Topology and its Applications 313 (2022) 107983

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On Lyubeznik type invariants[☆]

Thomas Reichelt[a], Uli Walther[b,*], Wenliang Zhang[c]

[a] *Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany*
[b] *Purdue University, Dept. of Mathematics, 150 N. University St., West Lafayette, IN 47907, USA*
[c] *Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, USA*

ARTICLE INFO

Article history:
Received 7 June 2021
Received in revised form 27 September 2021
Accepted 15 October 2021
Available online 29 December 2021

Keywords:
Local cohomology
Lyubeznik number
D-module
Perverse sheaf
de Rham cohomology
Eulerian
Čech–de Rham number
Hodge–de Rham

ABSTRACT

We discuss for an affine variety Y embedded in affine space X two sets of integers attached to $Y \subseteq X$ via local and de Rham cohomology spectral sequences. We investigate collapse of the spectral sequences, give topological interpretations, study them in small dimension, and consider to what extent one can attach them to projective varieties.

Contents

☆ TR was supported by DFG Emmy-Noether-Fellowship RE 3567/1-2. UW was supported by NSF grant DMS-2100288 and by Simons Foundation Collaboration Grant for Mathematicians #580839. WZ was supported by NSF grant DMS-1752081.

* Corresponding author.
E-mail addresses: treichelt@mathi.uni-heidelberg.de (T. Reichelt), walther@purdue.edu (U. Walther), wlzhang@uic.edu (W. Zhang).

https://doi.org/10.1016/j.topol.2021.107983
0166-8641/© 2021 Published by Elsevier B.V.

1. Introduction

Notation 1.1. Throughout we will use the following conventions: $\mathbb{K}$ will be a field of characteristic zero, and R and I are the default names for a regular $\mathbb{K}$-algebra and an ideal in R. We write

$$R_n := \mathbb{K}[x_1, \dots, x_n], \qquad X := \mathbb{A}^n_{\mathbb{K}} = \operatorname{Spec}(R_n)$$

for the special case of a polynomial ring over $\mathbb{K}$ in n indeterminates, and the associated affine space. Our default affine variety will be

$$Y := \operatorname{Var}(I) \subseteq X, \text{ with complement } U = X \smallsetminus Y,$$

and if I is homogeneous then

$$\tilde{Y} := \operatorname{Proj}(R_n/I) \subseteq \mathbb{P}^{n-1}_{\mathbb{K}}$$

will be the projective scheme associated to I, with complement $\tilde{U} := \mathbb{P}U = \mathbb{P}^{n-1}_{\mathbb{K}} \smallsetminus \tilde{Y}$. The homogeneous irrelevant ideal of R_n will be denoted $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ and d will stand for $\dim(Y)$.

The ring of $\mathbb{K}$-linear differential operators on R is denoted $D_{\mathbb{K}}(R)$, but if $R = R_n$ then we just write D_n for the Weyl algebra $D_{\mathbb{K}}(R_n)$. If X' is a smooth $\mathbb{K}$-variety, then we write $\mathcal{D}_{X',\mathbb{K}}$ for the sheaf of $\mathbb{K}$-linear differential operators on $\mathcal{O}_{X'}$, but if $\mathbb{K}$ is understood then we write just $\mathcal{D}_{X'}$. Unless specified otherwise, we consider left $\mathcal{D}_{X'}$-modules. $\diamond$

Hartshorne's seminal work [16] begins with

> "The idea of using differential forms and their integrals to define numerical invariants of algebraic varieties goes back to Picard and Lefschetz..."

and then outlines the development of this branch of mathematics until the writing of his article on algebraic de Rham cohomology. While originally the base field was the complex numbers $\mathbb{C}$, Hartshorne works in greater generality over fields $\mathbb{K}$ of characteristic zero. It has become clear since, particularly through the work of Lyubeznik [29], that Kashiwara's framework of D-modules is the right set-up for these investigations. This article is a contribution to this general theme, with the two main characters defined as follows.

If a variety Y' can be embedded into a smooth $\mathbb{K}$-variety X' of dimension n, one can define the de Rham homology and cohomology functors of Y' as

$$H^{\mathrm{dR}}_q(Y') := \mathbb{H}^{2n-q}_{Y'}(X', \Omega^\bullet_{X'}), \qquad H^q_{\mathrm{dR}}(Y') := \mathbb{H}^q(X', \hat{\Omega}^\bullet_{X'}).$$

Here, $\mathbb{H}(-)$ denotes hypercohomology functor on complexes of sheaves, $\Omega^\bullet_{X'}$ is the de Rham complex (relative to $\mathbb{K}$) of X', and the hat denotes completion along Y'. Hartshorne proves that these quantities do not depend on X' or on the chosen embedding of Y', and demonstrates many interesting facts about these two functors.

We focus on de Rham homology for a moment, under the assumption that X' is affine (and smooth). Then hypercohomology collapses to global sections since the modules in $\Omega^\bullet_{X'}$ are coherent, equal to exterior powers of the free $\mathcal{O}_{X'}$-module $\Omega^1_{X'}$ of rank n given by the Kähler differentials on X'. The set-theoretic sections-with-support functor on a coherent sheaf agrees with algebraic local cohomology. In particular, $\mathbb{H}^{2n-q}_{Y'}(X', \Omega^i_{X'})$ is just local cohomology $H^{2n-q}_{Y'}(\Omega^i_{X'})$ of the module of i-forms (identifying sheaves with their global sections).

The sheaf $\omega_X := \Omega^n_X$ has a natural *right* module structure over the sheaf $\mathcal{D}_X$ of $\mathbb{K}$-linear differential operators on X. The global sections of the sheaf of differential operators $\mathcal{D}_X$ on X are the elements of the Weyl algebra

$$D_n = R_n\langle \partial_1, \dots, \partial_n \rangle$$

where ∂_i stands for the partial differentiation operator $\frac{\partial}{\partial x_i}$. On the other hand, Ω^i_X is the free $\mathcal{O}_X$-module of rank $\binom{n}{i}$ generated by the symbols $\mathrm{d}x_I = \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_i}$ with $I \subseteq 2^{[n]}$ and $|I| = i$, and the global sections of ω_X are the elements of the right D_n-module $D_n/\partial \cdot D_n := D_n/(\partial_1, \dots, \partial_n)D_n$.

Let us write $\Omega^\bullet_{\mathcal{D},X}$ for the Koszul co-complex on $\mathcal{D}_X$ generated by left multiplication by the derivations $\partial_1, \dots, \partial_n$. This is a free resolution of right $\mathcal{D}_X$-modules for ω_X shifted right n steps, and yields an explicit form of the de Rham cohomology functors

$$H^i_{\mathrm{dR}}(-) := \mathbb{H}^{i-n}(X, \omega_X \otimes^L_{\mathcal{D}_X} (-)) = H^i(\Omega^\bullet_{\mathcal{D},X} \otimes_{\mathcal{D}_X} (-))$$

from the category of left $\mathcal{D}_X$-modules to the category of $\mathbb{K}$-vector spaces. Since the constituents of $\Omega^\bullet_{\mathcal{D},X}$ are $\mathcal{D}_X$-free and X is $\mathcal{D}_X$-affine, for each left $\mathcal{D}_X$-module $\mathcal{M}$ with global sections M one has

$$H^i_{\mathrm{dR}}(\mathcal{M}) = H^{i-n}((D_n/\partial \cdot D_n) \otimes^L_{D_n} M).$$

If $\mathcal{M}$ is holonomic, these vector spaces are $\mathbb{K}$-finite since they are the cohomology of the $\mathcal{D}$-module theoretic direct image functor under the map to a point, as one sees by inspecting the transfer module $\mathcal{D}_{pt\leftarrow X}$ (see [21, 1.3.1, 1.3.3]) and construction of direct images (see [21, p. 50]).

We return to de Rham homology $\mathbb{H}^{2n-q}_Y(X, \Omega^\bullet_X)$ with X equal to affine n-space, as always. Since Ω^j_X is finite free over $\mathcal{O}_X$, there is a natural identification of $H^i_Y(\Omega^j_X)$ with $\Omega^j_X \otimes_{\mathcal{O}_X} H^i_Y(\mathcal{O}_X)$. The complex

$$\dots \longrightarrow \Omega^{j-1}_X \otimes_{\mathcal{O}_X} H^i_Y(\mathcal{O}_X) \longrightarrow \Omega^j_X \otimes_{\mathcal{O}_X} H^i_Y(\mathcal{O}_X) \longrightarrow \Omega^{j+1}_X \otimes_{\mathcal{O}_X} H^i_Y(\mathcal{O}_X) \longrightarrow \dots$$

with differential induced by the usual exterior derivative is quasi-isomorphic to the complex $\Omega^\bullet_{\mathcal{D},X} \otimes_{\mathcal{D}_X} H^i_Y(\mathcal{O}_X)$.

Since X is affine, $\Gamma(X, -)$ induces a spectral sequence for hypercohomology,

$$H^p_{\mathrm{dR}}(H^q_Y(R_n)) \Longrightarrow \mathbb{H}^{p+q}_Y(X, \Omega^\bullet_X) = H^{\mathrm{dR}}_{2n-p-q}(Y) \tag{1.0.1}$$

that has been considered in [43, Lemma 2.16] in the complete local case, and in [20,3] in the context we are working in. We note that over the complex numbers, the abutment is naturally equal to the reduced singular cohomology of the open complement $U = U(I)$ given as $U(I) := X \setminus Y$, so there is a spectral sequence

$$E^2_{p,q} = H^p_{\mathrm{dR}}(H^q_Y(R_n)) \Longrightarrow \tilde{H}^{p+q-1}(U; \mathbb{C}) \tag{1.0.2}$$

to the reduced cohomology of U. For $I = \mathfrak{m}$, the abutment is $H^n_{\mathrm{dR}}(H^n_I(R_n)) = \mathbb{C}[2n]$, the reduced cohomology of the $(2n-1)$-sphere shifted by one. For details see for example [16, p. 67], [27, Thm. 3.1], or [20, Prop. 4.2].

The articles [43,3] proceed to show that the E_r-pages, $r \geq 2$, of these spectral sequences are isomorphic for all embeddings of Y. In consequence, the terms on pages $r \geq 2$ of (1.0.1) are numerical invariants of Y.

Definition 1.2. Let $Y = \operatorname{Var}(I)$ be an affine variety embedded in $X = \operatorname{Spec}(R_n)$ defined by the ideal $I \subseteq R_n = \mathbb{K}[x_1, \dots, x_n]$ over the field $\mathbb{K}$ of characteristic zero. For $r \geq 2$, the (r, p, q)-*Čech–de Rham number* of Y is the dimension

$$\rho^r_{p,q}(Y) := \dim_{\mathbb{K}}(E_r^{n-p,n-q})$$

of the corresponding entry in the spectral sequence (1.0.1). If $r = 2$ we denote $\rho^r_{p,q}(Y) = H^{n-p}_{\mathrm{dR}}(H^{n-q}_I(R_n))$ by just $\rho_{p,q}(Y)$. $\diamond$

Switala defined these for ideals in the power series ring [43, Dfn. 2.23]; they are well-defined by [43, Prop. 2.17] and [3, Thm. 1.1]. The dimensions $\rho^r_{p,q}$ are invariant under field extensions, and one can compute them algorithmically over any field of definition for I, see [36,37,44].

A related construction appeared in [29], where Lyubeznik shows that the socle dimensions of the E_2-terms of the Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{m}}(H^q_I(R_n)) \Longrightarrow H^{p+q}_{\mathfrak{m}}(R_n) \tag{1.0.3}$$

are independent of the closed embedding of $Y = \operatorname{Spec}(R_n/I)$ into any affine space $\mathbb{A}^n_{\mathbb{K}} = \operatorname{Spec}(R_n)$ and uses it to define numerical invariants

$$\lambda_{p,q}(R_n/I) := \dim_{\mathbb{C}} \operatorname{Hom}(R_n/\mathfrak{m}, H^p_{\mathfrak{m}}(H^{n-q}_I(R_n))).$$

These numbers, known as *Lyubeznik numbers* have been investigated for nearly three decades and are indeed functions of the ring R/I (and do not depend on the presentation of R/I as a quotient of a polynomial ring). For detailed information on the history and the *status quo* we refer to the survey articles [33,46].

In this article we develop further the theory of the Lyubeznik numbers on one side, and on the other describe a number of properties that the invariants introduced by Switala and Bridgland enjoy.

More precisely, in the next section we study vanishing of the Čech–de Rham numbers, explore them for small dimension of Y, and investigate the collapse of the corresponding spectral sequence. We identify classes of examples where this collapse happens on the E_2-page, and explain why this is so for subspace arrangements, by stringing together known results of Goresky–MacPherson, and Àlvarez–García–Zarzuela. We further explore the behavior of the Čech–de Rham numbers under Veronese maps and deduce that most of the Čech–de Rham numbers associated to the affine cone over a given projective variety $\tilde{Y}$ only depend on the class of the line bundle that the cone choice induces on $\tilde{Y}$.

In the third section we discuss Lyubeznik numbers. We elaborate on the results from [40] by establishing some classes of projective varieties $\tilde{Y}$ with Picard number one that have almost all Lyubeznik numbers of the affine cone Y independent of the chosen cone. This includes determinantal varieties, certain toric varieties, and horospherical varieties. We also prove for certain projective varieties of dimension four or less that their Lyubeznik numbers are independent of the embedding.

Some known facts.

Since we will have to refer to them a few times, we state here some results from the literature.

Remark 1.3. (1) If $\mathbb{K}$ is of characteristic zero, then local cohomology, algebraic de Rham cohomology, injective dimension, dimension, socle dimension all behave well under field extensions. Since all varieties are defined by a finite number of data, one can restrict all questions we discuss from the given field $\mathbb{K}$ to a field of definition for I, and then extend to $\mathbb{C}$. In particular, we can assume that $\mathbb{K} = \mathbb{C}$ whenever it is convenient.

(2) ([29, (4.4.iii)]) Suppose $Y \subseteq X = \mathbb{A}^n_{\mathbb{K}}$ is an affine variety. Then the local cohomology module $H^i_I(R_n)$ has support dimension at most $n - i$, and it vanishes if $i < c := \operatorname{codim}(Y, X)$. If Y is equi-dimensional and $i > c$, then $H^i_I(R_n)$ has support dimension less than $n - i$.

(3) ([29, Thm. 2.4] in the power series case; the polynomial case reduces to this) If M is a holonomic D_n-module, then $H^i_{\mathfrak{m}}(M)$ is a finite sum of copies of the (Artinian, indecomposable) injective hull $H^n_{\mathfrak{m}}(R_n)$ of $\mathbb{K} = R_n/\mathfrak{m}$. More generally, one has for all D_n-modules that

$$\operatorname{injdim}_R(M) \leq \dim \operatorname{Supp}(M).$$

Thus, all right derived functors of R_n-modules with derivation level greater than $n - i$ vanish on $H^i_I(R_n)$, and those of derivation level $n - i$ vanish if I is equi-dimensional and $i > c$.

(4) If G is a group acting linearly on R_n and stabilizes I, the local cohomology modules $H^i_I(R_n)$ become (strongly) equivariant D_n-modules. For details and references on equivariance of $\mathcal{D}$-modules, see for example [28, Section 2.1].

(5) Let $I \subseteq R_n = \mathbb{K}[x_1, \ldots, x_n]$ be a homogeneous ideal such that $\dim(R_n/I) \geq 2$. Assume that $\mathbb{K}$ is separably closed. Hartshorne proved in [15, Theorem 7.5] that if $\operatorname{Proj}(R_n/I)$ is connected then $H^n_I(R_n) = H^{n-1}_I(R_n) = 0$, and named this result the Second Vanishing Theorem. This theorem subsequently has been extended to the local settings as follows: Let R be either a complete regular local ring of dimension n that contains a separably closed coefficient field or an unramified complete regular local ring of dimension n in mixed characteristic with a separably closed residue field. Let $I \subseteq R$ be an ideal. Then $H^n_I(R) = H^{n-1}_I(R) = 0$ if and only if $\dim(R/I) \geq 2$ and the punctured spectrum of R/I is connected, [34,39,18,49].

(6) Over $\mathbb{C}$, the local cohomology groups $H^\bullet_I(R_n)$ are (up to shift) the global sections of the pushforward of the structure sheaf on the open set U to $\mathbb{C}^n$, which carry a natural mixed Hodge module structure. The corresponding perverse sheaves encode information on the intersection cohomology of U and this can be used to study Lyubeznik and Čech–de Rham numbers in characteristic zero, see [40]. $\diamond$

The following is a special case of a more general result comparing direct image to a point and restriction to a point.

Lemma 1.4 *([41, Lemma 3.3]). Suppose $\mathbb{K} = \mathbb{C}$. Assume that $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module on $X = \mathbb{C}^n$ and that its global sections M form a standard graded R_n-module. Suppose further that $\mathcal{M}$ is (strongly) equivariant as a $\mathcal{D}_X$-module with respect to the $\mathbb{C}^*$-action corresponding to this grading. Then its de Rham cohomology groups agree with the restriction groups to the origin of the holonomically dual module. In particular, the dimensions of these groups satisfy*

$$\dim_{\mathbb{C}}(H^i_{\mathrm{dR}}(\mathcal{M})) = \dim_{\mathbb{C}}(\operatorname{Hom}_{R_n}(R_n/\mathfrak{m}, H^{n-i}_{\mathfrak{m}}(\mathbb{D}(M))),$$

where $\mathbb{D}$ is the holonomic duality functor. $\square$

2. Čech–de Rham numbers

2.1. Basic structure results

Basic properties of the de Rham functor imply that $\rho^r_{p,q}$ is zero for p outside the interval $[0, n]$. On the other hand, local cohomology $H^j_I(R_n)$ is nonzero only when $\operatorname{codim}(I, R_n) \leq j \leq n$, and so $\rho^r_{p,q}$ is zero for q outside the interval $[0, \dim(Y)]$. Our first statement on these numbers is that they are confined to a triangular region:

Theorem 2.1. *The Čech–de Rham numbers satisfy for all $r \geq 2$ that*

$$\rho^r_{p,q}(Y) = 0 \qquad \text{if } p > \dim \operatorname{Supp}(H^{n-q}_I(R_n)).$$

In particular, this vanishing occurs whenever $p > q$.

Before entering the proof we set up some notation and collect several facts and from [7,21,25] on constructible sheaves and the Riemann–Hilbert correspondence. All spaces mentioned in the sequel are assumed to be algebraic varieties.

Remark 2.2. Let X be a smooth algebraic variety.

(1) For any algebraic map f between algebraic sets we denote, on the level of constructible sheaves, the usual direct and inverse image functors by f_* and f^{-1}, and the proper direct and exceptional inverse image functors by $f_!$ and $f^!$ respectively. For the sake of notational brevity, we mean by these symbols always the derived functors on the appropriate derived categories (so that, for example, we write j_* instead of Rj_* as a functor on the bounded derived category of constructible sheaves). This abuse of notation is common in the relevant literature.

(2) On the level of $\mathcal{D}$-modules, we will use f_+ and $f_!$ for the usual and proper direct image functors, and f^+ and $f^\dagger$ for the usual and exceptional inverse image functors. For reference and comparison, our $\mathcal{D}$-functors $f_+, f_!, f^+, f^\dagger$ are (in this sequence) denoted by $\int_f, \int_{f!}, f^\dagger, f^\star$ in [21].

(3) Let X' be a smooth variety. The Riemann–Hilbert correspondence sets up an equivalence between the derived category of bounded complexes of $\mathcal{D}_{X'}$-modules with holonomic cohomology, and the derived category of bounded complexes of constructible sheaves $D^b_{\text{c.s.}}(X')$. The correspondence is induced by the de Rham functor $\Omega^\bullet_{X'^{an}} \otimes^L_{\mathcal{D}_{X'^{an}}} (-)$ computed on the analytic space attached to X'.

Under this correspondence, taking cohomology of a complex of $\mathcal{D}_{X'}$-modules corresponds to an operation on complexes of constructible sheaves that is denoted ${}^p\mathcal{H}$ and called *taking perverse cohomology*. Perverse cohomology of a complex of constructible sheaves is not the same as the usual cohomology. The perverse cohomology of a complex is a *perverse sheaf*, but most perverse sheaves are not representable by a single module but are a proper complex (see [7, Def. 4.5.10, Thm. 7.2.5, 8.1.28]). We call *perverse exact* any functor on the derived category of the category of constructible sheaves that commutes with ${}^p\mathcal{H}$.

(4) Suppose $f \colon X' \longrightarrow X''$ is a morphism of smooth algebraic varieties. Under the Riemann–Hilbert correspondence, the functors for $\mathcal{D}$-modules correspond to those on constructible sheaves as follows:

$$\begin{aligned} DR_{X''} \circ f_+ &\simeq f_* \circ DR_{X'}; \qquad & DR_{X''} \circ f_! &\simeq f_! \circ DR_{X'}; \\ DR_{X'} \circ f^+ &\simeq f^! \circ DR_{X''}; \qquad & DR_{X'} \circ f^\dagger &\simeq f^{-1} \circ DR_{X''}\,. \end{aligned}$$

(The last two identifications are not misprints; for inverse images, the Riemann–Hilbert correspondence via the de Rham functor aligns a regular inverse image with an exceptional one).

(5) Consider an open embedding $j \colon U \hookrightarrow X$ and a closed embedding $i \colon Y \hookrightarrow X$ where Y is closed (and, *a fortiori*, constructible) and where U is the complement of Y in X. We have the following properties of induced functors for complexes of constructible sheaves:

- $i_! = i_*$ is perverse exact ([7, Thm. 5.2.4]) and exact (since i is a closed embedding);
- i^{-1} is exact ([7, Rmk. 2.3.8]) but usually not perverse exact;

- $i^!$ and j_* are usually neither exact nor perverse exact;
- $j^{-1} = j^!$ is exact and perverse exact ([7, Rmk. 2.3.8, Thm. 5.2.4]);
- $j_!$ is exact (clear from the definition) but usually not perverse exact.

(6) In the situation of item (5), we have the following distinguished triangles, Verdier dual to one another, in $D^b_{\text{c.s.}}(X)$:

$$i_!i^!F^\bullet \longrightarrow F^\bullet \longrightarrow j_*j^{-1} \xrightarrow{+1},$$
$$j_!j^{-1}F^\bullet \longrightarrow F^\bullet \longrightarrow i_!i^{-1}F^\bullet \xrightarrow{+1} .$$

(7) We will always denote by a_S the map from a space S to a point, which we sometimes denote with pt and occasionally identify with the vertex of a cone if a cone is present. $\diamond$

We now enter the

Proof of Theorem 2.1. It suffices to consider $r = 2$. We will use the Riemann–Hilbert correspondence to translate $\rho_{p,q} = \dim_{\mathbb{C}}(H^{n-p}_{\text{dR}}(H^{n-q}_I(R_n)))$ into the language of constructible sheaves. The de Rham functor takes the local cohomology $H^{n-q}_I(\mathcal{O}_X)$ to ${}^p\mathcal{H}^{n-q}h_!h^!\mathbb{C}_X[n] \simeq h_!({}^p\mathcal{H}^{-q}\omega_Y)$ where $h : Y \to X = \mathbb{A}^n_{\mathbb{C}}$ is the canonical embedding, $\mathbb{C}_X[n]$ is the constant sheaf on X shifted to the left by n and

$$\omega_Y = \mathbb{D}\mathbb{C}_Y \tag{2.1.1}$$

is the (topological) dualizing complex $R\operatorname{Hom}_{\text{c.s.}}(\mathbb{C}_Y, \mathbb{C}_Y)$ for constructible sheaves on Y. (We use $\mathbb{D}$ also to denote Verdier duality, the operation corresponding to holonomic duality under the Riemann–Hilbert correspondence).

The theorem will follow from a more general fact that can be seen as a companion result to [29, Thm. 2.4]:

Lemma 2.3. *Let M be a regular holonomic D_n-module. Then $H^{n-p}_{\text{dR}}(M) = 0$ if $p > \dim \operatorname{Supp}(M)$.*

Proof. Denote by $\mathcal{M}$ the $\mathcal{D}_X$-module corresponding to M, let Z be the support of M and write $\mathcal{C}_M$ for the perverse sheaf on X that corresponds to $\mathcal{M}$ under the Riemann–Hilbert correspondence. Then Z is closed and $\operatorname{Supp}(M) = \operatorname{Supp}(\mathcal{M}) = \operatorname{Supp}(\mathcal{C}_M)$. Write

$$i_Z \colon Z \hookrightarrow X, \qquad j_Z \colon (X \setminus Z) \hookrightarrow X.$$

There is an exact triangle

$$R\Gamma_Z \longrightarrow \text{id} \longrightarrow (j_Z)_+(j_Z)^\dagger \xrightarrow{+1}$$

for $\mathcal{D}_X$-modules, that corresponds via Riemann–Hilbert to

$$(i_Z)_!(i_Z)^! \longrightarrow \text{id} \longrightarrow (j_Z)_*(j_Z)^{-1} \xrightarrow{+1}$$

for constructible sheaves. (The advantage of the use of constructible sheaves is that one can talk about them on singular spaces).

Since $(j_Z)^{-1}(\mathcal{C}_M) = 0$, one has $(i_Z)_!(i_Z)^!(\mathcal{C}_M) = \mathcal{C}_M$. But i_Z is proper, so $(i_Z)_*(i_Z)^!(\mathcal{C}_M) = \mathcal{C}_M$ since $(i_Z)_* = (i_Z)_!$. This shows that the hypercohomology of $\mathcal{C}_M$ (computed on X) equals the hypercohomology of $(i_Z)^!(\mathcal{C}_M)$ (which is computed on Z).

For every perverse sheaf $\mathcal{F}$ on Z, the hypercohomology $\mathbb{H}^k(Z,\mathcal{F})$ vanishes for $k \notin [-\dim(Z), 0]$ (see e.g. [7, Cor. 5.2.18] and [7, Prop. 5.2.20]). The lemma follows now from

$$\dim_{\mathbb{C}}(H^{n-p}_{\mathrm{dR}}(M) = \dim_{\mathbb{C}}(\mathbb{H}^{-p}(X,\mathcal{C}_M)) = \dim_{\mathbb{C}}(\mathbb{H}^{-p}(Z,(i_Z)^!(\mathcal{C}_M))). \quad \square$$

Theorem 2.1 now follows with $M = H^{n-q}_I(R_n)$ since $\dim(\mathrm{Supp}(h_!({}^p\mathcal{H}^{-q}\omega_Y))) = \dim(\mathrm{Supp}(H^{n-q}_I(R_n)))$ $\leq q$ by Remark 1.3. $\square$

If one pictures the $\rho^r_{p,q}$ as a table, it thus takes the following general form, assuming that Y is embedded into $\mathbb{A}^n_{\mathbb{K}}$, cut out by the ideal $I \subseteq R_n$ of dimension d:

$$P^r(Y) = ((\rho^r_{p,q})) := \begin{pmatrix} \rho^r_{0,0} & \cdots & \cdots & \rho^r_{0,d} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \rho^r_{d,d} \end{pmatrix} \tag{2.1.2}$$

Here, p is the row index counting downward, q the column index counting towards the right, and the arrows of the Čech–de Rham spectral sequence point North to Northeast.

2.2. Degeneration

Switala raised in [43, Question 8.2] the following question for a complete local ring A with coefficient field $\mathbb{K}$ of characteristic zero:

"Does the Čech–de Rham homology (1.0.2) spectral sequence degenerate at E_2?"

One can ask a similar question for the affine scenario. We discuss interesting classes where this question has a positive answer.

Degeneration is certain if only one $H^j_I(R_n)$ is nonzero (whence I must be equi-dimensional of codimension j), for example for local complete intersections. Another example arises when I is equi-dimensional, and either has isolated singularities or is a local complete intersection outside a finite number of points. Indeed, then $H^{>\mathrm{codim}(I,R_n)}_I(R_n)$ is supported inside these points, hence these local cohomology modules are sums of copies of the R_n-injective hull of the residue field at these points. For such modules, de Rham cohomology is only nonzero in degree n, and that implies that for all differentials in (1.0.2) either the target or the source are zero. (So, in the table (2.1.2), the only nonzero terms are in the top row and rightmost column). See [20, Thm. 4.3] for more details. In light of Theorem 2.1, one obtains in the same way:

Corollary 2.4. *The Čech–de Rham spectral sequence degenerates on the E_2-page if $Y \subseteq X = \mathbb{C}^n$ is equi-dimensional and the singular locus has dimension at most* 1.

Proof. Nonzero entries on the E_2-page only exist then in rows 0 and 1, and in column d. $\square$

Example 2.5. Suppose $Y = \mathrm{Var}(I)$ is a complex subspace arrangement. Let P_Y be its *intersection lattice*, the collection of all possible intersections of the components of Y, ordered by inclusion. (This differs from standard notation in arrangement theory, where the order is the reverse). We agree that P_Y has a unique maximal element corresponding to the ambient space, but it may have several minimal elements as we do not insist that I be homogeneous (so, the arrangement may not be central).

It is well-known that the cohomology of the complement $\mathbb{C}^n \smallsetminus Y$ is determined by the combinatorics of P: building on work of Brieskorn, Orlik and Solomon [35] showed that the cohomology algebra of this complement is given by a purely combinatorial algebra constructed from the matroid of the arrangement.

Goresky and MacPherson [11, III, Thm. 1.3] proved that the Betti numbers of the complement can be computed as a sum of non-negative integers, one for each element of P_Y. Here, the integers for each flat $p \in \mathsf{P}$ are computed as Betti numbers of the simplicial complex $K(> p)$. (While Goresky and MacPherson phrase this in terms of relative homology for the pair $(K(\geq p), K(> p))$, the space $K(\geq p)$ is contractible and one can convert into an absolute homology without harm).

Àlvarez, García and Zarzuela established the degeneration on page two of a certain spectral sequence

$$E_2^{-i,j} = \varprojlim_{p \in \mathsf{P}_Y}{}^{(i)} H^j_{I_p}(R_n) \Longrightarrow H^{j-i}_I(R_n) \tag{2.2.1}$$

for the local cohomology groups $H^\bullet_I(R_n)$, the inverse limits being taken over the poset P_Y viewed as a category with a morphism for each containment. In [1, Thm. 1.2], the structure of the derived inverse limits is explained as direct sums of modules $H^j_{I_p}(R_n)$ with $\operatorname{codim}(I_p, R_n) = j$ and multiplicity given by the topological Betti numbers of $K(> p)$. In [1, Cor. 1.3], this is used to give a formula for the cohomology groups of the complement of Y, by translating the Goresky–MacPherson formula.

The affine complement of an affine space is homotopy equivalent to a sphere, hence applying the de Rham functor to a module of the form $H^j_{I_p}(R_n)$ gives exactly one (reduced) cohomology group. Thus, the entries of the E_2-page of the Čech–de Rham spectral sequence (1.0.2) correspond exactly to the composition factors of $H^{j-i}_I(R_n)$ in the spectral sequence (2.2.1) from [1] on one side, and to the direct summands for $H^\bullet(\mathbb{C}^n \smallsetminus Y)$ in [11] on the other. It follows that for complex subspace arrangements Y the Čech–de Rham spectral sequence collapses on the E_2-page. $\diamond$

In small dimensions we show that Switala's question has a positive answer as well.

Proposition 2.6. *If I is homogeneous and $\dim(\operatorname{Var}(I)) \leq 3$ then the Čech–de Rham spectral sequence degenerates at E_2.*

Proof. Let Y be of dimension 2 or less. If follows from Theorem 2.1 that no nonzero differential can exist in the spectral sequence.

Let now $\dim(Y) = 3$. Then Theorem 2.1 implies that then there might be at most one nonzero differential,

$$d_2 \colon H^{n-2}_{\mathrm{dR}}(H^{n-2}_I(R_n)) \longrightarrow H^n_{\mathrm{dR}}(H^{n-3}_I(R_n)), \tag{2.2.2}$$

linking $\rho_{2,2}$ and $\rho_{0,3}$.

Assume for the time being that Y is purely 3-dimensional. Remark 1.3 says that $\dim \operatorname{Supp}(H^{n-i}_I(R_n)) < i$ for $i < 3$. In particular, by Lemma 1.4, $\dim H^{n-2}_{\mathrm{dR}}(H^{n-2}_I(R_n))$ equals the socle dimension of $H^2_{\mathfrak{m}}(\mathbb{D} H^{n-2}_I(R_n)) = 0$. Thus, the degeneration of the spectral sequence is forced.

Now relax the equi-dimensionality condition and let Y_3 and Y' be the 3-dimensional and smaller dimensional components of Y respectively. Then $Y_3 \cap Y'$ is of dimension 1 or less, and the Mayer–Vietoris sequence implies that $H^{n-3}_{Y_3}(R_n) \oplus H^{n-3}_{Y'}(R_n) = H^{n-3}_Y(R_n)$ and that there is a short exact sequence

$$0 \longrightarrow H^{n-2}_{Y_3}(R_n) \oplus H^{n-2}_{Y'}(R_n) \longrightarrow H^{n-2}_Y(R_n) \longrightarrow C \longrightarrow 0$$

where C is a (graded) submodule of $H^{n-1}_{Y_3 \cap Y'}(R_n)$. In particular, the dimension of the support of C is one or less by Remark 1.3 and so $H^{\leq n-2}_{\mathrm{dR}}(C)$ is zero, being dual to the socle of $H^{\geq 2}_{\mathfrak{m}}(C) = 0$.

Applying the de Rham functor, the resulting long exact sequence shows that $H^{n-2}_{\mathrm{dR}}(H^{n-2}_{Y_3}(R_n) \oplus H^{n-2}_{Y'}(R_n))$ equals $H^{n-2}_{\mathrm{dR}}(H^{n-2}_{Y}(R_n))$. Then the map (2.2.2) is the direct sum of the corresponding d_2-morphisms for Y_3 and for Y' separately. But it is zero on $H^{n-2}_{\mathrm{dR}}(H^{n-2}_{Y_3}(R_n))$ since the source of d_2 is zero in that case, and it is zero on $H^{n-2}_{\mathrm{dR}}(H^{n-2}_{Y'}(R_n))$ since the target is zero in that case. □

2.3. *Affine complements*

In the next two Subsections 2.3.1 and 2.3.2, we investigate to what extent the cohomology of the affine complement, or its table of Čech–de Rham numbers, of a homogeneous variety Y is determined by the associated projective variety $\tilde{Y}$. We start in Subsection 2.3.1 with looking at the top cohomology group of the affine complement, and then investigate in Subsection 2.3.2 the affine complement under Veronese maps. In the process we review some algorithmic ideas that lead to a condition on the de Rham classes of graded $\mathcal{D}_X$-modules on affine space.

So, throughout, $\tilde{Y}$ is a projective variety and $Y \subseteq \mathbb{C}^n$ is a cone for $\tilde{Y}$.

2.3.1. *High cohomology groups of the affine complement*

Remark 2.7. Let $\tilde{Y}$ be a projective variety with cone $Y = \operatorname{Spec}(R_n/I)$. The following facts are due to Ogus [34] Let

$$f_Y := \min(k \in \mathbb{N} | H^\ell_I(R_n) \text{ is Artinian for all } \ell > k)$$

and

$$v_Y := \min(k \in \mathbb{N} | H^\ell_I(R_n) \text{ is zero for all } \ell > k).$$

(1) The number $n - f_Y$ is intrinsic to $\tilde{Y}$, it does not depend on the choice of the cone Y, [34, Thm. 4.1].
(2) The number $n - v_Y$ is intrinsic to $\tilde{Y}$, it does not depend on the choice of the cone Y, [34, Thm. 4.4] and the remark following it.

In particular,

$$\rho^r_{p,q} = 0 \quad \begin{cases} \text{if} & q < n - \nu_Y, \\ \text{or} & p > 0 \text{ and } q < n - f_Y. \end{cases} \qquad \diamond$$

We show next that in fact the top de Rham cohomology group of the affine cone complement is usually determined by $\tilde{Y}$.

Lemma 2.8. *Let $X = \mathbb{A}^n_{\mathbb{C}}$ and suppose $\tilde{Y} \subseteq \mathbb{P}X = \mathbb{P}^{n-1}_{\mathbb{C}}$ is defined by the homogeneous ideal $I \subseteq R_n := \Gamma(X, \mathcal{O}_X)$. Let $Y = \operatorname{Var}(I) \subseteq X$ and assume that Y has codimension at least two. Then the index and the dimension of the top non-vanishing de Rham cohomology group of $U := X \smallsetminus Y$ is encoded on $\tilde{Y}$.*

Proof. We recall Alexander duality, compare [22, V.6.6]: if $\mathbb{P}$ is a $\mathbb{C}$-orientable manifold and $\tilde{Y}$ a closed subset then the topological local cohomology group $H^i_{\tilde{Y}}(\mathbb{P}; \mathbb{C})$ is canonically identified with the $\mathbb{C}$-dual of the cohomology with compact support $H_c^{2\dim_{\mathbb{C}}\mathbb{P}-i}(\tilde{Y}; \mathbb{C})$. If $\tilde{Y}$ is, in addition, compact, the latter is just $H^{2\dim_{\mathbb{C}}\mathbb{P}-i}(\tilde{Y}; \mathbb{C})$.

On the other hand, [22, II.9.2] states the existence of a long exact sequence

$$H^i_{\tilde{Y}}(P; \mathcal{F}) \longrightarrow H^i(P; \mathcal{F}) \longrightarrow H^i(P \smallsetminus \tilde{Y}; \mathcal{F}) \xrightarrow{+1} \tag{2.3.1}$$

where $\mathcal{F}$ is a sheaf of Abelian groups on P and we ease notation by ignoring the pull-backs of $\mathcal{F}$ to $\tilde{Y}$ and its complement respectively. Notice that one can get this long exact sequence by applying hypercohomology to the first triangle in Remark 2.2 (4). We use these with $\mathbb{P} = \mathbb{P}X$ and $\tilde{Y}$ as above.

Via Poincaré duality, the map $H^i_{\tilde{Y}}(\mathbb{P}X;\mathbb{C}) \longrightarrow H^i(\mathbb{P}X;\mathbb{C})$ becomes $H^{2n-2-i}(\tilde{Y};\mathbb{C})^\vee \longrightarrow H^{2n-2-i}(\mathbb{P}X;\mathbb{C})^\vee$. This is the dual of $H^{2n-2-i}(\mathbb{P}X;\mathbb{C}) \longrightarrow H^{2n-2-i}(\tilde{Y};\mathbb{C})$ induced by restriction from $\mathbb{P}X$ to $\tilde{Y}$. The restriction $H^i(\mathbb{P}X;\mathbb{C}) \longrightarrow H^i(\tilde{Y};\mathbb{C})$ is injective[1] for $i \leq 2\dim\tilde{Y}$ and necessarily zero for $i > 2\dim_{\mathbb{C}}(\tilde{Y})$ since $\tilde{Y}$ is a CW-complex of dimension $2\dim_{\mathbb{C}}(\tilde{Y})$. Thus, one can determine from the topological Betti numbers of $\tilde{Y}$ alone the sizes of the kernels of the left-most morphisms in display (2.3.1). This in turn determines the sizes of the cohomology groups of $\mathbb{P}U := \mathbb{P}X \smallsetminus \tilde{Y}$.

As $\operatorname{codim}(Y, \mathbb{P}^{n-1}_{\mathbb{C}}) \geq 2$, U is simply connected by [12, Thm. 2.3]. Thus, the $\mathbb{C}^*$-fiber bundle $U \longrightarrow \mathbb{P}U$ has a Leray spectral sequence

$$H^i(\mathbb{P}U; H^j(\mathbb{C}^*;\mathbb{C})) \Longrightarrow H^{i+j}(U;\mathbb{C})$$

in which the coefficients on the left are global (in a trivial vector bundle). Let m be the largest index with $H^m(\mathbb{P}U;\mathbb{C}) \neq 0$. Since all differentials out of and into $H^m(\mathbb{P}U; H^1(\mathbb{C}^*;\mathbb{C})) \neq 0$ are zero, $m+1$ must be the largest index with $H^{m+1}(U;\mathbb{C}) \neq 0$ and $\dim_{\mathbb{C}} H^m(\mathbb{P}U;\mathbb{C}) = \dim_{\mathbb{C}} H^{m+1}(U;\mathbb{C})$. $\square$

Corollary 2.9. *Let Y be an affine variety defined by the homogeneous ideal $I \subseteq R_n = \mathbb{C}[x_1,\ldots,x_n]$. If $H^{>\ell}_I(R_n) = 0$ then the socle dimension s of $H^\ell_I(R_n)$ is encoded in the projective variety $\tilde{Y} = \mathbb{P}(Y)$ and does not depend on the choice of the cone Y.*

Proof. Let $X = \operatorname{Spec}(R_n)$ and set $U = X \smallsetminus Y$. By [27, Thm. 3.1], $s = \dim_{\mathbb{C}} H^{n+\ell-1}(U;\mathbb{C})$, and U has no higher non-vanishing singular cohomology groups. Then Lemma 2.8 implies that s is encoded on $\tilde{Y}$. $\square$

2.3.2. Integrals of Eulerian modules

We investigate next to what extent the $\rho^r_{p,q}$, or the abutment terms $H^{\mathrm{dR}}_\bullet(Y)$ of the Čech–de Rham spectral sequence are independent of the cone Y (*i.e.*, the line bundle $\mathcal{L}$ on $\tilde{Y}$ that induces the cone). In the following we show that replacing $\mathcal{L}$ by a power of itself does not change the $H^{\mathrm{dR}}_\bullet(Y)$.

For this we give an account on the main results on algorithmic computation of the integral of a D_n-module along $\partial_1,\ldots,\partial_n$. See [36,37,44] for details, and a generalization to the case when M is a bounded complex of finitely generated modules that has holonomic cohomology.

We define a grading $\operatorname{gr}^i_{\tilde{V}}(D_n) := \{P \in D_n \,|\, \deg(P) = i\}$ on D_n by setting

$$\deg(x_j) = 1 = -\deg(\partial_j)$$

for all $1 \leq j \leq n$. With it we define a filtration on D_n by

$$\tilde{V}^k(D_n) = \sum_{i \leq k} \operatorname{gr}^i_{\tilde{V}}(D_n).$$

Let M be a D_n-module, finitely generated by elements $m_1,\ldots,m_r$, and choose integers $s_1,\ldots,s_r$. Then define a filtration on M by setting

[1] The cohomology fundamental class of Y in $H^{2\dim_{\mathbb{C}}(\tilde{Y})}(\mathbb{P}X;\mathbb{C})$ evaluates on the homology class of a generic $\mathbb{P}^{n-1-\dim_{\mathbb{C}}(\tilde{Y})} \subseteq \mathbb{P}X$ to the 0-cycle given by the intersection of $\tilde{Y}$ with that generic subspace. But this intersection is the degree of $\tilde{Y}$, hence positive. Thus the restriction of the class represented by this subspace on $\mathbb{P}X$, a generator of $H^{2\dim_{\mathbb{C}}(\tilde{Y})}(\mathbb{P}X;\mathbb{C})$, to $\tilde{Y}$ is nonzero. But cohomology of projective space is a polynomial algebra in the hyperplane section, and if the $\dim_{\mathbb{C}}(\tilde{Y})$-power of the hyperplane restricts to a nonzero class on $\tilde{Y}$ then so do all smaller powers.

$$\tilde{V}^k(M) = \sum_{i=1}^{r} \tilde{V}^{k-s_i}(D_n) \cdot m_i.$$

Denote the operator $-\sum_{j=1}^{n} \partial_j \cdot x_j$ by $\tilde{E}$.

It is a result of Kashiwara [23] that when M is holonomic there is a *b-function for integration* $\tilde{b}_M(s)$. This is a univariate polynomial that satisfies

$$\tilde{b}_M(\tilde{E} + n + k) \cdot \tilde{V}^k(M) \subseteq \tilde{V}^{k-1}(M) \tag{2.3.2}$$

for all $k \in \mathbb{Z}$. We describe now ideas that lead to a proof for Proposition 2.12 below.

As before, let ω_n be the right D_n-module $(D_n/\partial \cdot D_n)$ where $\partial = \{\partial_1, \ldots, \partial_n\}$. This is a free rank one R_n-module, and can be naturally identified with the D_n-module $\operatorname{Ext}^n_{D_n}(R_n, D_n)$ (and with the global sections of the right $\mathcal{D}_X$-module of top differential forms $\mathcal{O}_X \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$ that is denoted Ω_X in [21]). Give it a $\tilde{V}$-filtration by placing the generator $1 + \partial \cdot D_n$ into $\tilde{V}$-level n.

The D-module theoretic direct image functor π_+ for the projection map $\pi \colon \mathbb{C}^n \longrightarrow \mathbb{C}^0$ can on global sections be identified with $\omega_n \otimes^L_{D_n} (-)$ shifted by n, computing the Tor-functors against ω_n. This derived tensor product can be viewed as the tensor product of ω_n with a free D_n-resolution $F^\bullet$ of the input module M, or of a free resolution $K^\bullet$ of ω_n with M, or of the tensor product of $K^\bullet$ with $F^\bullet$. There are natural morphisms from the last scenario to the two former ones that induce isomorphisms on cohomology.

One major difficulty in identifying $\pi_+(M)$ is that its homology consists of finite-dimensional vector spaces with no further module structure, while the modules that appear in the complex are infinite-dimensional vectors spaces with no further module structure.

A free resolution $F^\bullet$ of M is *$\tilde{V}$-strict* if each F^i is equipped with a $\tilde{V}$-filtration $\tilde{V}(F^i)$ such that every differential $\delta^i \colon F^i \longrightarrow F^{i+1}$ satisfies $\delta^i(\tilde{V}^k(F^i)) \subseteq \tilde{V}^k(F^{k+1})$ and moreover $\delta^i(F^i) \cap \tilde{V}^k(F^{i+1}) = \delta^i(\tilde{V}^k(F^i))$. It is a theorem of algorithmic algebraic analysis that finitely generated V-filtered D_n-modules do allow $\tilde{V}$-strict resolutions of finite length. The $\tilde{V}$-filtration on $F^\bullet$ induces a quotient filtration on $\omega_n \otimes_{D_n} F^\bullet$. This filtered complex may not be strict anymore, but still the morphisms will respect the filtration. The $\tilde{V}$-filtration on $\omega_n \otimes_{D_n} F^\bullet$ is bounded below while on $F^\bullet$ it is not. Moreover, $\operatorname{gr}^k_{\tilde{V}}(F^i)$ is infinite dimensional over $\mathbb{C}$, while each $\operatorname{gr}^k_{\tilde{V}}(\omega_n \otimes_{D_n} F^i)$ is $\mathbb{C}$-finite. Nonetheless, the $\mathbb{C}$-dimension of each $\omega_n \otimes_{D_n} F^i$ is still infinite.

Let ℓ be the largest and s the smallest integral root of the b-function $\tilde{b}_M(s)$.

Theorem 2.10 *(Integration Theorem [36,37]). With notation as introduced above, the morphisms*

$$\omega_n \otimes_{D_n} F^\bullet \hookleftarrow \tilde{V}^\ell(\omega_n \otimes_{D_n} F^\bullet) \twoheadrightarrow \tilde{V}^\ell(\omega_n \otimes_{D_n} F^\bullet)/\tilde{V}^{s-1}(\omega_n \otimes_{D_n} F^\bullet)$$

are quasi-isomorphisms.

In other words, every cohomology class of $\operatorname{Tor}^{D_n}_\bullet(\omega_n, M)$ *has a representative inside* $\tilde{V}^\ell(\omega_n \otimes_{D_n} F^\bullet)$*, and the complex* $\tilde{V}^{s-1}(\omega_n \otimes_{D_n} F^\bullet)$ *is exact.*

Note that the subquotient complex $\tilde{V}^\ell(\omega_n \otimes_{D_n} F^\bullet)/\tilde{V}^{s-1}(\omega_n \otimes_{D_n} F^\bullet)$ is, in contrast to $\omega_n \otimes F^\bullet$, $\mathbb{C}$-finite, reducing the computation of $\pi_+(M)$ to finite-dimensional linear algebra in this subquotient complex.

One can now just as well resolve ω_n and M, or just ω_n, and obtain other complexes that represent $\omega_n \otimes^L_{D_n} M$. A natural resolution for ω_n is the cohomological Koszul complex $K^\bullet$ on the left-multiplications on D_n by the various ∂_j. (So, $K^\bullet$ is the complex of global sections of $\Omega^\bullet_{\mathcal{D},X}$). The module K^ℓ has a natural generating set given by the size-ℓ-subsets of $1, \ldots, n$. We place these generators in $\tilde{V}$-level ℓ and extend $\tilde{V}$ to each K^ℓ by D_n-linearity. Since ∂_i is in $\tilde{V}$-level -1, this produces a $\tilde{V}$-strict resolution of ω_n. Having resolutions $K^\bullet, F^\bullet$ with $\tilde{V}$-filtration, there is an induced $\tilde{V}$-filtration on $K^\bullet \otimes_{D_n} F^\bullet$.

The complex $K^\bullet \otimes_{D_n} M$ is sometimes called the (affine, global) *de Rham complex* of M. If M is a space of functions on which one can differentiate, multiplication by ∂_i in $K^\bullet$ corresponds to differentiation by x_i in the usual de Rham complex.

Suppose now that $M = \bigoplus_{\ell\in\mathbb{Z}} M_\ell$ is a graded module over the graded ring D_n, with homogeneous generators $m_1,\dots,m_r$ of degrees $s_1,\dots,s_r$. Choosing the degrees of the generators as shifts (*i.e.*, $s_i = \deg(m_i)$) for the $\tilde{V}$-filtration on M one obtains a direct sum of the graded components of M,

$$\tilde{V}^k(M) = \sum_{i=1}^{r} \tilde{V}^{k-s_i}(D_n)\cdot m_i = \bigoplus_{\ell\leq k} M_\ell. \tag{2.3.3}$$

Since the twisted Euler operator $\tilde{E} = -\sum_{j=1}^n \partial_j x_j$ is $\tilde{V}$-homogeneous of degree zero, the defining equation (2.3.2) becomes

$$\tilde{b}_M(\tilde{E}+n+k)\cdot M_k = 0$$

for all $k\in\mathbb{Z}$.

For $\tilde{V}$-graded M one can arrange the resolution $F^\bullet$ to respect this grading, and $K^\bullet$ is graded in any case. If now η_F is a cohomology class generator in $H^i(\omega_n \otimes_{D_n} F^\bullet)$, one can lift it into $K^n \otimes_{D_n} F^i$ and then chase it into a class η_K of $K^\bullet \otimes_{D_n} M$, since Tor is a balanced functor. The grading of the resolutions involved implies that the $\tilde{V}$-level of this class in $K^\bullet \otimes_{D_n} M$ is the same as the $\tilde{V}$-level of η_F in $\omega_n \otimes_{D_n} F^\bullet$.

We recall the notion of an Eulerian D_n-module.

Definition 2.11 *([32]).* The graded D_n-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ is *Eulerian* if for every homogeneous $m\in M_i$ one has $(\sum_{j=1}^n x_j\partial_j)m = i\cdot m$.

In terms of $\tilde{E}$ this is equivalent to $(\tilde{E}+n+\deg(m))m = 0$. $\diamond$

Eulerian D_n-modules are a very special case of Brylinski's *monodromic* modules, which are those on which the Euler operator has a minimal polynomial. They include (iterated) local cohomology modules $H^{i_1}_{I_1}(\dots(H^{i_k}_{I_k}(R_n)\dots)$ for homogeneous ideals $I_1,\dots,I_k$.

Proposition 2.12. *Let M be a finitely generated Eulerian D_n-module. Then every nonzero cohomology class of $\omega_n \otimes^L_{D_n} M$ has degree zero.*

Proof. Since the module is Eulerian, we have $(\tilde{E}+n+\deg(m))m = 0$ for every homogeneous $m\in M$. We put the $\tilde{V}$-filtration on M that is induced by a finite set of homogeneous generators as in (2.3.3), with shifts $s_i = \deg(m_I)$. Then, a b-function for integration is given by $\tilde{b}(s) = s$. The conclusion is immediate from the Integration Theorem 2.10. $\square$

Remark 2.13. Let us call *quasi-Eulerian* a graded monodromic D_n-module M. Then one can easily generalize Proposition 2.12 to: if M is quasi-Eulerian then the degree of every cohomology class of $\omega_n \otimes^L_{D_n} M$ must be an integral root of the minimal polynomial of $\tilde{E}$ on M.

There is a version of the Integration Theorem for complexes of holonomic modules (more generally, for complexes that have a b-function for integration), see [44]. This allows a further generalization to finite graded complexes with quasi-Eulerian cohomology. $\diamond$

We now consider the Eulerian D_n-module that arises as the localization $M = R_n[1/f]$ of R_n at a homogeneous polynomial f. It is clear that this is an Eulerian module since the Euler operator E acts on a rational homogeneous function of degree k by multiplication with k. Thus, $K^\bullet \otimes_{D_n} M$ is $\tilde{V}$-graded and every class in $\omega_n \otimes^L_{D_n} M$ has native degree zero.

If one reads elements of $K^\ell \otimes M$ as differential ℓ-forms on M, this implies that the cohomology of $K^\bullet \otimes_{D_n} M$ is spanned as vector space by differential forms of degree zero: forms of the type

$$\sum_{\substack{|I|=\ell \\ I\subseteq\{1,\ldots,n\}}} \frac{g_I \mathrm{d}x_I}{f^{k_I}}$$

where $\mathrm{d}x_I = \wedge_{i\in I}\mathrm{d}x_i$, where g_I is a homogeneous element of R_n, and where $\deg(g_I) + \ell = k_I \cdot \deg(f)$. Similarly, integrating a graded complex $M^\bullet$ with Eulerian cohomology modules yields a de Rham complex of $M^\bullet$ with cohomology groups concentrated in degree zero.

Corollary 2.14. *If I is a homogeneous ideal in R_n then the de Rham cohomology of the affine complement $U(I) = X \smallsetminus \mathrm{Var}(I)$ of the affine variety $\mathrm{Var}(I) \subseteq X := \mathbb{C}^n$ is generated by chains of differential forms of degree zero. Moreover, the de Rham cohomology groups $H^i_{\mathrm{dR}}(H^j_I(R_n))$ all are concentrated in degree zero.* □

Proof. The Grothendieck comparison theorem asserts that the cohomology of $K^\bullet \otimes_{D_n} \check{C}^\bullet$ is the de Rham cohomology of $U(I)$. The rest follows from Proposition 2.12. □

Remark 2.15. Since multiplication by $\mathbb{C} \ni \lambda \neq 0$ is an isomorphism on $U(f)$, the de Rham cohomology of $U(f)$ of a divisor is spanned by homogeneous differential forms (homothety eigenvectors) for all homogeneous $f \in R_n$. Alex Dimca pointed out that path-connectedness of $\mathbb{C}^*$ implies that this multiplication is in fact homotopy equivalent to the identity, and thus does not change the class. Hence, the cohomology of $U(f)$ must be eigenvectors to eigenvalue 1, and thus of degree zero. Moreover, it was pointed out to us by A. Lőrincz that $\mathbb{C}^*$-equivariance can be used to obtain Proposition 2.12; see Lem. 2.1 and Cor. 2.2 in the recent preprint `arXiv:2105.00271` for a more general result. ◇

2.4. On Veronese maps

Throughout this subsection, $2 \leq d, n \in \mathbb{N}$. Let

$$v^d_n \colon X = \mathbb{C}^n \longrightarrow \mathbb{C}^N =: W \tag{2.4.1}$$

be the d-th Veronese morphism on the affine level, so $N = \binom{n+d-1}{n-1}$. If n, d are understood, we abbreviate v^d_n to just v. We set $X' := v(X) \subseteq W$, $W^\circ = W \smallsetminus \{0\}$, $X^\circ := X \smallsetminus \{0\}$ and $X'^\circ := X' \smallsetminus \{0\}$.

Let $R_n = \mathbb{C}[x_1, \ldots, x_n] = \mathcal{O}_X(X)$ and $R_N = \mathbb{C}[\{y_S \mid S \in \mathbb{N}^n, |S| = d\}] = \mathcal{O}_W(W)$. Let $I \subseteq R_n$ be a homogeneous ideal and Y the associated variety. Denote U the complement $X \smallsetminus Y$, and let Y', U' the images of Y, U under v. Let U_W be the complement $W \smallsetminus v(Y)$. In this subsection we will compare the cohomology of the affine complements of Y and Y'.

Note that $v^\# \colon R_N \longrightarrow R_n$ sends $y_S \mapsto x^S$ in multi-index notation. The d-th roots of unity $\boldsymbol{\mu}_d$ act diagonally on X, as well as on every other variety of a homogeneous ideal of R_n, by multiplication on each x_i. Moreover, v is the orbit map to this action, followed by inclusion into W. The image of v has a unique isolated singularity at the origin, and v is a $d:1$ covering of X'° by X°.

Note that $\boldsymbol{\mu}_d$ is the covering group of the map $U \longrightarrow U'$, and its order d is nonzero in $\mathbb{C}$. Under these circumstances, $H^\bullet(U';\mathbb{C})$ is the group of $\boldsymbol{\mu}_d$-invariants in $H^\bullet(U;\mathbb{C})$. Using the de Rham manifestation of $H^\bullet(U;\mathbb{C})$, in which we showed that every class has a representative that is of degree zero, the entire space $H^\bullet(U;\mathbb{C})$ is $\boldsymbol{\mu}_d$-invariant, so that

$$H^\bullet_{\mathrm{dR}}(U;\mathbb{C}) = H^\bullet_{\mathrm{dR}}(U';\mathbb{C}).$$

In what follows, we replace de Rham cohomology by singular cohomology, since we will have need to step outside the category of smooth algebraic varieties. Known comparison theorems over $\mathbb{C}$ assure functorial isomorphisms between these cohomology theories whenever both exist.

It will turn out to be useful to know the cohomology of $W \smallsetminus X' = W^\circ \smallsetminus X'^\circ$. Note that X'° has the homology of the homotopy $(2n-1)$-sphere X° and is a closed submanifold of the $2N$-dimensional manifold W°, the latter being homotopy equivalent to the $(2N-1)$-sphere $\mathbb{S}^{2N-1}$. Alexander duality gives an isomorphism $H^i_{X'^\circ}(W^\circ;\mathbb{C}) \simeq \operatorname{Hom}_{\mathbb{C}}(H^{2N-i}_c(X'^\circ;\mathbb{C}),\mathbb{C})$ with the dual of compactly supported cohomology, [22, Alexander Duality V.6.6]. But then X'° being a $2n$-dimensional real manifold yields by Poincaré duality that $\operatorname{Hom}_{\mathbb{C}}(H^{2N-i}_c(X'^\circ;\mathbb{C}),\mathbb{C}) \simeq H^{2n-2N+i}(X'^\circ;\mathbb{C})$, [4, I.(5.4)]. The latter is $\mathbb{C}$ for $i = 2N-1$ and $i = 2N-2n$, and zero otherwise. In the long exact sequence

$$\cdots \longrightarrow H^i_{X'^\circ}(W^\circ;\mathbb{C}) \longrightarrow H^i(W^\circ;\mathbb{C}) \longrightarrow H^i(W^\circ \smallsetminus X'^\circ;\mathbb{C}) \xrightarrow{+1},$$

we have $H^i_{X'^\circ}(W^\circ;\mathbb{C}) \neq 0$ only when $i = 2N-1, 2(N-n)$ and $H^i(W^\circ;\mathbb{C}) \neq 0$ only if $i = 2N-1, 0$. The map $\mathbb{C} = H^{2N-1}_{X'^\circ}(W^\circ;\mathbb{C}) \longrightarrow H^{2N-1}(W^\circ;\mathbb{C}) = \mathbb{C}$ is surjective (hence bijective) since $W^\circ \smallsetminus X'^\circ$ is homotopy equivalent to an open subset of a $(2N-1)$-sphere) and so $H^{2N-1}(W^\circ \smallsetminus X'^\circ;\mathbb{C}) = 0$. It follows that

$$H^i(W \smallsetminus X';\mathbb{C}) = H^i(W^\circ \smallsetminus X'^\circ;\mathbb{C}) = \begin{cases} \mathbb{C} & \text{if} \quad i = 0, 2(N-n)-1; \\ 0 & \qquad\qquad \text{else.} \end{cases} \tag{2.4.2}$$

Next we compute the cohomology of $U_W = W \smallsetminus Y', Y' = v(Y)$ where $Y = \operatorname{Var}(I)$ for some homogeneous ideal $I \subseteq R_n$. Since $U' = v(U)$ is an embedded submanifold of U_W with complex codimension $N-n$, we can consider the tubular neighborhood T' of U' that arises via the tubular neighborhood theorem as the total space of the normal bundle of U' in U_W. Then

$$U_W = W \smallsetminus Y' = (W \smallsetminus X') \cup U' = (W \smallsetminus X') \cup T',$$

with intersection $(W \smallsetminus X') \cap T' = T'^\circ$.

As U', U_W are complex manifolds, the removal of the zero section U' from T' leaves a space T'° homotopic to an oriented sphere bundle $\mathbb{S}^q \hookrightarrow T'^\circ \twoheadrightarrow U'$ where

$$q = 2(N-n)-1.$$

The q-sphere bundle T° yields a Gysin sequence

$$\ldots \longrightarrow H^i(T'^\circ;\mathbb{C}) \xrightarrow{\pi_*} H^{i-q}(U';\mathbb{C}) \xrightarrow{e\cup} H^{i+1}(U';\mathbb{C}) \xrightarrow{\pi^*} H^{i+1}(T'^\circ;\mathbb{C}) \longrightarrow \ldots$$

Here, $\pi\colon T'^\circ \longrightarrow U'$ is the fibration map, π^* is the pullback under this map, and e is the Euler class of the bundle T'° when restricted from relative cohomology to absolute cohomology on T. The map π_* is special to the situation of bundles with fibers homotopic to compact manifolds, and is induced by integration along the fibers in the following sense. For any oriented $\mathbb{R}^k$-bundle $E \longrightarrow B$ with $E^\circ = E \smallsetminus B$ there is a fundamental class $u \in H^k(E, E^\circ;\mathbb{Z})$ that restricts in each fiber to the canonical class in $H^k(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\};\mathbb{Z})$; this canonical class is the given orientation on the bundle (an orientation is a global section of the orientation bundle with fiber $H^k(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\};\mathbb{Z})$). The existence of the fundamental class is the content of the Thom isomorphism theorem for oriented vector bundles, and the cup product with u sets up an isomorphism $u\cup\colon H^j(E;\mathbb{Z}) \longrightarrow H^{j+k}(E, E^\circ;\mathbb{Z})$. The cap product with the Poincaré dual of u induces an isomorphism $H_j(E, E^\circ;\mathbb{Z}) \longrightarrow H_{j-k}(E;\mathbb{Z})$, the “integration along the fibers” above (compare [31, Ch. 9-12]). The image of u in $H^k(E;\mathbb{Z})$ is the Euler class (by definition). If the fiber dimension k is large, $H^k(E;\mathbb{Z}) = H^{k-1}(E;\mathbb{Z}) =$

0. In that case, the Euler class of the bundle must be zero and then u corresponds to the class in $H^{k-1}(E^\circ;\mathbb{Z})$ with the property that it restricts in each fiber to the canonical generator of $H^{k-1}(\mathbb{R}^k \smallsetminus \{0\};\mathbb{Z})$.

Our Gysin sequence above arises from the long exact sequence to the pair (T', T'°) with replacements coming from the Thom isomorphism and the fact that U', T' are homotopic.

Since $\dim_{\mathbb{C}}(U') = n$, $H^i(U';\mathbb{C}) = 0$ if $i \geq 2n$. On the other hand, the Euler class is of homological degree $q+1 = 2(N-n)$. Thus, if $2(N-n) \geq 2n$ then either the source or the target of the Euler map $H^{i-q}(U';\mathbb{C}) \xrightarrow{e\cup} H^{i+1}(U';\mathbb{C})$ is zero for every i. But $N = \binom{n+d-1}{n-1} \geq 2n$ for $n, d \geq 2$ unless $d = n = 2$, and usually much larger. Thus the Gysin sequence splits into isomorphisms

$$H^i(T'^{\circ};\mathbb{C}) \xrightarrow{\pi_*} H^{i-q}(U';\mathbb{C}) = H^i(T';\mathbb{C}) \qquad \text{if } i \geq 2n; \tag{2.4.3}$$

$$H^i(T';\mathbb{C}) = H^i(U';\mathbb{C}) \xrightarrow{\pi^*} H^i(T';\mathbb{C}^{\circ}) \qquad \text{if } i < 2n. \tag{2.4.4}$$

Note that the composition $H^i(T';\mathbb{C}) \longrightarrow H^i(T'^{\circ};\mathbb{C}) \longrightarrow H^i(U';\mathbb{C})$ is an isomorphism since $U' \hookrightarrow T'$ is a homotopy equivalence; so the left map is an isomorphism if and only if the right one is. Now consider the Mayer–Vietoris sequence to the pair $(W \smallsetminus X') \cup T' = U_W$ with $T'^{\circ} = (W \smallsetminus X') \cap T'$:

$$\cdots \longrightarrow H^i((W \smallsetminus X') \cup T';\mathbb{C}) \longrightarrow H^i(W \smallsetminus X';\mathbb{C}) \oplus H^i(T';\mathbb{C}) \longrightarrow H^i(T'^{\circ};\mathbb{C}) \longrightarrow \cdots$$

Here, each (component of a) map is the natural restriction, possibly with a (-1) factor.

If $i < 2n$, the map $H^i(T';\mathbb{C}) \longrightarrow H^i(T'^{\circ};\mathbb{C})$ in the Mayer–Vietoris sequence is therefore the identity by (2.4.4). It follows that in this range, $H^i((W \smallsetminus X') \cup T';\mathbb{C}) \longrightarrow H^i(W \smallsetminus X';\mathbb{C})$ is an isomorphism as well. But in that range, by (2.4.2), only $H^0(W \smallsetminus X';\mathbb{C})$ is nonzero and so $H^i((W \smallsetminus X') \cup T';\mathbb{C})$ is zero for $0 < i < 2n$.

If $2n-1 \leq i < q$, then $H^i((W \smallsetminus X') \cup T';\mathbb{C})$ vanishes since $H^i(W \smallsetminus X';\mathbb{C}) = H^i(T';\mathbb{C}) = H^i(T'^{\circ};\mathbb{C}) = 0$.

Let us look at the situation when $i = q$:

$$\underbrace{H^{q-1}(T'^{\circ};\mathbb{C})}_{=H^{-1}(U';\mathbb{C})=0} \longrightarrow H^q((W \smallsetminus X') \cup T';\mathbb{C}) \longrightarrow \underbrace{H^q(W \smallsetminus X';\mathbb{C})}_{=\mathbb{C}} \oplus \underbrace{H^q(T';\mathbb{C})}_{=0} \longrightarrow$$

$$\underbrace{H^q(T'^{\circ};\mathbb{C})}_{=H^0(U';\mathbb{C})=\mathbb{C}} \longrightarrow H^{q+1}((W \smallsetminus X') \cup T';\mathbb{C}) \longrightarrow \underbrace{H^{q+1}(W \smallsetminus X';\mathbb{C})}_{=0} \oplus \underbrace{H^{q+1}(T';\mathbb{C})}_{=0}.$$

If one restricts the morphism $H^q(W \smallsetminus X';\mathbb{C}) \longrightarrow H^q(T'^{\circ};\mathbb{C})$ to the intersection with a small ball around a generic point of Y', both spaces become homotopic to $\mathbb{S}^q$ and so the morphism $H^q(W \smallsetminus X';\mathbb{C}) \longrightarrow H^q(T'^{\circ};\mathbb{C})$ restricts to an isomorphism $\mathbb{C} \longrightarrow \mathbb{C}$. But since $H^q(W \smallsetminus X';\mathbb{C})$ and $H^q(T'^{\circ};\mathbb{C})$ are also equal to $\mathbb{C}$, the morphism $H^q(W \smallsetminus X';\mathbb{C}) \longrightarrow H^q(T'^{\circ};\mathbb{C})$ is an isomorphism. Thus, $H^q(U_W;\mathbb{C}) = H^{q+1}(U_W;\mathbb{C}) = 0$.

If $i > q$, $H^i(T';\mathbb{C}) = H^i(W \smallsetminus X';\mathbb{C}) = 0$. Thus, $H^{i-q}(U';\mathbb{C}) = H^i(T'^{\circ};\mathbb{C}) = H^{i+1}((W \smallsetminus X') \cup T';\mathbb{C})$.

We have proved

Proposition 2.16. *We use notation as defined at the start of Subsection 2.4. Let $T' \longrightarrow U'$ be the normal bundle of U' in W°. With $U_W := W \setminus Y' = (W \smallsetminus X') \cup T'$, and $q = \binom{n+d-1}{n-1} > 2n$ we have on the level of reduced cohomology for every $i \in \mathbb{Z}$ the isomorphisms*

$$\begin{tikzcd}
H^i(U';\mathbb{C}) \arrow[r, "\pi^*", "\simeq"'] & H^i(T';\mathbb{C}) \arrow[r, "e_0\cup", "\simeq"'] \arrow[dr, "\simeq", "e\cup"'] & H^{i+q}(T'^{\circ};\mathbb{C}) \arrow[r, "\delta^*", "\simeq"'] \arrow[d, "\delta^*", "\simeq"'] & H^{i+q+1}(U_W;\mathbb{C}) \\
 & & H^{i+q+1}(T', T'^{\circ};\mathbb{C}) &
\end{tikzcd}$$

Here $e \in H^{q+1}(T', T'^{\circ}; \mathbb{Z})$ is the Euler class of the bundle, $e_0 \in H^q(T'^{\circ}; \mathbb{Z})$ is its preimage, the vertical δ^* is the connecting morphism for the pair (T', T'°), and the horizontal δ^* is the connecting morphism for the Mayer–Vietoris spectral sequence for the cover $U_W = (W \setminus X') \cup T'$.

In particular, the singular reduced cohomology groups of the complements of the cones Y and Y' over $\tilde{Y}$ are the same up to a cohomological shift by $q + 1 = 2(N - n)$.

Corollary 2.17. *If $Y \subseteq X = \mathbb{C}^n$ is homogeneous and of equi-dimension three, then the Čech–de Rham numbers $\rho^r_{p,q}(Y)$ are invariant under Veronese maps of Y.*

Proof. With $Y' \subseteq W$ and notation on Y', U, U_W as set at the start of Subsection 2.4, the Čech–de Rham spectral sequence degenerates for dimensional reasons by Proposition 2.6, both for $Y \subseteq X$ and for $Y' \subseteq W$. According to Proposition 2.16, the two complements have the same reduced cohomology up to a shift by the relative dimension. Hence, up to that same shift, the two Čech–de Rham spectral sequences have the same abutment. The degeneration shows that the abutment determines the $\rho^2_{p,q}$, except for the numbers $\rho^2_{2,2}$ and $\rho^2_{3,0}$ for which it follows that their sum is equal to $\dim_{\mathbb{C}} H^{n-3}_{\mathrm{dR}}(U) = \dim_{\mathbb{C}} H^{N-3}_{\mathrm{dR}}(W_U)$. However, $\rho^2_{2,2}$ is the dimension of $H^{n-2}_{\mathrm{dR}}(H^{n-2}_I(R_n))$ and $H^{N-2}_{\mathrm{dR}}(H^{N-2}_I(R_N))$ respectively, and thus equals the socle dimension of $H^2_{\mathfrak{m}}(\mathbb{D}H^{n-2}_I(R_n))$ and $H^2_{\mathfrak{m}}(\mathbb{D}H^{N-2}_I(R_N))$ respectively, by Lemma 1.4. But equi-dimensionality and Remark 1.3 show that these latter integers are both zero. This implies invariance of $\rho^2_{3,0}$ and settles the case $r = 2$. But no higher nonzero differentials can exist by degeneration. □

We show next that, under less stringent conditions, most of the Čech-de Rham numbers of level two are, for cones over projective varieties, still unchanged under Veronese maps. The idea of the proof is inspired by the companion result Lemma 3.2 for Lyubeznik numbers, which in turn is based on results in [40].

Theorem 2.18. *Let $\tilde{Y}$ be a projective variety and suppose Y is a cone for $\tilde{Y}$, embedded as a closed subvariety of an affine space $X = \mathbb{C}^n$. Then, for $k \geq 2$, the Čech–de Rham numbers $\rho^2_{k,\ell}$ derived from Y agree with those derived from any Veronese of the pair $Y \subseteq X$.*

The proof will start with translating the $\rho_{k,\ell}$ into objects of constructible sheaves involving the Verdier dual ω_Y of the constant sheaf on Y, see (2.1.1). After some rewriting we use Lemma 1.4 to exchange a direct image functor to a point for the pullback to the origin and then use an adjunction triangle to reformulate them in terms of Y°, the complement of the origin in Y. We finally lift to $\tilde{Y}$ where an interpretation in terms of Chern classes makes it easy to compare the construction for Y to the corresponding construction for the Veronese embedding.

During the proof we shall use the following diagram of maps

$$\begin{tikzcd}
\{0\} \arrow[r, equal] \arrow[d, "i_Y"'] & \{0\} \arrow[d, "i"'] \\
Y \arrow[r, "h"] \arrow[u, bend right, "a_Y"'] & \mathbb{C}^n =: X \arrow[u, bend right, "a_X"'] \\
Y_0 \arrow[r, "h_0"] \arrow[u, "j_Y"] \arrow[d, "\pi"'] & \mathbb{C}^n \setminus \{0\} =: X^{\circ} \arrow[u, "j"'] \arrow[d, "p"] \\
\tilde{Y} \arrow[r, "g"] & \mathbb{P}^{n-1}
\end{tikzcd}$$

Proof of Theorem 2.18. The Čech–de Rham numbers of level 2 are given by

$$\rho_{k,\ell} := \dim H^{n-k}_{\mathrm{dR}}(H^{n-\ell}_I R_n) = \dim {}^p\mathcal{H}^{-k} a_*({}^p\mathcal{H}^{n-\ell} h_! h^! \underline{\mathbb{C}}_X[n]),$$

where $\underline{\mathbb{C}}_X[n]$ is the constant sheaf on X with stalk $\mathbb{C}$. (The shift from $-n+k$ on the left to $-k$ on the right occurs since the de Rham functor used in the Riemann–Hilbert correspondence arises from the "natural" algebraic de Rham functor—which goes along with the tensor product with ω—by analytification *and* a shift by n).

We have

$$\begin{aligned}
{}^p\mathcal{H}^{-k} a_*({}^p\mathcal{H}^{n-\ell} h_! h^! \underline{\mathbb{C}}_X[n]) &\overset{(a)}{\simeq} {}^p\mathcal{H}^{-k} a_*({}^p\mathcal{H}^{n-\ell} h_! h^! a_X^! \mathbb{C}_{pt}[-n]) \\
&\overset{(b)}{\simeq} {}^p\mathcal{H}^{-k} a_*({}^p\mathcal{H}^{n-\ell} h_! \omega_Y[-n]) \\
&\overset{(c)}{\simeq} {}^p\mathcal{H}^{-k} a_* h_!({}^p\mathcal{H}^{n-\ell} \omega_Y[-n]) \\
&\overset{(d)}{\simeq} {}^p\mathcal{H}^{-k} a_* h_*({}^p\mathcal{H}^{n-\ell} \omega_Y[-n]) \\
&\simeq {}^p\mathcal{H}^{-k} (a_Y)_*({}^p\mathcal{H}^{n-\ell} \omega_Y[-n]) \\
&\simeq {}^p\mathcal{H}^{-k} (a_Y)_*({}^p\mathcal{H}^{-\ell} \omega_Y) \\
&\overset{(e)}{\simeq} {}^p\mathcal{H}^{-k} (i_Y)^{-1}({}^p\mathcal{H}^{-\ell} \omega_Y)
\end{aligned} \tag{2.4.5}$$

The justifications are as follows: (a) holds since the real dimension of X is $2n$; (b) follows from the definition of ω_Y; (c) holds since h is a closed embedding and hence $h_!$ is perverse exact; (d) comes from $h_* = h_!$ for closed embeddings; (e) is Lemma 3.3 in [41] (quoted as Lemma 1.4 in the present article).

We have the following triangle from the inclusion of the origin into Y:

$$j_{Y!} j_Y^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y) \longrightarrow {}^p\mathcal{H}^{-\ell}\omega_Y \longrightarrow i_{Y!} i_Y^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y) \xrightarrow{+1}$$

and it induces the following long exact sequence

$$\begin{tikzcd}
& \dots \arrow[r] & {}^p\mathcal{H}^{-3} i_{Y!}(i_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y) \arrow[dll] \\
{}^p\mathcal{H}^{-2} j_{Y!}(j_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y) \arrow[r] & 0 \arrow[r] & {}^p\mathcal{H}^{-2} i_{Y!}(i_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y) \arrow[dll] \\
{}^p\mathcal{H}^{-1} j_{Y!}(j_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y) \arrow[r] & 0 \arrow[r] & {}^p\mathcal{H}^{-1} i_{Y!}(i_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y) \arrow[dll] \\
{}^p\mathcal{H}^{0} j_{Y!}(j_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y) \arrow[r] & {}^p\mathcal{H}^{-\ell}\omega_Y \arrow[r] & {}^p\mathcal{H}^{0} i_{Y!}(i_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y) \arrow[r] & 0
\end{tikzcd}$$

We now use that $k \geq 2$, which yields the following isomorphisms from the long exact sequence above:

$$\begin{aligned}
{}^p\mathcal{H}^{-k} i_{Y!}(i_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y) &\simeq {}^p\mathcal{H}^{-k+1} j_{Y!}(j_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y) \\
&\overset{(f)}{\simeq} {}^p\mathcal{H}^{-k+1} j_{Y!}({}^p\mathcal{H}^{-\ell}\omega_{Y^\circ}) \\
&\overset{(g)}{\simeq} {}^p\mathcal{H}^{-k+1} j_{Y!}({}^p\mathcal{H}^{-\ell}\pi^!\omega_{\tilde{Y}})
\end{aligned} \tag{2.4.6}$$

$$\begin{aligned} &\simeq {}^p\mathcal{H}^{-k+1} j_{Y!}({}^p\mathcal{H}^{-\ell+1}\pi^![-1]\omega_{\tilde{Y}}) \\ &\overset{(h)}{\simeq} {}^p\mathcal{H}^{-k} j_{Y!}\pi^!({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}}) \end{aligned}$$

with justifications as follows: (f) since j_Y is open and so $(j_Y)^{-1}$ is perverse exact; (g) is dual to the fact that $\mathbb{C}_{Y^\circ} = \pi^{-1}\mathbb{C}_{\tilde{Y}}$; (h) is because π is smooth so that $\pi^![-1]$ is perverse exact.

We have then

$$ {}^p\mathcal{H}^i a_{Y!}{}^p\mathcal{H}^{-k}(j_{Y!}\pi^!({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}})) \simeq {}^p\mathcal{H}^i a_{Y!}({}^p\mathcal{H}^{-k}(i_{Y!}(i_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y))) = 0 \qquad \text{for} \quad i \neq 0$$

since ${}^p\mathcal{H}^{-n+k} i_{Y!}(i_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y)$ is at most supported on a point. A spectral sequence argument shows therefore that

$$ {}^p\mathcal{H}^0 a_{Y!}{}^p\mathcal{H}^{-k} j_{Y!}\pi^!({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}}) \simeq {}^p\mathcal{H}^{-k} a_{Y!} j_{Y!}\pi^!({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}}). \tag{2.4.7}$$

Summarizing we have for $k \geq 2$ that

$$\begin{aligned} {}^p\mathcal{H}^{-k} a_*({}^p\mathcal{H}^{n-\ell} h_! h^! \underline{\mathbb{C}}_V[n]) &\overset{(i)}{\simeq} {}^p\mathcal{H}^{-k}(i_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y) \\ &\overset{(j)}{\simeq} {}^p\mathcal{H}^0 a_{Y!} i_{Y!}({}^p\mathcal{H}^{-k}(i_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y)) && (2.4.8) \\ &\overset{(k)}{\simeq} {}^p\mathcal{H}^0 a_{Y!}({}^p\mathcal{H}^{-k} i_{Y!}(i_Y)^{-1}({}^p\mathcal{H}^{-\ell}\omega_Y)) \\ &\overset{(l)}{\simeq} {}^p\mathcal{H}^{-k} a_{Y!} j_{Y!}\pi^!({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}}), && (2.4.9) \end{aligned}$$

where (i) follows from display (2.4.5); (j) follows since $a_Y \circ i_Y$ is the identity on $\{0\}$; (k) is since $i_{Y!}$ is perverse exact as i_Y is closed; (l) comes from displays (2.4.6) and (2.4.7).

Now let $\mathcal{L}$ be the quasi-coherent pullback of $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ via g. By abuse of notation we denote the total space of the corresponding line bundle by the same letter. Notice that $Y^\circ \simeq \mathcal{L} \smallsetminus \{\text{zero section}\}$. Consider the following diagram

$$\begin{tikzcd} \{0\} \arrow[d, "i_Y"'] & \tilde{Y} \arrow[l] \arrow[d, "\tilde{i}"] \arrow[dr, equal] & \\ Y & \mathcal{L} \arrow[l, "u"'] \arrow[r, "q"] & \tilde{Y} \\ Y^\circ \arrow[u, "j_Y"] \arrow[r, equal] & Y^\circ \arrow[u, "\tilde{j}"'] \arrow[ur, "\pi"'] & \end{tikzcd}$$

in which q is the bundle map, $\tilde{i}$ the embedding of the zero section, and u is the contraction of the zero section. We have

$$\begin{aligned} {}^p\mathcal{H}^{-k} a_{Y!} j_{Y!}\pi^!({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}}) &\simeq {}^p\mathcal{H}^{-k} a_{Y!} j_{Y!}\pi^{-1}[2]({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}}) \\ &\simeq {}^p\mathcal{H}^{-k+2} a_{Y!} j_{Y!}\pi^{-1}({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}}) \\ &\simeq {}^p\mathcal{H}^{-k+2} a_{Y!} j_{Y!}(\tilde{j})^{-1} q^{-1}({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}}) \\ &\simeq {}^p\mathcal{H}^{-k+2} a_{X!} q_! \, \tilde{j}_!(\tilde{j})^{-1} q^{-1}({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}}) \\ &\simeq {}^p\mathcal{H}^{-k+2} a_{X!}\pi_!\pi^{-1}({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}}) \\ &\simeq \mathbb{H}_c^{-k+2}(X, \pi_!\pi^{-1}({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}})) \end{aligned}$$

Here the first isomorphism comes from the fact that $\pi^{-1}[1] = \pi^![-1]$ is perverse exact, and the last one because cohomology of compact supports is the cohomology of the exceptional direct image functor.

From the closed embedding of $\tilde{Y}$ into $\mathcal{L}$ arises a triangle

$$\tilde{j}_!(\tilde{j})^{-1}q^{-1}({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}}) \longrightarrow q^{-1}({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}}) \longrightarrow \tilde{i}_!(\tilde{i})^{-1}q^{-1}({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}}) \xrightarrow{+1} \tag{2.4.10}$$

Applying $q_!$ we get

$$\pi_!\pi^{-1}\mathcal{G}_{-\ell+1} \longrightarrow q_!q^{-1}\mathcal{G}_{-\ell+1} \longrightarrow \mathcal{G}_{-\ell+1} \xrightarrow{+1}$$

where we have set $\mathcal{G}_{-\ell} := {}^p\mathcal{H}^{-\ell}\omega_{\tilde{Y}}$ and used $\pi = q \circ \tilde{j}$. We have $\mathcal{G} \simeq q_!q^!\mathcal{G} \simeq q_!q^{-1}\mathcal{G}[2]$ for any $\mathcal{G} \in \mathrm{Perv}(\tilde{Y})$ since q is smooth of relative dimension 1. This gives the triangle

$$\pi_!\pi^{-1}\mathcal{G}_{-\ell+1} \longrightarrow \mathcal{G}_{-\ell+1}[-2] \longrightarrow \mathcal{G}_{-\ell+1} \xrightarrow{+1}$$

As in [40, (1.3.1)], this triangle is dual to a triangle $\mathcal{F} \longrightarrow \mathcal{F}[2] \longrightarrow p_*\pi^!\mathcal{F} \xrightarrow{+1}$ where the first map is induced by

$$e \otimes 1 : \mathbb{C}_{\tilde{Y}} \otimes \mathcal{F} \longrightarrow \mathbb{C}_{\tilde{Y}}[2] \otimes \mathcal{F},$$

with $e \in \mathrm{Hom}_{D^b_{\mathrm{c.s.}}(\tilde{Y})}(\mathbb{C}_{\tilde{Y}}, \mathbb{C}_{\tilde{Y}}[2]) \simeq \mathrm{Hom}_{D^b_{\mathrm{c.s.}}(pt)}(\mathbb{C}, R\Gamma(\tilde{Y}; \mathbb{C}_{\tilde{Y}}[2])) \simeq H^2(\tilde{Y}; \mathbb{C})$ is the image of the Euler class of the vector bundle $\mathcal{L}$.

We get a long exact sequence

$$\longrightarrow \mathbb{H}_c^{-k+2}(\tilde{Y}, \pi_!\pi^{-1}\mathcal{G}_{-\ell+1}) \longrightarrow \mathbb{H}_c^{-k}(\tilde{Y}, \mathcal{G}_{-\ell+1})(-1) \longrightarrow \mathbb{H}_c^{-k+2}(\tilde{Y}, \mathcal{G}_{-\ell+1}) \longrightarrow \mathbb{H}_c^{-k+3}(\tilde{Y}, \pi_!\pi^{-1}\mathcal{G}_{-\ell+1}) \longrightarrow$$

In particular we get short exact sequences

$$0 \longrightarrow \mathbb{H}_c^{-k+1}(\tilde{Y}, \mathcal{G}_{-\ell+1})_{\mathcal{L}} \longrightarrow \mathbb{H}_c^{-k+2}(\tilde{Y}, \pi_!\pi^{-1}({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}})) \longrightarrow \mathbb{H}_c^{-k}(\tilde{Y}, \mathcal{G}_{-\ell+1})^{\mathcal{L}} \longrightarrow 0$$

where

$$\begin{aligned} \mathbb{H}_c^{-k+1}(\tilde{Y}, \mathcal{G}_{-\ell+1})_{\mathcal{L}} &:= \operatorname{coker}\left(\mathbb{H}_c^{-k-1}(\tilde{Y}, \mathcal{G}_{-\ell+1})(-1) \longrightarrow \mathbb{H}_c^{-k+1}(\tilde{Y}, \mathcal{G}_{-\ell+1})\right) \\ \mathbb{H}_c^{-k}(\tilde{Y}, \mathcal{G}_{-\ell+1})^{\mathcal{L}} &:= \ker\left(\mathbb{H}_c^{-k}(\tilde{Y}, \mathcal{G}_{-\ell+1})(-1) \longrightarrow \mathbb{H}_c^{-k+2}(\tilde{Y}, \mathcal{G}_{-\ell+1})\right) \end{aligned}$$

Putting everything together we get that

$$\rho_{k,\ell} = \dim \mathbb{H}_c^{-k+2}(\tilde{Y}, \pi_!\pi^{-1}({}^p\mathcal{H}^{-\ell+1}\omega_{\tilde{Y}})) = \dim \mathbb{H}_c^{-k+1}(\tilde{Y}, \mathcal{G}_{-\ell+1})_{\mathcal{L}} + \dim \mathbb{H}_c^{-k}(\tilde{Y}, \mathcal{G}_{-\ell+1})^{\mathcal{L}}$$

is unchanged under Veronese maps for $k \geq 2$, since the Euler class of a bundle power is a multiple of the original Euler class (and over $\mathbb{C}$, scaling preserves kernels and cokernels of linear maps). □

Question 2.19. Are all Čech-de Rham numbers invariant under Veronese maps? ◇

2.5. Incomplete linear series

We wish here to compare the Čech–de Rham numbers of a projective variety induced by cones that belong to the same line bundle. So, let $\tilde{Y} \subseteq \mathbb{P}^{n-1}_{\mathbb{C}}$ be a projective variety, let R_n be the coordinate ring of $\mathbb{P}^{n-1}_{\mathbb{C}}$ and $I \subseteq R_n$ the ideal defining $\tilde{Y}$. Denote $\mathfrak{m}$ the homogeneous maximal ideal of R_n and $\mathcal{L}$ the line bundle $\mathcal{O}_{\tilde{Y}}(1)$ induced by this embedding.

We assume that the n coordinate functions are linearly independent on $\tilde{Y}$, since otherwise a coordinate change can be used to replace $\mathbb{P}^{n-1}_{\mathbb{C}}$ by a smaller projective space for which Lyubeznik and Čech–de Rham numbers are the same as for the given embedding.

Then there is an exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}}(R_n/I) \longrightarrow R_n/I \longrightarrow \underbrace{\bigoplus_{k\in\mathbb{Z}} \Gamma(\tilde{Y}, \mathcal{L}^k)}_{=:\bar{S}} \longrightarrow H^1_{\mathfrak{m}}(R_n/I) \longrightarrow 0.$$

Set $V = (R_n)_1$ be the space of linear functions on $\mathbb{P}^{n-1}_{\mathbb{C}}$ and let $\bar{V} := \Gamma(\tilde{Y}, \mathcal{L})$ be the complete linear system attached to $\mathcal{L}$. Write $\bar{R}_n := \mathbb{C}[\bar{V}]$ and let $\bar{I} \subseteq \bar{R}_n$ be the ideal cutting out $\tilde{Y}$ inside the dual projective space $\mathbb{P}\bar{V}^*$.

As R_n/I is Noetherian, $H^i_{\mathfrak{m}}(R_n/I)$ is Artinian for all i, and so $\left(H^i_{\mathfrak{m}}(R_n/I)\right)_{\geq k_0}$ is zero for large k_0. It follows that the containments $(R_n/I)_{\geq k_0} \subseteq (\bar{R}_n/\bar{I})_{\geq k_0} \subseteq \bar{S}_{\geq k_0}$ are equalities. This implies that the d-th Veronese iterates of the two projective embeddings of $\tilde{Y}$ to R_n/I and $\bar{R}_n/\bar{I}$ are the same up to a projective coordinate change, provided that $d \geq k_0$. Thus, their cones yield identical Čech–de Rham numbers $\rho^2_{p,q}$, for $p > 1$.

By Theorem 2.18, at least for $p > 1$, the invariants $\rho_{p,q}$ derived from the cones inside V^* and $\bar{V}^*$ agree.

Corollary 2.20. *If two projective embeddings of $\tilde{Y}$ induce the same line bundle on $\tilde{Y}$ then the respective cones produce the same Čech–de Rham numbers $\rho^2_{p,q}$ respectively, at least for $p > 1$.*

3. Lyubeznik numbers

In this section we study the Lyubeznik numbers and their spectral sequence (1.0.3). After surveying some known facts we discuss to what extent a projective variety determines the Lyubeznik numbers of its cone(s). We look first specifically at varieties of Picard number 1, listing some examples and open questions. After that we discuss cases where in small dimension the Lyubeznik tables of all cones agree.

3.1. Basic properties

We should begin with drawing some parallels to the case of the Čech–de Rham numbers. Quite immediately, being defined as the socle dimensions of the E_2-terms in the Grothendieck spectral sequence (1.0.3), the Lyubeznik numbers vanish for $q \notin [\operatorname{codim}(I, R_n), n]$ and for $p \notin [0, n]$. In fact, similarly to the $\rho^r_{p,q}$, Lyubeznik numbers fit into a triangular region

$$\Lambda(Y) := \begin{pmatrix} \lambda_{0,0} & \cdots & \cdots & \lambda_{0,d} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_{d,d} \end{pmatrix},$$

(see [29]). In this picture, the differentials of the spectral sequence point South to Southeast.

For notational ease, we will hereafter indicate a zero entry in a Lyubenik table by a single dot.

Remark 3.1. The fact that the abutment of (1.0.3) is $H^n_{\mathfrak{m}}(R_n)$ implies that the entries

$$\lambda_{0,d} = \lambda_{1,d} = 0$$

always vanish unless the dimension of I is less than two, in which cases the Lyubeznik tables are (1) and $\begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \end{pmatrix}$ respectively. $\diamond$

The number $\lambda_{d,d}$ is never zero by [29] and related to connectedness issues. For example, if $\dim(Y) = 2$ then $\Lambda = \begin{pmatrix} \cdot & a-1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & a \end{pmatrix}$ where a is the number of connected components of the punctured spectrum of the ring defining the purely 2-dimensional part of I, [45,24]. By [47], $\lambda_{d,d}$ is the number of connected components of the Hochster–Huneke graph of the completed strict Henselization of R/I.

It was first observed in [10] that the Lyubeznik numbers encode interesting topological information also in higher dimension. However, it is often not easy to decode this information. Garcia and Sabbah concentrate on the case of an isolated singularity and find that the topology of the singularity link carries all information on Λ. Other relations to connectedness dimensions are discussed in the survey [33].

3.2. Lyubeznik numbers and projective schemes

Suppose $\tilde{Y}$ is a projective variety in $\mathbb{P}^{n-1}_{\mathbb{K}}$, with defining ideal $I \subseteq R_n = \mathbb{K}[x_1, \dots, x_n]$. Different embeddings of $\tilde{Y}$ give rise to different ideals in different polynomial rings, and thus potentially to different sets of Lyubeznik numbers. That this is indeed a possibility was shown to be the case in [40, Sections 2.2, 2.3] where a projective variety with two embeddings is constructed that produce (partially) different $\lambda_{p,q}$. On the other hand, if $\tilde{Y}$ is smooth or a $\mathbb{Q}$-homology manifold or analytically locally a set-theoretic complete intersection, then all cones for $\tilde{Y}$ yield the same Lyubeznik numbers [10,42,40].

A new angle was introduced in [40] by applying the theory of perverse sheaves and mixed Hodge modules to the problem. It is proved there that if I is homogeneous then the $\lambda_{p,q}(R/I)$ measure, for $p > 1$, the failure of a certain morphism of cohomology groups of certain perverse sheaves on $\tilde{Y}$ to be an isomorphism. I follows that, for the purpose of studying the Lyubeznik numbers with $p > 1$ of cones Y over a fixed $\tilde{Y}$, one can move freely between line bundles on $\tilde{Y}$ and cones, as cones that produce different Lyubeznik numbers for $p > 1$ must induce different line bundles on $\tilde{Y}$.

That examples of cones over $\tilde{Y}$ with varying Lyubeznik numbers exist over $\mathbb{C}$ is rather surprising at first, since similar examples cannot exist in any positive characteristic. Indeed, it is shown in [48] that Lyubeznik numbers in finite characteristic can be seen as eigenvalues of certain operators on sheaves that are intrinsic to the projective variety $\tilde{Y}$ associated to I.

All known examples of projective varieties with possibly varying Lyubeznik numbers of their cones come from varieties with Picard number at least two. This is not an accident as we show now.

Lemma 3.2. *Let Y be a cone over the projective variety $\tilde{Y} \subseteq \mathbb{P}^{n-1}_{\mathbb{C}}$. Let v^d_n be the d-th Veronese applied to the cone Y, and write $Y' = v^d_n(Y)$ for the new cone. Then for $p \geq 2$, the Lyubeznik numbers $\lambda_{p,q}(Y)$ and $\lambda_{p,q}(Y')$ agree.*

In particular, if the Picard number of $\tilde{Y}$ equals one, then the Lyubeznik numbers $\lambda_{\geq 2,q}(Y)$ to cones over $\tilde{Y}$ are independent of the cone.

Proof. Let ι_1, ι_2 be two embeddings of $\tilde{Y}$ into projective spaces $\mathbb{P}^{n-1}_{\mathbb{K}}, \mathbb{P}^{m-1}_{\mathbb{K}}$ and denote $Y_1 \subseteq X_1, Y_2 \subseteq X_2$ the two cones over $\tilde{Y}$, sitting in the respective affine spaces that belong to the two embeddings. Let $\mathcal{L}_1, \mathcal{L}_2$ be the associated line bundles on $\tilde{Y}$ obtained as pullbacks of $\mathcal{O}_{\mathbb{P}^{n-1}_{\mathbb{K}}}(1)$ and $\mathcal{O}_{\mathbb{P}^{m-1}_{\mathbb{K}}}(1)$ respectively. Then by [40, Prop. 1,2,3], the Lyubeznik numbers $\lambda_{p,q}$ of $\tilde{Y}$ that belong to Y_i and have $p \geq 2$ are determined by the (co)kernel sizes of the Chern classes of $\mathcal{L}_i$ on certain cohomology groups of $\tilde{Y}$ with rational coefficients. These cohomology groups themselves (see [40, Prop. 2]) do not depend on the bundles $\mathcal{L}_i$.

If ι_2 is the d-fold Veronese applied to ι_1 then the (first) Chern class of $\mathcal{L}_2$ is d times that of L_1. In particular, their kernels and cokernels on $\mathbb{Q}$-spaces are identical and the first claim follows.

Now suppose that the target of the natural map

$$\phi\colon \operatorname{Pic}(\tilde{Y}) \longrightarrow \operatorname{Pic}(\tilde{Y}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is $\mathbb{Q}$. If ι_1, ι_2 are both projective embeddings of $\tilde{Y}$, ampleness implies that $\phi(\mathcal{L}_i) > 0$. Then if $\phi(\mathcal{L}_1) = q_1$ and $\phi(\mathcal{L}_2) = q_2$, both positive rational numbers, we have for $k \gg 0$ with $kq_1, kq_2 \in \mathbb{N}$ that $\mathcal{L}_1^{k|q_2|} = \mathcal{L}_2^{k|q_1|}$. Then by the first part of the proof, ι_1, $\iota_1^{kq_2}$, $\iota_2^{kq_1}$ and ι_2 all yield the same Lyubeznik numbers $\lambda_{p,q}$ for $p \geq 2$ (where we write ι^ℓ for the ℓ-th Veronese of the embedding ι). □

For $p < 2$, we do not know how to compare the $\lambda_{p,q}$ of different cones.

Problem 3.3. Do *all* Lyubeznik numbers of all cones Y of $\tilde{Y}$ agree if the Picard number of $\tilde{Y}$ is one? ⋄

Here are three interesting sets of varieties to which the lemma applies.

3.2.1. Determinantal ideals

Proposition 3.4. *The Lyubeznik numbers $\lambda_{p,q}$ with $p \geq 2$ of (the cones over) the projective determinantal varieties $\tilde{Y}_{m,n,t}$ cut out by the $t \times t$ minors of an $m \times n$ matrix of indeterminates are unique.*

Proof. Let $A_{m,n,t}$ be the ring obtained as quotient of the polynomial ring $\mathbb{K}[x_{i,j}|1 \leq i \leq m, 1 \leq j \leq n]$ by the t-minors of the matrix $x := ((x_{i,j}))$.

The case $t = 1$ is trivial. If $t = 2$, the associated projective variety is the product of two projective spaces, and in particular smooth. By [10], or [42], the Lyubeznik numbers of $\tilde{Y}_{m,n,2}$ are independent of the embedding.

Now consider the case $t > 2$. By [5, Cor. 8.4], the divisor class group of $A_{m,n,t}$ is $\mathbb{Z}$, a generator being the ideal $\bar{I}_{m,n,t-1}$ of $A_{m,n,t}$ generated by the $(t-1)$-minors of the first $t-1$ rows (or columns) of x.

Since determinantal varieties are normal, they satisfy condition $(*)$ in [17, Page 130]. By [17, Exercise II.6.3], there is a short exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \operatorname{Cl}(\tilde{Y}) \longrightarrow \operatorname{Cl}(Y) \longrightarrow 0$, Y the cone over $\tilde{Y}$, where the last map factors through the class group $\operatorname{Cl}(Y \smallsetminus P)$ of the complement of the origin in Y. For $t \geq 2$ this implies that $\operatorname{Cl}(\tilde{Y}_{m,n,t}) = \mathbb{Z} \oplus \mathbb{Z}$. In this sequence, $1 \in \mathbb{Z}$ is sent to the generic hyperplane section of $\tilde{Y}$. In order to determine the Picard group of $\tilde{Y}_{m,n,t}$ we need by [17, Prop. II.6.15] to determine the Cartier classes of $\operatorname{Cl}(\tilde{Y}_{m,n,t})$. From the preceding, this amounts to checking which multiples of $\bar{I}_{m,n,t-1}$ are Cartier on the punctured spectrum of $A_{m,n,t}$. One sees easily that for $t = 2$, $\bar{I}_{m,n,t-1}$ is Cartier on the punctured spectrum. For $t > 2$ only its trivial power is Cartier: by the coordinate change expounded in [27], powers of $\bar{I}_{m,n,t-1}$ are locally principal on the open set $U_{x_{1,1}}$ if and only if corresponding powers of $\bar{I}_{m-1,n-1,t-2}$ are locally principal everywhere on $Y_{m-1,n-1,t-1}$; for $t = 3$ this is clearly not so. Hence the Picard group of $\tilde{Y}_{m,n,t}$ is $\mathbb{Z}$ for $t > 2$. Now use Lemma 3.2. □

Remark 3.5. In particular, the Lyubeznik numbers $\lambda_{p,q}$ of determinantal varieties computed by Lörincz and Raicu in [26] for the standard embedding equal those of any embedding, at least for $p \geq 2$. ⋄

Remark 3.6. Suppose G is a semisimple linear algebraic group, P a parabolic subgroup and w an element of the Weyl group of G. The Schubert variety $X_P(w) := BwP/P$ sits inside the homogeneous space G/P, and every line bundle on $X_P(w)$ is the restriction of a line bundle on G/P, [30]. In particular, the Picard group of $X_P(w)$ is (freely) generated by the Schubert divisors (the Schubert varieties inside $X_P(w)$ of codimension one), and the interior points of the positive Schubert cone are very ample [2, Prop. 2.2.8, Prop. 1.4.1]. ⋄

Problem 3.7. Compute the Lyubeznik numbers of Schubert varieties induced from embeddings interior to the positive Schubert divisor cone $\diamond$

3.2.2. Toric varieties

Suppose $\tilde{Y}$ is the toric variety attached to a complete fan Δ that is projective. If Δ is smooth, or at least simplicial, then the Picard group of $\tilde{Y}$ is a free Abelian group generated by the torus invariant (Cartier) divisors corresponding to the n rays of Δ, [6, Thm. 4.2.1]. The ambient lattice imposes $d := \dim(\tilde{Y})$ many independent relations on these divisors, so that $\mathrm{Pic}(\tilde{Y}) = \mathbb{Z}^{n-d}$. In order for this number to be 1, there is very little choice for Δ; it forces $\tilde{Y}$ to be a weighted projective space. These are $\mathbb{Q}$-homology manifolds and thus yield the same Lyubeznik numbers under all embeddings by [40].

However, singular fans fail the Picard rank formula above and can have Picard group $\mathbb{Z}$ with greater variety. The Picard group is free if the fan is full-dimensional by [6, Thm. 4.2.5], and equals the inverse limit of the quotient lattices $M/M(\sigma)$, taken modulo M by [6, Thms. 4.2.1,4.2.9].

Example 3.8. If Δ is a complete rational fan in $\mathbb{Z}^3$, one can use the description of the Picard group via support functions to show that if Δ has at most one simplicial cone, then the Picard group of the associated toric variety is rank one. For example, the fan over the sides of a cube leads to a projective three-fold with Picard number one. The generating support function takes the value zero on one square and one on the opposing square (see [9, Exa. 1.5.(3)]). Our next result shows that all projective toric threefolds have their Lyubeznik table independent of the embedding. $\diamond$

Theorem 3.9. *Let $\tilde{Y}$ be the projective variety attached to a complete projective fan in $\mathbb{Z}^3$ with Picard number $\tilde{p}+1$. Then for any cone Y over $\tilde{Y}$ its Lyubeznik numbers take the form*

$$\Lambda(Y) = \begin{pmatrix} \cdot & \cdot & \cdot & \tilde{p} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \tilde{p} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

We postpone the proof until the end of the final subsection.

Problem 3.10. Express Lyubeznik numbers of projective toric varieties (of Picard rank 1 or otherwise) in terms of fan data (and embedding polytopes, if necessary). $\diamond$

3.2.3. Horospherical varieties

Horospherical varieties are complex normal algebraic varieties on which a connected complex reductive algebraic group G acts with an open orbit that is isomorphic to a torus bundle over a flag variety; the dimension of this torus is referred to as the rank of the variety. In particular, toric and flag varieties are examples of horospherical varieties.

Any flag variety G/P with P a parabolic subgroup of G is smooth and projective. Their Lyubeznik numbers are hence all topological, by [42]. By [38, Thm. 0.1], a smooth projective horospherical variety of Picard number 1 must either be a homogeneous space or have horospherical rank one.

There are many singular horospherical varieties of Picard number one. For example, let G be a simple linear algebraic group and choose two dominant weights χ_1 and χ_2 that cannot be written as the sum of a common dominant weight with another dominant weight. Writing $V(\chi)$ for the simple G-module of weight χ, let $\tilde{Y}$ be the closure of the G-orbit of the sum of two highest weight vectors in $\mathbb{P}(V(\chi_1) \oplus V(\chi_2))$. It is a projective variety of horospherical rank one, it has Picard number one and is smooth only in very few cases, namely when χ_1 and χ_2 are fundamental weights ϖ_α and ϖ_β and (G, α, β) is in the list of [38, Thm. 1.7]. It has three G-orbits (one open and two closed), the singularities if they exist, are on the closed orbit(s).

By taking a longer list of weights $\chi_1, \ldots, \chi_n$ one can produce (usually singular) varieties of horospherical rank $n-1$.

In all these Picard rank 1 cases, the Lyubeznik numbers $\lambda_{p,q}$ with $p > 1$ of the cone of $\tilde{Y}$ are embedding independent, and can hence be computed from the embedding that arises from the definition.

Problem 3.11. Compute Lyubeznik numbers of horospherical varieties of Picard number one for the standard families. $\diamond$

3.3. Lyubeznik numbers in small dimension

We consider here to what extent the Lyubeznik numbers of varieties of small dimension are functions of the associated projective variety only. Some independence is known quite generally.

Remark 3.12. (1) If $\tilde{Y}$ is a projective scheme of dimension at most 1 over any field (not necessarily connected or equi-dimensional) then R/I is two-dimensional for any embedding, and so Λ is independent of embeddings by [45,24]
(2) Set $d-1 = \dim(\tilde{Y})$. Then $\lambda_{d,d}$ is independent of embeddings unconditionally by [47]. $\diamond$

We begin with some preparations involving Hartshorne's (local) algebraic de Rham cohomology.

Theorem 3.13. *Let $\tilde{Y} \subseteq \mathbb{P}^{n-1}_{\mathbb{K}}$ be a projective variety over a field $\mathbb{K}$ of characteristic 0, let $Y \subseteq \mathbb{A}^n_{\mathbb{K}}$ be the affine cone of $\tilde{Y}$, and let P be its vertex. Let $H^j_P(Y)$ denote the local de Rham cohomology of Y supported in $\{P\}$. Assume that the Picard group of $\tilde{Y}$ has rank 1. Then $\dim_k(H^j_P(Y))$ depends only on $\tilde{Y}$, but not on the embedding $\tilde{Y} \subseteq \mathbb{P}^n_{\mathbb{K}}$. More precisely, if $\tilde{Y} \subseteq \mathbb{P}^{n'-1}_{\mathbb{K}}$ is another embedding of $\tilde{Y}$ into a projective space and Y' is its affine cone with the vertex P', then $\dim_{\mathbb{K}}(H^j_P(Y)) = \dim_{\mathbb{K}}(H^j_{P'}(Y'))$.*

To prove Theorem 3.13, we need the following result of Hartshorne.

Theorem 3.14 *(Proposition III.3.2 in [16]). Let $\tilde{Y}, Y, P$ be the same as in Theorem 3.13. Then $H^0_P(Y) = 0$ and there are two exact sequences:*

$$0 \longrightarrow \mathbb{K} \longrightarrow H^0_{\mathrm{dR}}(\tilde{Y}) \longrightarrow H^1_P(Y) \longrightarrow 0$$

and

$$0 \longrightarrow H^1_{\mathrm{dR}}(\tilde{Y}) \longrightarrow H^2_P(Y) \longrightarrow H^0_{\mathrm{dR}}(\tilde{Y}) \longrightarrow H^2_{\mathrm{dR}}(\tilde{Y}) \longrightarrow H^3_P(\tilde{Y}) \longrightarrow H^1_{\mathrm{dR}}(\tilde{Y}) \longrightarrow H^3_{\mathrm{dR}}(\tilde{Y}) \longrightarrow \cdots$$

*where the maps $H^i_{\mathrm{dR}}(\tilde{Y}) \longrightarrow H^{i+2}_{\mathrm{dR}}(\tilde{Y})$ are given by the cup product with the Chern class $\xi \in H^2_{\mathrm{dR}}(\tilde{Y})$ of the hyperplane section (*i.e.*, the first Chern class of $\mathcal{O}_{\tilde{Y}}(1)$).* $\square$

Proof of Theorem 3.13. The case when $j \leq 1$ is clear from the long exact sequence above.

Since the Picard group of $\tilde{Y}$ has rank 1, any two very ample line bundles on $\tilde{Y}$ have a common power. It is thus sufficient to consider the case where the two ample line bundles in question are $\mathcal{L}$ and $\mathcal{L}^m$.

Let $\xi(\mathcal{L}) \in H^2_{dR}(\tilde{Y})$ be the first Chern class of $\mathcal{L}$, represented by a generic hyperplane section with the embedding given by $\mathcal{L}$. Then we have $\xi(\mathcal{L}^m) = m\xi(\mathcal{L})$. Since the cup product is linear and $\mathrm{char}(\mathbb{K}) = 0$, the maps $H^i_{\mathrm{dR}}(\tilde{Y}) \xrightarrow{\cup\xi} H^{i+2}_{\mathrm{dR}}(\tilde{Y})$ and $H^i_{\mathrm{dR}}(\tilde{Y}) \xrightarrow{\cup m\xi} H^{i+2}_{\mathrm{dR}}(\tilde{Y})$ have the same rank. Therefore $\dim_{\mathbb{K}}(\ker(H^i_{dR}(\tilde{Y}) \longrightarrow H^{i+2}_{\mathrm{dR}}(\tilde{Y})))$ and $\dim_{\mathbb{K}}(\mathrm{coker}(H^i_{dR}(\tilde{Y}) \longrightarrow H^{i+2}_{\mathrm{dR}}(\tilde{Y})))$ depend only on $\tilde{Y}$, but not on the choice of the embedding (or equivalently, not on the choice of ample line bundles $\mathcal{L}$). When $j \geq 2$ we have

$$\dim_{\mathbb{K}}(H_P^j(Y) = \dim_{\mathbb{K}}(\ker(H_{\mathrm{dR}}^{j-2}(\tilde{Y}) \longrightarrow H_{\mathrm{dR}}^j(\tilde{Y})))) + \dim_{\mathbb{K}}(\mathrm{coker}(H_{\mathrm{dR}}^{j-3}(\tilde{Y}) \longrightarrow H_{\mathrm{dR}}^{j-1}(\tilde{Y})))),$$

hence the conclusion holds for $\dim_k(H_P^j(Y))$ when $j \geq 2$. □

Corollary 3.15. *Assume that the Picard group of $\tilde{Y}$ has rank 1. Then $\lambda_{p,q}$ is independent of embeddings for all $q < n - f_Y$, with f_Y as in Remark 2.7.*

Proof. Assume $q < n - f_Y$. Since $\mathrm{Supp}(H_I^{n-q}(R_n)) \subseteq \{\mathfrak{m}\}$, [29] shows that $H_I^{n-q}(R_n) \cong H_{\mathfrak{m}}^n(R_n)^{\lambda_{0,q}}$ and $H_{\mathfrak{m}}^p(H_I^{n-q}(R_n)) = 0$ for $p \geq 1$. Hence $\lambda_{p,q} = 0$ for all $p \geq 1$ (and $q < n - f_Y$).

Let $D(-)$ denote the Matlis dual. Then $D(H_I^\ell(R_n)) \cong \hat{R_n}^{\lambda_{0,n-\ell}}$ whenever $H_I^\ell(R_n)$ is Artinian. On the other hand, [34, Proposition 2.2,Theorem 2.3] shows that, for $q < n - f_Y$,

$$D(H_I^{n-q}(R_n)) \cong H_P^q(\hat{X}, \mathcal{O}_{\hat{X}}) \cong \hat{R_n} \otimes H_P^q(Y)$$

where Y denotes the affine cone of $\tilde{Y}$ with vertex P and $\hat{X}$ denotes the formal completion of $\mathrm{Spec}\,(\hat{R_n})$ along the subscheme defined by I. This shows that $\dim_{\mathbb{K}}(H_P^q(Y)) = \lambda_{0,q}$. Hence $\lambda_{0,q}$ depends only on $\tilde{Y}$ by Theorem 3.13. □

Remark 3.16. An alternative way to look at Corollary 3.15 arises through Proposition 2.16: for $q > f_Y$, the multiplicities of $H_{\mathfrak{m}}^n(R_n)$ in $H_I^j(R_n)$ are exactly the Betti numbers $H^{n-1+j}(U)$ where U is the affine complement of Y, because of the spectral sequence (1.0.2). By Proposition 2.16 these do not change under Veronese maps. ◇

We now consider the effect of Serre's conditions (S_t) in R_n/I on the Lyubeznik numbers.

Remark 3.17. Assume that "$\tilde{Y}$ satisfies (S_t) locally everywhere", by which we mean that each local ring $\mathcal{O}_{\tilde{Y},\tilde{\mathfrak{y}}}$ of the projective scheme $\tilde{Y} = \mathrm{Proj}\,(R_n/I)$ satisfies Serre's condition (S_t).

Let Y be the cone $\mathrm{Spec}\,(R_n/I)$ as always and P the vertex; then the punctured cone $Y^\circ = Y \smallsetminus P$ is a bundle over $\tilde{Y}$. It follows that every local ring of Y° also is (S_t). So for each non-maximal prime ideal $\mathfrak{p}$ of R_n such that $\dim((R_n/I)_{\mathfrak{p}}) \geq t$, one has $\mathrm{depth}((R_n/I)_{\mathfrak{p}}) \geq t$.

In general, if $(A, \mathfrak{n}) \longrightarrow (A', \mathfrak{n}')$ is a faithfully flat morphism, then

$$\mathrm{depth}(A') = \mathrm{depth}(A) + \mathrm{depth}(A'/\mathfrak{n}A).$$

If A' is the strict Henselization A^{sh} or the completion $\hat{A}$ of A, then A' is faithfully flat over A. Therefore,

$$\mathrm{depth}\left(\left(((R_n/I)_{\mathfrak{p}})^\frown\right)^{sh}\right)^\frown \geq t. \quad \diamond$$

Lemma 3.18. *If $\tilde{Y}$ is equi-dimensional and locally everywhere (S_2) then the off-diagonal entries $\lambda_{i-1,i}$ vanish for $1 < i < d := \dim(\tilde{Y}) + 1$, and $H_I^{n-1}(R_n)$ is Artinian and injective.*

Proof. By Remark 3.17, for each non-maximal prime ideal $\mathfrak{p}$ of R_n with $\dim((R_n/I)_{\mathfrak{p}}) \geq 2$, we have $\mathrm{depth}\left(\left(((R_n/I)_{\mathfrak{p}})^\frown\right)^{sh}\right)^\frown \geq 2$. Hence the punctured spectrum of this ring is connected by [14, Thm. 2.2]. The Second Vanishing Theorem implies that $H_I^{>\mathrm{codim}(P,R_n)-2}(R_n)_{\mathfrak{p}} = 0$ for each prime ideal P such that $\dim((R_n/I)_{\mathfrak{p}}) \geq 2$. Therefore the support dimension of $H_I^i(R_n)$ with $n - 1 > i > n - d$ is at most equal to $n-i-2$ and so $H_{\mathfrak{m}}^{i-1} H_I^{n-i}(R_n) = 0$ by Grothendieck's vanishing theorem. For $H_I^{n-1}(R_n)$, localization shows in conjunction with the Hartshorne–Lichtenbaum theorem that its support is at best at P. By Lyubeznik's work, it is hence Artinian and injective. □

For the next three result we will use the following reduction.

Lemma 3.19. *Let $\tilde{Y}$ be an equi-dimensional projective variety of dimension at least two. If the Lyubeznik numbers for the cones over all connected components of $\tilde{Y}$ are independent of the choice of the cone then the same is true for $\tilde{Y}$ itself.*

Proof. Let Y', Y'' be two cones for $\tilde{Y}$ and let $\tilde{Y} = \tilde{Y}_1 \sqcup \tilde{Y}_2$ be a disconnection. The resulting cones Y_1', Y_1'' and Y_2', Y_2'' satisfy: $Y_1' \cap Y_2'$ and $Y_1'' \cap Y_2''$ both equal the origin. Let I', I'' be the defining ideals for Y', Y'' and denote the defining ideals of Y_1', Y_2', Y_1'', Y_2'' by $I_1', I_2' \subseteq R_{n'}$ and $I_1'', I_2'' \subseteq R_{n''}$ respectively. All these ideals have dimension three or more.

Then $H^q_{I'}(R_{n'}) = H^q_{I_1'}(R_{n'}) \oplus H^q_{I_2'}(R_{n'})$ and $H^q_{I''}(R_{n''}) = H^q_{I_1''}(R_{n''}) \oplus H^q_{I_2''}(R_{n''})$ for all $q < n-1$ as follows from the Mayer–Vietoris sequence.

It follows that, apart from $q = n, n-1$, the Lyubeznik numbers satisfy $\lambda_{p,q}(Y') = \lambda_{p,q}(Y_1') + \lambda_{p,q}(Y_2')$ and $\lambda_{p,q}(Y'') = \lambda_{p,q}(Y_1'') + \lambda_{p,q}(Y_2'')$. By the presumed embedding independence of $\Lambda(Y_1)$ and $\Lambda(Y_2)$, the same follows for $\Lambda(Y)$, except for columns $n, n-1$.

In column n all entries in all cases are zero by the Hartshorne–Lichtenbaum theorem. So is the diagonal entry $\lambda_{1,1}$ for all three ideals by equi-dimensionality. Thus, $\lambda_{0,1}(Y_i) = \lambda_{0,1}(Y_i') + \lambda_{0,1}(Y_i'') + 1$ for $i = 1, 2$ as follows from the Grothendieck spectral sequence (which implies that the alternating sum of all $\lambda_{p.q}$ is 1). Therefore, all Lyubeznik numbers of $\tilde{Y}$ are embedding independent. $\square$

Theorem 3.20. *Let $\tilde{Y}$ be an equi-dimensional projective scheme of dimension two, which*

(1) either satisfies locally everywhere Serre's condition S_2,
(2) or has Picard number one.

Then the Lyubeznik numbers of all affine cones Y over $\tilde{Y}$ agree.

Proof. Let Y be any cone over $\tilde{Y}$. It is a scheme of pure dimension 3, and thus by Remark 1.3, the Lyubeznik table of Y is

$$\begin{pmatrix} \cdot & \lambda_{0,1} & \lambda_{0,2} & \cdot \\ \cdot & \cdot & \lambda_{1,2} & \cdot \\ \cdot & \cdot & \cdot & \lambda_{2,3} \\ \cdot & \cdot & \cdot & \lambda_{3,3} \end{pmatrix}.$$

By Lemma 3.19, we can assume that $\tilde{Y}$ is connected. That assures that $\lambda_{0,1}$ is zero by the Second Vanishing Theorem [15, Theorem 7.5].

If $\tilde{Y}$ is (S_2) locally everywhere then by Lemma 3.18, $H^{n-2}_I(R_n)$ has support dimension zero and is the top local cohomology module, and so $\lambda_{1,2} = 0$. It follows from [29] that $H^{n-2}_I(R_n)$ is injective. By Corollary 2.9, the socle dimension $\lambda_{0,2}$ of this module is determined by the topology of $\tilde{Y}$. Finally, the convergence of the spectral sequence to $H^n_{\mathfrak{m}}(R_n)$ implies that $\lambda_{2,3} = \lambda_{0,2}$.

Suppose now that $\tilde{Y}$ has Picard number one. Then by Lemma 3.2, the $\lambda_{i,j}$ with $i > 1$ are a function of $\tilde{Y}$ alone. The only possibly nonzero differentials are:

- on page two the morphism $E_2^{0,n-2} \longrightarrow E_2^{2,n-3}$ and $E_2^{1,n-2} \longrightarrow E_2^{3,n-3}$;
- on page three the morphism $E_3^{0,n-1} \longrightarrow E_3^{3,n-3}$.

Convergence of the spectral sequence forces $E_2^{0,n-2} \longrightarrow E_2^{2,n-3}$ to be an isomorphism,[2] and the maps $E_2^{1,n-2} \longrightarrow E_2^{3,n-3}$ and $E_3^{0,n-1} \longrightarrow E_3^{3,n-3}$ to be injective. Moreover, the cokernel of $E_3^{0,n-1} \longrightarrow E_3^{3,n-3}$ must be one copy of $H^n_{\mathfrak{m}}(R_n)$.

Since all modules in $E_{\geq 2}^{p,q}$ are injective, socle dimensions are additive in short exact sequences. Thus, $\lambda_{0,2} = \lambda_{2,3}$, and $\lambda_{3,3} = \lambda_{1,2} + \lambda_{0,1} + 1 = \lambda_{1,2} + 1$. This settles the claim for $\lambda_{0,2}$. But $\lambda_{3,3}$ is a function of $\tilde{Y}$ by [47], and it follows that $\lambda_{1,2}$ is a function of $\tilde{Y}$ as well. $\square$

Theorem 3.21. *Let $\tilde{Y}$ be a projective complex scheme that is of equi-dimension three. Assume that every local ring $\mathcal{O}_{\tilde{Y},\tilde{y}}$ satisfies (S_2) and that the Picard group of $\tilde{Y}$ has rank 1. Then $\Lambda(Y)$ is independent of the choice of the cone Y for $\tilde{Y}$.*

Proof. By Lemma 3.19 we can assume that $\tilde{Y}$ is connected. This forces $\lambda_{0,1}(Y) = 0$ for any cone Y of $\tilde{Y}$ by the Second Vanishing Theorem [15, Theorem 7.5].

Using the equi-dimensionality and the (S_2)-property, the Lyubeznik table is by Remark 1.3 and Lemma 3.18 equal to

$$\Lambda = \begin{pmatrix} \cdot & \cdot & \lambda_{0,2} & \lambda_{0,3} & \cdot \\ \cdot & \cdot & \cdot & \lambda_{1,3} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \lambda_{2,4} \\ \cdot & \cdot & \cdot & \cdot & \lambda_{3,4} \\ \cdot & \cdot & \cdot & \cdot & \lambda_{4,4} \end{pmatrix},$$

Moreover, $H_I^{n-2}(R_n)$ is supported only in the origin, hence injective. By Corollary 2.9, its socle dimension is the dimension of the top de Rham group of the affine cone complement. By Proposition 2.16, this dimension is well-defined. Thus, $\lambda_{0,2}$ is a function of $\tilde{Y}$ alone, reflecting the de Rham group $H^{2n-2}(\mathbb{A}^n_{\mathbb{K}} \smallsetminus Y)$ independent of the choice of the cone.

Convergence of the spectral sequence forces, similarly to the proof of Theorem 3.20, that $\lambda_{3,4} = \lambda_{1,3} + \lambda_{0,2}$ and that $\lambda_{2,4} = \lambda_{0,3}$. By the Picard number condition, $\lambda_{2,4}$ is the same for every cone, and hence so is $\lambda_{0,3}$. Since $\lambda_{0,2}$ is a function of $\tilde{Y}$, and since $\lambda_{\geq 2,*}$ is independent of the embedding by the Picard number condition, the same is true for $\lambda_{1,3}$. $\square$

Theorem 3.22. *Let $\tilde{Y}$ be a projective complex scheme of equi-dimension four. Assume that $\tilde{Y}$ is locally everywhere (S_3), and that the Picard group of $\tilde{Y}$ has rank 1. Then $\Lambda(Y)$ is independent of the choice of the cone Y for $\tilde{Y}$.*

Proof. By Lemma 3.19 we can assume that $\tilde{Y}$ is connected. This forces $\lambda_{0,1}(Y) = 0$ for any cone Y of $\tilde{Y}$ by the Second Vanishing Theorem [15, Theorem 7.5].

Write $\tilde{Y} = \operatorname{Proj}(R/I)$ where $R = \mathbb{C}[x_1, \ldots, x_n]$. Since (S_3) implies (S_2), Remark 1.3 and Lemma 3.18 assure that the Lyubeznik table of R/I is

$$\Lambda = \begin{pmatrix} \cdot & \cdot & \lambda_{0,2} & \lambda_{0,3} & \lambda_{0,4} & \cdot \\ \cdot & \cdot & \cdot & \lambda_{1,3} & \lambda_{1,4} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \lambda_{2,4} & \lambda_{2,5} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{3,5} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{4,5} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{5,5} \end{pmatrix}.$$

Now take a prime $\mathfrak{p}$ of height $n-2$ that contains I. Then $\operatorname{depth}((R_n/I)_{\mathfrak{p}}) = 3$ and so by [8, Corollary 2.8], $(H_I^{(n-2)-3+1}(R_n))_{\mathfrak{p}} = 0$. Thus, $\dim(H_I^{n-4}(R_n)) \leq 1$ and $\lambda_{2,4} = 0$.

[2] This isomorphism property holds for any ideal I of dimension greater than two.

Localizing at primes of height $n-1$ yields, with the result of Dao and Takagi [8, Corollary 2.8], that $H_I^{n-2}(R_n)$ and $H_I^{n-3}(R_n)$ are Artinian. It follows that $\lambda_{1,3}=0$, and $f_Y \le n-3$. By Corollary 3.15, since the Picard number is one, $\lambda_{0,2}$ and $\lambda_{0,3}$ are independent of the embedding choice.

Convergence of the spectral sequence to $H^n_{\mathfrak{m}}(R_n)$ forces that

$$\lambda_{0,4}=\lambda_{2,5} \qquad \text{and} \qquad \lambda_{1,4}=\lambda_{3,5}-\lambda_{0,3} \qquad (\text{and} \qquad \lambda_{0,2}=\lambda_{4,5}).$$

As the Picard number is one, the $\lambda_{i,j}$ are independent of embeddings for all $i \ge 2$ and all j. This then fixes all $\lambda_{p,q}$. □

Proof of Theorem 3.9. Toric projective varieties are connected and locally the spectra of semigroup rings to saturated semigroups. They are hence normal, and so by Hochster's theorem $\tilde{Y}$ is Cohen–Macaulay, [19]. The coordinate ring R_n/I of the cone Y thus has a Lyubeznik table as in the proof of Theorem 3.21. Moreover, $H_I^{n-2}(R_n)$ is Artinian.

Additional vanishings are due to the (S_3)-condition on the punctured spectrum of Y. As in the proof of Theorem 3.22, localization at a prime of R_n of height $n-1$ shows with [8, Thm. 2.8] that the support of $H_I^{n-3}(R_n)$ is zero-dimensional, hence $\lambda_{1,3}=0$.

At this point, let us assume that $\tilde{Y}$ is not a hypersurface, and hence of codimension two or more.

If $\lambda_{0,2}$ is nonzero, it is therefore the dimension of $H^n_{\mathrm{dR}}(H_I^{n-2}(R_n)) = H^{2n-3}(U;\mathbb{C})$ where U is the affine complement of Y. By the spectral sequence argument in the proof of Lemma 2.8, it also equals the dimension of the top cohomology group $H^{2n-4}(\mathbb{P}U;\mathbb{C})$ of the projective complement $\mathbb{P}U$.

The long exact sequence (2.3.1) takes the form

$$H^{2n-4}_{\tilde{Y}}(\mathbb{P}^{n-1};\mathbb{C}) \longrightarrow H^{2n-4}(\mathbb{P}^{n-1};\mathbb{C}) \longrightarrow H^{2n-4}(\mathbb{P}U;\mathbb{C}) \longrightarrow H^{2n-3}_{\tilde{Y}}(\mathbb{P}^{n-1};\mathbb{C}) \longrightarrow \cdots$$

and by [22, V.6.6] $H^{2n-3}_{\tilde{Y}}(\mathbb{P}^{n-1};\mathbb{C})$ is dual to $H^1_c(\tilde{Y};\mathbb{C}) = H^1(\tilde{Y};\mathbb{C})$. But projective toric varieties (or more generally toric varieties to a fan with a full-dimensional cone) are simply connected by [9, 3.2]. So $H^1(\tilde{Y};\mathbb{C})$ and $H^{2n-3}_{\tilde{Y}}(\mathbb{P}^{n-1};\mathbb{C})$ are zero.

The morphism $H^{2n-4}_{\tilde{Y}}(\mathbb{P}^{n-1};\mathbb{C}) \longrightarrow H^{2n-4}(\mathbb{P}^{n-1};\mathbb{C})$ is via Alexander and Poincaré duality dual to the (injective) restriction morphism $H^2(\mathbb{P}^{n-1};\mathbb{C}) \longrightarrow H^2(\tilde{Y};\mathbb{C})$, hence itself surjective. It follows that $H^{2n-4}(\mathbb{P}U;\mathbb{C}) = H^{2n-3}(U;\mathbb{C}) = 0$, and hence $\lambda_{0,2}$ and $H_I^{n-2}(R_n)$ are both zero.

It now follows that actually the Artinian module $H_I^{n-3}(R_n)$ is the top local cohomology group of I, and $H^{2n-5}(\mathbb{P}U;\mathbb{C})$ is the top cohomology group of $\mathbb{P}U$. Repeating the above computations, we now have a long exact sequence

$$0 = H^{2n-5}(\mathbb{P}^{n-1};\mathbb{C}) \longrightarrow H^{2n-5}(\mathbb{P}U;\mathbb{C}) \longrightarrow H^{2n-4}_{\tilde{Y}}(\mathbb{P}^{n-1};\mathbb{C}) \longrightarrow H^{2n-4}(\mathbb{P}^{n-1};\mathbb{C}) \longrightarrow \cdots$$

in which the arrow $H^{2n-4}_{\tilde{Y}}(\mathbb{P}^{n-1};\mathbb{C}) \longrightarrow H^{2n-4}(\mathbb{P}^{n-1};\mathbb{C})$ is dual to the (injective) morphism $H^2(\mathbb{P}^{n-1};\mathbb{C}) \longrightarrow H^2(\tilde{Y};\mathbb{C})$, and where $H^{2n-5}(\mathbb{P}U;\mathbb{C}) = H^{2n-4}(U;\mathbb{C})$ is a vector space of dimension $\lambda_{0,3}$.

For projective toric varieties (and more generally, when all cones of the fan are top-dimensional), the Picard group of $\tilde{Y}$ is isomorphic to $H^2(\tilde{Y};\mathbb{C})$, [6, Thm. 12.3.2]. The long exact sequence above thus shows the equation $\lambda_{0,3}=\tilde{p}$.

Finally, by convergence of the spectral sequence, $\lambda_{4,4}=1$ and $\lambda_{3,4}=\lambda_{0,2}=0$ and $\lambda_{2,4}=\lambda_{0,3}$.

This settles the problem for all embeddings in which $\tilde{Y}$ is not a hypersurface. If in some embedding $\tilde{Y}$ happens to be a hypersurface, necessarily in $\mathbb{P}^4_{\mathbb{C}}$, its Lyubeznik table is trivial for this embedding, simply for lack of higher local cohomology. On the other hand, [13, Exp. XII, Cor 3.7] asserts that the Picard group of $\tilde{Y}$ is then cyclic, equal to that of $\mathbb{P}^4_{\mathbb{C}}$. Thus, $\tilde{p}$ is zero and we see that all Lyubeznik tables of $\tilde{Y}$ agree. □

Acknowledgements

We would like to thank Winfried Bruns, Boris Pasquier and Alex Dimca for helpful conversations. We are indebted to the referee for a careful reading of the manuscript, several suggestions for improvement, and for catching a number of misprints.

References

[1] Josep Àlvarez Montaner, Ricardo García López, Santiago Zarzuela Armengou, Local cohomology, arrangements of subspaces and monomial ideals, Adv. Math. 174 (1) (2003) 35–56, MR 1959890.
[2] Michel Brion, Lectures on the geometry of flag varieties, in: Topics in Cohomological Studies of Algebraic Varieties, in: Trends Math., Birkhäuser, Basel, 2005, pp. 33–85, MR 2143072.
[3] Nicole Bridgland, On the de Rham homology of affine varieties in characteristic 0, arXiv:2006.01334.
[4] Raoul Bott, Loring W. Tu, Differential Forms in Algebraic Topology, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York-Berlin, 1982, MR 658304.
[5] Winfried Bruns, Udo Vetter, Determinantal Rings, Monografías de Matemática [Mathematical Monographs], vol. 45, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1988, MR 986492.
[6] David A. Cox, John B. Little, Henry K. Schenck, Toric Varieties, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011, MR 2810322.
[7] Alexandru Dimca, Sheaves in Topology, Universitext, Springer-Verlag, Berlin, 2004, MR 2050072.
[8] Hailong Dao, Shunsuke Takagi, On the relationship between depth and cohomological dimension, Compos. Math. 152 (4) (2016) 876–888, MR 3484116.
[9] William Fulton, Introduction to toric varieties, in: The William H. Roever Lectures in Geometry, in: Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, MR 1234037.
[10] R. García López, C. Sabbah, Topological computation of local cohomology multiplicities, Collect. Math. 49 (1998) 317–324, Dedicated to the memory of Fernando Serrano, MR 1677136.
[11] Mark Goresky, Robert MacPherson, Stratified Morse theory, in: Ergebnisse der Mathematik und ihrer Grenzgebiete (3) (Results in Mathematics and Related Areas (3)), vol. 14, Springer-Verlag, Berlin, 1988, MR 932724.
[12] Claude Godbillon, Éléments de Topologie Algébrique, Hermann, Paris, 1971, MR 0301725.
[13] Alexander Grothendieck, Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2), in: Séminaire de Géométrie Algébrique du Bois Marie, in: Documents Mathématiques (Paris) [Mathematical Documents (Paris)], vol. 4, Société Mathématique de France, Paris, 2005, 1962, Augmenté d'un exposé de Michèle Raynaud. [With an exposé by Michèle Raynaud], with a preface and edited by Yves Laszlo, Revised reprint of the 1968 French original, MR 2171939.
[14] Robin Hartshorne, Complete intersections and connectedness, Am. J. Math. 84 (1962) 497–508, MR 142547.
[15] Robin Hartshorne, Cohomological dimension of algebraic varieties, Ann. Math. (2) 88 (1968) 403–450, MR 232780.
[16] Robin Hartshorne, On the de Rham cohomology of algebraic varieties, Publ. Math. IHÉS (45) (1975) 5–99, MR 432647.
[17] Robin Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York-Heidelberg, 1977, MR 0463157.
[18] C. Huneke, G. Lyubeznik, On the vanishing of local cohomology modules, Invent. Math. 102 (1) (1990) 73–93, MR 1069240.
[19] M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, Ann. Math. (2) 96 (1972) 318–337, MR 304376.
[20] Robin Hartshorne, Claudia Polini, Simple $\mathcal{D}$-module components of local cohomology modules, J. Algebra 571 (2021) 232–257, MR 4200718.
[21] Ryoshi Hotta, Kiyoshi Takeuchi, Toshiyuki Tanisaki, D-Modules, Perverse Sheaves, and Representation Theory, Progress in Mathematics, vol. 236, Birkhäuser Boston, Inc., Boston, MA, 2008, Translated from the 1995 Japanese edition by Takeuchi, MR 2357361.
[22] Birger Iversen, Cohomology of Sheaves, Universitext, Springer-Verlag, Berlin, 1986, MR 842190.
[23] Masaki Kashiwara, On the holonomic systems of linear differential equations. II, Invent. Math. 49 (2) (1978) 121–135, MR 511186.
[24] Ken-ichiroh Kawasaki, On the highest Lyubeznik number, Math. Proc. Camb. Philos. Soc. 132 (3) (2002) 409–417, MR 1891679.
[25] Masaki Kashiwara, Pierre Schapira, Sheaves on manifolds, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag, Berlin, 1994, With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original, MR 1299726.
[26] András C. Lőrincz, Claudiu Raicu, Iterated local cohomology groups and Lyubeznik numbers for determinantal rings, Algebra Number Theory 14 (9) (2020) 2533–2569, MR 4172715.
[27] Gennady Lyubeznik, Anurag K. Singh, Uli Walther, Local cohomology modules supported at determinantal ideals, J. Eur. Math. Soc. 18 (11) (2016) 2545–2578, MR 3562351.
[28] András C. Lőrincz, Uli Walther, On categories of equivariant $\mathcal{D}$-modules, Adv. Math. 351 (2019) 429–478, MR 3952575.
[29] Gennady Lyubeznik, Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra), Invent. Math. 113 (1) (1993) 41–55, MR 1223223.
[30] Olivier Mathieu, Formules de caractères pour les algèbres de Kac-Moody générales, Astérisque (159–160) (1988) 267, MR 980506.

[31] John W. Milnor, James D. Stasheff, Characteristic Classes, Annals of Mathematics Studies, vol. 76, Princeton University Press/University of Tokyo Press, Princeton, N.J./Tokyo, 1974, MR 0440554.
[32] Linquan Ma, Wenliang Zhang, Eulerian graded $\mathcal{D}$-modules, Math. Res. Lett. 21 (1) (2014) 149–167, MR 3247047.
[33] Luis Núñez Betancourt, Emily E. Witt, Wenliang Zhang, A survey on the Lyubeznik numbers, in: Mexican Mathematicians Abroad: Recent Contributions, in: Contemp. Math., vol. 657, Amer. Math. Soc., Providence, RI, 2016, pp. 137–163, MR 3466449.
[34] Arthur Ogus, Local cohomological dimension of algebraic varieties, Ann. Math. (2) 98 (1973) 327–365, MR 506248.
[35] Peter Orlik, Louis Solomon, Combinatorics and topology of complements of hyperplanes, Invent. Math. 56 (2) (1980) 167–189, MR 558866.
[36] Toshinori Oaku, Nobuki Takayama, An algorithm for de Rham cohomology groups of the complement of an affine variety via D-module computation, vol. 139, in: Effective Methods in Algebraic Geometry, Saint-Malo, 1998, 1999, pp. 201–233, MR 1700544.
[37] Toshinori Oaku, Nobuki Takayama, Algorithms for D-modules—restriction, tensor product, localization, and local cohomology groups, J. Pure Appl. Algebra 156 (2–3) (2001) 267–308, MR 1808827.
[38] Boris Pasquier, On some smooth projective two-orbit varieties with Picard number 1, Math. Ann. 344 (4) (2009) 963–987, MR 2507635.
[39] C. Peskine, L. Szpiro, Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck, Publ. Math. IHÉS (42) (1973) 47–119, MR 374130.
[40] Thomas Reichelt, Morihiko Saito, Uli Walther, Dependence of Lyubeznik numbers of cones of projective schemes on projective embeddings, Sel. Math. New Ser. 27 (1) (2021), Paper No. 6, 22. MR 4202748.
[41] Thomas Reichelt, Uli Walther, Weight filtrations on GKZ-systems, Preprint, arXiv:1809.04247, 2018.
[42] Nicholas Switala, Lyubeznik numbers for nonsingular projective varieties, Bull. Lond. Math. Soc. 47 (1) (2015) 1–6, MR 3312957.
[43] Nicholas Switala, On the de Rham homology and cohomology of a complete local ring in equicharacteristic zero, Compos. Math. 153 (10) (2017) 2075–2146, MR 3705285.
[44] Uli Walther, Algorithmic computation of de Rham cohomology of complements of complex affine varieties, vol. 29, in: Symbolic Computation in Algebra, Analysis, and Geometry, Berkeley, CA, 1998, 2000, pp. 795–839, MR 1769667.
[45] Uli Walther, On the Lyubeznik numbers of a local ring, Proc. Am. Math. Soc. 129 (6) (2001) 1631–1634, MR 1814090.
[46] Uli Walther, Wenliang Zhang, Local cohomology—an invitation, Preprint, arXiv:2106.09796, 2021.
[47] Wenliang Zhang, On the highest Lyubeznik number of a local ring, Compos. Math. 143 (1) (2007) 82–88, MR 2295196.
[48] Wenliang Zhang, Lyubeznik numbers of projective schemes, Adv. Math. 228 (1) (2011) 575–616, MR 2822240.
[49] Wenliang Zhang, The second vanishing theorem for local cohomology modules, arXiv:2102.12545, submitted for publication.