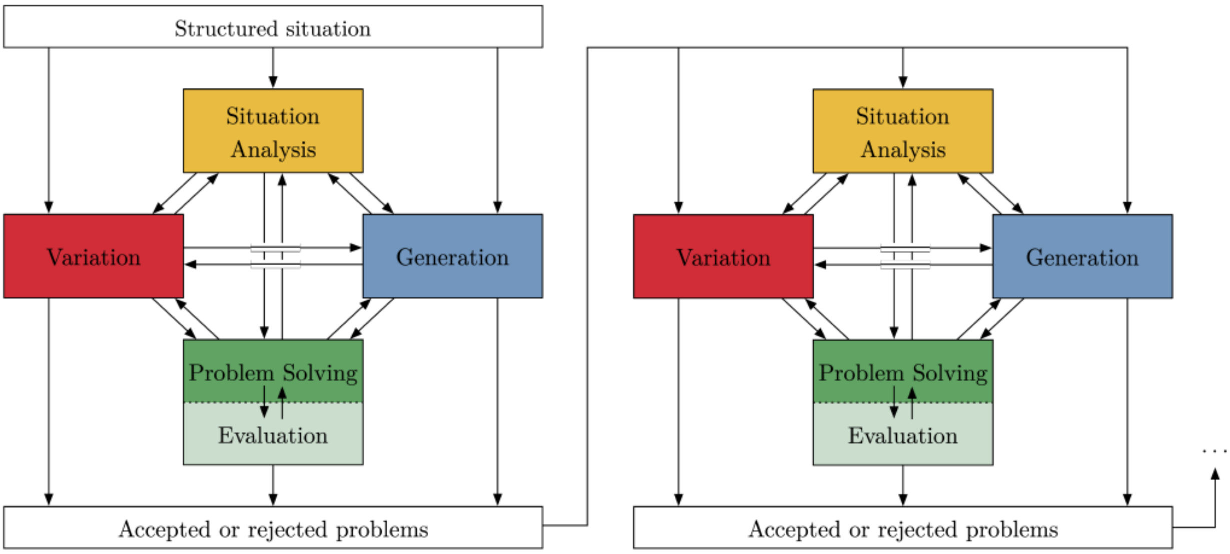


*Figure 2. Problem-posing phase model by Baumanns and Rott (2022)*

As is the case for different problem-solving models, different problem-posing models serve different goals. For example, Koichu and Kontorovich described a problem-posing process in which one high-quality problem gradually emerges from the pool of initial problem-posing ideas, whereas Baumanns and Rott's (2022) model attends to problem posing as a sequence of repeated problem-posing cycles where each problem posed is considered to be a separate product.

### Affective processes of problem posing

Regarding research on problem solving, the whole affective dimension, ranging from emotions to attitudes to beliefs (Philipp, 2007), with a focus on beliefs, has proven very useful and important (Schoenfeld, 1992). Regarding research on problem posing, however, the affective dimension has only recently been systematically addressed by means of a special issue in *Educational Studies in Mathematics* (Cai & Leikin, 2020). This special issue, encompassing for example studies dealing with teachers' beliefs (Li, Song, Hwang, & Cai, 2020) or students' motivation and self-efficacy (Voica, Singer, & Stan, 2020), can only be the starting point of systematic research on affect in mathematical problem posing. In our initial example, Prompt 7B capitalizes on affect in problem posing.

## TASK VARIABLES IN STUDYING PROBLEM POSING

### Focusing on task variables

There are many ways in which mathematics education research might investigate the cognitive and affective processes of problem posing in an effort to better incorporate problem posing in the teaching and learning of mathematics (Cai et al., 2015). In this paper, we focus on task variables to explore the affective and cognitive processes of problem posing as has been successfully done in research on problem solving. There are two main reasons for such a focus. The first is that we have prior research to draw from on task variables in problem solving (Goldin & McClintock, 1984). The second reason is that we have prior research to draw from on how specific characteristics of mathematical tasks of different natures can affect teachers' and students' responses, in terms of both thinking and instruction (e.g., Koichu & Zazkis, 2021; Liljedahl, Chernoff, & Zazkis, 2007; Zazkis & Mamolo, 2018).

In mathematical problem-solving research conducted over the past several decades, researchers have explored the effects of various task variables on students' problem solving. For example, several classifications of task variables related to problem solving are considered in Goldin and McClintock (1984): syntax variables, content and context variables, structure variables, and heuristic behavior variables. Syntax variables are factors dealing with how problem statements are written. These factors, such as problem length as well as numerical and symbolic forms within the problem, may contribute to ease or difficulty in reading comprehension. Content variables refer to the semantic elements of the problem, such as the mathematical topic or the field of application, whereas context variables refer to the problem representation and the format of information in the problem. Structure variables refer to factors involved in the solution process, such as problem complexity and factors related to specific algorithms or solution strategies. Finally, heuristic process variables refer to the interactions between the mental operations of the problem solver and the task. Considering heuristic variables separately from subject variables (factors that differ between the individuals solving the problem) is difficult because heuristic processes involve the problem solver's interactions with the task. However, the interaction between heuristic processes and the other task variables can have a significant impact on problem-solving ability.

**Problem-posing tasks**

Just as there are many types of problems and problem-solving tasks, there are many types of problem-posing tasks. Although researchers have proposed categorization schemes for problem posing (e.g., Baumanns & Rott, 2021; Stoyanova & Ellerton, 1996), in this paper, we adopt the idea of a problem-posing task as consisting of two parts: situations and prompts (Cai & Hwang, in press), as exemplified in the first section by means of an example in the context of geometry. The problem situation is what provides the context and data that the students may draw on (in addition to their own life experiences and knowledge) to craft problems. Figure 3 shows the various types of problem situations (Cai & Hwang, in press).

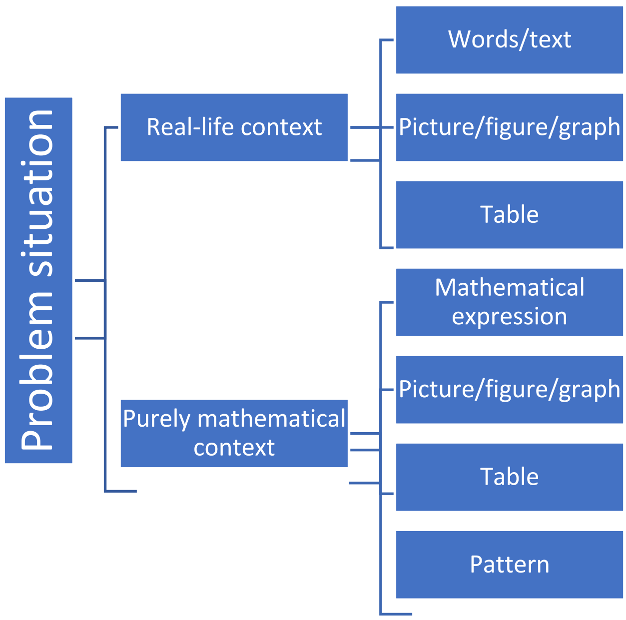

*Figure 3. Types of problem situations in problem-posing tasks (Cai & Hwang, in press).*

In addition to a problem situation that provides context and data for students to use in their posed problems, a problem-posing task must include a prompt that lets posers know what they are expected to do (Cai & Hwang, in press). Depending on the goal of the task, for the same problem-posing situation, there can be many kinds of prompts. Some possible prompts include:

- Pose as many mathematical problems as possible
- Pose problems of different levels of difficulty (e.g., "Pose one easy problem, one moderately difficult problem, and one difficult problem.")
- Given a sample problem, pose similar problems (or problems that are structurally different)

The choice of prompt can influence both the mathematical focus for the students and the level of challenge or affective engagement that the posing task presents. Indeed, from a research perspective, it is not yet well understood what prompts are best to pair with a given problem situation or what prompts are most suited to achieving a desired degree of challenge or to address particular learning goals. That is, research has not yet illuminated the connections between different kinds of problem-posing prompts and different cognitive processes in problem posers.

Admittedly, there are many different levels with which to approach research related to task variables and their associated processes in problem posing. In this paper, we describe three such levels: the individual level, group level, and classroom level.

**Problem-posing prompts at the individual level**

The first level with which we approach problem-posing research described in this paper is the individual level. Research on problem-solving tasks has established that different prompts can elicit different cognitive processes and impact students' problem-solving performance (Goldin & McClintock, 1984). Thus, it is reasonable to expect that the prompt in a problem-posing task also shapes students' engagement with the task. A few studies have investigated how different prompts in problem-posing tasks impact students' or teachers' problem-posing performance and processes (e.g., Silber & Cai, 2017). Silber and Cai (2017) compared preservice teachers' problem posing using structured prompts and free prompts, finding that the preservice teachers in the structured-posing condition more closely attended to the mathematical concepts in each task. Moreover, the effect of the prompt depends, in part, on the setup of the task. For example, in their review of problem posing in textbooks, Cai and Jiang (2017) identified four common types of problem-posing tasks: posing a problem that matches the given/specific kinds of arithmetic operations, posing variations of a question with the same mathematical relationship or structure, posing additional questions based on the given information and a sample question, and posing questions based on given information. A similar prompt (e.g., "Pose a mathematical problem.") could be used with many of these types of tasks, but its meaning to the student could be different for each type.

Leung and Silver (1997) developed and analyzed a Test of Arithmetic Problem Posing (TAPP) which they then used to examine how the presence of numerical information affected preservice teachers' problem-posing abilities. The instructions, which are the prompts we are focusing on, include: "(1) Consider possible combinations of the pieces of information given and pose mathematical problems related to the contexts; (2) Do not ask questions that are not mathematical problems; (3) Set up as many problems as you can think of; (4) Think of problems with a variety of difficulty levels. Do not solve them; (5) Set up a variety of problems rather than many problems of the same kind; (6) Include unusual problems that your peers might not be able to create; (7) You can change the given information and/or supply more information" (Leung & Silver, 1997, p. 8). The first prompt seems to be advice for the participants on how to pose problems. The second prompt emphasizes that the problem posed should be accepted by the community of mathematicians. The third through sixth prompts are related to the "many," "different kinds" or unusual, and "different difficulty levels" mentioned earlier. The last prompt tells the participants what they can do with the data (either change the given information or add more information). Responses were analyzed along two dimensions: quality and complexity. With respect to quality, the responses were classified as mathematical or nonmathematical, as plausible or implausible, and as containing sufficient or insufficient information. With respect to complexity, the responses were classified according to the arithmetic complexity of the solution of the posed problem (i.e., the number of steps to answer the question). Results from the TAPP indicated that the teachers performed better on tasks that included specific numerical information than on tasks without specific numerical information. This might "be due to their being able to 'use the numbers' in the given information rather than having to supply their own numbers or rather than engaging in the generation of qualitative reasoning problems which would not need to contain numerical information" (Leung & Silver, 1997, p. 20). This result provides some insight into how task variables can impact problem posing. Adapting the TAPP to examine how different characteristics of problem situations affect subjects' problem posing could offer a way to study the effect of other task variables on problem posing.

Zhang et al. (2022) replicated and extended the study by Leung and Silver (1997), focusing on elementary school students' problem posing. They examined the cognitive process of mathematical problem posing in three stages: a) input—understanding the task, b) processing—constructing the problem, and c) output—expressing the problem. They also found that the provision of specific numerical information in the problem-posing situation was associated with better problem-posing performance but only in the stages of understanding the task and constructing the problem stages. Students' performance in the stage of expressing posed problems did not show a significant difference with respect to provision of specific numerical information. A similar pattern was revealed for the problem-posing situations with or without contexts, favoring the task format with contexts. Students performed better in all three problem-posing stages on the problem-posing situations with contexts.

English (1998) compared third-grade children's problem-posing performance in formal and informal contexts. In the formal context, children were first asked to make up a story problem to given number sentences like 12 – 8 = 4 and were then asked to think of a completely different problem that could also be solved by the number sentences. Three kinds of informal contexts were presented to the children. The first informal context was a real-life situation presented in pictures. A photograph of children playing with sets of colored items was shown to the participants, then they were asked to make up story problems about something that could be seen in the photographs. The second informal text was a real-life situation presented in words—for example, a card with a statement like "Sarah has five dolls on one shelf and four toy cars on another." Then, the participants were asked to make the statement into a problem they could solve. The third informal context was a piece of literature supported with a list of numbers of native animals. English found that all children offered a significantly greater number of basic change/part-part-whole problems for their first attempt in the formal context, but many of them had difficulty creating a second problem for the given number sentence. Comparatively, they generated more compare problems in the informal context. Encouragingly, several participants even posed multistep problems in the informal context.

Silber and Cai (2021) presented two kinds of problem-posing tasks to undergraduate students taking a noncredited developmental mathematics course so they could be ready to take the foundational mathematics courses required for their major. One kind of problem-posing task consisted of a purely mathematical context presented in a linear graph (i.e., the Graph of a Line posing task). The other was a real-life context described in words only (i.e., Handshakes and Making Change) or in words and pie charts (i.e., Food Drive). Students were required to pose three problems for each context. The problems posed were categorized as mathematical questions, mathematical statements, or nonmathematical responses. The mathematical questions were further analyzed based on their solvability. Among the three real-life contexts, the Food Drive context seemed to be the most familiar context for the participants because they possibly had experienced it when they learned percentages and pie charts. The Making Change context seemed to be the second most familiar because it was often used as a model for addition and subtraction (cost + change = pay) and the model for the system of linear equations (e.g., ten coins [dimes and half-dollars] to pay $2.20). The Handshakes problem, which involves modelling (using points or circles to represent people and the line between any two points as a handshake), is usually used for patterns in algebra. The results obtained in Silber and Cai's (2021) study revealed that the percentages of problems that were solvable mathematical problems for Food Drive, Making Change, Handshakes, and Graph of a Line were 98%, 90%, 88%, and 52%, respectively. Thus, the familiarity level of the contexts might need to be taken into consideration in future studies.

**Effect of problem-posing prompts at the group level**

The second level with which to approach research related to task variables in problem posing is the group level, that is, how a small group poses mathematical problems and how the task variables affect group problem posing (e.g., Kontorovich et al., 2012).

As early as 1987, Kilpatrick pointed out that group work can provide a fruitful setting for mathematical problem posing because the dialogue between problem posers may have a synergetic effect. In his words,

> When students work together, they often identify problems that would be missed if they were working alone. A poorly formulated idea brought up by one student can be tossed around the group and reformulated to yield a fruitful problem. Students participate in a dialogue with others that mirrors the kind of internal dialogue that good problem formulators appear to have with themselves. (Kilpatrick, 1987, pp. 141-142)

Despite the broad attention that this seminal article has attracted in the mathematics education research community, research on problem posing in groups is still relatively rare. A Google Scholar search using the key words "group problem posing" +

"mathematics," "collaborative problem posing" + "mathematics," and "collective problem posing" + "mathematics" returns dozens of results (50, 137, and 95, respectively) as compared to the thousands of results returned by a parallel search in which "problem posing" is replaced with "problem solving." Furthermore, in many of the studies identified in the search, "group," "collaborative," or "collective" problem posing are mentioned merely as potential counterparts of "group," "collaborative," or "collective" problem solving, with the main attention given to the latter rather than to the former activity.

Armstrong (2014) alluded to collective problem posing as an emergent phenomenon in school discourse. She argued that the insufficient attention paid to group problem posing thus far could partially be explained by specific features of the mainstream line of research on problem posing as it had been developed since the 1990s. Namely, many of the problem-posing studies operate with written products of problem posing as a focus of analysis and value large-size pools of participants and large collections of problems posed that can be categorized in a variety of ways. Arguably, this focus, as useful as it is, leaves aside problem-posing processes and in turn leaves aside phenomena related to the dynamics of group work on problem-posing tasks, as has been suggested by Kilpatrick (1987). Indeed, Kilpatrick's provisional argument was about group processes that can lead inexperienced problem posers to formulating fruitful ideas rather than about the quantity of the resulting problems posed.

However, it is safe to say that research on problem posing at the group level is gradually growing. While recognizing that the critical mass of studies that would enable us to clearly identify trends has not yet accumulated, we can (tentatively) identify four different approaches to treating group work as a variable in problem-posing research.

In the first approach, the fact that students work on problem posing in small groups is provided as contextual information, but the findings are reported in an aggregated manner that hides within-group processes. A study by English and Watson (2015) serves as a characteristic example. The study explores the problem-posing products of 20 groups of fourth-grade students working in groups of four in the context of statistical literacy. The main results are reported per group, as the following quotation shows: "Of the 20 groups, 9 posed three or four different types of questions, 10 created two types, and 1 group, just one type" (English & Watson, 2015, p. 11). The between-group differences in this study are attributed to individual differences between students' pre-existing knowledge and preferences but not to the dynamic processes in the groups. Another example comes from Leung and Wu (1999), who first reflected on two problem-posing lessons as if each group was an individual student (e.g., "the six groups changed the problem in three ways"; p. 113) and then stopped on ideas of a particular student expressed in front of the class (of note is that this study can also be considered in the section on problem posing at the classroom level).

The second approach focuses on individual students in the context of small-group problem posing. For instance, one of the results of the study by Headrick et al. (2020) is that even when students are organized in small groups, they tend to individually pose problems to the teacher as opposed to their groupmates. Another study, by Koichu (2020), showed that students in small groups who face a multistage task including problem solving, exploration, and problem posing tend to distribute the load and work separately on different parts of the task so that problem posing essentially turns into an individual enterprise. These results do not contradict but rather complement the findings by Schindler and Bakker (2020), who found that a group setting can play a positive role in shaping an affective field of individual problem posers. In their case study of one student working in a small group on a series of problem-posing and problem-solving tasks, the student overcame the initial anxiety rooted in her prior experiences, increased her interest in problem posing, and became an open-minded and active participant in the project due to the group collaboration that provided her with the feeling of safety and appreciation. Furthermore, Ellerton (2015) pointed out that working in groups may either support or hinder the problem-posing progress of individuals. Her study suggests the importance of keeping a delicate balance between the collective and the individual in problem posing as well as the importance of learning how to give and take feedback on the problem-posing ideas of others in productive ways.

The third approach attends to the richness of problem-posing performance in small groups working on the same task while featuring summative rather than dynamic descriptions. Armstrong (2014) developed an original methodology (called "tapestries") that blurs the data but provides visual representations of collective patterns of problems posed. This methodology was used in a study with four groups of 12-year-old students to compare the across-group problem-posing products as related to the group problem-posing strategies and tactics. Armstrong introduced the term "group's personality" (p. 62) and compared the groups in the following manner: For example, a group that tended to deeply explore concepts and connect participants' ideas posed more problems than another group that tended to argue about every problem's formulation, aiming at reaching a consensus. In contrast, Cai (2012) compared two groups of preservice teachers working on a task in the context of numerical sequences by summarizing the main mathematical ideas developed in each group. Despite methodological differences, both studies converge to conclusions about the opportunities embedded in well-chosen problem-posing tasks that trigger rich mathematics discussions and learning.

Finally, the fourth approach is heavily informed by sociocultural perspectives on teaching and learning mathematics and therefore considers within-group problem-posing interactions as the main data to be analyzed as opposed to the written problems as the main data. For example, English, Fox, and Watters (2005) argued for the potential of problem posing and solving with mathematical modelling while systematically demonstrating how problem-solving and problem-posing ideas emerge and evolve in small-group discussions. In this study, the argument for the usefulness of combining problem posing, problem solving, and modelling relies not only on the demonstrated benefits of the chosen types of tasks for student learning of mathematics but also for the development of their collaborative learning skills. Meanwhile, an in-depth analysis of student interactions of low-track eighth-grade students who were engaged in small-group work on a problem-posing task in the context of geometry is the focus of a study by Agarwal (2020). The analysis of six groups revealed how the students shifted their actions and restructured their activity towards organizing for collective agency in mathematical problem solving while balancing risk-taking behaviors (e.g., there is a risk to be misunderstood or mocked) and agency-driven behaviors in favor of emotional courage and productive participation.

We conclude this section by reviewing a study by Kontorovich, Koichu, Leikin, and Berman (2012) that has an explicit focus on handling the complexity of problem posing in small groups. These authors present a confluence exploratory framework that aims to explain the emergence of problem-posing products from problem-posing processes as shaped by five facets: task organization, students' knowledge base, problem-posing heuristics and schemes, group dynamics and interactions, and individual considerations of aptness. The framework is presented in Figure 4.

This framework was used to make sense of the work of two groups of tenth-grade students who were given the Billiard Ball Mathematics Task (adapted from Silver, Mamona-Downs, Leung, & Kenney, 1996, and used in several additional studies). The analysis attempted to explain the quantity and quality of the resulting problems posed in each group by systematically attending to all the facets included in the framework. In particular, the analysis showed the role of group dynamics captured in terms of normalization, conformity, and innovation—three social processes well-known from the literature on group interaction and development in the context of coping with challenging (though not necessarily mathematical) tasks (e.g., Wit, 2007). The analysis also shed light on the importance of functional roles that group members assumed in a small-group discussion.

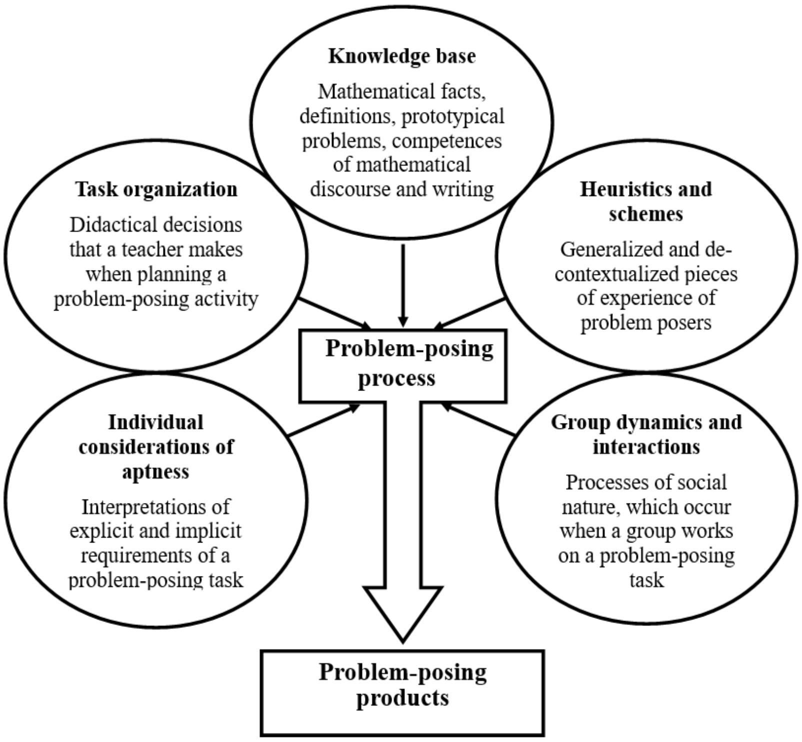


*Figure 4. A confluence framework of problem posing in small groups (Kontorovich et al., 2012).*

Along with the aforementioned studies by English et al. (2005) and Agarwal (2020), a study by Kontorovich et al. (2012) supports and empirically substantiates Kilpatrick’s (1987) vision of group work as a fruitful but immensely complex pedagogical setting for further promoting problem posing in school. Needless to say, more research on problem posing at the group level is needed.

**Effect of problem-posing prompts at the classroom level**

Finally, the third level with which to approach research related to task variables in problem posing is the classroom level. Mathematics can be taught through engaging in problem posing, and researchers have begun to explore what teaching mathematics through problem posing looks like and to develop problem-posing cases to illustrate problem-posing instruction (Cai & Hwang, 2020; Ellerton, 2015; Zhang & Cai, 2021). However, it is not yet clear how we should design the problem-posing tasks used in such instruction so as to create greater learning opportunities for students. For example, for a given situation in the classroom, students could be asked to pose three mathematical problems or to pose three problems with different difficulty levels such as easy, moderately difficult, and difficult (Cai & Hwang, 2002). How would such different prompts impact classroom instruction and students' learning?

The past two decades and especially recent years have seen increased research on implementing problem posing into classrooms (Cai & Hwang, 2020; Cai et al., 2015). Researchers have begun to explore what teaching mathematics through problem posing looks like (Çakır & Akkoç, 2020; Cai, 2022; Chen & Cai, 2020; Crespo & Sinclair, 2008; Ellerton, 2013; Christou et al., 2005; Klaassen & Doorman, 2015; Stoyanova & Ellerton, 1996; Zhang & Cai, 2021), develop problem-posing teaching cases to illustrate problem-posing instruction (e.g., Chen & Cai, 2020; Zhang & Cai, 2021), and examine the impact of problem-posing instruction on students and teachers (Akben, 2020; Bevan & Capraro, 2021; English, 1997, 1998; Li et al., 2020; Klaassen & Doorman, 2015; Kopparla et al., 2019; Suarsana, Lestari, & Mertasari, 2019; Yang & Xin, 2021).

In teaching through problem posing, students are encouraged to pose problems that may be meaningful to them personally or socially. Thus, classroom activity around problem posing involves the negotiation of socio-mathematical norms, such as in determining criteria for what counts as a mathematically interesting problem (Çakır & Akkoç, 2020; Crespo & Sinclair, 2008).

Cai (2022) proposed a problem-posing task-based instructional model (see Figure 5). In a lesson, there might be more than one problem-posing task or a combination of problem-solving and problem-posing tasks. This model describes using one problem-posing task to teach mathematics. The first step is to present a problem-posing situation (see specifics in Figure 3), the second step is to provide a problem-posing prompt, and the third step is for students to pose problems either individually or in group. The later steps in Figure 5 show how teachers can handle the posed problems based on the learning goals.

Zhang and Cai (2021) analyzed 22 problem-posing teaching cases based on the work of Merseth (2016) and Stein, Henningsen, Smith, and Silver (2009). They described a teaching case as the following:

> A teaching case includes major elements of a lesson and related analysis, but it is not a transcribed lesson. Teaching cases include narratives describing instructional tasks and related instructional moves for the tasks. Cases also

> include information about the underlying thinking of major instructional decisions as well as reflections on and discussions of those decisions. The development of teaching cases is based on real lessons and typical instructional events from the lessons. (Zhang & Cai, 2021, p. 962)

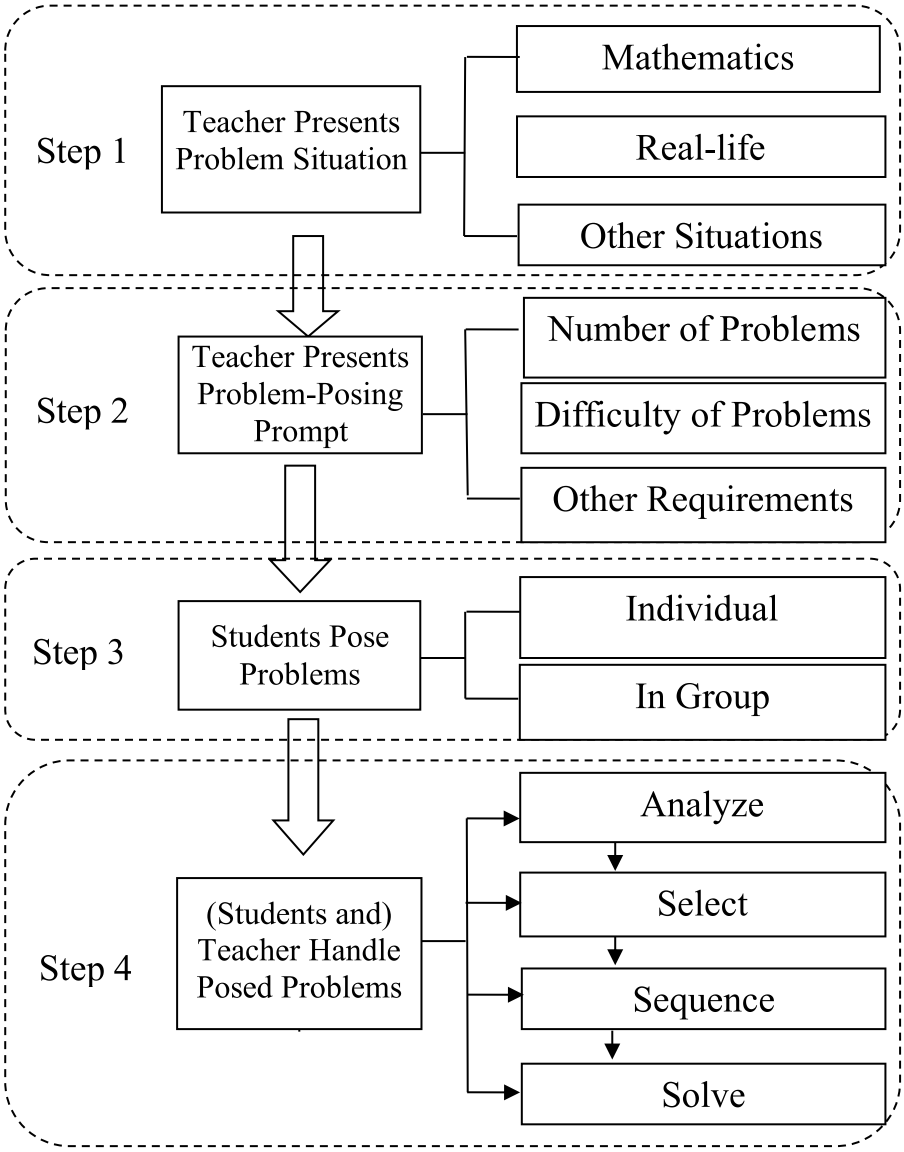


*Figure 5. A problem-posing task-based instructional model (Cai, 2022)*

In their analysis of problem-posing teaching cases, Zhang and Cai (2021) found that teachers used different prompts in their problem-posing tasks, such as posing a problem that matches the given or specific kinds of arithmetic operations, posing problems based on given information, and posing variations on a question with the same mathematical relationship or structure. Their analysis found no teaching case that explored the effect of different prompts for the same problem-posing situation. In fact, thus far, there are no studies that have studied the effect of different prompts on students' problem posing at the classroom level.

Using problem posing as an instructional intervention, researchers have found positive effects of problem posing not only on teachers' own development (Li et al., 2020) but also on students' learning along both cognitive and noncognitive measures (e.g., Akben, 2020; Bevan & Capraro, 2021). Although such positive effects of problem posing on both students and teachers are encouraging, none of these studies include information about the effects of problem-posing tasks with different prompts. In fact, as Klaassen and Doorman (2015) summarized, there has been no specific investigation of the effects of the variety of prompts researchers have used in classrooms, prompts such as:

- Write a problem to the story so that the answer to the problem is a specific one
- Write an appropriate problem for the specific information, such as the expression or picture
- Ask as many questions as you can, and try to put them in a suitable order
- Write a problem that you would find difficult to solve

As part of a larger research project, Cai, Muirhead, Cirillo, and Hwang (2021) began to explore how teachers view the impact of different prompts on students' problem posing. Each teacher was presented with three pairs of tasks, each of which uses the same problem situation but includes different prompts (Prompt A: Pose three different mathematical problems that can be solved using this information; Prompt B: Pose one easy mathematical problem, one moderately difficult mathematical problem, and one difficult mathematical problem that can be solved using this information). Each teacher was asked to anticipate their students' responses to Prompt A compared to Prompt B and to describe how these variations in the wording of the prompts might affect their students' responses. According to one sixth-grade teacher, Prompt A is less wordy and more accessible for students. However, the teacher thought that Prompt B engaged students more in their thinking because they must think about posing problems with different difficulty levels. Thus, Prompt B "forces" or "encourages" students to think more. The teacher also thought that in implementing problem posing in the classroom, teachers can scaffold problem-posing tasks with Prompt A to problem-posing tasks with Prompt B.

In practice, it does seem that encouraging students to pose different difficulty levels of problems has some advantages for eliciting deeper student thinking about certain kinds of problems (Cai & Hwang, 2002) and adjusting the level of challenge of the task relative to each student. For example, the prompt, "Create a problem that would be difficult for you to solve," can challenge each student to stretch toward the edge of their own ability. Although each student may still engage in the problem-posing task at a level that is appropriate for their existing mathematical understanding, such a prompt could result in the overall level of challenge increasing. Ultimately, we believe that the choice of problem-posing prompt has the potential to make a difference in how students engage with problem-posing tasks in the classroom.

## CONCLUSION

The purpose of this paper is to summarize some progress in problem-posing research related to processes and task variables. We end by presenting some research questions for the field of mathematics education.

As mentioned, problem-solving variables include syntax variables, content and context variables, structure variables, and heuristic behavior variables. Can all these types of variables be adapted to problem posing? Should the additional variables be considered to pinpoint not only similarity of problem solving and problem posing but also characteristic differences between these activities? How can systematic variation of problem-posing situations and prompts inform our understanding of the relationship between problem-posing processes and products and between problem posing and problem solving? How do student-related variables (e.g., knowledge, affect, experiences) interact with task-related problem-posing variables?

In addition, we not only need to continue the effort to examine the cognitive processes of problem posing related to task variables but also affective processes of problem posing. For example, how do beliefs influence problem-posing processes and posed problems? How do problem-posing activities influence students' (epistemological) beliefs regarding mathematics? Do problem-posing activities—used in teaching settings—impact students' sense-making and motivation regarding mathematics? Cai and Leikin (2020) provided additional research questions about affect in problem posing.

The literature offers characterizations of teacher knowledge needed to incorporate problem solving in teaching (e.g., Chapman, 2015). Similarly, we can ask: What teacher knowledge is needed for successful integration of problem posing in the classroom? (In other words, we can think of task-related variables, student-related variables, and teacher-related variables.)

In this paper, we have discussed the impact of task variables (specifically problem-posing prompts) on problem posing at the individual, group, and classroom levels. In addition to the discussion of the impact at different levels, there is also a need to understand how teachers handle posed problems (Cai, 2022; Zhang & Cai, 2021), because of its importance in integrating problem posing in mathematics learning and instruction. In problem solving, we have more or less clear criteria to measure success (i.e., successfully solved problems). In problem posing, it is much harder to determine whether a problem-posing process was successful or not (given that posed problems could be nonmathematical, repetitive, boring, not challenging, etc.). How do we measure "success" in problem posing? Compared to problem solving (where you can easily identify whether a problem has been solved), it is often unclear when a problem-posing process is finished or whether the posed problems are "good" or not. To effectively teach mathematics through problem posing, we have to address these questions in general and to develop strategies to deal with students' posed problems in particular.

Using a clinical interview methodology or a large-sample-size survey, we could examine how different types of problem-posing tasks with different situations and prompts influence students' problem-posing processes. Such research requires coordination and international collaboration, and the ideas presented in this paper will be a step towards establishing it.

**Acknowledgement**: Jinfa Cai would like to acknowledge the support of a grant from the USA National Science Foundation (NSF; Award #2101552). Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the NSF. He appreciates his colleagues, Dr. Michelle Cirillo, Faith Muirhead, and Stephen Hwang's collaboration on the P-PBL research project.

## References

Agarwal, P. (2020). Organizing for collective agency: Negotiating social and disciplinary risks of collaborative learning. In M. Gresalfi & I. S. Horn (Eds.), *The interdisciplinarity of* *the learning sciences, 14th International Conference of the Learning Sciences (ICLS) 2020* (Vol. 1, pp. 223-229). Nashville, TN: International Society of the Learning Sciences.

Akben, N. (2020). Effects of the problem-posing approach on students' problem solving skills and metacognitive awareness in science education. *Research in Science Educationa, 50,* 1143–1165. https://doi.org/10.1007/s11165-018-9726-7

Armstrong, A. (2014). Collective problem posing as an emergent phenomenon in middle school mathematics group discourse. In C. Nicol, P. Liljedahl, S. Oesterle, & D. Allan (Eds.), Proceedings of the joint meeting of PME 38 and PME-NA 36 (Vol. 2, pp. 57-64). Vancouver, Canada.

Baumanns, L. (in press). *Mathematical problem posing: Conceptual considerations and empirical investigations for understanding the process of problem posing*. Springer.

Baumanns, L., & Rott, B. (2021). Rethinking problem-posing situations: A review. *Investigations in Mathematics Learning*, *13*(2), 59-76.

Baumanns, L., & Rott, B. (2022). The process of problem posing: Development of a descriptive phase model of problem posing. *Educational Studies in Mathematics* (online first). https://doi.org/10.1007/s10649-021-10136-y

Bevan, D., & Capraro, M. M. (2021). Posing creative problems: A study of elementary students' mathematics understanding. *International Electronic Journal of Mathematics Education, 16*(3), em0654. https://doi.org/10.29333/iejme/11109

Brown, S. I., & Walter, M. I. (1983). *The art of problem posing*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Cai, J. (1998). An investigation of U.S. and Chinese students' mathematical problem posing and problem solving. *Mathematics Education Research Journal, 10*, 37–50.

Cai, J. (2012). *Problem posing as lenses of improving students' learning in classroom*. Newark, DE: University of Delaware.

Cai, J. (2022). What research says about teaching mathematics through problem posing. Paper to be submitted to *Education & Didactique*.

Cai, J., & Cifarelli, V. V. (2005). Exploring mathematical exploration: How two college students formulated and solved their own mathematical problems. *Focus on Learning Problems in Mathematics, 27*(3), 43–72.

Cai, J., & Hwang, S. (2002). Generalized and generative thinking in US and Chinese students' mathematical problem solving and problem posing. *The Journal of Mathematical Behavior, 21*(4), 401–421.

Cai, J., & Hwang, S. (2020). Learning to teach through mathematical problem posing: Theoretical considerations, methodology, and directions for future research. *International Journal of Educational Research, 102*, 101420.

Cai, J., Hwang, S., Jiang, C., & Silber, S. (2015). Problem-posing research in mathematics education: Some answered and unanswered questions. In F. M. Singer, N. F. Ellerton, & J. Cai (Eds.), *Mathematical problem posing: From research to effective practice* (pp. 3–34). Springer.

Cai, J., & Jiang, C. (2017). An analysis of problem-posing tasks in Chinese and US elementary mathematics textbooks. *International Journal of Science and Mathematics Education, 15*(8), 1521–1540.

Cai, J., & Leikin, R. (2020). Affect in mathematical problem posing: Conceptualization, advances, and future directions for research. *Educational Studies in Mathematics*, *105*, 287–301.

Cai, J. & Hwang, S. (in. press). Making mathematics challenging through problem posing in the classroom. In R. Leikin, C. Christou, A. Karp, D. Pitta-Pantazi, & R. Zazkis, *Mathematical challenges for all*. Springer, New York, NY.

Cai, J., Muirhead, F., Cirillo, M., & Hwang, S. (2021). *Supporting teachers to teach mathematics through problem posing: An early-stage longitudinal study* [NSF Grant Proposal]. University of Delaware.

Çakır, A., Akkoç, H. (2020). Examining socio-mathematical norms related to problem posing: A case of a gifted and talented mathematics classroom. *Educational Studies in Mathematics, 105,* 19–34.

Chapman, O. (2015). Mathematics teachers' knowledge for teaching problem solving. *LUMAT: International Journal on Math, Science and Technology Education*, *3*(1), 19–36.

Chen, T., & Cai, J. (2020). An elementary mathematics teacher learning to teach using problem posing: A case of the distributive property of multiplication over addition. *International Journal of Educational Research*, *102*, 101420.

Christou, C., Mousoulides, N., Pittalis, M., & Pitta-Pantazi, D. (2005). Problem solving and problem posing in a dynamic geometry environment. *The Montana Mathematics Enthusiast, 2*, 125–143.

Cifarelli, V. V., & Cai, J. (2005). The evolution of mathematical explorations in open-ended problem-solving situations. *Journal of Mathematical Behavior, 24*, 302–324.

Crespo, S., & Sinclair, N. (2008). What makes a problem mathematically interesting? Inviting prospective teachers to pose better problems. *Journal of Mathematics Teacher Education, 11,* 395–415.

Cruz, M. (2006). A mathematical problem–formulating strategy. *International Journal for Mathematics Teaching and Learning*, 79–90.

Ellerton, N. F. (1986). Children's made-up mathematics problems—A new perspective on talented mathematicians. *Educational Studies in Mathematics 17,* 261–271.

Ellerton, N. F. (2013). Engaging pre-service middle-school teacher-education students in mathematical problem posing: Development of an active learning framework. *Educational Studies in Mathematics, 83,* 87–101.

Ellerton, N. (2015). Problem posing as an integral component of the mathematics curriculum: A study with prospective and practicing middle-school teachers. In F. M. Singer, N. Ellerton & J. Cai (Eds.), *Problem posing: From research to effective practice* (pp. 513-546). New York, NY: Springer.

English, L. (1997). Promoting a problem-posing classroom. *Teaching Children Mathematics, 4*, 172-179.

English, L. D. (1998). Children's problem posing within formal and informal contexts. *Journal for Research in Mathematics Education, 29*(1), 83–106.

English, L. D., Fox, J. L., & Watters, J. J. (2005). Problem posing and solving with mathematical modeling. *Teaching Children Mathematics*, *12*(3), 156-163.

English, L. D., & Watson, J. M. (2015). Statistical literacy in the elementary school: Opportunities for problem posing. In M. Singer, N. Ellerton, & J. Cai (Eds.), *Mathematical problem posing* (pp. 241-256). New York, NY: Springer.

Felmer, P., Pehkonen, E., & Kilpatrick, J. (Eds.). (2016). *Posing and solving mathematical problems: Advances and new perspectives.* Springer.

Fernandez, M. L., Hadaway, N., & Wilson, J. W. (1994). Problem solving: Managing it all. *The Mathematics Teacher, 87*(3), 195–199.

Garofalo J., & Lester, F. K., Jr. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education, 16*(3), 163-176.

Goldin, G. A., & McClintock, C. E. (Eds.). (1984). *Task variables in mathematical problem solving*. Philadelphia, PA: Franklin Institute Press.

Headrick, L., Wiezel, A., Tarr, G., Zhang, X., Cullicott, C. E., Middleton, J. A., & Jansen, A. (2020). Engagement and affect patterns in high school mathematics classrooms that exhibit spontaneous problem posing: an exploratory framework and study. *Educational Studies in Mathematics*, *105*(3), 435-456.

Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 123–147). Hillsdale, NJ: Erlbaum.

Klaassen, K., & Doorman, M. (2015). Problem posing as providing students with content-specific motives. In F. M. Singer, N. F. Ellerton & J. Cai (Eds.), *Mathematical problem posing: From research to effective practice* (pp. 215-240). New York, NY: Springer.

Koichu, B. (2020). Problem posing in the context of teaching for advanced problem solving. *International Journal of Educational Research*, *102*, 101428.

Koichu, B., & Kontorovich, I. (2013). Dissecting success stories on mathematical problem posing: A case of the Billiard Task. *Educational Studies in Mathematics, 83*, 71–86.

Koichu, B., & Zazkis, R. (2021). *Mathematical encounters and pedagogical detours. Stories of disturbance and learning opportunities in teacher education*. Springer.

Kontorovich, I., Koichu, B., Leikin, R., & Berman, A. (2012). An exploratory framework for handling the complexity of students' mathematical problem posing in small groups. *Journal of Mathematical Behavior*, *31*(1), 149-161.

Kopparla, M., Bicer, A., Vela, K., Lee, Y., Bevan, D., Kwon, H., Caldwell, C., Capraro, M. M., & Capraro, R. M. (2019). The effects of problem-posing intervention types on elementary students' problem-solving. *Educational Studies, 45*(6), 708-725.

Lavy, l., & Bershadsky, I. (2003). Problem posing via "what if not?" strategy in solid geometry: A case study. *The Journal of Mathematical Behavior, 22*(4), 369–387.

Leung, S. S., & Silver, E. A. (1997). The role of task format, mathematics knowledge, and creative thinking on the arithmetic problem posing of prospective elementary school teachers. *Mathematics Education Research Journal, 9*(1), 5–24.

Leung, S. S., & Wu, R. X. (1999). Posing problems: Two classroom examples. *Mathematics Teaching in the Middle School*, *5*(2), 112-117.

Li, X., Song, N., Hwang, S., & Cai, J. (2020). Learning to teach mathematics through problem posing: teachers' beliefs and performance on problem posing. *Educational Studies in Mathematics, 105*, 325–347.

Liljedahl, P., Chernoff, E., & Zazkis, R. (2007). Interweaving mathematics and pedagogy in task design: A tale of one task. *Journal of Mathematics Teacher Education*, *10*(4), 239-249.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. Pearson.

Merseth, K. K. (2016). The early history of case-based instruction: Insights for teacher education today. *Journal of Teacher Education*, *42*(4), 243-249.

Pelczer, I., & Gamboa, F. (2009). Problem posing: Comparison between experts and novices. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 353–360). International Group for the Psychology of Mathematics Education.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect (Chapter 7) In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257–315). Charlotte: Information Age.

Pittalis, M., Christou, C., Mousoulides, N., & Pitta-Pantazi, D. (2004). A structural model for problem posing. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 49–56). Bergen, Norway: Bergen University College.

Poincaré, H. (1908). *Science et méthode* [Science and method]. Flammarion.

Pólya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.

Rott, B., Specht, B. & Knipping, C. A. (2021). descriptive phase model of problem-solving processes. *ZDM Mathematics Education, 53*, 737–752.

Schindler, M., & Bakker, A. (2020). Affective field during collaborative problem posing and problem solving: A case study. *Educational Studies in Mathematics.*

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically. In D. A. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: MacMillan.

Silber, S., & Cai, J. (2017). Pre-service teachers' free and structured mathematical problem posing, *International Journal of Mathematical Education in Science and Technology*, *48*(2), 163-184.

Silber, S., & Cai, J. (2021). Exploring underprepared undergraduate students' mathematical problem posing. *ZDM Mathematics Education, 53,* 877–889.

Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14*(1), 19–28.

Silver, E. A., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education, 27*(5), 521-539.

Silver, E. A., Mamona-Downs, J., Leung, S., & Kenney, P. A. (1996). Posing mathematical problems: An exploratory study. *Journal for Research in Mathematics Education*, 27(3), 293–309.

Singer, F. M., Ellerton, N., Cai, J. (2013). Problem-posing research in mathematics education: New questions and directions. *Educational Studies in Mathematics, 83*(1), 1-7.

Singer, F. M., Ellerton, N., & Cai, J. (Eds.). (2015). *Mathematical problem posing: From research to effective practice*. New York, NY: Springer.

Song, S. H., Yim, J. H., Shin, E. J., & Lee, H. H. (2007). Posing problems with use the 'What if not?' strategy in NIM game. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education, Vol. 4* (pp. 193–200). Seoul, South Korea: International Group for the Psychology of Mathematics Education.

Stein, M. K., Henningsen, M. A., Smith, M. S., & Silver, E. A. (2009). Implementing standards-based mathematics instruction: A casebook for professional development (2nd ed.). Teachers College Press.

Stoyanova, E., & Ellerton, N. F. (1996). A framework for research into students' problem posing. In P. Clarkson (Ed.), *Technology in mathematics education* (pp. 518-525). Melbourne: Mathematics Education Research Group of Australasia.

Suarsana, I. M., Lestari, I., & Mertasari, N. M. (2019). The effect of online problem posing on students' problem-solving abilitiy in mathematics. *International Journal of Instruction*, *12*(1), 809-820.

Voica, C., Singer, F.M. & Stan, E. (2020). How are motivation and self-efficacy interacting in problem-solving and problem-posing? *Educational Studies in Mathematics, 105*, 487–517.

Wit, A. (2007). Interacting in groups. In O. Hargie (Ed.), *The handbook of communication skills* (3rd ed., Vol. 1, pp. 383–402). Routledge.

Yang, X., & Xin, Y. P. (2021). Teaching problem posing to students with learning disabilities. *Learning Disability Quarterly*.

Zazkis, R., & Mamolo, A. (2018). From disturbance to task design, or a story of a rectangular lake. *Journal of Mathematics Teacher Education*, *21*(5), 501-516.

Zhang, H., & Cai, J. (2021). Teaching mathematics through problem posing: insights from an analysis of teaching cases. *ZDM–Mathematics Education*, 1-13.

Zhang, L., Cai, J., Song, N., Zhang, H., Chen, T., Zhang, Z., & Guo, F. (2022). Mathematical problem posing of elementary school students: The impact of task format and its relationship to problem solving. *ZDM-Mathematics Education*.

**Citation information for this paper**: Cai, J., Koichu, B., Rott, B., Zazkis, R., & Jiang, C. (2022). Mathematical problem posing: Task variables, processes, and products. In C. Fernandez, S. Llinares, A. Gutierrez, & N. Planas (Eds.), *Proceedings of the 45th of the Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 119-145). PME.