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To N. K. Nikolski on the occasion of his 80th birthday

PRESERVATION OF ABSOLUTELY CONTINUOUS SPECTRUM FOR CONTRACTIVE OPERATORS

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Contractive operators T that are trace class perturbations of a unitary operator U are treated. It is proved that the dimension functions of the absolutely continuous spectrum of T, T^*, and of U coincide. In particular, if U has a purely singular spectrum, then the characteristic function θ of T is a *two-sided inner* function, i.e., $\theta(\xi)$ is unitary a.e. on $\mathbb{T}$. Some corollaries to this result are related to investigations of the asymptotic stability of the operators T and T^* (the convergence $T^n \to 0$ and $(T^*)^n \to 0$, respectively, in the strong operator topology).

The proof is based on an explicit computation of the characteristic function.

Notation

$\mathbb{D}$	The open unit disk in the complex plane $\mathbb{C}$, $\mathbb{D} := \{z \in \mathbb{C} : \lvert z\rvert < 1\}$.
$\mathbb{T}$	The unit circle in $\mathbb{C}$, $\mathbb{T} = \partial\mathbb{D}$.
$\mathfrak{m}$	The normalized ($\mathfrak{m}(\mathbb{T}) = 1$) Lebesgue measure on $\mathbb{T}$.
$\mathbf{I}_{\mathfrak{D}}$, $\mathbf{I}$	The identity operator; in most situations, where it is clear from the context we will skip the index, denoting the space where the operator acts.
$\mathfrak{S}_1$	The trace class.

Ключевые слова: Trace class perturbations, contractive operators, dimension function, absolutely continuous spectrum.

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$\mathfrak{S}_2$ The Hilbert–Schmidt class.

H^2 The Hardy space H^2; we will also use the symbol $H^2(E)$ for the vector-valued H^2 functions with values in a Hilbert space E.

$z \to \xi \sphericalangle$ $z \in \mathbb{D}$ approaches $\xi \in \mathbb{T}$ nontangentially; the aperture of the nontangential approach regions is assumed to be fixed (but it is not essential).

All Hilbert spaces in this paper are separable, and all operators act between Hilbert spaces (or on the same Hilbert space). By a measure we always mean a finite Borel measure on $\mathbb{T}$.

The term a.e. always means a.e. with respect to the Lebesgue measure on $\mathbb{T}$. For a.e. with respect to a different measure the term μ-a.e. is used.

§1. Introduction and main results

Recall that a unitary operator U on a separable Hilbert space is unitarily equivalent to multiplication by the independent variable ξ in the von Neumann direct integral of Hilbert spaces,

$$\mathcal{N} := \int_{\mathbb{T}} \oplus E(\xi) \mathrm{d}\mu(\xi). \tag{1.1}$$

The *dimension function*

$$N(\xi) = N_{\mathrm{U}}(\xi) := \dim E(\xi)$$

is a unitary invariant of the operator U: together with the *spectral type* $[\mu]$ of μ, which is the class of all measures mutually absolutely continuous with μ, they completely determine the operator U up to unitary equivalence. The function N_{U} is often called the *spectral multiplicity function*, and we will use this term.

For definiteness, we assume that $N(\xi) = 0$ whenever the Lebesgue density $w = \mathrm{d}\mu/\mathrm{d}\mathfrak{m}$ of μ vanishes (note that $\mu(\{\xi \in \mathbb{T} \colon w(\xi) = 0\}) = 0$).

The multiplicity of the absolutely continuous (a.c.) part of U is, by definition, the function N a.e. with respect to the Lebesgue measure on $\mathbb{T}$.

Now, we introduce the notion of spectral multiplicity (of the a.c. spectrum) for a contraction. Recall that any contraction T can be uniquely decomposed in the direct sum $T = V \oplus T_0$, where T_0 is a completely nonunitary (c.n.u.) contraction, and V is unitary (either of these terms can be 0). The spectral multiplicity of the a.c. spectrum of V is precisely the dimension function N_V considered a.e. with respect to Lebesgue measure.

As for the c.n.u. part T_0, the rank of the defect functions $\Delta(\xi)$ and $\Delta_*(\xi)$, $\xi \in \mathbb{T}$, see (1.2) below, is often interpreted as the dimension functions for the a.c. spectrum of a c.n.u. contraction; in this paper we use this interpretation.

Let us recall the main definitions. Recall that a completely nonunitary contraction T_0 is uniquely determined (up to unitary equivalence) by its characteristic function $\theta = \theta_{T_0}$, cf. [15], which is an analytic operator-valued function on the unit disk $\mathbb{D}$ whose values are strict contractions $\theta(z)\colon \mathfrak{D} \to \mathfrak{D}_*$; here $\mathfrak{D}$ and $\mathfrak{D}_*$ are some auxiliary Hilbert spaces.

The characteristic function is defined up to constant unitary factors (possibly between different spaces) on both sides, so each such equivalence class corresponds to a collection of unitarily equivalent c.n.u. contractions. We should also mention that for a general contraction $T = V \oplus T_0$, its characteristic function coincides with the characteristic function of its purely contractive part T_0.

Recall also that any bounded analytic function F with values in $B(\mathfrak{D};\mathfrak{D}_*)$ has nontangential boundary values in the strong operator topology a.e. on $\mathbb{T}$, and that $F(z)$, $z \in \mathbb{D}$ can be represented as the Poisson extensions of these boundary values. So for the characteristic function θ we denote its boundary values by $\theta(\xi)$, $\xi \in \mathbb{T}$, and we will treat θ as a function defined on $\mathbb{D}$ and a.e. on $\mathbb{T}$.

For a characteristic function θ, its *defect functions* are defined a.e. on $\mathbb{T}$ as

$$\Delta := (\mathbf{I} - \theta^*\theta)^{1/2}, \quad \Delta_* := (\mathbf{I} - \theta\theta^*)^{1/2}. \tag{1.2}$$

Theorem 1.1. *Let U be a unitary operator (on a separable Hilbert space), and let K be a trace class operator such that $T = U + K$ is a contraction. If $T = V \oplus T_0$ is the decomposition of T into unitary and completely nonunitary parts, and θ is the characteristic function of T, then*

$$\operatorname{rank}\Delta(\xi) = \operatorname{rank}\Delta_*(\xi), \tag{1.3}$$

$$N_U(\xi) = N_V(\xi) + \operatorname{rank}\Delta(\xi) \tag{1.4}$$

a.e. on $\mathbb{T}$.

We should mention that there is a large body of work studying the absolutely continuous spectrum in the case when the perturbed operator is not unitary/selfadjoint, see for example [8–10, 14, 17]. However, these papers were mostly concerned with the existence of the wave operators, and we are not sure if it is possible to easily get our result from there. In particular our result covers the case when the spectrum of the perturbed operator is the whole closed unit disk, and a typical assumption in results about wave operators is the “thinness” of the spectrum.

Even if we assume that the spectrum is not the whole unit disk (for example if the unitary operator has purely singular spectrum [11]), a rigorous translation from one language to the other would be not much simpler than our self-contained presentation; and we would need to use some highly nontrivial results from very technical papers.

Corollaries 1.2, 1.3, 1.4 below concern the asymptotic stability of the perturbed operator. Some of these results might be known to experts. Let, like in Theorem 1.1, $T = U + K$ be a contraction, let U be unitary, and let $K \in \mathfrak{S}_1$. Let also $T = V \oplus T_0$ be the decomposition of T into unitary and completely nonunitary parts, and let θ be the characteristic function of T.

Corollary 1.2. *If U has purely singular spectrum (i.e., if μ is purely singular), then θ is a two-sided inner function, meaning that $\theta(\xi)$ is a unitary operator a.e. on $\mathbb{T}$.*

Corollary 1.3. *If U has purely singular spectrum, then T_0 and T_0^* are asymptotically stable, meaning that $T_0^n \to 0$ and $(T_0^*)^n \to 0$ in the strong operator topology as $n \to \infty$.*

Corollary 1.4. *If T is asymptotically stable, i.e., if $T^n \to 0$ in the strong operator topology as $n \to \infty$, then U has purely singular spectrum.*

Note that in the corollaries above the spectrum of T does not fill the unit disk (see [11] for Corollaries 1.2, 1.3, and [16] for Corollary 1.4), so the results about wave operators can be used to prove the corollaries. However, as we mentioned above, such a proof relies on some highly technical nontrivial papers. Moreover, we expect that presented with all the details it will not be significantly shorter than our self-contained paper.

Remark 1.5. We should mention that under the assumptions of any of the above corollaries the operator T_0 belongs to the class of so-called C_0-contractions, meaning that there exists a function $\varphi \in H^\infty$ such that $\varphi(T_0) = 0$. The theory for this operator class is well developed, but not directly relevant for our paper, so, we will omit further discussion.

Our proof of the main result (Theorem 1.1) is slightly lengthy but mostly elementary: after some simple operator-theoretic arguments, we reduce everything to a particular case, see Lemma 2.1 below. We then express the characteristic function θ in terms of the Cauchy–Herglotz transform of some $\mathfrak{S}_1$-valued measure, see §3. The proof of the theorem is then obtained by analyzing the boundary values of θ, which is pretty straightforward, see §4.

We prove Corollaries 1.2 through 1.4 in Subsection 4.3.

§2. Some reductions

Recall that an operator T is called a *strict contraction* if $\|Tx\| < \|x\|$ for all $x \neq 0$; clearly in this case $\|T\| \leqslant 1$.

Lemma 2.1. *Let $T = U + K$, where U is unitary, $K \in \mathfrak{S}_1$, and $\|T\| \leqslant 1$ (all operators act on a Hilbert space $\mathcal{H}$). Then T can be represented as*

$$T = U_1 + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^* U_1, \tag{2.1}$$

where U_1 is unitary, $U - U_1 \in \mathfrak{S}_1$, $\mathbf{B}\colon \mathfrak{D} \to \mathcal{H}$ is an isometry from an auxiliary Hilbert space $\mathfrak{D}$, and $\Gamma = \Gamma^* \geqslant \mathbf{0}$ is a strict contraction such that $\mathbf{I} - \Gamma \in \mathfrak{S}_1$.

In the proof of the above Lemma 2.1 we will use the following trivial fact.

Lemma 2.2. *Let $\|T\| \leqslant 1$, and let $\|Tx\| = \|x\| \neq 0$. Then for any $y \perp x$ we have $Ty \perp Tx$.*

We leave the proof of this lemma as an exercise for the reader.

We will need one more simple lemma.

Lemma 2.3. *Let $R = \mathbf{I} + K$, where $K \in \mathfrak{S}_1$. Then one can write a polar decomposition $R = V|R|$, where $|R| := (R^*R)^{1/2}$ and V is unitary, such that $|R| - \mathbf{I} \in \mathfrak{S}_1$ and $V - \mathbf{I} \in \mathfrak{S}_1$.*

Remark. The term $|R|$ in the polar decomposition is uniquely determined. The unitary operator V is uniquely determined if and only if $\ker R = \{0\}$ (since R is a Fredholm operator of index 0, this happens if and only if $\operatorname{Ran} R$ is the whole space). Formally the above Lemma 2.3 means that for some choice of the unitary operator V we have $\mathbf{I} - V \in \mathfrak{S}_1$; while this is not essential for the proof, one can see from the proof that in fact, *all* possible choices of V satisfy $\mathbf{I} - V \in \mathfrak{S}_1$.

Proof of Lemma 2.3. First, we consider the case when R is invertible (which happens if and only if $\ker R = \{0\}$). In this case, $|R|$ is trivially invertible and V is unique and is defined by $V = R|R|^{-1}$.

We know that $R \in \mathbf{I} + \mathfrak{S}_1$, so trivially $|R|^2 = R^*R \in \mathbf{I} + \mathfrak{S}_1$, and so $|R| \in \mathbf{I} + \mathfrak{S}_1$. Since $|R|$ is invertible, it is easy to see that $|R|^{-1} \in \mathbf{I} + \mathfrak{S}_1$, and therefore $V = R|R|^{-1} \in \mathbf{I} + \mathfrak{S}_1$.

Now, we consider the general case. Since for a compact K the operator $\mathbf{I}+K$ is Fredholm of index 0, the range of R is closed, and $\dim\ker R = \dim\ker R^* < \infty$ (and $\operatorname{Ran} R = (\ker R^*)^\perp$). Take any invertible operator $R_1\colon \ker R \to \ker R^*$ (such an operator exists and has finite rank, because $\dim\ker R = \dim\ker R^* < \infty$). Define $\widetilde{R} := R + R_1$. By the construction, $\widetilde{R}$ is invertible, and maps $(\ker R)^\perp$ onto $\operatorname{Ran} R = (\ker R^*)^\perp$ and $\ker R$ onto $\ker R^*$.

Note also that $\widetilde{R} - \mathbf{I} \in \mathfrak{S}_1$.

If we denote by R_0 the restriction of R to $(\ker R)^\perp$ (with target space restricted to $\operatorname{Ran} R = (\ker R^*)^\perp$), we can see that $|\widetilde{R}|$ in the decomposition

$(\ker R)^\perp \oplus \ker R$ has the block diagonal form

$$|\widetilde{R}| = \begin{pmatrix} |R_0| & 0 \\ 0 & |R_1| \end{pmatrix}. \tag{2.2}$$

Consider the polar decomposition $\widetilde{R} = V|\widetilde{R}|$; since $\widetilde{R}$ is invertible, V is uniquely defined by $V = \widetilde{R}|\widetilde{R}|^{-1}$. As we discussed above in the beginning of the proof, since $\widetilde{R}$ is invertible, we have $V \in \mathbf{I} + \mathfrak{S}_1$. We also know that $|\widetilde{R}| \in \mathbf{I} + \mathfrak{S}_1$, and so $|R| \in \mathbf{I} + \mathfrak{S}_1$, because $|R|$ differs from $|\widetilde{R}|$ by a finite rank block $|R_1|$, see (2.2) above.

The fact that $\widetilde{R}$ maps $(\ker R)^\perp$ onto $\operatorname{Ran} R = (\ker R^*)^\perp$ and $\ker R$ onto $\ker R^*$ and the block diagonal structure (2.2) imply that V also maps $(\ker R)^\perp$ onto $\operatorname{Ran} R = (\ker R^*)^\perp$ and $\ker R$ onto $\ker R^*$. Therefore $R = V|R|$, so we have constructed the desired polar decomposition. □

Proof of Lemma 2.1. We will prove a “dual” formula to (2.1), namely the formula

$$T = U_1 + U_1\mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^*; \tag{2.3}$$

applying this formula to the adjoint $T^* = U^* + K^*$ and then taking the adjoint we will get (2.1).

The identity $U + K = U(\mathbf{I} + U^*K)$ means that it is sufficient to prove (2.3) for the particular case where $U = \mathbf{I}$.

So, let $T = \mathbf{I} + K$ with $\|T\| \leqslant 1$, and let $K \in \mathfrak{S}_1$. Denote $\mathfrak{D}_1 := (\ker K)^\perp$. Clearly

$$(\mathbf{I} + K)x = x \quad \forall x \in \mathfrak{D}_1^\perp,$$

and therefore by Lemma 2.2

$$(\mathbf{I} + K)\mathfrak{D}_1 \subset \mathfrak{D}_1.$$

Then $K\mathfrak{D}_1 \subset \mathfrak{D}_1$, and we can treat K as an operator on $\mathfrak{D}_1$.

So, we restrict our attention to $\mathfrak{D}_1$. Denote

$$R = (\mathbf{I} + K)\Big|_{\mathfrak{D}_1}.$$

By Lemma 2.3 we can write a polar decomposition $R = V|R|$ of R with unitary V such that

$$|R| - \mathbf{I}_{\mathfrak{D}_1} \in \mathfrak{S}_1, \quad V - \mathbf{I}_{\mathfrak{D}_1} \in \mathfrak{S}_1.$$

Denote $\mathfrak{D}_2 := \mathfrak{D}_1 \ominus \ker(|R| - \mathbf{I}_{\mathfrak{D}_1})$. Then trivially, $\Gamma = |R|\Big|_{\mathfrak{D}_2}$ is a strict contraction on $\mathfrak{D}_2$, $\Gamma = \Gamma^*$, and $\Gamma - \mathbf{I}_{\mathfrak{D}_2} \in \mathfrak{S}_1$.

Now gathering everything together we see that

$$\mathbf{I}_{\mathcal{H}} + K = U_1 + U_1 P_{\mathfrak{D}_2}(\Gamma - \mathbf{I}_{\mathfrak{D}_2}) P_{\mathfrak{D}_2},$$

where

$$U_1 x = \begin{cases} Vx & x \in \mathfrak{D}_1, \\ x & x \in \mathcal{H} \setminus \mathfrak{D}_1. \end{cases}$$

Note that $\mathfrak{D}_1$ is a reducing subspace for U_1, and so the condition $V - \mathbf{I}_{\mathfrak{D}_1} \in \mathfrak{S}_1$ implies that $U_1 - \mathbf{I}_{\mathcal{H}} \in \mathfrak{S}_1$. Thus we have proved formula (2.3) for the case of $U = \mathbf{I}$ (with $\mathfrak{D} = \mathfrak{D}_2$ and $\mathbf{B}$ being the embedding of $\mathfrak{D}_2$ into $\mathcal{H}$).

If $\mathfrak{D}$ is an abstract space, $\dim \mathfrak{D} = \dim \mathfrak{D}_2$, then taking an isometry

$$\mathbf{B}\colon \mathfrak{D} \to \mathcal{H}, \quad \operatorname{Ran} \mathbf{B} = \mathfrak{D}_2,$$

we can rewrite the above identity as

$$\mathbf{I}_{\mathcal{H}} + K = U_1 + U_1 \mathbf{B}(\mathbf{B}^* \Gamma \mathbf{B} - \mathbf{I}_{\mathfrak{D}}) \mathbf{B}^*$$

so the general "abstract" form of (2.3) is proved for $U = I$.

As we discussed in the beginning of the proof, this proves (2.3) for the general case, and so formula (2.1). Lemma 2.1 is proved. □

§3. Characteristic functions

3.1. Operator-valued spectral measures and spectral representation. An operator-valued measure $\boldsymbol{\mu}$ on $\mathbb{T}$ is a countably additive function defined on Borel subsets of $\mathbb{T}$ with values in the set of nonnegative selfadjoint operators. This definition means that an operator-valued measure is always *finite*, i.e., that $\boldsymbol{\mu}(\mathbb{T})$ is a bounded operator.

Let U be a unitary operator on $\mathcal{H}$, let an operator $\mathbf{B}\colon \mathfrak{D} \to \mathcal{H}$ have trivial kernel, and let $\operatorname{Ran} \mathbf{B}$ be star-cyclic for U. Define the operator-valued spectral measure $\boldsymbol{\mu} = \boldsymbol{\mu}_U$ (with values in $B(\mathfrak{D})$) as

$$\boldsymbol{\mu}(E) = \mathbf{B}^* \mathcal{E}(E) \mathbf{B}, \tag{3.1}$$

for any Borel $E \subset \mathbb{T}$; here $\mathcal{E} = \mathcal{E}_U$ is the (projection-valued) spectral measure of U. An equivalent definition is that $\boldsymbol{\mu}$ is unique operator-valued measure such that

$$\mathbf{B}^* U^n \mathbf{B} = \int_{\mathbb{T}} \xi^n \mathrm{d}\boldsymbol{\mu}(\xi) \quad \forall n \in \mathbb{Z},$$

or equivalently,

$$\mathbf{B}^* (\mathbf{I} - zU^*)^{-1} \mathbf{B} = \int_{\mathbb{T}} \frac{1}{1 - z\overline{\xi}} \mathrm{d}\boldsymbol{\mu}(\xi) =: \mathcal{C}\boldsymbol{\mu}(z) \quad \forall z \in \mathbb{C} \setminus \mathbb{T}. \tag{3.2}$$

The operator U is unitarily equivalent to the multiplication operator M_ξ by the independent variable ξ in the weighted space $L^2(\boldsymbol{\mu})$.

We recall that the weighted space $L^2(\boldsymbol{\mu})$ with the operator-valued measure $\boldsymbol{\mu}$ is defined as follows. First the inner product in $L^2(\boldsymbol{\mu})$ is introduced on functions of the form $f = \varphi\mathbf{x}$, where φ is a scalar-valued measurable function and $\mathbf{x} \in \mathfrak{D}$:

$$(\varphi\mathbf{x}, \psi\mathbf{y})_{L^2(\boldsymbol{\mu})} := \int_{\mathbb{T}} \varphi(\xi)\overline{\psi(\xi)}\,(\mathrm{d}\boldsymbol{\mu}(\xi)\mathbf{x}, \mathbf{y})_{\mathfrak{D}}\,.$$

This inner product is then extended by linearity to the set of all (finite) linear combinations of such functions. Such linear combinations (of course, modulo the class of functions of norm 0) form an inner product space, and its completion is, by definition, the weighted space $L^2(\boldsymbol{\mu})$.

The unitary operator $\mathcal{V}\colon \mathcal{H} \to L^2(\boldsymbol{\mu})$ such that $M_\xi = \mathcal{V}U\mathcal{V}^*$ is also well known. Namely, for $x \in \mathfrak{D}$,

$$\mathcal{V}[\varphi(U)\mathbf{B}x] = \varphi(\,\cdot\,)x \in L^2(\boldsymbol{\mu}).$$

3.2. Trace class operator-valued measures, spectral representation and the spectral multiplicity function. The representation of a unitary operator as a multiplication operator in the weighted space $L^2(\boldsymbol{\mu})$ with operator-valued measure looks like "abstract nonsense", and the model looks more complicated than the original object. However, when the measure $\boldsymbol{\mu}$ takes values in the set $\mathfrak{S}_1$ of trace class operators, all objects are significantly simplified.

If $\boldsymbol{\mu}$ takes values in the set $\mathfrak{S}_1$ of trace class operators, we can define the scalar-valued measure μ as $\mu := \operatorname{tr}\boldsymbol{\mu}$. In this case the operator-valued measure $\boldsymbol{\mu}$ can be represented as

$$\mathrm{d}\boldsymbol{\mu} = W\mathrm{d}\mu,$$

where $\|W(\xi)\| \leqslant \|W(\xi)\|_{\mathfrak{S}_1} = 1$ μ-a.e. on $\mathbb{T}$.

It is not hard to see that in this case the measure $\mu = \operatorname{tr}\boldsymbol{\mu}$ is a scalar spectral measure of the operator U that can be used in the von Neumann direct integral (1.1). The inner product in the weighted space $L^2(\boldsymbol{\mu})$ can be computed (for measurable functions f and g) as

$$(f, g)_{L^2(\boldsymbol{\mu})} = \int_{\mathbb{T}} \big(W(\xi)f(\xi), g(\xi)\big)_{\mathfrak{D}}\,\mathrm{d}\mu(\xi).$$

The weighted space $L^2(\boldsymbol{\mu})$ in this case consists of all measurable functions for which $\|f\|_{L^2(\boldsymbol{\mu})} < \infty$ (the obvious quotient space over the set of functions of norm 0 should be taken).

It is also not hard to see that in this case the dimension function $N_U(\xi)$ can be computed as

$$N_U(\xi) = \operatorname{rank} W(\xi), \quad \mu\text{-a.e.}$$

(recall that we assume that $\operatorname{Ran} \mathbf{B}$ is star-cyclic for U).

We have presented precisely the fact we will need; an interested reader can find more details in [6].

3.3. Characteristic function via the Cauchy–Herglotz integral of a trace class measure. Define the Cauchy transforms,

$$\mathcal{C}\boldsymbol{\mu}(z) := \int_{\mathbb{T}} \frac{\mathrm{d}\boldsymbol{\mu}(\xi)}{1 - z\overline{\xi}}, \quad \mathcal{C}_1\boldsymbol{\mu}(z) := \int_{\mathbb{T}} \frac{z\overline{\xi}}{1 - z\overline{\xi}} \mathrm{d}\boldsymbol{\mu}(\xi), \quad \mathcal{C}_2\boldsymbol{\mu}(z) := \int_{\mathbb{T}} \frac{1 + z\overline{\xi}}{1 - z\overline{\xi}} \mathrm{d}\boldsymbol{\mu}(\xi).$$

In [7] we obtained formulas (3.4), (3.5) below for the characteristic function $\theta = \theta_\Gamma$ of the operator

$$T_\Gamma = U + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^* U, \tag{3.3}$$

where $\mathbf{B}$ is an isometry acting from $\mathfrak{D} \to \mathcal{H}$ and Γ (and therefore Γ^*) is a strict contraction.

Recall that for a contraction Γ the *defect operator* D_Γ is defined by $D_\Gamma := (\mathbf{I} - \Gamma^*\Gamma)^{1/2}$.

The characteristic function $\theta = \theta_\Gamma$ of T_Γ was proved to be given by

$$\theta_\Gamma(z) = -\Gamma + D_{\Gamma^*} F_1(z)\Big(\mathbf{I} - (\Gamma^* - \mathbf{I})F_1(z)\Big)^{-1} D_\Gamma \tag{3.4}$$

$$= -\Gamma + D_{\Gamma^*}\Big(\mathbf{I} - F_1(z)(\Gamma^* - \mathbf{I})\Big)^{-1} F_1(z) D_\Gamma, \tag{3.5}$$

where $F_1(z) = \mathcal{C}_1\boldsymbol{\mu}(z)$. Here the measure $\boldsymbol{\mu}$ was given by (3.1), or equivalently by (3.2).

This formula was proved in [7] for the case of finite rank perturbations. However, the only place where the finite rank was used in the proof was in the definition of the measure $\boldsymbol{\mu}$, which was in that case expressed explicitly via the scalar spectral measure in the von Neumann direct integral (1.1) and the matrix of the operator $\mathbf{B}$. Such an explicit expression is not possible in the general case, but what one really needs for the proof of the formula, is the identity

$$z\mathbf{B}^*(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}U^*\mathbf{B} = \mathcal{C}_1\boldsymbol{\mu}(z) =: F_1(z). \tag{3.6}$$

For convenience, we include the proof of (3.4), (3.5), and (3.6) in §5.

Moving forward, we would like to express the characteristic function θ in terms of the Cauchy–Herglotz integral $\mathcal{C}_1\widetilde{\boldsymbol{\mu}}$ of some $\mathfrak{S}_1$-valued measure $\widetilde{\boldsymbol{\mu}}$: this will allow us to express the defect functions $\Delta := (\mathbf{I} - \theta^*\theta)^{1/2}$ and $\Delta_* := (\mathbf{I} - \theta\theta^*)^{1/2}$.

First, we express θ in terms of $F_2 := \mathcal{C}_2\boldsymbol{\mu}$. Using the fact that $\Gamma = \Gamma^*$, we can rewrite (3.4) as

$$\begin{aligned}\theta_\Gamma &= D_\Gamma\left(-D_\Gamma^{-1}\Gamma D_\Gamma^{-1} + F_1\left(\mathbf{I} - (\Gamma - \mathbf{I})F_1\right)^{-1}\right)D_\Gamma \\ &= D_\Gamma^{-1}\left(-\Gamma + \Gamma(\Gamma - \mathbf{I})F_1 + D_\Gamma^2 F_1\right)\left(\mathbf{I} - (\Gamma - \mathbf{I})F_1\right)^{-1}D_\Gamma;\end{aligned}$$

note that while the operator D_Γ^{-1} is unbounded, it is densely defined, and the above identity can be understood as an identity for bilinear forms on a dense linear submanifold $\mathfrak{D} \times \operatorname{Ran} D_\Gamma \subset \mathfrak{D} \times \mathfrak{D}$.

Since $F_1 = (F_2 - \mathbf{I})/2$, we can continue

$$\begin{aligned}\theta_\Gamma &= D_\Gamma^{-1}\Big(-2\Gamma + (\mathbf{I} - \Gamma)(F_2 - \mathbf{I})\Big)\Big(2\mathbf{I} - (\Gamma - \mathbf{I})(F_2 - \mathbf{I})\Big)^{-1}D_\Gamma \\ &= D_\Gamma^{-1}\Big(-(\mathbf{I} + \Gamma) + (\mathbf{I} - \Gamma)F_2\Big)(\mathbf{I} - \Gamma)D_\Gamma^{-1} \\ &\qquad\qquad \Big[D_\Gamma^{-1}\Big((\mathbf{I} + \Gamma) - (\Gamma - \mathbf{I})F_2\Big)(\mathbf{I} - \Gamma)D_\Gamma^{-1}\Big]^{-1};\end{aligned}$$

note that for a strict contraction $\Gamma = \Gamma^* \geqslant \mathbf{0}$ the operator $(\mathbf{I} - \Gamma)D_\Gamma^{-1} = (\mathbf{I} - \Gamma)^{1/2}(\mathbf{I} + \Gamma)^{-1/2}$ is bounded, so the above expression is again well defined.

Using the identity

$$D_\Gamma^{-1}(\mathbf{I} - \Gamma)(\mathbf{I} + \Gamma)D_\Gamma^{-1} = \mathbf{I},$$

we can rewrite θ_Γ as

$$\theta_\Gamma = (\boldsymbol{\beta}F_2\boldsymbol{\beta} - \mathbf{I})(\boldsymbol{\beta}F_2\boldsymbol{\beta} + \mathbf{I})^{-1},$$

where

$$\boldsymbol{\beta} = \boldsymbol{\beta}^* := D_\Gamma^{-1}(\mathbf{I} - \Gamma).$$

Define the measure $\widetilde{\boldsymbol{\mu}} := \boldsymbol{\beta}\boldsymbol{\mu}\boldsymbol{\beta}$. Then, trivially, $\mathcal{C}_2\widetilde{\boldsymbol{\mu}} = \boldsymbol{\beta}\mathcal{C}_2\boldsymbol{\mu}\boldsymbol{\beta} = \boldsymbol{\beta}F_2\boldsymbol{\beta}$, so

$$\theta = \frac{\mathcal{C}_2\widetilde{\boldsymbol{\mu}} - \mathbf{I}}{\mathcal{C}_2\widetilde{\boldsymbol{\mu}} + \mathbf{I}} \tag{3.7}$$

(we write it as a fraction to emphasize that the terms commute). Note that for $z \in \mathbb{D}$ we have $\operatorname{Re}(\mathcal{C}_2\widetilde{\boldsymbol{\mu}}(z) + \mathbf{I}) \geqslant \mathbf{I}$, so the operator $\mathcal{C}_2\widetilde{\boldsymbol{\mu}}(z) + \mathbf{I}$ is invertible and the right-hand side of (3.7) is well defined.

Finally, under our assumptions that $\Gamma = \Gamma^*$ is a strict contraction and $\mathbf{I} - \Gamma \in \mathfrak{S}_1$, the formula

$$\boldsymbol{\beta} = D_\Gamma^{-1}(\mathbf{I} - \Gamma) = (\mathbf{I} - \Gamma)^{1/2}(\mathbf{I} + \Gamma)^{-1/2}$$

shows that $\boldsymbol{\beta} \in \mathfrak{S}_2$ (the Hilbert–Schmidt class). Therefore, the measure $\widetilde{\boldsymbol{\mu}}$ is $\mathfrak{S}_1$-valued.

§4. Proof of the main results

4.1. Proof of Theorem 1.1: the principal case. In this subsection we prove Theorem 1.1 for the main special case when

$$T = U + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^* U,$$

where $\operatorname{Ran}\mathbf{B}$ is star-cyclic for U and $\Gamma = \Gamma^* \geqslant \mathbf{0}$ is a strict contraction, $\mathbf{I} - \Gamma \in \mathfrak{S}_1$. The general case can easily be obtained from this by using Lemma 2.1, see Subsection 4.2 below.

4.1.1. *Some technical lemmas.* The function θ is defined in the open unit disk $\mathbb{D}$. Since it is a bounded analytic operator-valued function, it possesses nontangential boundary values

$$\theta(\xi) := \lim_{z \to \xi \sphericalangle} \theta(z), \quad \xi \in \mathbb{T}$$

(in the strong operator topology) a.e. on $\mathbb{T}$.

Lemma 4.1. *The function $\mathbf{I} - \theta$ is invertible a.e. on $\mathbb{T}$.*

Proof. One can see from (3.7) that

$$(\mathbf{I} - \theta(z))^{-1} = (\mathcal{C}_2 \widetilde{\boldsymbol{\mu}}(z) + \mathbf{I})/2 = \mathcal{C}\widetilde{\boldsymbol{\mu}}(z) \tag{4.1}$$

for $z \in \mathbb{D}$. As we discussed at the end of §3, the measure $\widetilde{\boldsymbol{\mu}}$ is $\mathfrak{S}_1$-valued, so it can be represented as

$$\mathrm{d}\widetilde{\boldsymbol{\mu}} = W \mathrm{d}\mu,$$

where the scalar measure μ is given by $\mu = \operatorname{tr}\widetilde{\boldsymbol{\mu}}$. In this case

$$\|W(\xi)\|_{\mathfrak{S}_2} \leqslant \|W(\xi)\|_{\mathfrak{S}_1} = 1 \quad \mu\text{-a.e. on } \mathbb{T}.$$

The space $\mathfrak{S}_2$ is a Hilbert space, so by Lemma 4.2 below the nontangential boundary values of $\mathcal{C}\widetilde{\boldsymbol{\mu}}$ exist a.e. on $\mathbb{T}$. Note that for our purposes it is sufficient that the boundary values exist in the strong operator topology, while Lemma 4.2 states that the boundary values exist in the (much stronger) topology of $\mathfrak{S}_2$.

Formula (4.1) means that for all $z \in \mathbb{D}$ we have

$$\mathcal{C}\widetilde{\boldsymbol{\mu}}(z)(\mathbf{I} - \theta(z)) = (\mathbf{I} - \theta(z))\mathcal{C}\widetilde{\boldsymbol{\mu}}(z) = \mathbf{I},$$

and taking the nontangential boundary values we conclude that the same identities are fulfilled a.e. on $\mathbb{T}$. But this exactly means that $\mathbf{I} - \theta$ is invertible a.e. on $\mathbb{T}$. □

Lemma 4.2. *Let μ be a (finite) Borel measure on $\mathbb{T}$, and let $f \in L^2(\mu; E)$ (where E is a Hilbert space). Then the nontangential boundary values*

$$[\mathcal{C} f\mu](\xi) = \lim_{z\to\xi\sphericalangle} [\mathcal{C} f\mu](z), \quad \xi \in \mathbb{T}$$

(in the norm topology of E) exist a.e. on $\mathbb{T}$.

Proof. We use the well-known result ([1, Theorem 1.1], see also [3, Proposition 10.2.3]) that for a measure μ the operator $\mathcal{V}_\mu$,

$$\mathcal{V}_\mu f(z) := \frac{[\mathcal{C} f\mu](z)}{[\mathcal{C}\mu](z)}, \quad f \in L^2(\mu),$$

is a bounded operator from $L^2(\mu)$ to the Hardy space H^2, $\|\mathcal{V}_\mu\|_{L^2(\mu)\to H^2} \leqslant C(\mu)$.[1]

The operator $\mathcal{V}_\mu$ is defined on scalar-valued functions, but the same formula defines an operator on the vector-valued space $L^2(\mu; E)$. Take $f \in L^2(\mu; E)$. Applying the scalar estimate to each coordinate of f, we conclude that

$$\|\mathcal{V}_\mu f\|_{H^2(E)} \leqslant C(\mu)\|f\|_{L^2(\mu;E)}, \quad f \in L^2(\mu; E).$$

It is well known that for $g \in H^2(E)$ the nontangential boundary values (in the norm topology of E) exist a.e. on $\mathbb{T}$. It is also well known that (finite and nonzero) nontangential boundary values of $\mathcal{C}\mu$ exist a.e. on $\mathbb{T}$. Since for $f \in L^2(\mu; E)$ we have

$$[\mathcal{C} f\mu](z) = \mathcal{V}_\mu f(z) \cdot [\mathcal{C}\mu](z),$$

we immediately get the conclusion of the lemma. □

4.1.2. *Computing the defect functions.* Recall that the spectral measure $\widetilde{\boldsymbol{\mu}}$ is represented as $\mathrm{d}\widetilde{\boldsymbol{\mu}} = W \mathrm{d}\mu$, where $\mu = \operatorname{tr} \widetilde{\boldsymbol{\mu}}$. Denote by w the Lebesgue density of μ (i.e., of its absolutely continuous part), $w := \mathrm{d}\mu/\mathrm{d}\mathfrak{m}$.

Proposition 4.3. *The defect functions Δ and Δ_* can be computed as*

$$\Delta(\xi)^2 = (\mathbf{I} - \theta(\xi)^*)W(\xi)w(\xi)(\mathbf{I} - \theta(\xi)), \tag{4.2}$$

$$\Delta_*(\xi)^2 = (\mathbf{I} - \theta(\xi))W(\xi)w(\xi)(\mathbf{I} - \theta(\xi)^*) \tag{4.3}$$

a.e. on $\mathbb{T}$.

[1] In fact, it is well known and not hard to show that for a probability measure μ the operator $\mathcal{V}_\mu$ is a contraction. Simple scaling then allows one to get the estimate $\|\mathcal{V}_\mu\|_{L^2(\mu)\to H^2} \leqslant \mu(\mathbb{T})^{-1/2}$.

Proof. Let $\mathcal{P}\widetilde{\boldsymbol{\mu}}$ be the Poisson extension of the measure $\widetilde{\boldsymbol{\mu}}$. Trivially,

$$\mathcal{P}\widetilde{\boldsymbol{\mu}} = \operatorname{Re}\widetilde{F}_2,$$

where $\widetilde{F}_2 = \mathcal{C}_2\widetilde{\boldsymbol{\mu}}$. The representation $\mathrm{d}\widetilde{\boldsymbol{\mu}} = W\mathrm{d}\mu$ implies that the nontangential boundary values of $\mathcal{P}\widetilde{\boldsymbol{\mu}}$ exist and coincide with Ww a.e. on $\mathbb{T}$; the nontangential boundary values exist a.e. in the $\mathfrak{S}_2$ norm, cf. Lemma 4.2 above, but for our purposes taking limits in the strong operator topology will suffice.

So, we have

$$\mathcal{P}\widetilde{\boldsymbol{\mu}} = \operatorname{Re}\widetilde{F}_2 = \operatorname{Re}[(\mathbf{I}+\theta)(\mathbf{I}-\theta)^{-1}].$$

Computing we get (for $z\in\mathbb{D}$)

$$\begin{aligned}\mathcal{P}\widetilde{\boldsymbol{\mu}} &= \operatorname{Re}[(\mathbf{I}+\theta)(\mathbf{I}-\theta)^{-1}] = \frac{1}{2}[(\mathbf{I}+\theta)(\mathbf{I}-\theta)^{-1} + (\mathbf{I}-\theta^*)^{-1}(\mathbf{I}+\theta^*)]\\ &= \frac{1}{2}(\mathbf{I}-\theta^*)^{-1}\left[(\mathbf{I}-\theta^*)(\mathbf{I}+\theta) + (\mathbf{I}+\theta^*)(\mathbf{I}-\theta)\right](\mathbf{I}-\theta)^{-1}\\ &= \frac{1}{2}(\mathbf{I}-\theta^*)^{-1}[2\mathbf{I}-2\theta^*\theta](\mathbf{I}-\theta)^{-1}\\ &= (\mathbf{I}-\theta^*)^{-1}[\mathbf{I}-\theta^*\theta](\mathbf{I}-\theta)^{-1}.\end{aligned}$$

Taking the nontangential boundary values, we obtain

$$Ww = (\mathbf{I}-\theta^*)^{-1}[\mathbf{I}-\theta^*\theta](\mathbf{I}-\theta)^{-1} = (\mathbf{I}-\theta^*)^{-1}\Delta^2(\mathbf{I}-\theta)^{-1}$$

a.e. on $\mathbb{T}$. Since the function $\mathbf{I}-\theta$ is invertible a.e. on $\mathbb{T}$ by Lemma 4.1, this identity is equivalent to (4.2).

To get (4.3), we simply need to repeat the above calculation with the order of θ and θ^* interchanged, namely

$$\begin{aligned}\mathcal{P}\widetilde{\boldsymbol{\mu}} &= \operatorname{Re}[(\mathbf{I}+\theta)(\mathbf{I}-\theta)^{-1}] = \frac{1}{2}[(\mathbf{I}-\theta)^{-1}(\mathbf{I}+\theta) + (\mathbf{I}+\theta^*)(\mathbf{I}-\theta^*)^{-1}]\\ &= \frac{1}{2}(\mathbf{I}-\theta)^{-1}\left[(\mathbf{I}+\theta)(\mathbf{I}-\theta^*) + (\mathbf{I}-\theta)(\mathbf{I}+\theta^*)\right](\mathbf{I}-\theta^*)^{-1}\\ &= \frac{1}{2}(\mathbf{I}-\theta)^{-1}[2\mathbf{I}-2\theta\theta^*](\mathbf{I}-\theta^*)^{-1}\\ &= (\mathbf{I}-\theta)^{-1}[\mathbf{I}-\theta\theta^*](\mathbf{I}-\theta^*)^{-1}.\end{aligned}$$

Taking boundary values again, we get

$$Ww = (\mathbf{I}-\theta)^{-1}\Delta_*^2(\mathbf{I}-\theta^*)^{-1}$$

which is equivalent to (4.3) because $\mathbf{I}-\theta$ is invertible a.e. on $\mathbb{T}$. ☐

4.1.3. *Completion of the proof of the principal case.* As we discussed above in Subsection 3.2, the dimension function N can be computed as $N(\xi) = \operatorname{rank} W(\xi)$ μ-a.e. By Lemma 4.1 the function $\mathbf{I} - \theta$ is invertible a.e. on $\mathbb{T}$, so the conclusion of Theorem 1.1 (in the case we are considering) immediately follows from identities (4.2), (4.3). $\square$

4.2. Proof of Theorem 1.1: the general case. According to Lemma 2.1, the operator T can be represented as

$$T = U_1 + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^* U_1$$

where, as in Subsection 4.1, we have $U - U_1 \in \mathfrak{S}_1$, $\Gamma = \Gamma^*$ is a strict contraction, and $\mathbf{I} - \Gamma \in \mathfrak{S}_1$. Note that $\operatorname{Ran} \mathbf{B}$ is not necessarily star-cyclic for U. Denote

$$\mathcal{H}_0 := \overline{\operatorname{span}}\{U_1^n \operatorname{Ran} \mathbf{B} : n \in \mathbb{Z}\}, \quad \mathcal{H}_1 := \mathcal{H}_0^\perp.$$

The subspaces $\mathcal{H}_0$, $\mathcal{H}_1$ are reducing for both U_1 and T; moreover $T|_{\mathcal{H}_1} = U_1|_{\mathcal{H}_1}$ is trivially unitary and $T|_{\mathcal{H}_0}$ is a completely nonunitary contraction on $\mathcal{H}_0$, see [7, Lemma 1.4].

Denote $V := U_1|_{\mathcal{H}_1}$, $U_0 := U_1|_{\mathcal{H}_0}$, $T_0 := T|_{\mathcal{H}_0}$ (with the target space also restricted to the spaces $\mathcal{H}_0$, $\mathcal{H}_1$ respectively). Clearly

$$T_0 = U_0 + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^* U_0$$

and $\operatorname{Ran} \mathbf{B}$ is star-cyclic for U_0. Recall that, as we discussed in §1, the characteristic function of the contraction T coincides with the characteristic function of its completely nonunitary part T_0. Therefore, by the discussion in Subsection 4.1,

$$\operatorname{rank} \Delta(\xi) = \operatorname{rank} \Delta_*(\xi) = N_{U_0}(\xi) \quad \text{a.e. on } \mathbb{T},$$

which gives us (1.3). Adding $N_V(\xi)$ to both parts, and noticing that

$$N_V(\xi) + N_{U_0}(\xi) = N_{U_1}(\xi),$$

we get

$$\operatorname{rank} V(\xi) + \operatorname{rank} \Delta(\xi) = N_{U_1}(\xi) \quad \text{a.e. on } \mathbb{T}.$$

The above formula is exactly identity (1.4) with $N_{U_1}(\xi)$ instead of $N_U(\xi)$. But $U - U_1 \in \mathfrak{S}_1$, so by the classical Kato–Rosenblum theorem (or, more precisely Birman–Krein theorem[2] [2]) the identity $N_U(\xi) = N_{U_1}(\xi)$ holds true a.e. on $\mathbb{T}$, so (1.4) is fulfilled. Thus Theorem 1.1 is proved in full generality. $\square$

[2]The statement that we are using, about the preservation of the absolutely continuous parts of a unitary trace class perturbations of unitary operators, first appeared in [2], which should be a proper reference. It also can be obtained via linear fractional transformation from an appropriate version (difference of resolvents is of trace class) of the Kato–Rosenblum theorem for selfadjoint operators due to S. T. Kuroda [4, 5].

4.3. Proof of the corollaries. As in Theorem 1.1, let U be a unitary operator, let $K \in \mathfrak{S}_1$, and let $T = U + K$. Further let θ be its characteristic function and let $T = V \oplus T_0$ be the decomposition of T into unitary and c.n.u. parts.

Recall that a bounded analytic operator-valued function θ on the unit disk is called *inner* if its boundary values $\theta(\xi)$ are isometries a.e. on $\mathbb{T}$. The function θ is called $*$-inner (or co-inner) if the function $z \mapsto \theta(\overline{z})^*$ is inner, which means that the operators $\theta(\xi)^*$ are isometries a.e. on $\mathbb{T}$.

Finally, the function θ is called *two-sided inner* if it is both inner and $*$-inner, which means that the boundary values $\theta(\xi)$ are unitary a.e. on $\mathbb{T}$.

Proof of Corollary 1.2. Let U have purely singular spectrum, which means that $N_U(\xi) = 0$ a.e. on $\mathbb{T}$. Then equation (1.4) informs us that $\operatorname{rank} \Delta(\xi) = 0$ a.e. on $\mathbb{T}$, and by (1.3), we see that also $\operatorname{rank} \Delta_*(\xi) = 0$ a.e. on $\mathbb{T}$. Therefore, by the definition of the defect functions Δ and Δ_*, we see that $\theta(\xi)$ is unitary a.e. on $\mathbb{T}$, i.e., that θ is double inner. □

For the proofs of Corollaries 1.3 and 1.4, recall the following result.

Proposition 4.4 (see, e.g., [15, Proposition VI.3.5])**.** *For a c.n.u. contraction T_0 we have:*

(i) *T_0 is asymptotically stable if and only if its characteristic function θ is $*$-inner;*

(ii) *T_0^* is asymptotically stable if and only if its characteristic function θ is inner.*

Proof of Corollary 1.3. This follows immediately from Corollary 1.2 and the "if" direction of both items in Proposition 4.4. □

Proof of Corollary 1.4. Let T be asymptotically stable.

Then its unitary part is trivial ($V = 0$). In particular, the dimension function of its absolutely continuous part $N_V(\xi)$ is trivial, i.e., $N_V(\xi) = 0$ for a.e. $\xi \in \mathbb{T}$.

From the asymptotic stability of T we further deduce that T is a c.n.u. contraction. In particular, $T = T_0$ is asymptotically stable. So Proposition 4.4 implies that θ is $*$-inner. Therefore, we have $\operatorname{rank} \Delta_*(\xi) = 0$ for a.e. $\xi \in \mathbb{T}$ and so by (1.3) $\operatorname{rank} \Delta(\xi) = 0$ for a.e. $\xi \in \mathbb{T}$.

Invoking (1.4), we see that $N_U(\xi) = N_V(\xi) + \operatorname{rank} \Delta(\xi) = 0$ for a.e. $\xi \in \mathbb{T}$. □

§5. Appendix: Derivation of the characteristic function

Mainly for the sake of self-containment, we include a proof of the formulas for the characteristic function in (3.4) and (3.5) following that of [7, Theorem 4.2] where the formula was proved in the matrix case. We also prove (3.6) at the end of this section.

Recall that for a contraction T its defect operators D_T and D_{T^*} are given by

$$D_T = (\mathbf{I} - T^*T)^{1/2}, \quad D_{T^*} = (\mathbf{I} - TT^*)^{1/2},$$

and the defect spaces are defined as

$$\mathfrak{D}_T = \operatorname{Clos}\operatorname{Ran} D_T, \quad \mathfrak{D}_{T^*} = \operatorname{Clos}\operatorname{Ran} D_{T^*}.$$

Recall that according to [15, Chapter VI] the abstract characteristic function $\widetilde{\theta} = \widetilde{\theta}_T$ of the operator T is an analytic function in the unit disk $\mathbb{D}$ whose values are strict contractions $\widetilde{\theta}(z) \colon \mathfrak{D}_T \to \mathfrak{D}_{T^*}$ which is given by the formula

$$\widetilde{\theta}_T(z) = (-T + zD_{T^*}(\mathbf{I}_{\mathcal{H}} - zT^*)^{-1}D_T)\Big|_{\mathfrak{D}_T}.$$

Usually in the literature the characteristic function is treated as the equivalence class of all functions obtained from $\widetilde{\theta}$ by right and left multiplication by constant unitary operators. But sometimes it is more convenient, as we will do, to pick a concrete representation in this equivalence class. Namely, if $\mathfrak{D}$ and $\mathfrak{D}_*$ are abstract Hilbert spaces of appropriate dimensions, and

$$V \colon \mathfrak{D}_T \to \mathfrak{D}, \quad V_* \colon \mathfrak{D}_{T^*} \to \mathfrak{D}_* \tag{5.1}$$

are unitary operators (the so-called coordinate operators), then, according to, e.g., [12, Theorem 1.2.8] or [13, Theorem 1.11], the representation of the characteristic function corresponding to the identification (5.1) is given by

$$\theta(z) = V_*\widetilde{\theta}(z)V^* = V_*(-T + zD_{T^*}(\mathbf{I}_{\mathcal{H}} - zT^*)^{-1}D_T)V^*. \tag{5.2}$$

Consider a contraction $T = T_\Gamma = U + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^*U$ from (3.3) where $\mathbf{B}$ is an isometry acting from $\mathfrak{D}$ to $\mathcal{H}$. In this case

$$D_T = U^*\mathbf{B}D_\Gamma\mathbf{B}^*U, \quad D_{T^*} = \mathbf{B}D_{\Gamma^*}\mathbf{B}^*. \tag{5.3}$$

If Γ (and therefore Γ^*) is a strict contraction, the defect spaces are

$$\mathfrak{D}_T = \operatorname{Ran}(U^*\mathbf{B}) = U^*\operatorname{Ran}\mathbf{B}, \quad \mathfrak{D}_{T^*} = \operatorname{Ran}\mathbf{B},$$

so

$$V = \mathbf{B}^*U, \quad V_* = \mathbf{B}^* \tag{5.4}$$

is a natural choice for the coordinate operators (this is exactly the choice that was made in [7]). Note that in this case $\mathfrak{D}_* = \mathfrak{D}$.

By the definition of T, using (5.4) we get

$$V_*TV^* = \mathbf{B}^*TU^*\mathbf{B}\Big|_{\mathfrak{D}} = \mathbf{B}^*\mathbf{B}\Gamma\mathbf{B}^*UU^*\mathbf{B}\Big|_{\mathfrak{D}} = \Gamma,$$

and therefore (5.2) can be rewritten as

$$\theta(z) = -\Gamma + \mathbf{B}^*zD_{T^*}(\mathbf{I}_{\mathcal{H}} - zT^*)^{-1}D_TU^*\mathbf{B}. \tag{5.5}$$

For much of the remainder of this section we include the space on which identity operators act, for clarification.

We continue to express the inverse for $z \in \mathbb{D}$:

$$
\begin{aligned}
(\mathbf{I}_{\mathcal{H}} - zT^*)^{-1} &= \big((\mathbf{I}_{\mathcal{H}} - zU^*)[\mathbf{I}_{\mathcal{H}} - z(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}U^*\mathbf{B}(\Gamma^* - \mathbf{I}_{\mathfrak{D}})\mathbf{B}^*]\big)^{-1} \\
&= X(z)^{-1}(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1},
\end{aligned}
$$

where

$$
X(z) := \mathbf{I}_{\mathcal{H}} - z(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}U^*\mathbf{B}(\Gamma^* - \mathbf{I}_{\mathfrak{D}})\mathbf{B}^*.
$$

Here, we note that both $\mathbf{I}_{\mathcal{H}} - zT^*$ and $\mathbf{I}_{\mathcal{H}} - zU^*$ are invertible for $z \in \mathbb{D}$ (because $\|zT^*\|, \|zU^*\| \leqslant |z| < 1$) so $X(z)$ is invertible as well. To obtain the expression for $X(z)^{-1}$, we apply Lemma 5.1 below with $P, Q\colon \mathfrak{D} \to \mathcal{H}$ given by $P = z(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}U^*\mathbf{B}$ and $Q^* = (\Gamma^* - \mathbf{I}_{\mathfrak{D}})\mathbf{B}^*$ to get

$$
\begin{aligned}
X(z)^{-1} = \mathbf{I}_{\mathcal{H}} &+ z(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}U^*\mathbf{B} \\
&\times \Big[\mathbf{I}_{\mathfrak{D}} - z(\Gamma^* - \mathbf{I}_{\mathfrak{D}})\mathbf{B}^*(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}U^*\mathbf{B}\Big]^{-1}(\Gamma^* - \mathbf{I}_{\mathfrak{D}})\mathbf{B}^*;
\end{aligned}
$$

note that Lemma 5.1 also implies that the expression in brackets is invertible for $z \in \mathbb{D}$.

Recalling that $F_1(z) = z\mathbf{B}^*(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}U^*\mathbf{B}$ by (3.6), we obtain

$$
\begin{aligned}
(\mathbf{I}_{\mathcal{H}} - zT^*)^{-1} = (\mathbf{I}_{\mathcal{H}} &- zU^*)^{-1} \\
&+ z(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}U^*\mathbf{B}\Big[\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z)\Big]^{-1} \\
&\times (\Gamma^* - \mathbf{I}_{\mathfrak{D}})\mathbf{B}^*(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1};
\end{aligned}
\tag{5.6}
$$

the expression in brackets is invertible for $z \in \mathbb{D}$, because it is exactly the expression in brackets is the above formula for $X(z)^{-1}$.

Now we substitute (5.6) and (5.3) in (5.5), and again use that $F_1(z) = z\mathbf{B}^*(\mathbf{I}_{\mathcal{H}} - zU^*)^{-1}U^*\mathbf{B}$. After straightforward but somewhat tedious calculations, we arrive at

$$
\begin{aligned}
\theta(z) &= -\Gamma + D_{\Gamma^*}\Big(F_1(z) + F_1(z)\Big[\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z)\Big]^{-1}(\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z)\Big)D_{\Gamma} \\
&= -\Gamma + D_{\Gamma^*}F_1(z)\Big[\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z)\Big]^{-1} \\
&\quad \times \Big(\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z) + (\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z)\Big)D_{\Gamma} \\
&= -\Gamma + D_{\Gamma^*}F_1(z)\Big[\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z)\Big]^{-1}D_{\Gamma},
\end{aligned}
$$

which is exactly (3.4).

Equation (3.5) is an immediate consequence of (3.4). Indeed, we clearly have

$$\Big[\mathbf{I}_{\mathfrak{D}} - F_1(z)(\Gamma^* - \mathbf{I}_{\mathfrak{D}})\Big]F_1(z) = F_1(z)\Big[\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z)\Big],$$

or, equivalently,

$$F_1(z)\Big[\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z)\Big]^{-1} = \Big[\mathbf{I}_{\mathfrak{D}} - F_1(z)(\Gamma^* - \mathbf{I}_{\mathfrak{D}})\Big]^{-1}F_1(z).$$

The following lemma, which we have used above, can be regarded as a particular case of the so-called Woodbury inversion formula, see [18], although formally in [18] only the case of matrices was treated.

Lemma 5.1. *Let $\mathcal{K}$ be a separable Hilbert space and consider operators P, $Q\colon \mathcal{K} \to \mathcal{H}$. The operators $\mathbf{I}_{\mathcal{H}} - PQ^*$ and $\mathbf{I}_{\mathcal{K}} - Q^*P$ are simultaneously invertible. In this case, we have the inversion formula*

$$(\mathbf{I}_{\mathcal{H}} - PQ^*)^{-1} = \mathbf{I}_{\mathcal{H}} + P(\mathbf{I}_{\mathcal{K}} - Q^*P)^{-1}Q^*.$$

Proof. Assume that $\mathbf{I}_{\mathcal{K}} - Q^*P$ is invertible and compute

$$\begin{aligned}
&(\mathbf{I}_{\mathcal{H}} - PQ^*)(\mathbf{I}_{\mathcal{H}} + P(\mathbf{I}_{\mathcal{K}} - Q^*P)^{-1}Q^*)\\
&\qquad = \mathbf{I}_{\mathcal{H}} - PQ^* + P(\mathbf{I}_{\mathcal{K}} - Q^*P)^{-1}Q^* - PQ^*P(\mathbf{I}_{\mathcal{K}} - Q^*P)^{-1}Q^*\\
&\qquad = \mathbf{I}_{\mathcal{H}} + P\left(-\mathbf{I}_{\mathcal{K}} + (\mathbf{I}_{\mathcal{K}} - Q^*P)(\mathbf{I}_{\mathcal{K}} - Q^*P)^{-1}\right)Q^*\\
&\qquad = \mathbf{I}_{\mathcal{H}}.
\end{aligned}$$

So, $\mathbf{I}_{\mathcal{H}} + P(\mathbf{I}_{\mathcal{K}} - Q^*P)^{-1}Q^*$ is the right inverse of $\mathbf{I}_{\mathcal{H}} - PQ^*$. To show that it is also the left inverse (and therefore the inverse), one can either reduce $(\mathbf{I}_{\mathcal{H}} + P(\mathbf{I}_{\mathcal{K}} - Q^*P)^{-1}Q^*)(\mathbf{I}_{\mathcal{H}} - PQ^*)$ in a similar way, or simply take the adjoint of the above computation and then swap the roles of P and Q.

Vice versa, to prove the invertibility of $\mathbf{I}_{\mathcal{K}} - Q^*P$ from that of $\mathbf{I}_{\mathcal{H}} - PQ^*$, we simply swap the roles of P and Q^* and those of $\mathcal{H}$ and $\mathcal{K}$, respectively, and apply the formulas we have proved. □

Proof of equation (3.6)**.** This identity follows easily from the definition (3.2) of the operator-valued measure $\boldsymbol{\mu}$. Indeed, since

$$zU^*(\mathbf{I} - zU^*)^{-1} = (\mathbf{I} - zU^*)^{-1} - \mathbf{I},$$

we see that

$$\begin{aligned}
z\mathbf{B}^*(\mathbf{I} - zU^*)^{-1}U^*\mathbf{B} &= \mathbf{B}^*(\mathbf{I} - zU^*)^{-1}\mathbf{B} - \mathbf{B}^*\mathbf{B}\\
&= \int_{\mathbb{T}} \frac{\mathrm{d}\boldsymbol{\mu}(\xi)}{1 - z\overline{\xi}} - \int_{\mathbb{T}} \mathrm{d}\boldsymbol{\mu}(\xi) = \int_{\mathbb{T}} \frac{z\overline{\xi}}{1 - z\overline{\xi}}\mathrm{d}\boldsymbol{\mu}(\xi).
\end{aligned}$$ □

Remark. Equation (3.6) also follows via a "high brow" approach invoking the functional calculus. Namely, for a rational function φ (with no poles on $\mathbb{T}$), equation (3.2) implies

$$\mathbf{B}^*\varphi(U)\mathbf{B} = \int_{\mathbb{T}} \varphi(\xi)\mathrm{d}\boldsymbol{\mu}(\xi).$$

Taking

$$\varphi(\xi) = \varphi_z(\xi) = \frac{z\xi^{-1}}{1 - z\xi^{-1}},$$

and using the fact that $\xi^{-1} = \bar{\xi}$ for $\xi \in \mathbb{T}$, we immediately obtain (3.6).

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