

Irreducibility of Recombination Markov Chains in the Triangular Lattice

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Abstract

In the United States, regions (such as states or counties) are frequently divided into districts for the purpose of electing representatives. How the districts are drawn can have a profound effect on who's elected, and drawing the districts to give an advantage to a certain group is known as gerrymandering. It can be surprisingly difficult to detect when gerrymandering is occurring, but one algorithmic method is to compare a current districting plan to a large number of randomly sampled plans to see whether it is an outlier. Recombination Markov chains are often used to do this random sampling: randomly choose two districts, consider their union, and split this union up in a new way. This approach works well in practice and has been widely used, including in litigation, but the theory behind it remains underdeveloped. For example, it's not known if recombination Markov chains are irreducible, that is, if recombination moves suffice to move from any districting plan to any other.

Irreducibility of recombination Markov chains can be formulated as a graph problem: for a planar graph G , is the space of all partitions of G into k connected subgraphs (k districts) connected by recombination moves? While the answer is yes when districts can be as small as one vertex, this is not realistic in real-world settings where districts must have approximately balanced populations. Here we fix district sizes to be $k_1 \pm 1$ vertices, $k_2 \pm 1$ vertices, $\dots$ for fixed $k_1, k_2, \dots$, a more realistic setting. We prove for arbitrarily large triangular regions in the triangular lattice, when there are three simply connected districts, recombination Markov chains are irreducible. This is the first proof of irreducibility under tight district size constraints for recombination Markov chains beyond small or trivial examples. The triangular lattice is the most natural setting in which to first consider such a question, as graphs representing states/regions are frequently triangulated. The proof uses a sweep-line argument, and there is hope it will generalize to more districts, triangulations satisfying mild additional conditions, and other redistricting Markov chains.

1 Introduction

In the United States, regions (such as cities, counties, or states) are frequently divided into *districts* for the purpose of electing officials to positions ranging from a local school board to the U.S. House of Representatives. The way these districts are drawn can have a large effect on who is elected. Drawing the lines of these districts so as to give an advantage to a certain individual, group, or political party is known as *gerrymandering*. It can be surprisingly difficult to detect if gerrymandering is occurring or whether outcomes considered ‘unfair’ are a

result of other factors, such as the spatial distribution of voters (for example, see [24]).

One method to detect gerrymandering is to see where the current or proposed districting plan lies within the context of all possible districting plans for a region. Districting plans are generally expected to have contiguous, compact districts, even when not explicitly required by law, though there are a variety of competing notion of compactness (see, for example, [23]). Other legal requirements, such as respecting communities of interest, avoiding county splits, or incorporating incumbency, are considered in certain jurisdictions as well. However, no matter the restrictions placed on a districting plan, the number of possible districting plans for any state or region is far too large to be studied in its entirety. For example, the number of ways to divide a 9×9 square grid into 9 contiguous, equally-sized districts is more than 700 trillion [31].

Because of this, random sampling of political districting plans has become an important tool to help understand the space of possible plans, beginning with the work of Chen and Rodden [13, 14]. A collection of randomly sampled districting plans has come to be called an *ensemble*. If a current or proposed districting plan is an outlier with respect to the ensemble, this may be evidence it is gerrymandered.

A variety of methods for creating ensembles of randomly sampled districting plans exist (see Related Work, below). In this paper, the focus is on recombination Markov chains. These chains create random districting plans by repeatedly choosing two random districts; merging these two districts together; and splitting this union up in a new way so that population balance and other constraints are still satisfied. Recombination Markov chains have been used in a variety of academic papers [3, 4, 6, 9, 10, 11, 15, 18, 21, 22, 26, 41], technical reports [7, 19, 20, 36, 37, 38], and court cases [5, 12, 25], including in 2021–2022 litigation in Pennsylvania, South Carolina, and Texas.

One common problem with all recombination Markov chains is that it's not known whether they're irreducible. That is, it's not known whether recombination moves suffice to reach all districting plans, or if there are some plans that cannot be created by the repeated merging and splitting process. This could po-

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tentially create problems if ensembles are created by sampling from only the reachable subset of plans rather than the entire space of plans. While there is no evidence so far that any real-world examples of recombination Markov chains are not irreducible, proofs have largely remained elusive.

1.1 Dual Graphs, balanced partitions, and irreducibility. Recombination Markov chains work with a discretization of the real-world political districting problem by considering *dual graphs*: graphs whose nodes are small geographic units (such as census blocks, census block groups, or voting precincts) and whose edges show geographic adjacency [27]. While political districts may occasionally split census blocks or voting precincts, this is rare, and it is generally agreed that considering districting plans built out of only whole geographic units is both a good approximation to considering all districting plans as well as necessary to make the problem tractable. This means districting plans are partitions of the nodes of the dual graphs into k connected subgraphs, where k is the number of districts. In real-world examples, nodes also have attached populations and districts must be population-balanced. This is usually operationalized by ensuring the population of each district doesn't differ by more than an ε multiplicative factor from its ideal population, which is the total population divided by the number of districts. In practice, ε is typically about 1-2%; if tighter population balance is required, this can be achieved by making targeted local changes after the fact. Because population can vary widely between different geographic units, this means districts will frequently have different numbers of nodes.

A Markov chain consists of states (here, each state is a partition of the dual graph into k connected subgraphs) and rules governing probabilistic transitions between these states (for us, recombination steps). A Markov chain is *irreducible* if the transitions of the chain suffice to go from any state of the Markov chain to any other state. The set of states of a Markov chain is known as the *state space*, and if the chain is irreducible the state space is said to be *connected*.

In beginning to study the irreducibility of recombination Markov chains, researchers have introduced a simplification: assume each node has the same population (see, for example, [2]). This means one can determine if a districting plan is population-balanced by looking only at the number of nodes in each district, rather than the populations at those nodes. In this setting, the *ideal size* of a district is n/k , the total number of vertices divided by the number of districts. In [2], authors show (1) when districts sizes can get arbitrarily large or small, recombination Markov chains are irreducible;

(2) when the underlying graph is Hamiltonian, recombination Markov chains are irreducible when districts are allowed to get as small as one vertex or as large as twice their ideal size; and (3) there exist Hamiltonian planar graphs on which recombination is not irreducible when districts sizes are constrained to be at least $(2/3)n/k$ and at most $(4/3)n/k$. This last result shows how imposing tight size constraints makes it much harder to reach all possible districting plans. The only known positive irreducibility results under tight size constraints are for double-cycle graphs and grid-with-a-hole graphs, where the large amount of structure present makes the proofs nearly trivial [11]. No positive irreducibility results are known for recombination Markov chains beyond these trivial examples when district sizes are constrained any tighter than in result (2) above.

The result of [2] is related to one approach to irreducibility sometimes taken in practice: Initially allow districts to get arbitrarily small, and then gradually tighten population constraints ('cool' the system) until the districts are as balanced in size or population as desired. However, if the state space is disconnected when restricted to partitions whose districts are close to their ideal sizes, there is no guarantee this process ends in each connected component of the state space with the correct relative probabilities and certain parts of the state space aren't oversampled or undersampled as a result. Additionally, this process is computationally expensive and is often skipped in practice.

Because of this, irreducibility results under tight size constraints are very desirable. The negative result of [2] mentioned above implies general results for planar graphs are impossible. However, their examples are far from the types of planar graphs that might be encountered in real-world applications, which don't usually have faces with long boundary cycles. However, there are also some negative irreducibility results even for grids when district sizes are constrained to take on an exact value. For example, see Figure 1(a) for an example from Jamie Tucker-Foltz that is rigid under recombination moves when districts are restricted to be size 3: any attempt to merge two districts and split the resulting union into two equal-sized pieces will always produce this exact partition [40]. It's worth noting such examples also exist in the triangular lattice, which will be our focus. For example, in a triangle with three vertices along each side (six vertices total), recombination is not irreducible when districts are constrained to be size exactly two; see Figure 1(b). This indicates that even in simple graphs such as grids, requiring exact sizes for districts is too much to ask for.

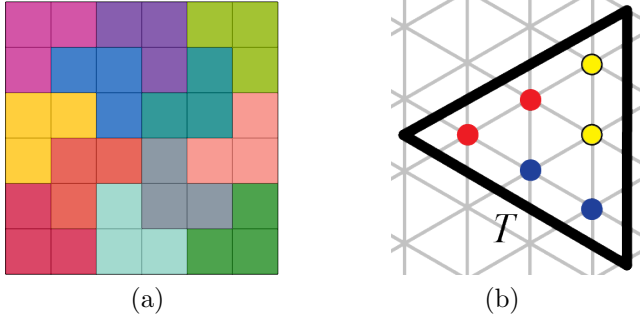


Figure 1: (a) From Jamie Tucker-Foltz, a partition of a 6×6 grid into 12 districts of size exactly 3 that is rigid under recombination moves [40]. (b) A partition of a triangular subgraph T of the triangular lattice into 3 districts of size exactly 2 that is rigid under recombination moves.

1.2 Results. In both examples in Figure 1, irreducibility can easily be achieved by allowing districts sizes to get one larger or one smaller than their prescribed ideal sizes. This motivates the following definitions used in this work. We assume districts have prescribed sizes $k_1, k_2, \dots$. A partition is *balanced* if each district i has exactly k_i vertices, and a partition is *nearly balanced* if the partition is not balanced but the number of vertices in district i is at least $k_i - 1$ and at most $k_i + 1$. While the case $k_1 = k_2 = \dots$ is most interesting, equality of district sizes is not required for our results.

This paper provides the first irreducibility results for an infinite class of graphs for the recombination Markov chain whose state space is all balanced and nearly balanced partitions. The graphs we consider, chosen to strike a balance between having enough structure for proofs to be possible while still being motivated by real world examples, are subsets of the triangular lattice. The triangular lattice was chosen instead of the more frequently used square lattice because dual graphs derived from geography frequently have mostly triangular faces: it's rare to have four or more geographic domains (such as census blocks or voting precincts) meet at a single point, making faces with a boundary cycle of length four or more rare in the corresponding dual graphs. Triangulations also have a nice feature that is incredibly important for our proofs: The neighbors of any vertex form a cycle.

Our main result is that for arbitrarily large triangular subgraphs of the triangular lattice (e.g., Figure 1(b)), recombination Markov chains for three districts are irreducible on the state space of balanced and nearly balanced partitions. There is one small caveat: we require districts to be simply connected rather than

just connected, and this is crucially used at several points in our proofs. However, this is not an unreasonable restriction: it's rare in practice to see one district completely encircling another. We also require the minor technical condition that the districts are not too small: if the triangular subgraph has side length n , then k_i must be at least $n + 1$ for all $i = 1, 2, 3$. We also assume $n \geq 5$. We summarize this result in the following theorem.

THEOREM 1. *Let T be a triangular subset of the triangular lattice with side length $n \geq 5$. Let k_1, k_2 , and k_3 be integers satisfying $k_1 + k_2 + k_3 = n(n+1)/2 = |V(T)|$ and each $k_i \geq n + 1$. Let Ω be all partitions of T into three simply connected pieces P_1, P_2 , and P_3 where $|P_i| \in [k_i - 1, k_i + 1]$ for $i = 1, 2, 3$. Recombination Markov chains on Ω are irreducible.*

This is a significant step beyond previous irreducibility results. We go beyond small, computationally-verified examples to an infinite class of graphs, and do so under extremely tight conditions on the district size: districts never get more than one vertex larger or smaller than their prescribed size. It is surprising that relaxing district sizes by just one vertex is sufficient for proofs to be successful. It is hoped this first step showing irreducibility for arbitrarily large graphs, and the ideas and approaches contained in this paper, can be used as a springboard for further irreducibility results. Future work includes generalizing this result to more than three districts, other subsets of the triangular lattice, and perhaps eventually all planar triangulations. These all now seem plausibly within reach.

1.3 Related Work.

Extensions of Recombination Markov Chains: While recombination Markov chains have been widely used in practice to create ensembles and evaluate potential gerrymandering, they have also formed the basis for further explorations. For example, [4] gives a multi-level version of the recombination Markov chain that, in addition to computational speed-ups, can help preserve communities of interest, such as counties. Additionally, [10] uses the recombination Markov chain to find districting plans that have many majority-minority districts by repeatedly running short ‘bursts’ of the recombination chain from carefully chosen starting points. Since new algorithms are being created with recombination Markov chains as a key underlying process, it's essential we continue to develop a rigorous understanding of recombination Markov chains, as we do here.

Flip Markov Chains: Another type of Markov chain that has been used for sampling districting plans

is flip Markov chains [33, 28, 29]. In these chains, only a single vertex is reassigned to a new district in each step. Flip moves are a subset of recombination moves: any flip move can be achieved by a recombination step that merges two districts and splits them up such that only one vertex has changed its district assignment. Flip moves are not known to connect all possible districting plans under tight population balance constraints. In [1], authors show in 2-connected graphs when districts can get arbitrarily large or small, flip moves connect all districting plans. In [29], the authors note their flip Markov chain may not be irreducible (and give an example of this) and instead restrict their attention to the connected pieces of the state space.

In our work, because flip moves are a subset of recombination moves and are simpler, we frequently focus on flip moves. Most results will apply to both recombination moves and flip moves. The main exception is the *Cycle Recombination Lemma* (Lemma 5) which requires a recombination step rather than a flip step. Recombination steps are also used at the end of our sweep line process to ultimately reach a ground state, and to move between ground states. If alternate proofs of these lemmas using flip steps rather than recombination moves is found, our results would also imply flip Markov chains are irreducible in the same settings. We believe such a result is likely possible, but we have not pursued it because our focus is on recombination moves.

While Markov chains have been most often used to create random samples, requiring running the chains for many steps, flip Markov chains have also been used to detect ‘careful crafting’ of districting plans by detecting whether a plan is an outlier with respect to the stationary distribution [17, 16]. These methods do not require the Markov chains to be irreducible, but provide a single significance value rather than a more robust understanding of the space of possible districting plans.

Other Methods for Generating Ensembles: It should be noted Markov chains are not the only methods employed to generate ensembles of districting plans. For example, the first papers introducing the idea of ensembles created them by randomly merging precincts to form the correct number of districts, and then exchanging precincts between districts to achieve population balance [13, 14]; it is challenging to know which distribution of districting plans this samples from. A technique known as Sequential Monte Carlo [35] generates random districting plans by iteratively sampling one district at a time using spanning tree methods and reweighting at each step to ensure convergence to a desired target distribution. The authors of [32, 30] use a two-stage method to generate districting plans that allows the incorporation of a notion of fairness into the

district selection process.

Proving Irreducibility: The proof of irreducibility we give here has some features in common with the irreducibility proofs in [8, 39]. The first shows a Markov chain on simply connected subgraphs of the triangular lattice is irreducible, and the second does the same in the presence of a fixed vertex that is constrained to always be in the subgraph. As in [8], the main idea we use is a sweep-line procedure, adjusting the districting in a left-to-right fashion; sweep line approaches are common throughout the field of computational geometry. As in both [8, 39], the idea of *towers* we use is inspired by the towers of [34].

2 Proof Overview

The proof proceeds largely from first principles. The most complicated mathematics used are breadth-first search trees and some facts about boundaries of planar sets. Despite this, an incredible attention to detail and extensive careful constructions are still required to account for the intricacies that are possible in the partitions we consider.

Let G_Δ be the infinite graph whose edges and vertices are those of the triangular lattice. Let T be a triangular subgraph of G_Δ . We let n denote the number of vertices along one side of (equilateral) triangle T , meaning T contains $n(n+1)/2$ vertices total; Figure 1(b) gives an example when $n = 3$ and Figure 2 gives an example when $n = 8$. For simplicity, we assume T is always oriented so that it has a vertical edge on its right side, as in both figures.

Let k_1 , k_2 , and k_3 be such that each $k_i \geq n + 1$ and $k_1 + k_2 + k_3 = n(n + 1)/2$. We are interested in partitions of the vertices of T into three simply connected subgraphs P_1 , P_2 , and P_3 , of sizes $k_1 \pm 1$, $k_2 \pm 1$, and $k_3 \pm 1$, respectively. In analogy to the redistricting motivation, we will call each of P_1 , P_2 , and P_3 a *district* of this partition. In an abuse of notation, we will let P_i represent both the vertices in district i as well as the induced subgraph of T on this vertex set. Because it is a partition, the three sets P_1 , P_2 , and P_3 must be disjoint and their union must be T . To avoid cumbersome language, throughout this paper we will say ‘partition’ to mean a partition of T into three simply connected districts. If $|P_1| = k_1$, $|P_2| = k_2$, and $|P_3| = k_3$, the partition is *balanced*. If the partition is such that $k_i - 1 \leq |P_i| \leq k_i + 1$ for $i = 1, 2, 3$ but it is not balanced, we say the partition is *nearly balanced*.

We will consider the state space Ω consisting of the balanced and nearly-balanced partitions of T into three districts. We will examine the graph G_Ω whose vertices are the partitions in Ω where an (undirected) edge exists between σ and τ if one district of σ is identical

to one district of τ . These are exactly the transitions allowed by recombination Markov chains, which recombine two districts but leave a third untouched. If G_Ω is connected, this implies recombination Markov chains are irreducible, and our main theorem is equivalent to showing G_Ω is connected. We now outline our approach for proving this theorem; all details can be found in the full version of this paper.

2.1 Ground States and Sweep Lines. We show, for every balanced or nearly balanced partition, there exists a sequence of moves transforming it into one of six ground states. Throughout this paper we consider a left-to-right, top-to-bottom ordering of the vertices of T , where the leftmost vertex of T is first; followed by the vertices in the next column, ordered from top to bottom; followed by the vertices in the third column, ordered from top to bottom; etc. Using this ordering, ground state σ_{123} has the first k_1 vertices in P_1 , the next k_2 vertices in P_2 , and the final k_3 vertices in P_3 . Other ground states σ_{132} , σ_{213} , σ_{231} , σ_{312} , and σ_{321} are defined similarly. Because each $k_i \geq n + 1$, this is always a valid partition. It is straightforward to see the six ground states are connected to each other by one or two recombination moves. Because for every recombination step, the reverse step is also valid, for irreducibility it suffices to show every balanced or nearly balanced partition can be transformed into one of these ground states.

Without loss of generality, we suppose T 's single leftmost vertex, which we call C_1 , is in P_1 , and we are trying to reach the ground state σ_{123} . We let C_i be the first column which contains vertices not in P_1 ; see Figure 2(a) for an example. We will show how to (1) increase the number of vertices in $C_i \cap P_1$ and (2) if necessary, transform the result from a nearly balanced to a balanced partition without decreasing $C_i \cap P_1$. In both (1) and (2), any vertices left of C_i remain unaffected. Figure 2 (b,c) gives an example of what this process looks like. After repeating this process for gradually increasing i , we eventually reach a state where there are no vertices in P_1 right of C_i . In this case, a small number of recombination steps suffice to reach σ_{123} .

We begin by outlining some key definitions and lemmas, and then give a high-level overview of how (1) and (2) are achieved. The lemmas presented here lack some formality for ease of presentation; formal statements and complete proofs of these lemmas can be found in the full version of this paper.

2.2 Key facts and lemmas. Let $bd(T)$ be the vertices in T that are adjacent to a vertex outside of T . We

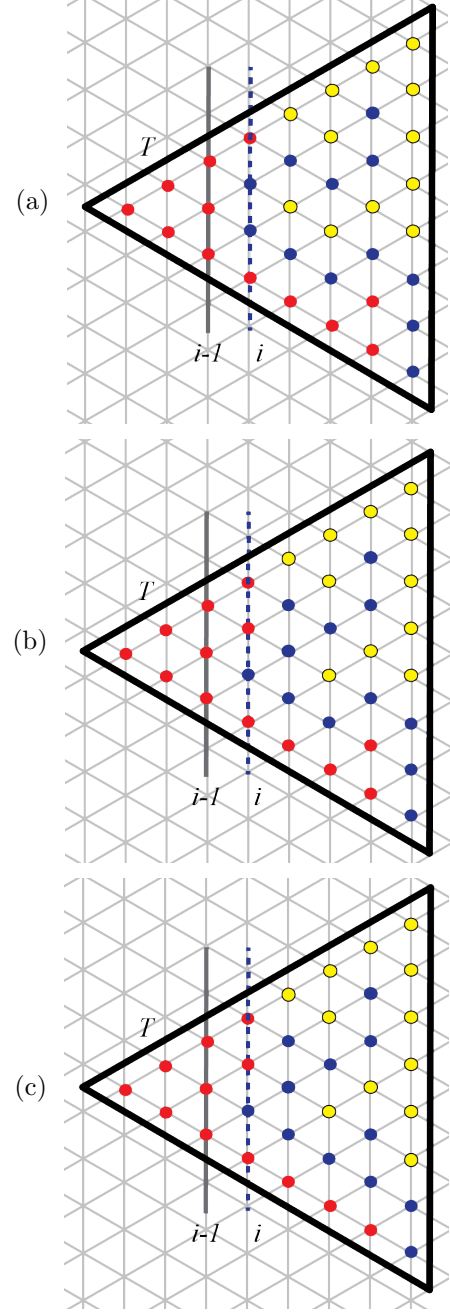


Figure 2: An example of our sweep line process on a partition where $k_1 = k_2 = k_3 = 12$. P_1 is red, P_2 is blue, and P_3 is green. (a) A balanced partition where the first $i - 1$ columns are in P_1 but the i^{th} column is not entirely within P_1 yet. (b) After applying a tower move, the number of vertices of P_1 in the i^{th} column has increased, but the partition is now nearly balanced instead of balanced, with $|P_1| = k_1 + 1$ and $|P_3| = k_3 - 1$. (c) After applying Case A of our rebalancing procedure by making changes near where P_2 and P_3 are adjacent in $bd(T)$, we reach a balanced partition.

let $N(v)$ be all neighbors of vertex v in G_Δ , and note $N(v)$ is always a cycle of length 6. For $i \in \{1, 2, 3\}$, we say v 's i -neighborhood is $N(v) \cap P_i$, that is, all vertices in P_i that are adjacent to v . This i -neighborhood is connected if $N(v) \cap P_i$ has only one connected component. While our overall proof is about recombination Markov chains, flip moves (where one vertex is assigned to a new district) are a subset of recombination moves, and we will often focus on flip moves because it makes our arguments easier. The following flip lemma describes when flip moves are possible. The simplicity of this flip lemma is a large reason why it is convenient to be working in the triangular lattice.

LEMMA 1. (FLIP LEMMA, INFORMAL) *If P is a partition of T into three simply connected districts, and $v \in P_i$ has a connected i -neighborhood and a connected, nonempty j -neighborhood for $j \in \{1, 2, 3\}$, $j \neq i$, then removing v from P_i and adding it to P_j produces another partition of T into three simply connected districts.*

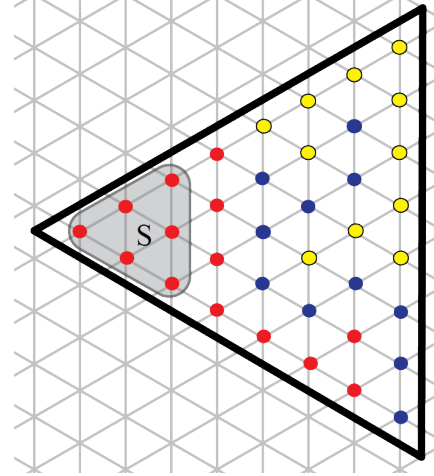
The following lemma suggests in most cases, when a vertex can be removed from P_i , it can always be added to one of the two other districts, P_j or P_l with $j, l \in \{1, 2, 3\}$ and i, j, l all distinct.

LEMMA 2. (ALTERNATION LEMMA, INFORMAL) *Let P be a partition of T into three simply connected districts, and suppose $v \in P_i$ has a connected i -neighborhood, is not in $bd(T)$, and is adjacent to at least one vertex in a different district. Then v 's j -neighborhood or l -neighborhood is connected and nonempty, for $j \neq l$.*

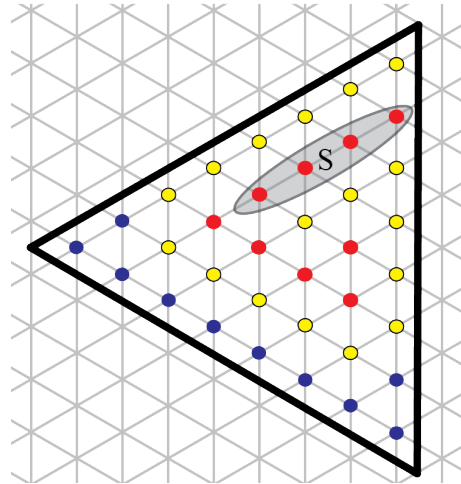
This is called the Alternation Lemma because its proof involves showing districts j and l cannot alternate too much in $N(v)$: having an ordered sequence of four vertices a, b, c, d in $N(v)$ with $a, c \in P_j$ and $b, d \in P_l$ is impossible because then P_j and P_l cannot both be connected.

Not every vertex can be removed from one district and added to another while maintaining simple connectivity of all districts. We say a simply connected subgraph $S \subset P_i$ is *shrinkable* if it contains a vertex that can be removed from P_i and added to a different district. Most sets will be shrinkable, but there are two notable exceptions: If S does not contain any vertices adjacent to other districts, or if S is a path ending with a single vertex in $bd(T)$. Figure 3 gives an example of each.

The following lemma gives sufficient conditions for $S \subseteq P_i$ to be shrinkable, and was crafted exactly to avoid the two non-shrinkable examples of Figure 3. Note we only consider the S that can be produced by removing a simply connected subset of P_i ; this ensures,



(a)



(b)

Figure 3: Two examples of simply connected subsets S (grey) of P_1 (red) that are not shrinkable. In (a), S is not shrinkable because no vertex of S is adjacent to any district besides P_i . In (b), S is not shrinkable because the removal of any vertex except the rightmost will disconnect P_1 , and while the rightmost vertex of S can be removed from P_1 , adding it to P_2 (blue) produces something not connected and adding it to P_3 (yellow) produces something that is not simply connected.

for example, that S is not entirely contained in the interior of P_i .

LEMMA 3. (SHRINKABILITY LEMMA, INFORMAL) *If $S \subseteq P_i$ is simply connected and $P_i \setminus S$ is simply connected, the following two conditions are each sufficient for S to be shrinkable:*

- (I) $S \cap bd(T) = \emptyset$.
- (II) S is adjacent to a different district, and P_i contains at least two vertices in $bd(T)$.

While the Shrinkability Lemma allows us to find a single vertex to remove, we cannot repeatedly remove vertices from the same district because this will produce partitions that are not balanced or nearly balanced. Instead, if we wish to remove multiple vertices from a particular district, we must alternate with adding new vertices to that district somewhere else. It is important the vertices we are adding are not adjacent to the vertices we are removing, otherwise we can't know any real progress is being made. The following Unwinding Lemma gets at this idea, where we have $S_1 \subseteq P_1$ that we want to add to P_2 and $S_2 \subseteq P_2$ that we want to add to P_1 . This lemma is only applied in the case where $|P_3| = k_3 - 1$, so adding a vertex to P_3 to bring it up to its ideal size is also considered a successful outcome. It is called the Unwinding Lemma because S_1 and S_2 are frequently long, winding arms of P_1 and P_2 , respectively, that we wish to contract so our partition is less intertwined.

LEMMA 4. (UNWINDING LEMMA, INFORMAL) *Let $S_1 \subseteq P_1$ and $S_2 \subseteq P_2$ be shrinkable and not adjacent. There exists a sequence of moves after which (1) a vertex has been added to P_3 , (2) all vertices in S_1 have been added to P_2 , or (3) all vertices in S_2 have been added to P_1 .*

Finally, at times we will need to work with $S_1 \subseteq P_1$ and $S_2 \subseteq P_2$ that are adjacent. This arises when S_1 and S_2 are both inside some cycle C , where all vertices of C are in P_1 except for one, x , which is in P_2 . If y is one of the vertices in C adjacent to x and y is a cut vertex of P_1 , the case where other approaches fail is when one component of $P_1 \setminus y$ is inside C . We would like y 's 1-neighborhood to be connected but the component S_1 of $P_1 \setminus y$ that is inside C prevents that from happening; see Figure 4(a) for an example. Instead of removing vertices from S_1 one at a time, we rearrange the entire interior of C with a single recombination step.

LEMMA 5. (CYCLE RECOMBINATION LEM., INFORMAL) *Let C be a cycle of vertices in P_1 with one vertex, x ,*

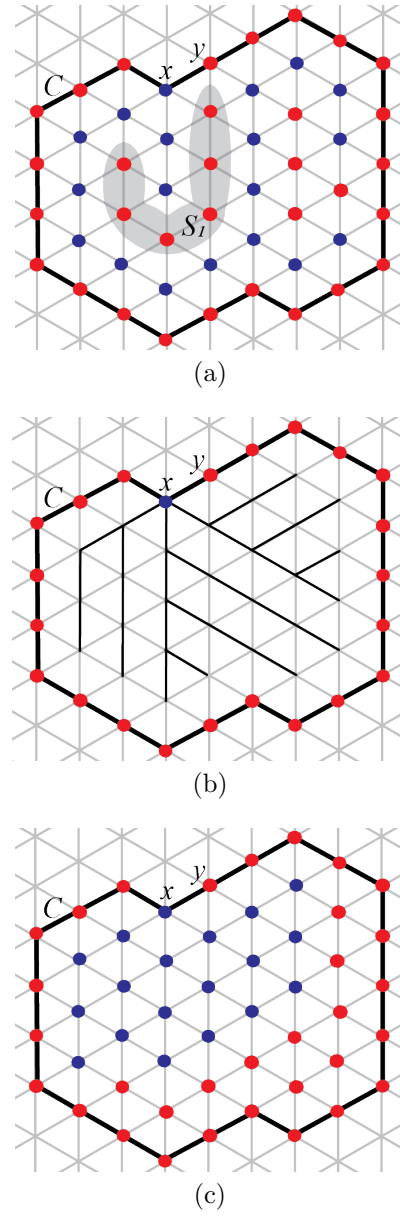


Figure 4: An example application of the Cycle Recombination Lemma. (a) An example satisfying the hypotheses of the lemma: A cycle C of vertices in P_1 (red) plus one vertex x in P_2 (blue), such that x 's neighbor y in C is a cut vertex of P_1 and $P_1 \setminus y$ has a component S_1 (grey) within C . The district assignments of vertices outside C are not shown. (b) All district assignments within C are erased, and we build a breadth first search tree of the vertices within C . (c) If initially there were m vertices of P_1 within C , the last m vertices added to the BFS tree are added to P_1 while the remaining vertices are added to P_2 . After this process y will have a connected 1-neighborhood.

in P_2 . Suppose no vertices of P_3 are inside C . There exists a single recombination step, changing only district assignments within C , after which x 's neighbor y in C has a connected 1-neighborhood (at least when looking in or inside C).

The main idea of this recombination step is to erase all district assignments within C and build a breadth first search tree of the interior of C . If initially there were m vertices of P_1 within C , the last m vertices added to the BFS tree are added to P_1 while the remaining vertices are added to P_2 . An example of this process is shown in Figure 4. Because the vertices in $N(y)$ that are inside C monotonically increase in their distance from x , after this recombination step $N(y)$ will consist of x , followed by some vertices in P_2 , followed by some vertices in P_1 , followed by y 's other neighbor in C . While the lemma does not say anything about the parts of $N(y)$ that are outside C , we will apply it in cases where knowing y 's 1-neighborhood in or inside C is connected implies y 's entire 1-neighborhood is connected.

2.3 Advancing toward Ground State: Towers.

In our sweep line procedure, the two main steps are (1) increase the number of vertices in $C_i \cap P_1$ and (2) if necessary, transform the result from a nearly balanced to a balanced partition without decreasing $C_i \cap P_1$. The way we achieve (1) is using *towers*. For a particular vertex in C_i , we may want to add it to P_1 but be unable to because doing so produces a partition with districts that are not simply connected. Let v_1 be a vertex in C_i that is not in P_1 but adjacent to a vertex of $P_1 \cap C_i$, and suppose without loss of generality that $v_1 \in P_2$; see Figure 5(a) for an example. This means v_1 has three neighbors in P_1 , and it's the middle of its other three neighbors that is crucial for determining whether v_1 can be added to P_1 or not. If it can't, we then examine this middle neighbor, which must have a similar neighborhood structure to v_1 . This process is repeated and must eventually end at a vertex that can be added to the district of the vertex before it in the tower. Flip moves are then made all the way back up the tower, ultimately producing a configuration in which v_1 can be added to P_1 . We do not state our tower lemma, or even formally define a tower, because of the technical details involved, but depict a sample application of the tower procedure in Figure 5.

2.4 Rebalancing: Cases. Performing a tower move as described in the previous section increases the number of vertices of P_1 in C_i , but can also move us from a balanced partition to a nearly balanced partition, because the number of vertices in P_1 has increased by one. Before proceeding further, we must return to a balanced

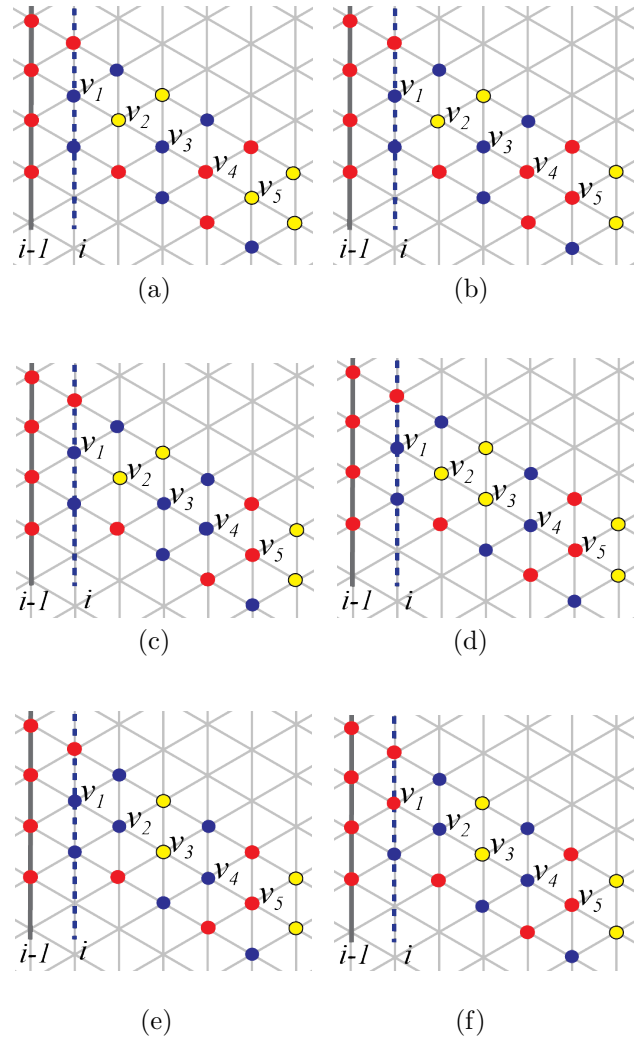


Figure 5: An example where we wish to add $v_1 \in P_2$ (blue) to P_1 (red), but cannot because this would disconnect P_2 . Instead we look at the middle of v_1 's three neighbors not in P_1 , which we call v_2 , and see if it can be added to P_2 ; in this example, it can't because doing so would create a cycle in P_2 . We continue along the line spanned by v_1 and v_2 until we find a vertex that can be added to the district of the vertex before it, which we prove must eventually happen. (a) In this example, vertex v_5 can be added to the district of v_4 , and the result of this move is shown in (b). Now v_4 can be added to the district of v_3 , and the result is (c). The same procedure for v_3 , v_2 , and v_1 gives the results shown in (d), (e), and (f), respectively. The end result is that there is one additional vertex of P_1 (v_1) in column i .

partition, and do so without decreasing the number of vertices in $P_1 \cap \mathcal{C}_i$. This is the most challenging part of the proof.

Without loss of generality, we suppose we have $|P_2| = k_2$ and $|P_3| = k_3 - 1$. We can use the Shrinkability Lemma to always find a vertex of P_1 that can be removed from P_1 and added to a different district; however, it may be that all such vertices can only be added to P_2 , not P_3 , and additionally all vertices that can be removed from P_2 can only be added to P_1 . In these cases we need to do some rearranging with P_1 (in columns $i+1$ and greater) and P_2 before finding a vertex that can be added to P_3 to reach a balanced partition. It is in this rearranging that the Unwinding Lemma and the Cycle Recombination Lemma play crucial roles.

Our proof considers four main cases for the rebalancing process, depending on the type of adjacency between P_2 and P_3 : (A) There exists $a \in P_2 \cap bd(T)$ and $b \in P_3 \cap bd(T)$ that are adjacent; (B) $P_2 \cap bd(T) = \emptyset$; (C) $P_3 \cap bd(T) = \emptyset$; and (D) No vertex of P_2 is adjacent to any vertex of P_3 . See Figure 6 for cartoonish examples of the four cases. We prove these four cases are disjoint and cover all possibilities, and consider each separately, though there are certainly common elements between their proofs. Cases (A) and (D) are the most straightforward because we only need to consider reassigning vertices near $bd(T)$, while (B) and especially (C) are more challenging because we must work in the interior of T , far from $bd(T)$.

2.5 Reaching a Ground State. After performing our sweep line procedure, for some i we have P_1 occupying all of the first $i - 1$ columns, some of $\mathcal{C}_i$, and none of columns $i + 1$ or greater. At this point we describe a sequence of steps alternating recombining P_2 and P_3 with moving a vertex of P_1 higher in column i . The end result is a partition where the vertices of P_1 in $\mathcal{C}_i$ occupy all of the topmost positions in $\mathcal{C}_i$, as they must in the ground state σ_{123} . One final recombination step for P_2 and P_3 reaches the ground state σ_{123} . Because the ground states are all easily connected by recombination moves, this proves there exists a sequence of moves from any balanced partition to any other balanced partitions.

The proof outlined so far assumes we begin at a balanced partition. Some additional work is required to show any nearly balanced partition can be transformed into a balanced partition. Similar lemmas and approaches to the rebalancing step described above are used to do so, completing the proof.

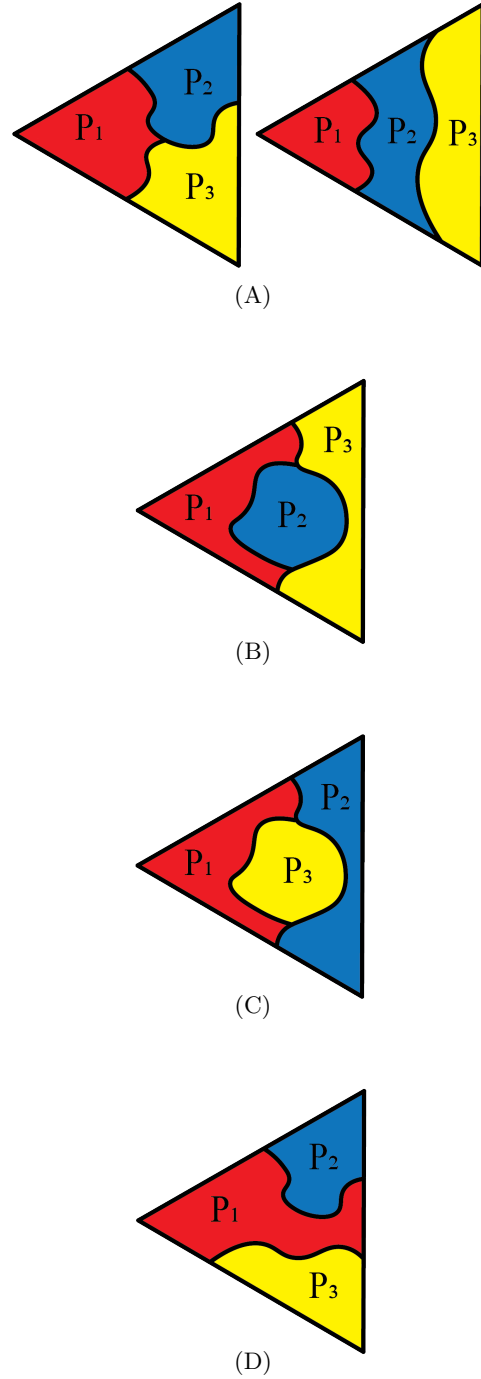


Figure 6: Cartoon representations of the relationship between P_1 (red), P_2 (blue), and P_3 (yellow) in the four rebalancing cases we consider: (A) There exists $a \in P_2 \cap bd(T)$ and $b \in P_3 \cap bd(T)$ that are adjacent; (B) $P_2 \cap bd(T) = \emptyset$; (C) $P_3 \cap bd(T) = \emptyset$; and (D) No vertex of P_2 is adjacent to any vertex of P_3 . There are two possibilities in Case (A), but we handle them simultaneously.

3 Conclusion and Next Steps

This paper gives the first proof of irreducibility for recombination Markov chains under tight district size constraints beyond small or trivial examples. Now that we know such results are possible, there are several next steps to consider. First, we believe there are ample opportunities to simplify and shorten the nearly 65-page proof included in the full version of this paper. The focus of this paper was getting a complete proof rather than getting the most concise, elegant proof, so improvements can likely be made, though the outline given above should give the reader some ideas of the inherent difficulties involved. Additionally, generalizing this proof so that it applies to flip Markov chains as well as recombination Markov chains seems an achievable next step; the main difficulty will be adapting the Cycle Recombination Lemma to obtain the same result but using only flip steps. More significant next steps include generalizing the proof to more than three districts or other subregions of the triangular lattice. We believe both are possible, but significant additional work will be required.

A major next step would be to prove a similar result beyond the triangular lattice. The class of bounded-degree Hamiltonian planar triangulations seems the most likely candidate for success. Restricting our attention to triangulations is helpful because the neighbors of any given vertex form a cycle, meaning we can easily understand when a vertex can be removed from one district and added to another. Triangulations are also relevant to real-world redistricting applications, as the dual graphs representing states or regions are frequently triangulated or nearly triangulated. Hamiltonicity makes the definition of a ground state straightforward, and was also used in some of the results of [1, 2], suggesting its usefulness. Several of the arguments included here would break down in the presence of large degree vertices, which is why we propose degree restrictions. However, there will be significant challenges in moving beyond the triangular lattice, as the assumption that the underlying graph is a regular lattice pervades nearly all of the proof.

In conclusion, this result is a major advancement that holds promise for inspiring future results. Knowing that recombination Markov chains are irreducible is a necessary first step to developing the theory behind them. For example, now that we know recombination Markov chains are irreducible on the triangular lattice, we know they will converge to a unique stationary distribution. This means we can begin to consider questions such as how long this convergence takes: a long-term goal of this research community is to be able to say something rigorous about the mixing and/or

relaxation times of recombination Markov chains, and knowing the chains are irreducible is a necessary first step. Gaining a rigorous understand of these Markov chains and their behavior is essential so that we can have confidence in the conclusions about gerrymandering they are used to produce.

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