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F^e-modules with applications to D-modules

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A R T I C L E   I N F O

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Using a theory of F^e-modules (a natural extension of Lyubeznik's F-module theory), we extend results on Matlis dual of F-finite F-modules to D-submodules of F^e-finite F^e-modules and apply these results to address the Lyubeznik-Yildirim conjecture in mixed characteristic.

1. Introduction

The theory of F-modules, whose roots can be found in [15,5,7], is introduced in [10]. Since its introduction, it has been proven indispensable in the study of rings of prime characteristic p (see, for instance, [4,1,11,16,12]). Replacing the Peskine-Szpiro functor $F(-)$ by its e-th iteration $F^e(-)$, one obtains the theory of $F^e(-)$-modules (details can be found in §2).

The motivations behind this article are two-fold:

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(†) It follows from [8, 7.4] that there exists an F-finite F-module which admits a simple D-submodule that is *not* an F-submodule (*cf.* Example 2.5). The structure of such D-submodules warrants further investigations.

(‡) Let R be the completion of $\mathbb{Z}[x_1, \dots, x_6]$ at the maximal ideal $(2, x_1, \dots, x_6)$ and let I be the monomial ideal associated with the minimal triangulation of the projective plane. Then it is proved in [3, 4.5] that the support of the Matlis dual of $H^4_I(R)$ is $\mathrm{Spec}(R/(2))$, a proper subset of $\mathrm{Spec}(R)$, which provides a counterexample to [13, Conjecture 1]. It is natural to ask whether $\mathrm{Spec}(R/(\pi))$ is always contained in the support of the Matlis dual of $H^j_I(R)$ whenever $R = V[[x_1, \dots, x_n]]$ and $V = (V, \pi V)$ is a complete DVR of mixed characteristic $(0, p)$.

One of our results regarding (†) is the following.

Theorem 1.1. *Let R be a noetherian regular ring of characteristic p which is a finitely generated R^p-module. If $\mathcal{N}$ is a simple D_R-submodule of an F^e-finite F^e-module $\mathcal{M}$, then there exists a positive integer e' such that $\mathcal{N}$ is an $F^{e'e}$-submodule of $\mathcal{M}$.*

Since each F^e-module is naturally a D_R-module (Remark 2.7), it is feasible to consider D-submodules of an F^e-module in the statement of Theorem 1.1. Example 2.5 shows that, in general, it is necessary to have $e' > 1$, even when $e = 1$. This provides one of the justifications for the necessity of considering F^e-modules (with $e > 1$).

As a consequence of our Theorem 1.1, we have the following result concerning (‡).

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a noetherian regular local ring of finite type over a regular local ring A such that A is module-finite over A^p. Let $\mathcal{N}$ be an arbitrary (not necessarily simple) D_R-submodule of an F^e-finite F^e-module. Assume that (0) is not an associated prime of $\mathcal{N}$. Then*

$$\mathrm{Supp}_R(\mathcal{N}^\vee) = \mathrm{Spec}(R),$$

where $\mathcal{N}^\vee$ denotes the Matlis dual of $\mathcal{N}$.

Theorem 1.3. *Let $R = V[[x_1, \dots, x_d]]$ be a formal power series ring over a complete DVR $(V, \pi V, k)$ of mixed characteristic and I be an ideal of R. Assume that $[k : k^p] < \infty$. If*

(1) either $\mathrm{Coker}(H^j_I(R) \xrightarrow{\pi} H^j_I(R)) \neq 0$
(2) or $\mathrm{Ker}(H^j_I(R) \xrightarrow{\pi} H^j_I(R)) \neq 0$,

then

$$\mathrm{Spec}(R/\pi R) \subseteq \mathrm{Supp}_R(H^j_I(R)^\vee),$$

where $H^j_I(R)^\vee$ denotes the Matlis dual of $H^j_I(R)$.

Theorem 1.2 is a natural extension of the main theorem in [13]. Without any further assumptions on I, Theorem 1.3 is the best possible since "$\subseteq$" can be "$=$" in general; see Remark 4.2 for details.

This article is organized as follows. In §2, we collect some necessary preliminaries on F^e-modules and D-modules; in §3, we prove Theorem 1.1 and its corollary; in §4, we apply results proved in §3 to the investigation of the support of Matlis dual of D-modules, especially local cohomology modules.

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2. Background and some results on D-modules and F^e-modules

Let A be a commutative ring with identity. A ($\mathbb{Z}$-linear) differential operator of order 0 is the multiplication by an element of A. A differential operator of order $\leq \ell$ is an additive map $\delta : A \to A$ such that the commutator $[\delta, \tilde{a}] = \delta \circ \tilde{a} - \tilde{a} \circ \delta$ is a differential operator of order $\leq \ell - 1$, where $\tilde{a} : A \to A$ is the multiplication by $a \in A$, for every $a \in A$. These differential operators form a ring, denoted by $D_{A|\mathbb{Z}}$ or simply D_A.

If $k \subseteq A$ is a subring, then the ring of k-linear differential operators, denoted by $D_{A|k}$, is the subring of D_A consisting of k-linear elements of D_A. Given any element $f \in A$, A_f carries a natural $D_{A|k}$-module structure. Consequently, the local cohomology modules $H^j_{\mathfrak{a}}(A)$ carry a natural $D_{A|k}$-module structure for each ideal $\mathfrak{a}$ in A.

Assume now that A contains a field of characteristic p, and let A^{p^e} be the subring of A consisting of all the p^e-th powers of all elements in A for each positive integer e. Then, every differential operator $\delta \in D_A$ of order $\leq p^e - 1$ is A^{p^e}-linear; that is $\delta \in \operatorname{Hom}_{A^{p^e}}(A, A)$. Let k be a perfect subfield of A (*e.g.* $k = \mathbb{Z}/p\mathbb{Z}$). Assume that A is a finite $k[A^p]$-module, then

$$D_A = D_{A|k} = \bigcup_e \operatorname{Hom}_{A^{p^e}}(A, A).$$

$\operatorname{Hom}_{A^{p^e}}(A, A)$ is also denoted by $D^{(e)}$ in the literature.

When $A = B[x_1, \dots, x_n]$ or $A = B[[x_1, \dots, x_n]]$ where B is a commutative ring with identity, the ring $D_{A|B}$ can be described explicitly as follows. Set $\partial_i^{[t]} := \frac{1}{t!}\frac{\partial^t}{\partial x_i^t}$; that is

$$\partial_i^{[t]}(x_i^s) = \begin{cases} 0 & s < t \\ \binom{s}{t} x_i^{s-t} & s \geq t \end{cases}$$

Then $D_{A|B}$ is the ring extension of A generated by $\partial_i^{[t]}$ for all i and all $t \geq 1$. Furthermore, if B is a perfect field of characteristic p, then $D^{(e)}$ is the ring extension of A generated by $\partial_i^{[t]}$ for all i and all $t \leq p^e - 1$.

Remark 2.1. Given this explicit descriptions of the rings of differential operators, one can check the following (*cf.* [2, 2.1] for details). Assume that $R = V[x_1, \ldots, x_n]$ or $R = V[[x_1, \ldots, x_n]]$ where $V = (V, \pi V, k)$ is a DVR with a uniformizer π. Set $\bar{R} = R/(\pi)$. Given each $D_{R|V}$-module M, the multiplication map $M \xrightarrow{\pi} M$ is $D_{R|V}$-linear as $\pi \in V$; consequently, the submodule $\operatorname{Ann}_M(\pi)$ and the quotient module $M/\pi M$ are naturally $D_{R|V}$-modules and $D_{\bar{R}|k}$-modules. The short exact sequence $0 \to R \xrightarrow{\pi} R \to \bar{R} \to 0$ induces a long exact of local cohomology modules

$$\cdots \to H^{j-1}_I(\bar{R}) \to H^j_I(R) \xrightarrow{\pi} H^j_I(R) \to H^j_I(\bar{R}) \to \cdots$$

which is an exact sequence in the category of $D_{R|V}$-modules, for each ideal I of R. In particular, the modules $\operatorname{Coker}(H^j_I(R) \xrightarrow{\pi} H^j_I(R))$ and $\operatorname{Ker}(H^j_I(R) \xrightarrow{\pi} H^j_I(R))$ are naturally $D_{R|V}$-modules and $D_{\bar{R}|k}$-modules. Consequently, the natural maps

$$\operatorname{Coker}(H^j_I(R) \xrightarrow{\pi} H^j_I(R)) \to H^j_I(\bar{R}),\ H^{j-1}_I(\bar{R}) \to \operatorname{Ker}(H^j_I(R) \xrightarrow{\pi} H^j_I(R))$$

are morphisms in the category of $D_{R|V}$-modules and morphisms in the category of $D_{\bar{R}|k}$-modules.

Let A be a commutative ring that contains a field of characteristic p. Then there is a natural functor on the category of A-modules called the Peskine-Szpiro functor and defined as follows. Let $F^e_* A$ denote the A-module whose underlying abelian group is the same as A and whose A-module structure is induced by the e-th Frobenius $A \xrightarrow{a \mapsto a^{p^e}} A$. The Psekine-Szpiro functor $F^e_A(-)$ on the category of A-modules is defined by

$$F^e_A(M) = F^e_* A \otimes_A M.$$

Remark 2.2. Assume R is a noetherian regular ring of characteristic p. Then a classical theorem due to Kunz ([9]) asserts that the Peskine-Szpiro functor F^e_R is an exact functor.

Moreover, assume that R is a finite generated R^p-module. Then the category of R-modules is equivalent to the category of $D^{(e)}$-modules ([1, Proposition 2.1]). The functor from the category of R-modules to the category of $D^{(e)}$-modules is precisely the Peskine-Szpiro functor F^e_R. Since one can identify $\operatorname{Hom}_{R^{p^e}}(R, R)$ with $\operatorname{Hom}_R(F^e_* R, F^e_* R)$, the $D^{(e)}$-module structure on $F^e_R(M) = F^e_* R \otimes_R M$ is induced by the action on $F^e_* R$. We refer the reader to [1] for details.

The following result, [1, Proposition 2.3], will be useful in the sequel.

Theorem 2.3. *Let R be a noetherian regular ring of characteristic p. Assume that R is a finitely generated R^p-module. Then the Peskine-Szpiro functor F^e_R is an equivalence of the category of D_R-modules with itself.*

Considerations on the Peskine-Szpiro functor have proven to be fruitful in the investigation of rings of prime characteristic p. When R is regular, the theory of F-modules is introduced in [10]. Since this theory is readily adapted to the e-th Peskine-Szpiro functor $F^e(-)$, we opt to explain here the theory of F^e-modules.

For the rest of the section, R denotes a noetherian regular ring of prime characteristic p.

Definition 2.4. Let e be a positive integer.

(1) An R-module $\mathcal{M}$ is an F^e*-module* if there is an R-module isomorphism

$$\theta : \mathcal{M} \to F^e(\mathcal{M}) = F^e_* R \otimes_R \mathcal{M}$$

called the structure isomorphism.

When $e = 1$, we will write F instead of F^1 whenever the context is clear.

(2) If $(\mathcal{M}, \theta_{\mathcal{M}})$ and $(\mathcal{N}, \theta_{\mathcal{N}})$ are F^e-modules, then an F^e*-module morphism* from $(\mathcal{M}, \theta_{\mathcal{M}})$ to $(\mathcal{N}, \theta_{\mathcal{N}})$ consists of the following commutative diagram:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\varphi"] \arrow[d, "\theta_{\mathcal{M}}"] & \mathcal{N} \arrow[d, "\theta_{\mathcal{N}}"] \\ F^e(\mathcal{M}) \arrow[r, "F^e(\varphi)"] & F^e(\mathcal{N}) \end{tikzcd}$$

We will simply write this F^e-module morphism as $\varphi : \mathcal{M} \to \mathcal{N}$ whenever the context is clear.

(3) A *generating morphism* of an F^e-module is an R-module homomorphism $\beta : M \to F^e(M)$, where M is an R-module, such that $\mathcal{M}$ is the direct limit of the top row of the following commutative diagram

$$\begin{tikzcd} M \arrow[r, "\beta"] \arrow[d, "\beta"] & F^e(M) \arrow[r, "F^e(\beta)"] \arrow[d, "F^e(\beta)"] & F^{2e}(M) \arrow[r] \arrow[d] & \cdots \\ F^e(M) \arrow[r, "F^e(\beta)"] & F^{2e}(M) \arrow[r, "F^{2e}(\beta)"] & F^{3e}(M) \arrow[r] & \cdots \end{tikzcd}$$

and the structure isomorphism $\theta : \mathcal{M} \to F^e(\mathcal{M})$ is induced by the vertical morphism in the diagram.

(4) An F^e-module $\mathcal{M}$ is F^e*-finite* if it admits a generating morphism $\beta : M \to F^e(M)$ where M is a finitely generated R-module.

We will denote the category of F^e-modules by $\mathscr{F}^e$.

Results on F-modules in the literature, *e.g.* [10] and [1], can be readily extended to F^e-modules by simply replacing the functor $F(-)$ with the functor $F^e(-)$. Before proceeding to properties of F^e-modules, we would like to explain one of the motivations behind introducing these modules and hopefully to answer the natural question: why not just work with F-modules?

Example 2.5. Let $R = \mathbb{F}_{11}[x, y, z]$ and let $f = x^7 + y^7 + z^7$. Denote $H^1_{(f)}(R)$ by $\mathcal{H}$. Then [8, 7.4] shows that

$$\ell_{\mathscr{D}}(\mathcal{H}) > \ell_{\mathscr{F}}(\mathcal{H}),$$

where $\ell_{\mathscr{D}}(\mathcal{H})$ (or $\ell_{\mathscr{F}}(\mathcal{H})$, respectively) denotes the length of $\mathcal{H}$ in the category of $\mathscr{D}$-modules (or in $\mathscr{F}$, respectively). Let H be a simple D-submodule of $\mathcal{H}$. If H is an F-submodule of $\mathcal{H}$, then it follows from [10, Theorem 2.8] that H is an F-finite F-submodule of $\mathcal{H}$ and consequently $\mathcal{H}/H$ is an F-finite F-module. Note $\ell_{\mathscr{D}}(\mathcal{H}/H) > \ell_{\mathscr{F}}(\mathcal{H}/H)$. Continuing this process, after at most $\ell_{\mathscr{F}}(\mathcal{H})$ steps, one can see that there is an F-finite F-module $\mathcal{H}'$ (a quotient of $\mathcal{H}$ in the category of F-modules) such that $\mathcal{H}'$ admits a simple D-submodule that is not an F-submodule of $\mathcal{H}'$. (Similarly, one can also deduce that $\mathcal{H}$ admits a $\mathscr{D}$-submodule which is not an F-submodule.)

Example 2.5 shows that the theory of F^e-modules may be applicable to $\mathscr{D}$-submodules of an F-finite F-module which may not be F-submodules in general.

Remark 2.6. Assume that $(\mathcal{M}, \theta)$ is an F^e-module for a positive integer e. Then, for every positive integer t, the composition

$$\mathcal{M} \xrightarrow{\theta} F^e(\mathcal{M}) \xrightarrow{F^e(\theta)} \cdots \to F^{te}(\mathcal{M})$$

is also an R-module isomorphism. Hence $\mathcal{M}$ is an F^{te}-module for every positive integer t. In particular, an F-module is also an F^e-module for every positive integer e. Consequently, all local cohomology modules $H^j_I(R)$ (and iterated local cohomology modules) are F^e-modules for every positive integer e.

Assume that $(\mathcal{M}, \theta)$ is an F^e-module for a positive integer e. Then so is $F^t(\mathcal{M})$ for every positive integer t since $F^t(\mathcal{M}) \cong F^t(F^e(\mathcal{M})) = F^e(F^t(\mathcal{M}))$.

Let e, f be positive integers such that $e|f$. Then $\mathscr{F}^e$ can be naturally viewed as a subcategory of $\mathscr{F}^f$. Let $\mathcal{M}$ be an F^e-module. By an F^f-submodule $\mathcal{N}$ of $\mathcal{M}$ we mean a sub-object of $\mathcal{M}$ when $\mathcal{M}$ is viewed as an object in $\mathscr{F}^f$.

Remark 2.7. Every F^e-module admits a natural D-module structure. This follows from Remark 2.2. Let δ be a differential operator. Then there exists an positive integer t such that its order (as a differential operator) is less than te. Let α_t denote the composition

$$\mathcal{M} \xrightarrow{\theta} F^e(\mathcal{M}) \xrightarrow{F^e(\theta)} \cdots \to F^{te}(\mathcal{M})$$

Given an arbitrary element $m \in \mathcal{M}$, write $\alpha_t(m) = \sum_i r_i \otimes m_i$. Then, for every element $m \in \mathcal{M}$, set

$$\delta \cdot m := \alpha_t^{-1}(\sum_i (\delta \cdot r_i) \otimes m_i)$$

Whenever we view an F^e-module as a D_R-module, we always refer to the D_R-module structure specified in the previous paragraph. Under this D_R-module structure, an F^e-module morphism between any two F^e-modules is also a D_R-module morphism.

We now collect some results on F^e-modules which are natural analogues of corresponding results on F-modules in the literature.

Remark 2.8. Let R be a noetherian regular ring of characteristic p that is module-finite over R^p. Let $\mathcal{M}$ be an F^e-module for a positive integer e.

(1) The F^e-finite modules form a full abelian subcategory of the category of F^e-modules which is closed under formation of submodules, quotient modules and extensions. When $e = 1$, this is [10, Theorem 2.8]. When e is an arbitrary positive integer, the same proof goes through (by replacing $F(-)$ with $F^e(-)$).
(2) The structure isomorphism $\theta : \mathcal{M} \to F^e(\mathcal{M})$ is D_R-linear, where the D_R-module structure is as in Remark 2.7. When $e = 1$, this is [1, Lemma 2.4]. When e is an arbitrary positive integer, the same proof goes through (by replacing $F(-)$ with $F^e(-)$).
(3) Assume further that R is of finite type over a regular local ring A such that A is module-finite over A^p. Then every F^e-finite F^e-module has finite length in $\mathscr{F}^e$ and in the category of D_R-modules, for each positive integer e. When $e = 1$, this is [10, Theorem 3.2] and [1, Theorem 2.5], respectively. When e is an arbitrary positive integer, the same proofs go through (by replacing $F(-)$ with $F^e(-)$).

3. Interactions between F^e-modules and D-modules

The main goal of this section is to prove Theorem 1.1. We begin with the following observation.

Proposition 3.1. *Let R be a noetherian regular ring of finite type over a regular local ring A such that A is module-finite over A^p and let $\mathcal{M}$ be an F^e-finite F^e-module. Let $\mathcal{N}$ be a D_R-submodule of $\mathcal{M}$. Assume that*[2] *$F^e(\mathcal{N}) \subseteq \mathcal{N}$. Then $\mathcal{N}$ is an F^e-submodule of $\mathcal{M}$.*

Proof. Since $F^e(\mathcal{N})$ is naturally a D_R-submodule of $\mathcal{M}$ (due to Theorem 2.3) and $F^e(\mathcal{N}) \subseteq \mathcal{N}$, we have a descending chain of D_R-submodules of $\mathcal{M}$:

[2] Here we identify $F^e(\mathcal{N})$ with an R-submodule of $\mathcal{M}$ under the isomorphism $F^e(\mathcal{M}) \cong \mathcal{M}$.

$$\mathcal{N} \supseteq F^e(\mathcal{N}) \supseteq F^{2e}(\mathcal{N}) \supseteq \cdots .$$

Since $\mathcal{M}$ has finite length in the category of D_R-modules (Remark 2.8), this chain must terminate in finitely many steps; that is $F^{te}(\mathcal{N}) = F^{(t+1)e}(\mathcal{N}) = F^{te}(F^e(\mathcal{N}))$ for an integer t. Hence $\mathcal{N} \cong F^e(\mathcal{N})$ which completes the proof. $\square$

We are now in position to prove Theorem 1.1, whose proof is inspired by the proof of [10, Theorem 5.6].

Proof of Theorem 1.1. Since $\mathcal{M}$ is an F^e-module, $F^{te}(\mathcal{N}) \subseteq F^{te}(\mathcal{M}) \cong \mathcal{M}$ for each positive integer t. We will view $F^{te}(\mathcal{N})$ as a D-submodule of $\mathcal{M}$. It follows from Theorem 2.3 that $F^{te}(\mathcal{N})$ is also a simple D-submodule of $\mathcal{M}$ for every positive integer t. Let t be the least positive integer such that

$$\mathcal{N} + F^e(\mathcal{N}) + \cdots + F^{(t-1)e}(\mathcal{N}) = \mathcal{N} \oplus F^e(\mathcal{N}) \oplus \cdots \oplus F^{(t-1)e}(\mathcal{N})$$

that is, t is the least positive integer such that

$$(\mathcal{N} + F^e(\mathcal{N}) + \cdots + F^{(t-1)e}(\mathcal{N})) \cap F^{te}(\mathcal{N}) \neq \emptyset.$$

Set

$$\mathcal{L} := \mathcal{N} + F^e(\mathcal{N}) + \cdots + F^{(t-1)e}(\mathcal{N}) = \mathcal{N} \oplus F^e(\mathcal{N}) \oplus \cdots \oplus F^{(t-1)e}(\mathcal{N}).$$

By the construction of $\mathcal{L}$, one sees that $\mathcal{L}$ is a semi-simple D_R-module.

We claim that $\mathcal{L}$ is an F^e-submodule of $\mathcal{M}$ and we reason as follows. Since $F^{te}(\mathcal{N})$ is also a simple D_R-module by Theorem 2.3 and $F^{te}(\mathcal{N}) \cap \mathcal{L} \neq \emptyset$, we have

$$F^{te}(\mathcal{N}) \subset \mathcal{L}.$$

Consequently $F^e(\mathcal{L}) \subseteq \mathcal{L}$. It follows from Proposition 3.1 that $\mathcal{L}$ is an F^e-submodule of $\mathcal{M}$ and hence is also an F^e-finite F-finite module by Remark 2.8.

This shows that $\mathcal{N}$ is a simple D_R-submodule of an F^e-finite F^e-module $\mathcal{L}$ such that $\mathcal{L} = \mathcal{N} \oplus \cdots \oplus F^{(t-1)e}(\mathcal{N})$, where $\mathcal{N}, \ldots, F^{(t-1)e}(\mathcal{N})$ are simple D-submodules of $\mathcal{L}$. Since $\mathcal{L}$ is a semi-simple D_R-module stable under $F^e(-)$ and $F^e(-)$ is an equivalence on the category of D-modules (Theorem 2.3), the functor $F^e(-)$ cycles through its direct summands $\mathcal{N}, \ldots, F^{(t-1)e}(\mathcal{N})$. Therefore, there exists a positive integer e' such that $\mathcal{N} \cong F^{e'e}(\mathcal{N})$. This finishes the proof. $\square$

Corollary 3.2. *Let R be as in Theorem 1.1. Assume that $\mathcal{N}$ is a D_R-module quotient of an F^e-finite F^e-module $\mathcal{M}$. Then there exists a positive integer e' such that $\mathcal{N}$ is an $F^{e'}$-finite $F^{e'}$-module.*

Proof. We will use induction on the length of $\mathcal{M}$ as a D_R-module; note that $\mathcal{M}$ has finite length in the category of D_R-modules according to Remark 2.8.

When $\mathcal{M}$ is a simple D_R-module, then either $\mathcal{N} = 0$ or $\mathcal{N} = \mathcal{M}$. The conclusion is clear.

Let $\ell_{\mathscr{D}}(\mathcal{M})$ denote the D_R-module length of $\mathcal{M}$. Assume now $\ell_{\mathscr{D}}(\mathcal{M}) \geq 2$ and the theorem has been proved for all F-finite F-modules with D_R-module length $\leq \ell_{\mathscr{D}}(\mathcal{M})-1$. Since $\mathcal{N}$ is a D_R-module quotient, there is a D_R-submodule $\mathcal{L}$ of $\mathcal{M}$ such that $\mathcal{N} = \mathcal{M}/\mathcal{L}$. Since $\ell_{\mathscr{D}}(\mathcal{L}) < \infty$, there is a simple D_R-submodule $\mathcal{L}'$ of $\mathcal{L}$. Since $\mathcal{L}'$ is a simple D_R-submodule of $\mathcal{M}$, by Theorem 1.1 $\mathcal{L}'$ is an F^{te}-submodule of $\mathcal{M}$ for a positive integer t. Consequently, $\mathcal{M}/\mathcal{L}'$ is an F^{te}-finite F^{te}-finite module. Set $\bar{\mathcal{M}} := \mathcal{M}/\mathcal{L}'$ and $\bar{\mathcal{L}} := \mathcal{L}/\mathcal{L}'$. Since $\ell_{\mathscr{D}}(\bar{\mathcal{M}}) < \ell_{\mathscr{D}}(\mathcal{M})$, by induction $\bar{\mathcal{M}}/\bar{\mathcal{L}}$ is an $F^{e'}$-finite $F^{e'}$-module for a positive integer e'. Since $\mathcal{N} = \mathcal{M}/\mathcal{L} \cong \bar{\mathcal{M}}/\bar{\mathcal{L}}$, this completes the proof. $\square$

Remark 3.3. When $R = k[x_1, \dots, x_n]$ where k is a field of characteristic p, one can also develop the notions of graded F^e-modules and graded F^e-finite F^e-modules and to extend results on graded F-modules to graded F^e-modules. For instance, one can show that a graded F^e-finite F^e-module is also an Eularian graded D_R-module; the interested reader is referred to [14] for the notion of Eulerian graded D-modules. We opt not to pursue this in the current article.

4. Applications to Matlis dual

Prompted by the work of Hellus in [6], Lyubeznik and Yildirim conjectured (in [13, Conjecture 1]) that, if R is a noetherian regular local ring and $H^j_I(R) \neq 0$ where I is an ideal of R, then $\mathrm{Supp}_R(H^j_I(R)^\vee) = \mathrm{Spec}(R)$. Here $H^j_I(R)^\vee$ denotes the Matlis dual of $H^j_I(R)$. This conjecture is proved in [13] in characteristic p. In mixed characteristic, this conjecture is shown to be false as stated ([3, 4.5]). One may notice that the example in [3, 4.5] (*cf.* Remark 4.2) is the kernel of multiplication by the uniformizer π on a local cohomology module $H^j_I(R)$; where (R, V) is a formal power series ring over a complete discrete valuation ring V with a uniformizer π. The main purpose of this section is to prove Theorem 1.3 which generalizes [3, 4.5].

We begin with the following extension of [13, Theorem 1.1].

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a complete regular local ring of characteristic p and let $\mathcal{M}$ be an F^e-finite F^e-module for a positive integer e. Assume that $(0) \notin \mathrm{Ass}_R(\mathcal{M})$. Then*

$$\mathrm{Supp}(\mathrm{Hom}_R(\mathcal{M}, E_R(R/\mathfrak{m}))) = \mathrm{Spec}(R),$$

where $E_R(R/\mathfrak{m})$ denotes the injective hull of $R/\mathfrak{m}$.

Proof. Once one replaces the functor $F(-)$ by $F^e(-)$, the proof is the same as the one of [13, Theorem 1.1]. To avoid duplication, we opt not to repeat the details here. $\square$

For each R-module M, we will denote its Matlis dual, $\mathrm{Hom}_R(M, E_R(R/\mathfrak{m}))$, by $M^\vee$.

We now can prove our Theorem 1.2 which is a natural extension of Theorem 4.1 to D_R-submodules of F^e-finite F^e-modules.

Proof of Theorem 1.2. Let $\mathcal{L}$ be a simple D_R-submodule of $\mathcal{N}$. Since $\mathrm{Ass}_R(\mathcal{L}) \subseteq \mathrm{Ass}_R(\mathcal{N})$ and $(0) \notin \mathrm{Ass}_R(\mathcal{N})$, we have $(0) \notin \mathrm{Ass}_R(\mathcal{L})$. The short exact sequence $0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{N}/\mathcal{L} \to 0$ induces a short exact sequence

$$0 \to (\mathcal{N}/\mathcal{L})^\vee \to \mathcal{N}^\vee \to \mathcal{L}^\vee \to 0.$$

If $\mathrm{Supp}(\mathcal{L}^\vee) = \mathrm{Spec}(R)$, then it follows from the short exact sequence above that $\mathrm{Supp}(\mathcal{N}^\vee) = \mathrm{Spec}(R)$. We are now reduced to proving that $\mathrm{Supp}(\mathcal{L}^\vee) = \mathrm{Spec}(R)$.

Since $\mathcal{L}$ is a simple D_R-submodule of an F^e-finite F^e-module, it follows from Theorem 1.1 that $\mathcal{L}$ is an $F^{e'e}$-submodule of $\mathcal{M}$. Since $\mathcal{M}$ is F^e-finite (hence $F^{e'e}$-finite) and $\mathcal{L}$ is an $F^{e'e}$-submodule, $\mathcal{L}$ must be $F^{e'e}$-finite as well because of Remark 2.8. Now Theorem 4.1 finishes the proof. $\square$

We now apply Theorem 1.2 to prove Theorem 1.3.

Proof of Theorem 1.3. The short exact sequence $0 \to R \xrightarrow{\pi} R \to R/\pi R \to 0$ induces a long exact sequence

$$\cdots \to H^{j-1}_I(\bar{R}) \to H^j_I(R) \xrightarrow{\pi} H^j_I(R) \to H^j_I(\bar{R}) \to \cdots$$

which implies

(1) an injection $\mathrm{Coker}\left(H^j_I(R) \xrightarrow{\pi} H^j_I(R)\right) \hookrightarrow H^j_I(\bar{R})$, and
(2) a surjection $H^{j-1}_I(\bar{R}) \to \mathrm{Ker}\left(H^j_I(R) \xrightarrow{\pi} H^j_I(R)\right)$.

Note that $\mathrm{Coker}(H^j_I(R) \xrightarrow{\pi} H^j_I(R)), \mathrm{Ker}(H^j_I(R) \xrightarrow{\pi} H^j_I(R)), H^j_I(\bar{R}), H^{j-1}_I(\bar{R})$ carry a natural $D_{\bar{R}|k}$-module structure, and that both the injection and the surjection are $D_{\bar{R}|k}$-linear (*cf.* Remark 2.1). This makes

(1) $\mathrm{Coker}(H^j_I(R) \xrightarrow{\pi} H^j_I(R))$ a $D_{\bar{R}|k}$-submodule of $H^j_I(\bar{R})$ which is an $F_{\bar{R}}$-finite $F_{\bar{R}}$-module, and
(2) $\mathrm{Ker}(H^j_I(R) \xrightarrow{\pi} H^j_I(R))$ a $D_{\bar{R}|k}$-module quotient of $H^{j-1}_I(\bar{R})$ which is an $F_{\bar{R}}$-finite $F_{\bar{R}}$-module.

Assume $\mathrm{Coker}(H^j_I(R) \xrightarrow{\pi} H^j_I(R)) = H^j_I(R)/\pi H^j_I(R) \neq 0$. Then it follows that $H^j_I(\bar{R}) \neq 0$. Hence $H^j_I(R)/\pi H^j_I(R)$ satisfies the assumptions in Theorem 1.2; our assumption on k ensures that $\bar{R}$ satisfies the hypothesis in Theorem 1.2. Consequently

$$\operatorname{Spec}(\bar{R}) = \operatorname{Supp}_{\bar{R}} \operatorname{Hom}_{\bar{R}}(H^j_I(R)/\pi H^j_I(R), \bar{E}),$$

where $\bar{E}$ denotes the injective hull of k as an $\bar{R}$-module.

Since $\bar{E} \cong \operatorname{Hom}_R(\bar{R}, E)$ where $E = E_R(k)$, by the adjunction between $\otimes$ and Hom, we have

$$\operatorname{Hom}_{\bar{R}}(H^j_I(R)/\pi H^j_I(R), \bar{E}) \cong \operatorname{Hom}_R(H^j_I(R)/\pi H^j_I(R), E).$$

It follows that

$$\operatorname{Spec}(\bar{R}) = \operatorname{Supp}_R \operatorname{Hom}_R(H^j_I(R)/\pi H^j_I(R), E)$$

where $\operatorname{Spec}(\bar{R})$ is considered a closed subset of $\operatorname{Spec}(R)$.

The surjection $H^j_I(R) \to H^j_I(R)/\pi H^j_I(R)$ induces an injection $(H^j_I(R)/\pi H^j_I(R))^\vee \hookrightarrow H^j_I(R)^\vee$. Therefore,

$$\operatorname{Spec}(\bar{R}) \subseteq \operatorname{Supp}_R(H^j_I(R)^\vee).$$

Assume $\operatorname{Ker}(H^j_I(R) \xrightarrow{\pi} H^j_I(R)) \neq 0$ and set $\mathcal{K} := \operatorname{Ker}(H^j_I(R) \xrightarrow{\pi} H^j_I(R))$. Then $\mathcal{K}$ is a $D_{\bar{R}|k}$-module quotient of $H^{j-1}_I(\bar{R})$ and $H^{j-1}_I(\bar{R})$ is an $F_{\bar{R}}$-finite $F_{\bar{R}}$-module. According to Corollary 3.2, $\mathcal{K}$ is an $F^{e''}$-finite $F^{e''}$-module and consequently

$$\operatorname{Supp}_R(\mathcal{K}^\vee) = \operatorname{Spec}(\bar{R})$$

by Theorem 4.1. The injection $\mathcal{K} \hookrightarrow H^j_I(R)$ induces a surjection $H^j_I(R)^\vee \twoheadrightarrow \mathcal{K}^\vee$ which proves that $\operatorname{Supp}_R(\mathcal{K}^\vee) \subseteq \operatorname{Supp}_R(H^j_I(R)^\vee)$. This completes the proof. $\square$

Remark 4.2. Let R be the completion of $\mathbb{Z}[x_1, \ldots, x_6]$ at the maximal ideal $\mathfrak{m} = (2, x_1, \ldots, x_6)$. Let I the monomial ideal associated with the minimal triangulation of the real projective plane. It is proved in [3, 4.5] that

$$H^4_I(R) \cong \operatorname{Hom}_R(R/(2), H^7_{\mathfrak{m}}(R))$$

and consequently $\operatorname{Supp}_R(H^4_I(R)^\vee) = \operatorname{Spec}(R/(2))$.

Therefore, without any further assumptions, the conclusion in Theorem 1.3 is the best possible.

In light of Theorem 1.3, we would like to ask the following.

Question 4.3. Let R be a complete unramified regular local ring of mixed characteristic and $(V, \pi V)$ be its coefficient ring.

(1) Is it always true that

$$\operatorname{Spec}(R/\pi R) \subseteq \operatorname{Supp}_R(H^j_I(R)^\vee)$$

for each ideal I and each integer j?

(2) Can one characterize the local cohomology modules $H^j_I(R)$ such that

$$\operatorname{Supp}_R(H^j_I(R)^\vee) = \operatorname{Spec}(R)?$$

Data availability

No data was used for the research described in the article.

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