Toda Flows, Gradient Flows, and Total Positivity

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Abstract—We outline various connections between the Toda flows, gradient flows, and the theory of total positivity.

1. Description

We consider firstly the full symmetric Toda flow on the space of $n \times n$ real symmetric matrices. It may viewed under an appropriate mapping as a flow on an adjoint orbit of the unitary group. The flow takes the Lax form

$$\dot{L}(t) = [L(t), \pi(L(t))], \tag{1}$$

see e.g. [5], where the flow is shown to be an integrable Hamiltonian flow. Here the projection π is onto the skew-symmetric part in the skew-symmetric/upper-triangular decomposition. Of particular interest is the tridiagonal case, which corresponds to the (finite nonperiodic) Toda lattice [12]. In this case the system has Hamiltonian

$$H(q_1, \dots, q_n, p_1, \dots, p_n) := \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}, \tag{2}$$

and may be interpreted as a system of n points on a line of unit mass governed by an exponential potential.

We relate the general symmetric flow to the theory of total positivity, see [4], where we showed in particular that this flow on an adjoint orbit preserves positivity, and is a gradient flow with the respect to the Kähler metric via applying a certain twist map. This extends earlier work of [3], and is complementary to work of [2] on flows with respect to the normal metric. This is related to our general work on positivity-preserving flows in [4]. We consider several aspects of the geometry, including the role of various metrics in relating the symmetric and tridiagonal flows.

2. Adjoint orbits and total positivity

We review adjoint orbits of $\mathfrak{u}_n$ and introduce its totally nonnegative part, following [4]. Let U_n denote the group of $n \times n$ unitary matrices, and let $\mathfrak{u}_n$ denote its Lie algebra of all $n \times n$ skew-Hermitian matrices. An *adjoint orbit* of $\mathfrak{u}_n$ is a subset of matrices with fixed spectrum (necessarily purely imaginary). Explicitly, let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ be such that $\lambda_1 \geq \cdots \geq \lambda_n$, and let $\mathsf{Diag}(\lambda)$ denote the diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$. Then we let

$$\mathcal{O}_\lambda := \{g(\mathrm{i}\,\mathsf{Diag}(\lambda))g^{-1} : g \in \mathrm{U}_n\}$$

denote the adjoint orbit of $\mathfrak{u}_n$ of all matrices with spectrum $\mathrm{i}\lambda$. We observe that the space of $n \times n$ real symmetric matrices embeds into $\mathfrak{u}_n$ as the space of purely imaginary skew-Hermitian matrices, via the map

$$L \mapsto \mathrm{i}L. \tag{3}$$

The adjoint orbit $\mathcal{O}_\lambda$ is isomorphic to the partial flag variety $\mathrm{GL}_n(\mathbb{C})/P_\lambda$, where P_λ is the parabolic subgroup consisting of block upper-triangular matrices with diagonal blocks of sizes equal to the multiplicities of λ. Lusztig [7, 8] introduced the totally positive and totally nonnegative parts of a partial flag variety, which may be explicitly described in terms of adjoint orbits as follows. A *minor* of a matrix is the determinant of a $k \times k$ submatrix for some $k \geq 0$. A *left-justified minor* is a minor using the first k columns (and an arbitrary subset of k rows). We call an element U_n *totally positive* (respectively, totally nonnegative) if all of its left-justified minors are positive (respectively, nonnegative), and denote the set of all such elements by $\mathrm{U}_n^{>0}$ (respectively, $\mathrm{U}_n^{\geq 0}$). For example,

$$\mathrm{U}_2^{\geq 0} = \left\{ \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} : \alpha \in [0, \tfrac{\pi}{2}] \right\}. \tag{4}$$

We define the *totally positive* and *totally nonnegative* parts of $\mathcal{O}_\lambda$ by

$$\mathcal{O}_\lambda^{>0} := \{g(\mathrm{i}\,\mathsf{Diag}(\lambda))g^{-1} : g \in \mathrm{U}_n^{>0}\},$$
$$\mathcal{O}_\lambda^{\geq 0} := \{g(\mathrm{i}\,\mathsf{Diag}(\lambda))g^{-1} : g \in \mathrm{U}_n^{\geq 0}\}.$$

We note that $\mathcal{O}_\lambda^{\geq 0}$ is the closure of $\mathcal{O}_\lambda^{>0}$ in the Euclidean topology. We say that a flow on $\mathcal{O}_\lambda$ *preserves positivity* if starting at any point in $\mathcal{O}_\lambda^{\geq 0}$, the flow remains in $\mathcal{O}_\lambda^{\geq 0}$ for all positive time.

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3. The tridiagonal case

We consider the Hamiltonian (2). Following Flaschka [6], we set

$$a_i := \tfrac{1}{2}e^{\frac{q_i-q_{i+1}}{2}}\ (1 \le i \le n-1), \quad b_i := -\tfrac{1}{2}p_i\ (1 \le i \le n).$$

Then the Hamiltonian equations become (with $a_0, a_n := 0$)

$$\dot{a}_i = a_i(b_{i+1} - b_i), \quad \dot{b}_i = 2(a_i^2 - a_{i-1}^2). \tag{5}$$

We also let L be the tridiagonal matrix

$$L := \mathrm{i}\begin{bmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_n \end{bmatrix}, \tag{6}$$

so that

$$\pi_{\mathfrak{u}}(-\mathrm{i}L) = \begin{bmatrix} 0 & -a_1 & 0 & \cdots & 0 \\ a_1 & 0 & -a_2 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $\pi_{\mathfrak{u}}$ is projection onto the skew-Hermitian part in the skew-symmetric/upper-triangular decomposition of $\mathfrak{gl}_n(\mathbb{C})$. Then we can write the flow (5) of the Toda lattice in the Lax form

$$\dot{L}(t) = [L(t), \pi_{\mathfrak{u}}(-\mathrm{i}L(t))], \tag{7}$$

which defines a flow on the adjoint orbit O_λ. Note that (7) is the flow corresponding to (1) under the embedding (3) of the space of $n \times n$ real symmetric matrices into $\mathfrak{u}_n$.

Above, L is assumed to be i times a real symmetric tridiagonal matrix, but more generally, we can consider any $L \in O_\lambda$. We call the flow (7) defined on the tridiagonal part of $\mathfrak{u}_n$ the *tridiagonal symmetric Toda flow*, and call the flow defined on all of $\mathfrak{u}_n$ the *full symmetric Toda flow*, studied in [5].

Let π_{U} denote projection onto the unitary part in the unitary/upper-triangular decomposition of $\mathrm{GL}_n(\mathbb{C})$ (also known as the QR- or Iwasawa decomposition). Symes [10, 11] found an explicit solution to (7), which can be verified directly.

Theorem 1 [10, 11] *Let $L(t)$ be a solution to the full symmetric Toda flow (7), with $L_0 \in \mathfrak{u}_n$. Then*

$$L(t) = \pi_{\mathrm{U}}(\exp(-t\mathrm{i}L_0))^{-1} L_0\, \pi_{\mathrm{U}}(\exp(-t\mathrm{i}L_0)).$$

Bloch [1] observed that for $L \in \mathfrak{u}_n$ tridiagonal, we have

$$\pi_{\mathfrak{u}}(-\mathrm{i}L) = [L, -\mathrm{i}\,\mathsf{Diag}(n-1, \dots, 1, 0)],$$

and proved the following:

Theorem 2 [1, 2] *Set $N = -\mathrm{i}\,\mathsf{Diag}(n-1, \dots, 1, 0) \in \mathfrak{u}_n$, and let $L_0 \in \mathfrak{u}_n$ be tridiagonal. Then the tridiagonal symmetric Toda flow beginning at L_0 can be written as*

$$\dot{L}(t) = [L(t), [L(t), N]].$$

In particular, the tridiagonal symmetric Toda flow restricted to O_λ is the gradient flow with respect to N in the normal metric.

4. The twist map

Following [4], we can explicitly describe the tridiagonal part of $O_\lambda^{\ge 0}$.

Definition 1 *Define the involution $\phi : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ by*

$$\phi(g)_{i,j} = (-1)^{i+j}(g^{-1})_{i,j} \quad \textit{for } 1 \le i, j \le n.$$

Then ϕ preserves $\mathrm{U}_n^{\ge 0}$. This induces the involution

$$\vartheta_\lambda : O_\lambda^{\ge 0} \to O_\lambda^{\ge 0},$$
$$g(\mathrm{i}\,\mathsf{Diag}(\lambda))g^{-1} \mapsto \phi(g)(\mathrm{i}\,\mathsf{Diag}(\lambda))\phi(g)^{-1} \quad (g \in \mathrm{U}_n^{\ge 0}).$$

We call ϑ_λ the *twist map*. For example, we see from (4) that when $n = 2$, the map ϕ (and hence ϑ_λ) is the identity.

Definition 2 *Let $\lambda \in \mathbb{R}^n$ be weakly decreasing. We define the spaces of isospectral Jacobi matrices $\mathcal{J}_\lambda^{>0}$ (respectively, $\mathcal{J}_\lambda^{\ge 0}$) to be set of tridiagonal matrices L as in (6) with spectrum $\mathrm{i}\lambda$ and such that a_i is positive (respectively, nonnegative) for all $1 \le i \le n-1$.*

Theorem 3 [4] *The spaces of Jacobi matrices $\mathcal{J}_\lambda^{>0}$ and $\mathcal{J}_\lambda^{\ge 0}$ are the tridiagonal parts of, respectively, $O_\lambda^{>0}$ and $O_\lambda^{\ge 0}$.*

Using ideas of Moser [9], we can parametrize $\mathcal{J}_\lambda^{>0}$ using the twist map ϑ_λ.

Theorem 4 [4] *Let $\lambda \in \mathbb{R}^n$ be strictly decreasing (otherwise, $\mathcal{J}_\lambda^{>0}$ is empty). Let $S_{>0}^{n-1}$ denote the subset of the unit sphere in $\mathbb{R}^n$ of vectors with positive coordinates. For $x \in S_{>0}^{n-1}$, define the rescaled Vandermonde matrix*

$$V(\lambda, x) := (\lambda_i^{j-1} x_i)_{1 \le i,j \le n} \in \mathrm{GL}_n(\mathbb{C}).$$

Then we have the diffeomorphism

$$S_{>0}^{n-1} \to \mathcal{J}_\lambda^{>0},$$
$$x \mapsto \vartheta_\lambda(\pi_{\mathrm{U}}(V(\lambda, x))(\mathrm{i}\,\mathsf{Diag}(\lambda))\,\pi_{\mathrm{U}}(V(\lambda, x))^{-1}).$$

When we restrict the domain of the twist map ϑ_λ from $O_\lambda^{\ge 0}$ to the tridiagonal subset $\mathcal{J}_\lambda^{\ge 0}$, it specializes to a map constructed by Bloch, Flaschka, and Ratiu [3] in general Lie type, and denoted ι. (We emphasize that in general, the image $\vartheta_\lambda(\mathcal{J}_\lambda^{\ge 0})$ is not contained in $\mathcal{J}_\lambda^{\ge 0}$.) The context in which the map ι appeared in [3] is similar to the one in the current discussion, namely, in order to realize the Toda flow on $\mathcal{J}_\lambda^{\ge 0}$ as a gradient flow compatible with the torus action. The ultimate goal in [3] was to prove convexity of $\mathcal{J}_\lambda^{\geq 0}$, by mapping it onto its moment polytope. It turns out that the usual moment map (i.e. projection onto the diagonal) is neither injective nor surjective on $\mathcal{J}_\lambda^{\geq 0}$, but if we first apply the map ι, we obtain a homeomorphism onto the moment polytope which restricts to a diffeomorphism from $\mathcal{J}_\lambda^{>0}$ onto its interior. The perspective of positivity gives a new and conceptually simpler way to construct the map ι.

5. The general symmetric case and Kähler flows

There are three natural Riemannian metrics on the adjoint orbit $\mathcal{O}_\lambda$: the Kähler, normal, and induced metrics. By Theorem 2, the tridiagonal Toda lattice flow is a gradient flow in the normal metric. In [4], we consider the full symmetric Toda flow in the Kähler metric:

Theorem 5 [4] *Let $\lambda \in \mathbb{R}^n$ be strictly decreasing, and set $N := -\mathrm{i}\,\mathsf{Diag}(\lambda) \in \mathfrak{u}_n$.*

(i) The full symmetric Toda flow on $\mathcal{O}_\lambda$ preserves positivity in both the positive and negative time directions. That is, if $L(t)$ evolves according to (7) beginning at $L_0 \in \mathcal{O}_\lambda^{\geq 0}$, then $L(t) \in \mathcal{O}_\lambda^{\geq 0}$ for all $t \in \mathbb{R}$.

(ii) The full symmetric Toda flow restricted to $\mathcal{O}_\lambda^{\geq 0}$ is the twisted gradient flow with respect to N in the Kähler metric. That is, if $L(t)$ evolves according to (7) beginning at $L_0 \in \mathcal{O}_\lambda^{\geq 0}$, then $\vartheta_\lambda(L(t))$ is the gradient flow with respect to N in the Kähler metric beginning at $\vartheta_\lambda(L_0) \in \mathcal{O}_\lambda^{\geq 0}$.

We observe that because the matrix N above is diagonal, the twisted flow $\vartheta_\lambda(L(t))$ is contained in a torus orbit of $\mathcal{O}_\lambda$.

We can in fact describe the flow (7) quite explicitly by writing

$$-\mathrm{i}L(t) = \Big(\sum_{k\in K}(\lambda_k - \lambda_{k+1})P_k(t)\Big) + \lambda_n I_n, \qquad (8)$$

where $P_k(t)$ is the orthogonal projection onto the subspace spanned by the eigenvectors of $-\mathrm{i}L(t)$ corresponding to the eigenvalues $\lambda_1, \dots, \lambda_k$. Explicitly, if $P_k(0)$ is orthogonal projection onto the subspace V_k, then $P_k(t)$ is orthogonal projection onto $\exp(t\mathrm{i}N)V_k$.

Much more can be said about the role of positivity in Toda flows, and in gradient flows on adjoint orbits and their corresponding partial flag varieties; see [4]. Forthcoming work will discuss relating the tridiagonal and symmetric Toda flows via the Plücker embedding, building on [5].

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