Eigenstripping, Spectral Decay, and Edge-Expansion on Posets

Jason Gaitonde ✉
Cornell University, Ithaca, NY, USA

Max Hopkins ✉
University of California, San Diego, La Jolla, CA, USA

Tali Kaufman ✉
Bar-Ilan University, Ramat-Gan, Israel

Shachar Lovett ✉
University of California, San Diego, La Jolla, CA, USA

Ruizhe Zhang ✉
The University of Texas at Austin, TX, USA

Abstract

Fast mixing of random walks on hypergraphs (simplicial complexes) has recently led to myriad breakthroughs throughout theoretical computer science. Many important applications, however, (e.g. to LTCs, 2-2 games) rely on a more general class of underlying structures called *posets*, and crucially take advantage of non-simplicial structure. These works make it clear that the global expansion properties of posets depend strongly on their underlying architecture (e.g. simplicial, cubical, linear algebraic), but the overall phenomenon remains poorly understood. In this work, we quantify the advantage of different poset architectures in both a spectral and combinatorial sense, highlighting how *regularity* controls the spectral decay and edge-expansion of corresponding random walks.

We show that the spectra of walks on expanding posets (Dikstein, Dinur, Filmus, Harsha APPROX-RANDOM 2018) concentrate in strips around a small number of approximate eigenvalues controlled by the regularity of the underlying poset. This gives a simple condition to identify poset architectures (e.g. the Grassmann) that exhibit strong (even exponential) decay of eigenvalues, versus architectures like hypergraphs whose eigenvalues decay linearly – a crucial distinction in applications to hardness of approximation and agreement testing such as the recent proof of the 2-2 Games Conjecture (Khot, Minzer, Safra FOCS 2018). We show these results lead to a tight characterization of edge-expansion on expanding posets in the ℓ_2-regime (generalizing recent work of Bafna, Hopkins, Kaufman, and Lovett (SODA 2022)), and pay special attention to the case of the Grassmann where we show our results are tight for a natural set of sparsifications of the Grassmann graphs. We note for clarity that our results do not recover the characterization of expansion used in the proof of the 2-2 Games Conjecture which relies on ℓ_∞ rather than ℓ_2-structure.

2012 ACM Subject Classification Theory of computation → Expander graphs and randomness extractors

Keywords and phrases High-dimensional expanders, posets, eposets

Digital Object Identifier 10.4230/LIPIcs.APPROX/RANDOM.2022.16

Category RANDOM

Related Version *Full Version*: https://arxiv.org/abs/2205.00644 [24]

Funding *Jason Gaitonde*: Supported by NSF Award CCF-1408673 and AFOSR Award FA9550-19-1-0183.
Max Hopkins: Supported by NSF Award DGE-1650112.
Tali Kaufman: Supported by ERC and BSF.
Shachar Lovett: Supported by NSF Award CCF-1909634.

1 Introduction

Random walks on high dimensional expanders (HDX) have been the object of intense study in theoretical computer science in recent years. Starting with their original formulation by Kaufman and Mass [31], a series of works on the spectral structure of these walks [33, 16, 2] led to significant breakthroughs in approximate sampling [4, 2, 3, 12, 13, 11, 22, 28, 41, 9], CSP-approximation [1, 6], error-correcting codes [29, 30], agreement testing [19, 15, 32], and more. Most of these works focus on the structure of expansion in *hypergraphs* (also called *simplicial complexes*). However, hypergraphs are not always the appropriate object – recent breakthroughs in locally testable [17] and quantum LDPC codes [42, 40, 39] rely crucially on *cubical* structure not seen in hypergraphs, while many agreement testing results like the proof of the 2-2 Games Conjecture [44] relies on *linear algebraic* rather than simplicial structure.

In this work, we study a generalized notion of HDX on *partially ordered sets* (posets) introduced by Dikstein, Dinur, Filmus, and Harsha (DDFH) [16] called *expanding posets* (eposets). Random walks on eposets capture a broad range of structures beyond their hypergraph analogs, including natural sparsifications of the Grassmann graphs recently crucial to the resolution of the 2-2 Games Conjecture [44, 37, 21, 20, 8, 36]. While originally a *global* notion of expansion, Kaufman and Tessler (KT) [34] recently extended the study of eposets by introducing *local-to-global* analysis to the setting and by identifying *regularity* as a key parameter controlling expansion. The authors strengthened local-to-global theorems for strongly regular posets like the Grassmann, giving the first general formulation for characterizing expansion based on an eposet's underlying architecture.

While analysis of the second eigenvalue is certainly important, a deeper understanding of the spectral structure of eposets is required for applications like the proof of the 2-2 Games Conjecture. Our main focus in this work lies in characterizing the spectral and combinatorial behavior of walks on eposets *beyond the second eigenvalue.* Strengthening DDFH and work of Bafna, Hopkins, Kaufman, and Lovett (BHKL) [6], we prove that at a coarse level (walks on) eposets exhibit the same spectral and combinatorial characteristics as expanding hypergraphs (e.g. spectral stripping, expansion of pseudorandom sets). On the other hand, as in KT, we show that the finer-grained properties of these objects are controlled by the underlying poset's regularity, including the *rate of decay* of the spectrum and combinatorial expansion of associated random walks. This gives a strong separation between structures like hypergraphs with weak (linear) eigenvalue decay, and Grassmann-based eposets with strong (exponential) decay (a crucial property in the proof of the 2-2 Games Conjecture [44]).

1.1 Background

We briefly overview the theory of expanding posets and higher order random walks (see Section 2 for details). A d-dimensional graded poset is a set X equipped with a partial order "$<$" and a ranking function $r : X \to [d]$ that respects "$<$" and partitions X into levels $X(0) \cup \ldots \cup X(d)$. When $x < y$ and $r(y) = r(x) + 1$, we write $x \lessdot y$. We assume throughout this work that our posets are *downward regular*: there exists a regularity function $R(k, i)$ such that every k-dimensional element is greater than exactly $R(k, i)$ i-dimensional elements.

Graded posets come equipped with a natural set of operators called the *up* and *down* operators that lift or lower functions $f : X(i) \to \mathbb{R}$ by averaging:

$$U_i f(x) = \mathop{\mathbb{E}}_{y \lessdot x}[f(y)] \qquad D_i f(y) = \mathop{\mathbb{E}}_{x \gtrdot y}[f(x)].$$

Composing the averaging operators leads to a natural notion of random walks on the underlying poset called *higher order random walks* (HD-walks). The simplest example is the *upper (lower) walk* $D_{i+1}U_i$ ($U_{i-1}D_i$) which moves between elements $x, x' \in X(i)$ via a common element $y \in X(i+1)$ (X_{i-1}) with $y > x, x'$ $(y < x, x')$. We also consider longer variants of the upper and lower walks called *canonical walks* $\widehat{N}_k^i = D_{k+1} \circ \ldots \circ D_{k+i} \circ U_{k+i-1} \circ \ldots \circ U_k$ and $\check{N}_k^i = U_k \circ \ldots \circ U_{k-i} \circ D_{k-i+1} \circ \ldots \circ D_k$ which similarly walk between k-dimensional elements in $X(k)$ via a shared element in $X(k+i)$ or $X(k-i)$ respectively.

Following DDFH [16], we call a poset a (δ, γ)-expander for $\delta \in [0,1]^{d-1}$ and $\gamma \in \mathbb{R}_+$ if the upper and lower walks are spectrally similar up to a laziness factor:

$$\|D_{i+1}U_i - (1-\delta_i)I - \delta_i U_{i-1} D_i\| \leqslant \gamma.$$

This generalizes standard spectral expansion which can be equivalently defined as looking at the spectral norm of $A_G - U_0 D_1$, where A_G (the adjacency matrix) is exactly the non-lazy upper walk.[1] While most of our results hold in general, we assume a weak non-laziness condition on our underlying posets that holds in most cases of interest (see Definition 13).

1.2 Results

We now give an overview of our results, splitting this section into three parts for readability: spectral stripping, characterizing edge expansion, and applications to the Grassmann.

Eigenstripping. We start with our generalized spectral stripping theorem.

▶ **Theorem 1** (Spectrum of HD-Walks (informal Corollary 20)). *Let M be an HD-walk on the kth level of a (δ, γ)-eposet. Then the spectrum of M is highly concentrated in $k+1$ strips:*

$$\mathsf{Spec}(M) \in \{1\} \cup \bigcup_{i=1}^{k} [\lambda_i(M) - e, \lambda_i(M) + e]$$

where $e \leqslant O_{k,\delta}(\gamma)$. Moreover, the span of eigenvectors in the ith strip approximately correspond to functions lifted from $X(i)$ to $X(k)$.

This substantially simplifies and improves an analogous result of BHKL [6] on expanding hypergraphs, which had sub-optimal error dependence of $O_k(\gamma^{1/2})$. The main improvement stems from an optimal spectral stripping result for arbitrary inner product spaces of independent interest. Theorem 1 follows by showing that the HD-Level-Set Decomposition, a natural basis on eposets introduced by DDFH [16], gives such an approximate eigendecomposition.

In full generality, the approximate eigenvalues in Theorem 1 depend on the eposet parameters δ, and can be fairly difficult to interpret. However, we show that under weak assumptions (see Section 2) the eigenvalues can be associated with the regularity of the underlying poset. We state the result just for lower walks for simplicity:

▶ **Theorem 2** (Regularity Controls Spectral Decay (informal Theorem 22)). *The approximate eigenvalues of the lower walk $\check{N}_k^{k-i}$ on a (δ, γ)-eposet are controlled by the poset's regularity function: $\lambda_j(\check{N}_k^{k-i}) \in \frac{R(i,j)}{R(k,j)} \pm O_{k,\delta}(\gamma)$.*

This generalizes work of Kaufman and Tessler [34] on the second eigenvalue, and reveals a major distinction among poset architectures: posets with higher regularity enjoy faster decay of eigenvalues.[2] Theorem 2 gives a new method of identifying poset architectures exhibiting

[1] For a broad range of posets, this is equivalent (up to constants) to *local-spectral expansion*, a notion of high dimensional expansion introduced by Dinur and Kaufman [19], as originally proved for simplicial complexes by DDFH [16], and later extended to a larger class of posets by Kaufman and Tessler [34].

[2] We note that Theorem 1 can also be obtained by combining our spectral stripping result with recent independent work of Dikstein, Dinur, Filmus, and Harsha [16, Section 8.4.1].

strong spectral decay in the sense that for any $\delta > 0$, the lower walk only contains $O_\delta(1)$ approximate eigenvalues larger than δ. This property is crucial for both the run-time of approximation algorithms on HDX [6] and the proof of the 2-2 Games Conjecture [44].

Characterizing Edge Expansion. Much of our motivation for studying the spectrum of HD-walks is to understand the *edge expansion* of subsets S, denoted $\Phi(S)$ (see Section 5 for formal definition). Characterizing edge-expansion in HD-walks has recently proven crucial to understanding both algorithms for [5, 6] and hardness of unique games [44]. On expanding hypergraphs, it is known that *links* give the canonical example of small non-expanding sets.

▶ **Definition 3** (Link). *Let X be a d-dimensional graded poset. The k-dimensional link, called a "k-link," of an element $\sigma \in X$ is the set of rank k elements greater than σ*[3]*, i.e. $X^k_\sigma = \{y \in X(k) : y > \sigma\}$. When k is clear from context, we write X_σ for X^k_σ for simplicity.*

BHKL [6] proved that on hypergraphs, the expansion of links is exactly controlled by their corresponding spectral strip. While their proof of this fact relied crucially on simplicial structure, we show via a more general analysis that the result can be recovered for eposets.

▶ **Theorem 4** (Expansion of Links (informal Theorem 29)). *Let X be a (δ,γ)-eposet and M an HD-walk on $X(k)$. Then for all $0 \leqslant i \leqslant k$ and $\tau \in X(i)$, $\Phi(X_\tau) = 1 - \lambda_i(M) \pm O_{M,k,\delta}(\gamma)$.*

Conversely one might ask: *are all non-expanding sets explained by links*? Following BHKL [6], given a set S, consider the function defined on a link $\tau \in X(i)$ by $L_{S,i}(\tau) := \mathop{\mathbb{E}}_{X_\tau}[\mathbb{1}_S] - \mathbb{E}[\mathbb{1}_S]$. Two standard formulations of "non-expansion is explained by links" correspond to $L_{S,i}$ having noticeable ℓ_2 or ℓ_∞-norm for a non-expanding set S. Thus, we say S is *pseudorandom* if $L_{S,i}$ is small with respect to one of these norms for all $i \leqslant \ell$ (see Section 4 for precise definitions). We prove that pseudorandom sets expand near-optimally.

▶ **Theorem 5** (Pseudorandom Sets Expand (informal Theorem 33)). *Let X be a (δ,γ)-eposet and M a walk on $X(k)$. Then the expansion of any (ε, i)-pseudorandom set S is at least:*

$$\Phi(S) \geqslant 1 - \lambda_{i+1} - O_\delta(R(k,i)\varepsilon) - O_{k,\delta,M}(\gamma).$$

The main technical component behind Theorem 5 is a result called a "level-i" inequality (cf. Theorem 26) which asserts that pseudorandomness controls the projection of the indicator of a subset S onto eigenstrips. This strictly generalizes the result for simplicial complexes in [6] where $R(k,i) = \binom{k}{i}$, and is tight for other important settings such as the Grassmann (discussed below). Theorem 5 and Theorem 26 can also be viewed as another separation between eposet architectures, this time in terms of *combinatorial* properties.

Applications: q-Eposets and the Grassmann Graphs. We conclude with applications of our results to a particularly important class of eposets called "q-eposets." Just like standard high dimensional expanders arise from expanding subsets of the complete complex (hypergraph), q-eposets arise from expanding subsets of the Grassmann Poset:

▶ **Definition 6** (Grassmann Poset). *The Grassmann Poset is a graded poset $(X, <)$ where X is the set of all subspaces of $\mathbb{F}_q^n$ of dimension at most d, the partial ordering "$<$" is given by inclusion, and the rank function is given by dimension.*

[3] In the literature, a link is often defined to be all such elements, not just those of rank k. We adopt this notation since we are mostly interested in working at a fixed level of the complex.

We call a (downward-closed) subset of the Grassmann poset a q-simplicial complex, and an expanding q-simplicial complex a q-eposet (see Section 2 for exact details). Using our machinery for general eposets, we prove a tight level-i inequality for (ε, ℓ)-pseudorandom sets $S \subseteq X(k)$ (see Theorem 37): for all $1 \leqslant i \leqslant \ell$,

$$|\langle \mathbb{1}_S, \mathbb{1}_{S,i} \rangle| \leqslant \left(\binom{k}{i}_q \varepsilon + O_{q,k}(\gamma) \right) \langle \mathbb{1}_S, \mathbb{1}_S \rangle,$$

where $\mathbb{1}_{S,i}$ is the ith level of the HD-Level-Set Decomposition and $\binom{k}{i}_q = \frac{(1-q^k)\cdots(1-q^{k-i+1})}{(1-q^i)\cdots(1-q)}$ is the Gaussian binomial coefficient. We also prove this bound cannot be improved by any constant factor, even in the ℓ_∞-regime. Furthermore, it is well known the dependence on k is necessary [37], even if one is willing to suffer a worse dependence on the pseudorandomness ε. This differs from simplicial complexes where the dependence can be removed in the ℓ_∞-regime [36, 7, 25]. Still, it is possible that the dependence on k can be removed by *changing the definition of pseudorandomness*, as was done on the Grassmann poset via finer-grained local structure called "zoom-in zoom-outs" [44]. The existence of a notion of locality based on the underlying poset structure that gives rise to k-independent bounds in the ℓ_∞-regime is an interesting open problem.

Finally, we give applications of these results to edge-expansion in an important class of walks that give rise to the well-studied *Grassmann graphs*.

▶ **Definition 7** (Grassmann Graphs). *The Grassmann Graph $J_q(n, k, t)$ is the graph on k-dimensional subspaces of $\mathbb{F}_q^n$ where $(V, W) \in E$ exactly when $\dim(V \cap W) = t$.*

Note that non-lazy upper walk on the Grassmann poset is exactly the Grassmann graph $J_q(n, k, k-1)$. In Section 6, we show how to express any $J_q(n, k, t)$ (in fact, for any q-simplicial complex) as a sum of standard higher order random walks. This leads to a set of natural sparsifications of the Grassmann graphs that may be of independent interest for agreement testing, PCPs, and hardness of approximation. For simplicity, on a given q-simplicial complex X, we refer to these "sparsified" Grassmann graphs as $J_{X,q}(n, k, t)$ for the moment. The level-i inequality then implies for a (ε, ℓ)-pseudorandom set $S \subseteq X(k)$ (Corollary 40):

$$\Phi(S) \geqslant 1 - \mathbb{E}[\mathbb{1}_S] - \varepsilon \sum_{i=1}^{\ell} \binom{t}{i}_q - q^{-(\ell+1)j} - O_{q,k}(\gamma).$$

In practice, t is generally thought of as being $\Omega(k)$ (or even $k - O(1)$), which results in a k-dependent bound. It remains an open problem whether a k-independent version can be proved for any q-eposet beyond the Grassmann poset itself.

1.3 Related Work

Higher Order Random Walks. Higher order random walks were introduced in 2016 by Kaufman and Mass [31]. Their spectral structure was later elucidated in a series of works by Kaufman and Oppenheim [33], DDFH [16], Alev, Jeronimo, and Tulsiani [1], Alev and Lau [2], and finally BHKL [6]. With the exception of DDFH, all of these works focused on hypergraphs rather than general posets. Our spectral stripping theorem for eposets essentially follows from combining eposet machinery developed by DDFH with our improved variant of BHKL's general spectral stripping theorem.

Among the myriad applications of higher order random walks described above, our work is closest to that of Bafna, Barak, Kothari, Schramm, and Steurer [5], and BHKL [6], who used the spectral and combinatorial structure of HD-walks to build new algorithms for unique games. The analysis in this paper also lends itself to the algorithmic techniques developed in those works, but we are unaware of interesting examples beyond those in BHKL.

High Dimensional Expansion Beyond Hypergraphs. Most works listed above focus only on the setting of hypergraphs. However, recent years have also seen the nascent development and application of expansion beyond this setting [18, 42, 40, 39, 26], including the seminal work of DDFH [16] on expanding posets as well as more recent breakthroughs on locally testable and quantum codes [17, 42]. While DDFH largely viewed eposets as having similar structure, we strengthen the case that different underlying poset architectures exhibit different properties. This complements the results of Kaufman and Tessler [34], who showed that expanding posets with strong regularity conditions such as the Grassmann exhibit more favorable properties with respect to the second eigenvalue. Our results provide a statement of the same flavor looking at the entire spectrum, along with additional separations in more combinatorial settings. A related connection between poset regularity and the approximate spectrum of walks was independently developed by DDFH in a recent update of their seminal work [16].

Expansion and Unique Games. One motivation behind this work is towards building a more general framework for understanding the structure underlying the Unique Games Conjecture [35], a standard hardness assumption in complexity for many combinatorial optimization problems (see e.g. Khot's survey [38]). In 2018, Khot, Minzer, and Safra [44] made a major breakthrough towards the UGC in proving the weaker 2-2 Games Conjecture, completing a long line of work in this direction [37, 21, 20, 8, 36, 44]. The key to the proof is the "Grassmann expansion hypothesis," which states that any non-expanding set in the Grassmann graph $J_q(d, k, k-1)$ is non-trivially concentrated inside a local-structure called "zoom-in zoom-outs." As noted in the previous section, this result differs from our analysis in two key ways: it lies in the ℓ_∞-regime, and must be totally independent of dimension.

Unfortunately, little progress has been made towards the UGC since, as KMS' proof of the Grassmann expansion hypothesis is quite complicated and highly tailored to the exact structure of the Grassmann, making it difficult to generalize to related conjectures [8]. However, just as the ℓ_2-regime analysis of DDFH and BHKL recently lead to a dimension independent bound in the ℓ_∞-regime for standard HDX [7, 25], we expect the groundwork laid in this paper will be important for proving generalized dimension independent expansion hypotheses in the ℓ_∞-regime beyond the special case of the Grassmann graphs.

2 Preliminaries

Graded Posets. A partially ordered set (poset) $P = (X, <)$ is a set of elements X endowed with a partial order "$<$". A graded poset has a rank function $r : X \to \mathbb{N}$ satisfying:

1. r preserves "$<$": if $y < x$, then $r(y) < r(x)$.
2. r preserves cover relations: if x is the smallest element greater than y, then $r(x) = r(y)+1$.

The function r partitions X into subsets by rank $X(0) \cup \ldots \cup X(d)$, where $\max_X(r) = d$, and $X(i) = r^{-1}(i)$. We refer to a poset with maximum rank d as "d-dimensional", and elements in $X(i)$ as "i-faces". Throughout this work, we consider d-dimensional graded posets that: (i) have a unique minimal element, and (ii) are "pure": all maximal elements have rank d. Many graded posets of interest, like pure simplicial complexes and the Grassmann poset, satisfy certain regularity conditions which will be crucial to our analysis.

▶ **Definition 8** (Regularity). *A d-dimensional graded poset is downward regular if for all $i \leqslant d$ there exists some constant $R(i)$ such that every element $x \in X(i)$ covers exactly $R(i)$ elements $y \in X(i-1)$.*

A d-dimensional graded poset is middle-regular if for all $0 \leqslant i \leqslant k \leqslant d$, there exists a constant $m(k,i)$ such that for any $x_k \in X(k)$ and $x_i \in X(i)$ satisfying $x_k > x_i$, there are exactly $m(k,i)$ chains[4] *of elements $x_k > x_{k-1} > \ldots > x_{i+1} > x_i$ where each $x_j \in X(j)$.*

A poset is regular if it is both downward and middle regular.

We will assume all posets we discuss in this work are regular from this point forward. Regular posets also have the nice property that for any dimensions $i < k$, there exists a higher order regularity function $R(k,i)$ such that any $x \in X(k)$ is greater than exactly $R(k,i)$ elements in $X(i)$ (see Appendix A). We define $R(i,i) = 1$ and $R(j,i) = 0$ whenever $j < i$ for convenience.

Measured Posets and The Random Walk Operators. A *measured poset* is a graded poset X endowed with a distribution $\Pi = (\pi_0, \ldots, \pi_d)$, where each marginal π_i is a distribution over $X(i)$. We focus on the case where Π is induced entirely from π_d. That is, $\forall\, 0 \leqslant i < d$:

$$\pi_i(x) = \frac{1}{R(i+1,i)} \sum_{y > x} \pi_{i+1}(y).$$

In other words, each lower dimensional distribution π_i may be induced through the following process: an element $y \in X(i+1)$ is selected with respect to π_{i+1}, and an element $x \in X(i)$ such that $x < y$ is then chosen uniformly at random.

The averaging operators U and D are defined analogously to their notions on simplicial complexes, with the main change being the use of the general regularity function $R(i+1,i)$:

$$U_i f(y) = \frac{1}{R(i+1,i)} \sum_{x < y} f(x) \qquad D_{i+1} f(x) = \frac{1}{\pi_{i+1}(X_x)} \sum_{y > x} \pi_{i+1}(y) f(y),$$

where for $i < k$ and $x \in X(i)$, the appropriate normalization factor is

$$\pi_k(X_x) = \sum_{y \in X(k): y > x} \pi_k(y) = R(k,i)\pi_i(x).$$

In Appendix A, we show that the up operators compose nicely, and in particular that:

$$U_i^k f(y) := U_{k-1} \circ \ldots \circ U_i f(y) = \frac{1}{R(k,i)} \sum_{x \in X(i): x < y} f(x).$$

As with simplicial complexes, the down and up operators are adjoint with respect to the standard inner product on measured posets: for any $f : X(k) \to \mathbb{R}$ and $g : X(k-1) \to \mathbb{R}$,

$$\langle f, U_{k-1} g \rangle_{X(k)} = \langle D_k f, g \rangle_{X(k-1)}, \quad \text{where} \quad \langle f, g \rangle_{X(k)} = \sum_{\tau \in X(k)} \pi_k(\tau) f(\tau) g(\tau).$$

We omit $X(k)$ from the notation when clear from context. This useful fact allows us to define basic self-adjoint notions of higher order random walks just like on simplicial complexes.

[4] Such objects are sometimes called flags, e.g. in the case of the Grassmann poset.

Higher Order Random Walks. Let C_k denote the set of functions $f : X(k) \to \mathbb{R}$. Following prior work, we define natural sets of random walk operators via the averaging operators.

▶ **Definition 9** (Walk Operators [31, 16, 1]). *Given a measured poset (X, Π), a k-dimensional pure walk $Y : C_k \to C_k$ on (X, Π) (of height $h(Y)$) is a composition $Y = Z_{2h(Y)} \circ \cdots \circ Z_1$, where each Z_i is a copy of D or U, and there are $h(Y)$ of each type.*

Let $\mathcal{Y}$ be a family of pure walks $Y : C_k \to C_k$ on (X, Π). We call an affine combination $M = \sum_{Y \in \mathcal{Y}} \alpha_Y Y$ a k-dimensional HD-walk on (X, Π) if it is stochastic and self-adjoint. The height of M, denoted $h(M)$, is the maximum height of any pure $Y \in \mathcal{Y}$ with a non-zero coefficient. The weight of M, denoted $w(M)$, is $|\alpha|_1$.

▶ **Definition 10** (Canonical Walk). *Given a d-dimensional measured poset (X, Π) and parameters $k + j \leqslant d$, the upper canonical walk is $\widehat{N}_k^j := D_k^{k+j} U_k^{k+j}$, while for $j \leqslant k$ the lower canonical walk is $\check{N}_k^j := U_{k-j}^k D_{k-j}^k$, where $U_\ell^k = U_{k-1} \dots U_\ell$, and $D_\ell^k = D_{\ell+1} \dots D_k$.*

Since the non-zero spectrum of $\widehat{N}_k^j$ and $\check{N}_{k+j}^j$ are equivalent (c.f. [2]), we focus in this work mostly on the upper walks which we write simply as N_k^j.

For certain specially structured posets, we will also study an important class of HD-walks known as (partial) *swap walks*. We will introduce these well-studied walks momentarily.

Expanding Posets and the HD-Level-Set Decomposition. DDFH [16] observed that one can use the averaging operators to define an extension of spectral expansion to graded posets:

▶ **Definition 11** (Eposet [16]). *Let (X, Π) be a measured poset, $\delta \in [0, 1]^{d-1}$, and $\gamma < 1$. X is an (δ, γ)-eposet if for all $1 \leqslant i \leqslant d - 1$:*

$$\|D_{i+1}U_i - (1 - \delta_i)I - \delta_i U_{i-1} D_i\| \leqslant \gamma.$$

Much of our analysis in this work will be based off of an elegant approximate Fourier decomposition for eposets introduced by DDFH [16].

▶ **Theorem 12** (HD-Level-Set Decomposition, Theorem 8.2 [16]). *Let (X, Π) be a d-dimensional (δ, γ)-eposet with γ sufficiently small. For all $0 \leqslant k \leqslant d$, let $H^0 = C_0, H^i = Ker(D_i), V_k^i = U_i^k H^i$. Then $C_k = V_k^0 \oplus \ldots \oplus V_k^k$. In other words, every $f \in C_k$ has a unique decomposition $f = f_0 + \ldots + f_k$ such that $f_i = U_i^k g_i$ for $g_i \in Ker(D_i)$.*

It is well known that the HD-Level-Set Decomposition is approximately an eigenbasis for HD-walks on simplicial complex [16, 1, 6]. We will show this statement extends to all eposets (extending DDFH's similar analysis of the upper walk N_k^1).

We will further assume for simplicity throughout this work an additional property of eposets we called (approximate) non-laziness.

▶ **Definition 13** (β-Non-Lazy Eposets). *Let (X, Π) be a d-dimensional measured poset. We call (X, Π) β-non-lazy if for all $1 \leqslant i \leqslant d$, $\max_{\sigma \in X(i)} \{\mathbb{1}_\sigma^T U_{i-1} D_i \mathbb{1}_\sigma\} \leqslant \beta$.*

This condition asserts that no element in the poset carries too much weight, even upon conditioning. All of our results hold for general eposets,[5] but their form is significantly more interpretable when the poset is additionally non-lazy. In fact, most γ-eposets of interest are $O(\gamma)$-non-lazy. It is easy to see for instance that any "γ-local-spectral" expander satisfies this condition, an equivalent notion of expansion to γ-eposets under suitable regularity conditions [34]. We discuss this further in Appendix A.

[5] The one exception is the lower bound of Theorem 4.

The Grassmann Poset and q-Eposets. At the moment, there are only two known families of expanding posets of significant interest in the literature: those based on pure simplicial complexes (the downward closure of a k-uniform hypergraph), and pure q-simplicial complexes (the analogous notion over subspaces). The ℓ_2-structure of the former is studied in detail in [6]; we focus on the latter which is less-studied, but responsible for a number of important results including the resolution of the 2-to-2 Games Conjecture [44].

▶ **Definition 14** (q-Simplicial Complex). *Let $G_q(n,d)$ denote the d-dimensional subspaces of $\mathbb{F}_q^n$. A weighted, pure q-simplicial complex (X,Π) is given by a family of subspaces $X \subseteq G_q(n,d)$ and a distribution Π over X. We will usually consider the downward closure $X(0) \cup \ldots \cup X(d)$, where $X(i) \subseteq G_q(n,i)$ consists of all i-dimensional subspaces contained in some element in $X = X(d)$. Further, on each level $X(i)$, Π induces a natural distribution π_i:*

$$\forall V \in X(i) : \pi_i(V) = \frac{1}{\binom{d}{i}_q} \sum_{W \in X(d): W \supset V} \pi_d(W),$$

where $\pi_d = \Pi$ and $\binom{d}{i}_q = \frac{(1-q^d)\cdots(1-q^{d-i+1})}{(1-q^i)\cdots(1-q)}$ is the Gaussian binomial coefficient.

Taking $X = G_q(n,d)$ yields the Grassmann poset, the q-analog of the complete simplicial complex. The Grassmann poset is well known to be a expander in this sense (see e.g. [43]) – in fact it is a 0-eposet with parameters

$$\delta_i = \frac{(q^i-1)(q^{n-i+1}-1)}{(q^{i+1}-1)(q^{n-i}-1)}, \tag{1}$$

the q-analog of the eposet parameters for the complete complex [16]. With this in mind, let's define a special class of eposets based on q-simplicial complexes.

▶ **Definition 15** (γ-q-Eposet [16]). *A pure, d-dimensional weighted q-simplicial complex (X,Π) is a γ-q-eposet if it is a (δ,γ)-eposet satisfying $\delta_i = q\frac{q^i-1}{q^{i+1}-1}$ for all $1 \leqslant i \leqslant d-1$.*

Constructing bounded-degree q-eposets (a problem proposed by DDFH [16]) remains an interesting open problem. Kaufman and Tessler [34] recently made some progress in this direction, but the expansion parameter of their construction is fairly poor (around 1/2).

Finally, in our applications to the Grassmann we consider a particularly important class of walks called *partial-swap walks*, which are non-lazy variants of the upper canonical walks.

▶ **Definition 16** (Partial-Swap Walk). *Let (X,Π) be a weighted, d-dimensional q-simplicial complex. The partial-swap walk S_k^j is the restriction of the canonical walk N_k^j to faces whose intersection has dimension $k-j$. In other words, if $|V \cap W| > k-j$ then $S_k^j(V,W) = 0$, and otherwise $S_k^j(V,W) \propto N_k^j(V,W)$.*

When applied to the Grassmann poset itself, it is clear by symmetry that the partial-swap walk S_k^j returns exactly the Grassmann graph $J_q(d,k,k-j)$. On the other hand, it is not immediately obvious these objects are even HD-walks when applied to a generic q-simplicial complex. We prove this is the case in Section 6.

3 Eigenstripping and the Spectra of HD-Walks

We now discuss HD-walks' spectral structure. It turns out that on expanding posets, these walks exhibit almost exactly the same properties as on the special case of simplicial complexes studied in [33, 16, 1, 6]: a walk's spectrum lies concentrated in strips corresponding to levels of the HD-Level-Set Decomposition. The key to proving this lies in a more general theorem characterizing the spectral structure of any inner product space admitting an "approximate eigendecomposition" [6]. We prove a significantly simpler, tight variant of this result.

▶ **Theorem 17** (Eigenstripping). *Let M be a self-adjoint operator over an inner product space V, and suppose $V = V^1 \oplus \ldots \oplus V^k$ satisfies $\|Mf - \lambda_i f\| \leqslant c_i \|f\|$ for all $f \in V^i$ for parameters $\lambda_1 \geqslant \ldots \geqslant \lambda_n$ and $c_i \geqslant 0$. Then as long as $c_i + c_{i+1} < \lambda_i - \lambda_{i+1}$, the spectrum of M is concentrated around each λ_i:*

$$\mathsf{Spec}(M) \subseteq \bigcup_{i=1}^{k} [\lambda_i - c_i, \lambda_i + c_i]$$

Proof. For each i, consider the (self-adjoint) operator $M_i^2 = (M - \lambda_i I)^2$. We claim it is enough to show that M_i^2 has exactly $\dim(V^i)$ eigenvalues less than c_i^2 in absolute value. To see why, observe that the eigenvalues of M_i^2 are exactly $(\mu - \lambda_i)^2$ for each μ in $\mathsf{Spec}(M)$ (with matching multiplicities), and therefore that any eigenvalue $\mu_i \in \mathsf{Spec}(M_i^2)$ less than c_i^2 implies the existence of a corresponding eigenvalue of M in $[\lambda_i \pm c_i]$. If each M_i^2 has $\dim(V^i)$ eigenvalues less than c_i^2, then M has at least $\dim(V^i)$ eigenvalues in each interval $[\lambda_i \pm c_i]$. Moreover, since these intervals are disjoint by assumption and $\sum \dim(V^i) = \dim(V)$, this must account for all eigenvalues of M.

To prove the claim, we apply the Courant-Fischer theorem [23], which asserts that the kth smallest eigenvalue of self-adjoint operator A is

$$\lambda_{n-k+1} = \min_{U} \left\{ \max_{f \in U} \left\{ \frac{\langle f, Af \rangle}{\langle f, f \rangle} \right\} \,\middle|\, \dim(U) = k \right\}.$$

Taking $U = V^i$, $A = M_i^2$ and $k = \dim(V^i)$ (noting that $\langle f, M_i^2 f \rangle = \|(M - \lambda_i I)f\|^2$ by self-adjointness) with the approximate eigendecomposition assumption yields the claim. ◀

Note that this result is also trivially tight for any true eigendecomposition. We remark that similar strategies have been used in the numerical analysis literature (see e.g. [27]).

Thus it is enough to prove that the HD-Level-Set Decomposition is an approximate eigenbasis. This follows similarly as for local-spectral expanders [6], though somewhat more care is required to deal with general eposet parameters. First, an inductive application of [16, Claim 8.8] (itself a repeated application of Definition 11) shows that functions in the HD-Level-Set Decomposition are close to being eigenvectors (see full version for details).

▶ **Proposition 18.** *Let (X, Π) be a (δ, γ)-eposet, and Y the pure balanced walk of height j, with down operators at positions $(i_1, \ldots, i_j)$. For $1 \leqslant \ell \leqslant k$, let $f_\ell = U_\ell^k g_\ell$ for some $g_\ell \in H^\ell$, and let*

$$\delta_j^k = \prod_{i=k-j}^{k} \delta_i, \;\; \gamma_j^k = \gamma \sum_{i=-1}^{j-1} \delta_i^k,$$

where $\delta_i^k = 1$ for any $i < 0$ for notational convenience. Then f_ℓ is an approximate eigenvector of Y:

$$\left\| Y f_\ell - \prod_{s=1}^{j} \left(1 - \delta_{k-2s+i_s-\ell}^{k-2s+i_s}\right) f_\ell \right\| \leqslant \|g_\ell\| \sum_{s=1}^{j} \gamma_{k-2s+i_s-\ell}^{k-2s+i_s} \prod_{t=1}^{s-1} \left(1 - \delta_{k-2t+i_t-\ell}^{k-2t+i_t}\right) \leqslant (j+k)j\gamma \, \|g_\ell\| \,.$$

When $\gamma = 0$, this implies that the HD-Level-Set decomposition is a true eigendecomposition. Since balanced walks are simply affine combinations of pure walks, this immediately implies a similar result for the more general case.

Before proceeding, for a d-dimensional (δ, γ)-eposet, and $0 \leqslant \ell \leqslant k < d$, define:

$$\rho_\ell^k = \prod_{i=1}^{k-\ell} \left(1 - \delta_{k-\ell-i}^{k-i}\right), \quad \rho_{\min} = \min_{0 \leqslant \ell \leqslant k} \{\rho_\ell^k\}. \tag{2}$$

The parameter ρ_ℓ^k arises throughout much of our work, and while it is difficult to interpret on general eposets, we prove it has a very natural form as long as non-laziness holds.

▷ Claim 19 (ρ_ℓ^k for Regular Eposets). Let (X,Π) be a regular, γ-non-lazy[6] d-dimensional (δ,γ)-eposet. Then for any $i \leqslant k < d$, we have:

$$\rho_i^k \in \frac{1}{R(k,i)} \pm err,$$

where $err \leqslant O\left(\frac{i^3k^2R_{\max}}{\delta_i(1-\delta_{i-1})}\gamma\right)$. Likewise as long as $\gamma \leqslant O\left(\frac{\max_i\{\delta_i(1-\delta_{i-1})\}}{i^3k^2R_{\max}^2}\right)$ we have $\rho_{\min}^{-1} \leqslant O(R_{\max})$, where $R_{\max} := \max_{0\leqslant i\leqslant k}\{R(k,i)\}$.

This gives a nice generalization of the interpretation of ρ_i^k on hypergraphs, where $\rho_i^k = \binom{k}{i}^{-1}$ [16]. We prove this claim in Appendix A. For simplicity, we will assume throughout the rest of this work that our eposets are γ-non-lazy, which is true for most cases of interest (see Appendix A). All results hold in the more general case using ρ_i^k unless otherwise noted.

Combining Proposition 18 and [16, Lemma 8.11] immediately implies that the HD-Level-Set Decomposition is an approximate eigendecomposition:

► **Corollary 20.** *Let (X,Π) be a (δ,γ)-eposet and let $M = \sum_{Y\in\mathcal{Y}}\alpha_Y Y$ be an HD-walk. For $1\leqslant \ell\leqslant k$, if $f_\ell = U_\ell^k g_\ell$ for some $g_\ell \in H^\ell$, then for $\gamma \leqslant O\left(\frac{\max_i\{\delta_i(1-\delta_{i-1})\}}{k^5R_{max}^2}\right)$:*

$$\left\| Mf_\ell - \left(\sum_{Y\in\mathcal{Y}}\alpha_Y\lambda_{Y,\delta,\ell}\right)f_\ell\right\| \leqslant c\gamma\,\|f_\ell\|,$$

where $\lambda_{Y,\delta,\ell}$ is the corresponding eigenvalues of the pure balanced walk Y on a $(\delta,0)$-eposet (the form of which are given in Proposition 18), and $c \leqslant O\left((h(M)+k)h(M)R(k,\ell)w(M)\right)$.

Theorem 17 then immediately implies that for any self-adjoint walk (e.g. canonical or swap walk), the true spectrum is concentrated around these approximate eigenvalues.

A straightforward, but useful example application of Corollary 20 immediately yields the approximate spectrum of a basic higher order random walk.

► **Corollary 21** (Spectrum of Lower Canonical Walks)**.** *Let (X,Π) be a (δ,γ)-eposet. The approximate eigenvalues of the canonical lower walk $\check{N}_k^{k-\ell}$ are:*

$$\lambda_j(\check{N}_k^{k-\ell}) = \prod_{s=1}^{k-\ell}(1-\delta_{k-s-j}^{k-s}).$$

Similar to the case of ρ_i^k, while this is difficult to interpret in the general setting, the eigenvalues have a very natural form on non-lazy eposets given by the regularity parameters.

► **Theorem 22.** *Let (X,Π) be a γ-non-lazy (δ,γ)-eposet. The approximate eigenvalues of the canonical lower walk $\check{N}_k^{k-i}$ are $\lambda_j(\check{N}_k^{k-i}) \in \frac{R(i,j)}{R(k,j)} \pm c\gamma$, where $c \leqslant O\left(\frac{i^4k^2R_{max}}{\delta_i(1-\delta_{i-1})}\gamma\right)$.*

The proof requires machinery developed in Section 5 and is given in Appendix A.

[6] One can prove this claim more generally for any β-non-laziness, but most γ-eposets of interest are additionally γ-non-lazy, so this simplified version is generally sufficient.

4 Pseudorandomness and the HD-Level-Set Decomposition

Now that we know the spectral structure of HD-walks, we shift to studying their combinatorial structure. In particular, we will focus on how natural notions of pseudorandomness control the projection of functions onto the HD-Level-Set Decomposition. As much of this theory generalizes arguments of BHKL, we defer the proofs in this section to the full version.

We start with the definition of pseudorandomness in the ℓ_2-regime, which measures the variance of a set across links.

▶ **Definition 23** (ℓ_2-Pseudorandom Functions [6]). *A function $f \in C_k$ is $(\varepsilon_1, \ldots, \varepsilon_\ell)$-$\ell_2$-pseudorandom if its variance across i-links is small for all $1 \leqslant i \leqslant \ell$:*

$$\mathrm{Var}(D_i^k f) \leqslant \varepsilon_i |\mathbb{E}[f]|.$$

In their work on simplicial complexes, BHKL [6] observed a close connection between ℓ_2-pseudorandomness, the HD-Level-Set Decomposition, and the spectra of the lower canonical walks. Using the approximate eigendecomposition developed in the previous section in Corollary 21, it turns out that the same connection holds in general for eposets.

▶ **Theorem 24.** *Let (X, Π) be a (δ, γ)-eposet with $\gamma \leqslant O\left(\frac{\max_i\{\delta_i(1-\delta_{i-1})\}}{k^5 R_{max}^2}\right)$. If $f \in C_k$ has HD-Level-Set Decomposition $f = f_0 + \ldots + f_k$, then for any $\ell \leqslant k$, $\mathrm{Var}(D_\ell^k f)$ is controlled by its projection onto $V_k^0 \oplus \ldots \oplus V_k^\ell$ in the following sense:*

$$\mathrm{Var}(D_\ell^k f) \in \sum_{j=1}^{\ell} \lambda_j(\check{N}_k^{k-\ell}) \langle f, f_j \rangle \pm c_k \gamma \|f\|^2,$$

where $c_k \leqslant O(k^{5/2} R_{max})$ and $\lambda_j(\check{N}_k^{k-\ell}) = \prod_{s=1}^{k-\ell}(1 - \delta_{k-s-j}^{k-s})$.

While ℓ_2-pseudorandomness is useful in its own right (e.g. for local-to-global algorithms for unique games [5, 6]), there is also significant interest in a stronger ℓ_∞-variant in the hardness of approximation literature [36, 44].

▶ **Definition 25** (ℓ_∞-Pseudorandom Functions [6]). *A function $f \in C_k$ is $(\varepsilon_1, \ldots, \varepsilon_\ell)$-$\ell_\infty$-pseudorandom if for all $1 \leqslant i \leqslant \ell$ its local expectation is close to its global expectation:*

$$\left\| D_i^k f - \mathbb{E}[f] \right\|_\infty \leqslant \varepsilon_i.$$

In their recent work on ℓ_2-structure of expanding simplicial complexes, BHKL prove a basic reduction from ℓ_∞ to ℓ_2-pseudorandomness that allows for an analogous level-i inequality for this notion as well. We show the same result holds for general eposets by applying Theorem 24 with Claim 19 to obtain a level-i inequality for pseudorandom sets (see the full version for the more general version). The key idea is to lower bound the variance of D_i^k by the ith component in the expansion of variance given by Theorem 24.

▶ **Theorem 26.** *Let (X, Π) be a (δ, γ)-eposet with $\gamma \leqslant O\left(\frac{\max_i\{\delta_i(1-\delta_{i-1})\}}{k^5 R_{max}^2}\right)$ and suppose that for $S \subseteq X(k)$, $\mathbb{1}_S$ is $(\varepsilon_1, \ldots, \varepsilon_\ell)$-$\ell_\infty$ pseudorandom. Then $\mathbb{1}_S$ is also $(\varepsilon_1, \ldots, \varepsilon_\ell)$-$\ell_2$ pseudorandom, and for all $1 \leqslant i \leqslant \ell$:*

$$|\langle \mathbb{1}_S, \mathbb{1}_{S,i} \rangle| \leqslant (R(k,i)\varepsilon_i + c\gamma)\, \mathbb{E}[\mathbb{1}_S],$$

where $c \leqslant O\left(\frac{k^5 R_{max}^2}{\max_i\{\delta_i(1-\delta_{i-1})\}}\right)$.

This recovers the tight inequality for simplicial complexes given in [6] where $R(k,i) = \binom{k}{i}$, as well as providing the natural q-analog for q-simplicial complexes where $R(k,i) = \binom{k}{i}_q$.

5 Expansion of HD-walks

It is well known that higher order random walks on simplicial complexes are not small-set expanders. BHKL gave an exact characterization of this phenomenon for local-spectral expanders: they showed that the expansion of any i-link with respect to an HD-walk M is almost exactly $1 - \lambda_i(M)$. Moreover, using the level-i inequality from the previous section, BHKL proved a tight converse to this result in an ℓ_2-sense: *any* non-expanding set must have high variance across links. This gave a complete ℓ_2-characterization of non-expanding sets on local-spectral expanders, and lay the structural groundwork for new algorithms for unique games over HD-walks. In this section, we extend these results to general expanding posets.

▶ **Definition 27** (Weighted Edge Expansion). *Let M be a k-dimensional HD-Walk on a graded poset (X, Π). Let $M(v, X(k)\backslash S) = \sum\limits_{y \in X(k)\backslash S} M(v, y)$ where $M(v, y)$ is the transition probability from v to y. The weighted edge expansion of a subset $S \subset X(k)$ with respect to M is*

$$\Phi(S) = \mathop{\mathbb{E}}_{v \sim \pi_k|_S} \left[M(v, X(k)\backslash S) \right].$$

Before we prove the strong connections between links and expansion, we need to introduce an important property of HD-walks, monotonic eigenvalue decay.

▶ **Definition 28** (Monotonic HD-walk). *Let (X, Π) be a (δ, γ)-eposet. We call an HD-walk M monotonic if its approximate eigenvalues $\lambda_i(M)$ (given in Corollary 20) are non-increasing.*

Most HD-walks of interest (e.g. pure walks, partial-swap walks on simplicial or q-simplicial complexes, etc.) are monotonic. This property will be crucial to understanding expansion. To start, let's see how it allows us to upper bound the expansion of links.

▶ **Theorem 29** (Local Expansion vs Global Spectra). *Let (X, Π) be a (δ, γ)-eposet and M be a k-dimensional monotonic HD-walk. Then for all $0 \leqslant i \leqslant k$ and $\tau \in X(i)$, it holds that $\Phi(X_\tau) \in 1 - \lambda_i(M) \pm c\gamma$, where $c \leqslant O\left(\frac{k^5 R_{max}^2 (h(M)+k) h(M) w(M)}{\delta_{k-i}^k (1-\delta_{i-1})} \right)$.*

The key to proving Theorem 29 is to show that the weight of an i-link lies almost entirely on level i of the HD-Level-Set Decomposition. To show this, we rely on another connection between regularity and eposet parameters for non-lazy posets.

▷ Claim 30. Let (X, Π) be a d-dimensional (δ, γ)-eposet. Then for every $1 \leqslant k \leqslant d$ and $0 \leqslant i \leqslant k$, the following relation between the eposet and regularity parameters holds:

$$\lambda_i(N_k^1) \in \frac{R(k, i)}{R(k+1, i)} \pm \left(\gamma_{k-i}^k + R(k, i) \delta_{k-i}^k \gamma \right).$$

We prove this relation in Appendix A. With this in hand, we can show links project mostly onto their corresponding level. We defer the proof to the full version, but the key idea is to express the (non)-expansion of the link both directly using the regularity parameters and using the approximate eigenvalues in the HD-Level-Set decomposition to argue that the only way these quantities can be equal is if the desired conclusion holds.

▶ **Lemma 31.** *Let (X, Π) be a d-dimensional (δ, γ)-eposet with $\gamma \leqslant O\left(\frac{\max_i \{\delta_i (1-\delta_{i-1})\}}{k^5 R_{max}^2} \right)$. Then for all $0 \leqslant i \leqslant k < d$ and $\tau \in X(i)$, for all $j \neq i$:*

$$\left| \frac{\langle \mathbb{1}_{X_\tau, i}, \mathbb{1}_{X_\tau, j} \rangle}{\langle \mathbb{1}_{X_\tau}, \mathbb{1}_{X_\tau} \rangle} \right| \leqslant O\left(\frac{k^3 R_{max}}{\delta_{k-i}^k (1 - \delta_{i-1})} \gamma \right).$$

We note that the above is the only result in our work that truly relies on non-laziness (it is used only to replace ρ with regularity in all other results). It is possible to recover the upper bound in Theorem 29 for general eposets via arguments used in [6], but the lower bound remains open for concentrated posets. With that in mind, we now prove Theorem 29.

Proof of Theorem 29. By the previous lemma, we have

$$\left|\frac{\langle \mathbb{1}_{X_\tau}, \mathbb{1}_{X_\tau,j}\rangle}{\langle \mathbb{1}_{X_\tau}, \mathbb{1}_{X_\tau}\rangle}\right| \leqslant O\left(\frac{1}{\delta_{k-i}^k(1-\delta_{i-1})}\cdot\left(\frac{k^3}{\rho_{\min}}\gamma + R(k,i)\gamma\right)\right).$$

Expanding out $\bar{\Phi}(\mathbb{1}_{X_\tau})$ then gives:

$$\begin{aligned}\bar{\Phi}(\mathbb{1}_{X_\tau}) &= \frac{1}{\langle \mathbb{1}_{X_\tau}, \mathbb{1}_{X_\tau}\rangle}\sum_{j=0}^{i}\langle \mathbb{1}_{X_\tau}, M\mathbb{1}_{X_\tau,j}\rangle \\ &\leqslant \frac{1}{\langle \mathbb{1}_{X_\tau}, \mathbb{1}_{X_\tau}\rangle}\sum_{j=0}^{i}\lambda_i(M)\langle \mathbb{1}_{X_\tau}, \mathbb{1}_{X_\tau,j}\rangle + c_2\gamma \\ &\leqslant \lambda_i(M)\frac{\langle \mathbb{1}_{X_\tau}, \mathbb{1}_{X_\tau,i}\rangle}{\langle \mathbb{1}_{X_\tau}, \mathbb{1}_{X_\tau}\rangle} + err_1 \\ &\leqslant \lambda_i(M) + err_2.\end{aligned}$$

where $c_2, err_1, err_2 \leqslant O\left(\frac{k}{\delta_{k-i}^k(1-\delta_{i-1})}\left(\frac{k^2(h(M)+k)h(M)w(M)}{\rho_{\min}}\gamma + R(k,i)\gamma\right)\right)$ and the last step follows from the previous lemma. The conclusion then follows from applying Claim 19. ◀

Thus, for sufficiently nice expanding posets, the expansion of any i-link with respect to an HD-walk is almost exactly $1-\lambda_i(M)$. As HD-walks are generally poor expanders (have large $\lambda_1(M)$), Theorem 29 implies that links are examples of small, non-expanding sets. Following BHKL, we now prove a converse to this result: any non-expanding set must be explained by some structure inside links. To give a precise statement, we need the following definition:

▶ **Definition 32** (Stripped Threshold Rank [6])**.** *Let (X,Π) be a (δ,γ)-eposet and M a k-dimensional HD-walk with γ small enough that the HD-Level-Set Decomposition has a corresponding decomposition of disjoint eigenstrips $C_k = \bigoplus W_k^i$. The ST-Rank of M with respect to η is the number of strips containing an eigenvector with eigenvalue at least η:*

$$R_\eta(M) = |\{W_k^i : \exists f \in V^i, Mf = \lambda f, \lambda > \eta\}|.$$

With this definition, we provide a converse to Theorem 29 in both ℓ_2 and ℓ_∞ senses:

▶ **Theorem 33.** *Let (X,Π) (δ,γ)-eposet, M a k-dimensional, monotonic HD-walk, and γ small enough that the eigenstrip intervals of Theorem 17 are disjoint. For any $\eta > 0$, let $r = R_\eta(M) - 1$. Then the expansion of a set $S \subset X(k)$ of density α is at least:*

$$\Phi(S) \geqslant 1-\alpha-(1-\alpha)\eta - \sum_{i=1}^{r}(\lambda_i(M)-\eta)R(k,i)\varepsilon_i - c\gamma$$

where S is $(\varepsilon_1,\ldots,\varepsilon_r)$-pseudorandom and $c \leqslant O\left(\frac{k^5 R_{max}^2(h(M)+k)h(M)w(M)}{\max_i\{\delta_i(1-\delta_{i-1})\}}\right)$.

The argument is similar to [6] for simplicial complexes and relies on similar manipulations to Theorem 29, so we defer the proof to the full version. Theorem 33 recovers the analogous result for simplicial complex in [6] with the appropriate value $R(k,i) = \binom{k}{i}$. BHKL also prove this special case is tight in multiple senses; see the discussion there for more details.

6 The Grassmann and q-eposets

In this section, we specialize to expanding subsets of the Grassmann poset. We will show that our analysis is tight in this regime.

Spectra. We start by examining the spectrum of HD-walks on the Grassmann and q-eposets, focusing on the most widely used walks in the literature, the canonical and partial-swap walks. To start, recall that the Grassmann poset itself is a 0-eposet. Plugging the parameters in Equation (1) into Proposition 18, along with an analogous calculation for q-eposets, gives a nice exact form for the spectra of canonical walks.

▶ **Corollary 34** (N_k^j Spectra). *Let $X = G_q(n,d)$ be the Grassmann Poset, $k+j \leqslant d$, and $f_\ell = U_\ell^k g_\ell$ for some $g_\ell \in H^\ell$. Then $N_k^j f_\ell = \lambda_\ell f_\ell$, where*

$$\lambda_\ell = q^{\ell j} \frac{\binom{k+j-\ell}{j}_q}{\binom{k+j}{j}_q} \frac{\binom{n-k-\ell}{j}_q}{\binom{n-k}{j}_q} \approx q^{-\ell j}.$$

Similarly, let (X,Π) be a d-dimensional γ-q-eposet with $\gamma \leqslant q^{-\Omega(k^2)}$, $k+j \leqslant d$, and $f_\ell = U_\ell^{k-1} g_\ell$ for some $g_\ell \in H^\ell$. Then:

$$\left\| N_k^j f_\ell - \frac{\binom{k+j-\ell}{j}_q}{\binom{k+j}{j}_q} f_\ell \right\| \leqslant O\left(j(j+k) \binom{k}{\ell}_q \right) \gamma \|f_\ell\|$$

The first equation above recovers a very simple proof of classical results to this effect (see e.g. [14]). Note that for small enough γ, Theorem 17 implies that the true spectra is then concentrated around these values as well. These eigenvalues are, as one would expect, the natural q-analog of the corresponding eigenvalues on simplicial complexes.

Moreover, this carries over to the class of partial-swap walks, which were originally analyzed by AJT on simplicial complexes [1]. To see this, we first need to show (in Appendix B) these walks are indeed HD-walks, which follows from the q-analog of statements in AJT [1].

▶ **Lemma 35.** *Let (X,Π) be a pure, measured q-simplicial complex. Then:*

$$N_k^j = \sum_{i=0}^{j} q^{i^2} \frac{\binom{j}{i}_q \binom{k}{k-i}_q}{\binom{k+j}{k}_q} S_k^i, \text{ and } \quad S_k^j = \frac{1}{q^{j^2}\binom{k}{k-j}_q} \sum_{i=0}^{j} (-1)^{j-i} q^{\binom{j-i}{2}} \binom{j}{i}_q \binom{k+i}{i}_q N_k^i.$$

This is unsurprisingly the q-analog of the analogous statement on simplicial complexes (see [1, Corollary 4.13]). Finally, combining the previous result with Corollary 20 and Corollary 34, it is possible to show that the eigenvalues of partial-swap walks on q-simplicial complexes are given by the natural q-analog of the simplicial complex case (see the full version for details):

▶ **Corollary 36.** *Let (X,Π) be a d-dimensional γ-q-eposet with γ sufficiently small, $k+j \leqslant d$, and $f_\ell = U_\ell^k g_\ell$ for some $g_\ell \in H^\ell$. Then:*

$$\left\| S_k^j f_\ell - \frac{\binom{k-j}{\ell}_q}{\binom{k}{\ell}_q} f_\ell \right\| \leqslant O\left(\left(\frac{q}{q-1}\right)^{\min(j,k-j)+2} k^2 \binom{k}{\ell}_q \right) \gamma \|f_\ell\|$$

Again, since the swap walks are self-adjoint Theorem 17 implies that for small enough γ the true spectra is closely concentrated around these values as well.

Pseudorandom Functions and Small Set Expansion. With an understanding of the spectra of HD-walks on q-simplicial complexes, we move to studying its combinatorial structure. By direct computation, it is not hard to show that on q-eposets, $\rho_\ell^k = \binom{k}{\ell}_q^{-1}$ (Claim 19 would only imply this is approximately true). As a result, we get a level-i inequality for q-simplicial complexes that is the natural q-analog of BHKL's inequality for basic simplicial complexes.

▶ **Theorem 37.** *Let (X,Π) be a γ-q-eposet with $\gamma \leqslant q^{-\Omega(k^2)}$ and let $S \subseteq X(k)$. If $\mathbb{1}_S$ is $(\varepsilon_1,\dots,\varepsilon_\ell)$-$\ell_\infty$-pseudorandom, then for $c \leqslant q^{O(k^2)}$,*

$$|\langle \mathbb{1}_S, \mathbb{1}_{S,i}\rangle| \leqslant \left(\binom{k}{i}_q \varepsilon_i + c\gamma\right)\mathbb{E}[\mathbb{1}_S] \qquad \forall\, 1 \leqslant i \leqslant \ell.$$

For large enough q,γ^{-1}, this result is exactly tight. The key to showing this fact is to examine a local structure unique to the Grassmann called *co-links*. The co-link of an element $W \in X(k')$, is all of the subspaces *contained in* W, i.e. $\bar{X}_W = \{V \in X(k) : V \subseteq W\}$. Just like links, it turns out that co-links of dimension i (that is $k' = d-i$) also come through levels 0 through i of the complex, although this is somewhat trickier to see. The essential observation is that co-links satisfy enough symmetries to explicitly construct a function in C_i whose image under U_i^k yields the desired function; see the full version for the construction.

▶ **Lemma 38** (HD-Level-Set Decomposition of Co-Links)**.** *Let $X = G_q(d,k)$ and $\mathcal{S} = \overline{X}_W$ be a co-link of dimension i for $W \in X(d-i)$. Then, we have $\mathbb{1}_{\mathcal{S}} \in V_k^0 \oplus \dots \oplus V_k^i$.*

Using this fact, we can show that our level-i inequality is exactly tight on co-links, deferring the proof to Appendix B.

▶ **Proposition 39.** *Let $X = G_q(d,k)$ be the Grassmann poset. For any $i \leqslant k \in \mathbb{N}$ and $c < 1$, there exist large enough q,d and a (i,ε_i)-pseudorandom set $S \subset X(k)$ such that*

$$\langle \mathbb{1}_S, \mathbb{1}_{S,i}\rangle > c\binom{k}{i}_q \varepsilon_i \langle \mathbb{1}_S, \mathbb{1}_S\rangle.$$

Finally, Theorem 37 directly implies that for the canonical and partial-swap walks, sufficiently pseudorandom functions expand near perfectly.

▶ **Corollary 40** (q-Eposets Edge-Expansion)**.** *Let (X,Π) be a d-dimensional γ-q-eposet, $S \subset X(k)$ a subset whose indicator function $\mathbb{1}_S$ is $(\varepsilon_1,\dots,\varepsilon_\ell)$-pseudorandom. Then the edge expansion of S with respect to the canonical walk N_k^j is bounded by:*

$$\Phi_{\pi_k}(N_k^j, S) \geqslant 1 - \mathbb{E}[\mathbb{1}_S] - \sum_{i=1}^{\ell} \frac{\binom{k+j-i}{j}_q}{\binom{k+j}{j}_q}\binom{k}{i}_q \varepsilon_i - q^{-(\ell+1)j} - q^{O(k^2)}\gamma.$$

Further, the edge expansion of S with respect to the partial-swap walk S_k^j is bounded by:

$$\Phi_{\pi_k}(S_k^j, S) \geqslant 1 - \mathbb{E}[\mathbb{1}_S] - \sum_{i=1}^{\ell} \binom{k-j}{i}_q \varepsilon_i - q^{-(\ell+1)j} - q^{O(k^2)}\gamma.$$

Note that S_k^j on q-eposets is a generalization of the Grassmann Graphs (and are equivalent when X is the Grassmann Poset). While our definition of pseudorandomness is weaker than that of [44] and therefore necessarily depends on the dimension k, we take the above as evidence that the framework of expanding posets may be important for making further progress on the Unique Games Conjecture. In particular, combined with recent works removing this k-dependence on simplicial complexes [7, 25], it seems plausible that the framework of expanding posets may lead to a more general understanding of the structure underlying the Unique Games Conjecture.

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A Eposet Parameters and Regularity

In this section, we discuss connections between notions of regularity, the averaging operators, and eposet parameters. To start, we'll show that downward and middle regularity (which are defined only on adjacent levels of the poset) imply extended regularity between any two levels.

▶ **Proposition 41.** *Let (X, Π) be a d-dimensional regular measured poset. Then for any $i < k \leqslant d$, there exist regularity constant $R(k, i)$ such that for any $x_k \in X(k)$, there are exactly $R(k, i)$ elements $x_i \in X(i)$ such that $x_k > x_i$.*

Proof. Given any element $x_k \in X(k)$, downward regularity promises there are exactly $\prod_{j=i+1}^{k} R(j)$ unique chains $x_k < x_{k-1} < \ldots < x_{i+1} < x_i$. By middle regularity, any fixed $x_i \in X(i)$ which appears in this fashion appears in exactly $m(k, i)$ chains. Noting that $x_i < x_k$ if and only if x_i appears in such a chain, the total number of $x_i < x_k$ must be exactly $R(k, i) = \frac{\prod_{j=i+1}^{k} R(j)}{m(k,i)}$. ◀

A similar argument shows that regularity allows the up operators to compose in the natural way.

▶ **Proposition 42.** *Let (X, Π) be a d-dimensional regular measured poset. Then for any $i < k \leqslant d$ we have:*

$$U_i^k f(x_k) = \frac{1}{R(k, i)} \sum_{x_i < x_k} f(x_i)$$

Proof. Expanding out $U_i^k f(y)$ gives:

$$U_i^k f(x_k) = \frac{1}{\prod\limits_{j=i+1}^{k} R(j)} \sum_{x_{k-1} < x_k} \cdots \sum_{x_i < x_{i+1}} f(x_i)$$

The number of times each x_i appears in this sum is exactly the number of chains starting at x_k and ending at x_i, so by middle regularity:

$$\begin{aligned} \frac{1}{\prod\limits_{j=i+1}^{k} R(j)} \sum_{x_{k-1} < x_k} \cdots \sum_{x_{i+1} < x_i} f(x_i) &= \frac{m(k, i)}{\prod\limits_{j=i+1}^{k} R(j)} \sum_{x_i < x_k} f(x_i) \\ &= \frac{1}{R(k, i)} \sum_{x_i < x_k} f(x_i). \end{aligned}$$ ◀

We'll now take a look at the connection between eposet parameters and regularity. It is convenient to first start with a lemma stating that non-laziness is equivalent to bounding the maximum transition probability of the lower walk.

▶ **Lemma 43.** *Let (X,Π) be a d-dimensional measured poset. Then for any $0 < i \leqslant d$, the maximum laziness of the lower walk is also the maximum transition probability:*

$$\max_{\sigma \in X(i)} \left\{ \mathbb{1}_\sigma^T U_{i-1} D_i \mathbb{1}_\sigma \right\} = \max_{\sigma,\tau \in X(i)} \left\{ \mathbb{1}_\sigma^T U_{i-1} D_i \mathbb{1}_\tau \right\}.$$

Proof. Assume that $\tau \neq \sigma$. Then the transition probability from τ to σ is exactly

$$\begin{aligned} \mathbb{1}_\sigma^T U_{i-1} D_i \mathbb{1}_\tau &= \frac{\pi_\tau(\sigma \backslash \tau)}{R(i,i-1)} \\ &\leqslant \frac{1}{R(i,i-1)} \sum_{\tau \lessdot \sigma} \pi_\tau(\sigma \backslash \tau) \\ &= \mathbb{1}_\sigma^\tau U_{i-1} D_i \mathbb{1}_\sigma, \end{aligned}$$

which implies the result. ◀

We now prove our two claims relating the eposet parameters to regularity.

▷ Claim 44. Let (X,Π) be a d-dimensional (δ,γ)-eposet. Then for every $1 \leqslant k \leqslant d$ and $0 \leqslant i \leqslant k$, the following approximate relation between the eposet and regularity parameters holds:

$$\lambda_i(N_k^1) \in \frac{R(k,i)}{R(k+1,i)} \pm \left(\gamma_{k-i}^k + R(k,i)\delta_{k-i}^k \gamma \right)$$

where we recall $\lambda_i(N_k^1) = 1 - \prod\limits_{j=i}^{k} \delta_j$.

Proof. We require a refinement of [16, Claim 8.8] given in [6, Lemma A.1]:[7]

$$D_{k+1} U_i^{k+1} = (1 - \delta_{k-i}^k) U_i^k + \delta_{k-i}^k U_{i-1}^k D_i + \sum_{j=-1}^{k-i-1} U_{k-j-1}^k \Gamma_j U_i^{k-j-1} \tag{3}$$

where $\sum \|\Gamma_j\| \leqslant \gamma_{k-i}^k$. The idea is now to examine the "laziness" of the two sides of this equality. In other words, given a starting k-face τ, what is the probability that the resulting i-face σ satisfies $\sigma < \tau$?

To start, we'll argue that the laziness of the lefthand side is exactly $\frac{R(k,i)}{R(k+1,i)}$. This follows from noting that there are $R(k,i)$ i-faces σ satisfying $\sigma < \tau$, and $R(k+1,i)$ options after taking the initial up-step of the walk to $\tau' > \tau$. After the down-steps, the resulting i-face is uniformly distributed over these $R(k+1,i)$ options $\sigma < \tau'$, and since every $\sigma < \tau < \tau'$, all original $R(k,i)$ lazy options are still viable after the up-step to τ'.

Analyzing the right-hand side is a bit trickier. The initial term $(1-\delta_{k-i}^k)U_i^k$ is completely lazy, so it contributes exactly $(1-\delta_{k-i}^k) = \lambda_i(N_k^1)$. We'll break the second term into two steps: walking from $X(k)$ to $X(i)$ via U_i^k, then from $X(i)$ to $X(i)$ via the lower walk $U_{i-1}D_i$. Starting at a k-face τ, notice that after applying the down step U_i^k we are uniformly spread over $\sigma < \tau$. Computing the laziness then amounts to asking what the probability of staying

[7] Formally the result is only stated for simplicial complexes in [6], but the same proof holds for eposets.

in this set is after the application of UD, which one can naively bound by the maximum transition probability times the set size $R(k,i)$. By non-laziness, the maximum transition probability is at most γ (see Lemma 43).

The third term can be handled similarly. The first down step U^k_{k-j-1} spreads τ evenly across $\sigma < \tau$ in $X(k-j-1)$. The resulting i-face σ' after applying $\Gamma_j U_i^{k-j-1}$ is less than τ if and only if the intermediary $(k-j-1)$-face after applying Γ_j is less than τ, which is bounded by the spectral norm $\|\Gamma_j\|$.[8]

Putting everything together, since both sides of Equation (3) must have equivalent laziness, we get that $\lambda_i(N^1_k)$ must be within $\sum \|\Gamma_j\| + \delta^k_{k-i} R(k,i)\gamma$ as desired. ◁

Claim 19 and Theorem 22 can both be proving an analogous theorem for the upper walk.

▷ Claim 45 (Regularity and Upper Walk Spectrum). Let (X, Π) be a d-dimensional (δ,γ)-eposet. Then for any $j \leqslant i \leqslant k < d$, we have:

$$\lambda_j(N_i^{k-i}) \in \frac{R(i,j)}{R(k,i)} \pm err,$$

where $err \leqslant O\left(\frac{i^4 k^2 R_{\max}}{\delta_i(1-\delta_{i-1})}\gamma\right)$.

Proof. This follows almost immediately from the fact that i-links lie almost entirely on the ith eigenstrip (Lemma 31). In particular, it is enough to examine the expansion of i-links with respect to the upper canonical walk N_i^{k-i}. On the one hand, for any $j \leqslant i$ and $\tau \in X(j)$ we have:

$$\begin{aligned}
\bar{\Phi}(X^i_\tau) &= \frac{\langle \mathbb{1}_{X^i_\tau}, N_i^{k-i}\mathbb{1}_{X^i_\tau}\rangle}{\langle \mathbb{1}_{X^i_\tau}, \mathbb{1}_{X^i_\tau}\rangle} \\
&= \frac{\langle U^k_j \mathbb{1}_\tau, U^k_j \mathbb{1}_\tau\rangle}{\langle U^i_j \mathbb{1}_\tau, U^i_j \mathbb{1}_\tau\rangle} \\
&= \frac{R(i,j)^2}{R(k,i)^2}\frac{\langle \mathbb{1}_{X^k_\tau}, \mathbb{1}_{X^k_\tau}\rangle}{\langle \mathbb{1}_{X^i_\tau}, \mathbb{1}_{X^i_\tau}\rangle} \\
&= \frac{R(i,j)}{R(k,i)}\frac{\langle \mathbb{1}_\tau, \mathbb{1}_\tau\rangle}{\langle \mathbb{1}_\tau, \mathbb{1}_\tau\rangle} \\
&= \frac{R(i,j)}{R(k,i)}.
\end{aligned}$$

where we have applied the fact that $\langle X^\ell_\tau, X^\ell_\tau\rangle = R(\ell,j)\langle \mathbb{1}_\tau, \mathbb{1}_\tau\rangle$. On the other hand, by Lemma 31 we also have that:

$$\begin{aligned}
\bar{\Phi}(\mathbb{1}_{X^i_\tau}) &= \frac{1}{\langle \mathbb{1}_{X^i_\tau}, \mathbb{1}_{X^i_\tau}\rangle}\sum_{\ell=0}^{i}\langle \mathbb{1}_{X^i_\tau}, N_i^{k-i}\mathbb{1}_{X^i_\tau,\ell}\rangle \\
&\in \frac{1}{\langle \mathbb{1}_\tau, \mathbb{1}_\tau\rangle}\sum_{\ell=0}^{i}\lambda_j(N_i^{k-i})\langle \mathbb{1}_{X^i_\tau}, \mathbb{1}_{X^i_\tau,\ell}\rangle + c\gamma \\
&\in \lambda_j(N_i^{k-i})\frac{\langle \mathbb{1}_\tau, \mathbb{1}_{\tau,j}\rangle}{\langle \mathbb{1}_\tau, \mathbb{1}_\tau\rangle} + \sum_{j=0}^{i} err_1 \\
&\in \lambda_j(N_i^{k-i}) + err_2
\end{aligned}$$

where as in the proof of Theorem 29, $c, err_1, err_2 \leqslant O\left(\frac{i^4k^2R_{\max}}{\delta^i_{i-j}(1-\delta_{j-1})}\gamma\right)$. ◁

[8] We note that Γ_j is not stochastic, but it is self-adjoint and an easy exercise to see that the analogous reasoning still holds.

Claim 19 follows immediately from observing that $\rho_i^k = \lambda_i(N_i^{k-i})$ (by Proposition 18). Theorem 22 follows from observing that $\widehat{N}_i^{k-i}$ and $\check{N}_k^{k-i}$ have the same approximate eigenvalues (similarly by Proposition 18).

Finally we close out the section by discussing the connection between non-laziness and a variant of eposets called local-spectral expanders [34].

▶ **Definition 46** (Local-Spectral Expander [19, 34]). *A d-dimensional measured poset (X, Π) is a γ-local-spectral expander if the graph underlying every link[9] of dimension at most $d-2$ is a γ-spectral expander.[10]*

Under suitable regularity conditions (see [34]), local-spectral expansion is equivalent to the notion of expanding posets used in this work. A simple argument shows that γ-local-spectral expanders are γ-non-lazy.

▶ **Lemma 47.** *Let (X, Π) be a d-dimensional γ-local-spectral expander, and $0 < i < d$. The laziness of the lower walk on level i is at most:*

$$\max_{\sigma \in X(i)} \left\{ \frac{\langle \mathbb{1}_\sigma, U_{i-1} D_i \mathbb{1}_\sigma \rangle}{\langle \mathbb{1}_\sigma, \mathbb{1}_\sigma \rangle} \right\} \leqslant \gamma.$$

Proof. Through direct computation, the laziness probability of the lower walk at $\sigma \in X(i)$ is exactly

$$\frac{\langle \mathbb{1}_\sigma, U_{i-1} D_i \mathbb{1}_\sigma \rangle}{\langle \mathbb{1}_\sigma, \mathbb{1}_\sigma \rangle} = \frac{1}{R(i, i-1)} \sum_{\tau \lessdot \sigma} \pi_\tau(\sigma \backslash \tau)$$

It is therefore enough to argue that $\pi_\tau(\sigma \backslash \tau) \leqslant \gamma$, as the graph underlying the link X_τ is a γ-spectral expander. Recall that an equivalent formulation of this definition states that:

$$\| A_\tau - UD_\tau \| \leqslant \gamma,$$

where A_τ is the standard (non-lazy upper) walk and UD_τ is the lower walk on the graph underlying X_τ. This implies that the weight of any vertex v in the graph is at most γ, as:

$$\frac{\langle \mathbb{1}_v, UD_\tau \mathbb{1}_v \rangle}{\langle \mathbb{1}_v, \mathbb{1}_v \rangle} = \frac{\langle \mathbb{1}_v, (UD_\tau - A_\tau) \mathbb{1}_v \rangle}{\langle \mathbb{1}_v, \mathbb{1}_v \rangle} \leqslant \| A_\tau - UD_\tau \| \leqslant \gamma$$

where we have used the fact that A_τ is non-lazy by definition. Since $\pi_\tau(\sigma \backslash \tau)$ is exactly the weight of the vertex $\sigma \backslash \tau$ in X_τ, this completes the proof. ◀

B Deferred Proofs

Proof of Lemma 35. We follow the structure and notation of [1, Lemma 4.11]. Assume that the canonical walk starts at a subspace $V \in X(k)$, and walks up to $W \in X(k+j)$. We wish to analyze the probability that upon walking back down to level k, a subspace V' with intersection $k-i$ is chosen, i.e. $dim(V \cap V') = k - i$. Let such an event be denoted $\mathcal{E}_i(W)$. It follows from elementary q-combinatorics (see e.g. [10, Lemma 9.3.2]) that

$$\Pr_{V' \subset W} [\mathcal{E}_i(W) \mid W] = q^{i^2} \frac{\binom{j}{i}_q \binom{k}{k-i}_q}{\binom{k+j}{k}_q},$$

[9] Here the link of τ is not just its top level faces, but the complex given by taking this set, removing τ from each face, and downward closing.

[10] A graph is a γ-spectral expander if its weighted adjacency matrix has no non-trivial eigenvalues greater than γ in absolute value.

where $V' \in X(k)$ is drawn uniformly from the k-dimensional subspaces of W. To relate this process to the swap walk S_k^i, note that while the swap walk (as defined) only walks up to $X(k+i)$, walking up to $X(k+j)$ and conditioning on intersection i, a process called the *i-swapping j-walk* by [1], is exactly the same due to symmetry (via the regularity condition, see [Proposition 4.9 of [1]] for a more detailed explanation). Thus consider the i-swapping j-walk, and let T_i' denote the variable standing for the subspace chosen by the walk. Conditioned on picking the same W as the canonical walk in its ascent, we may relate T_i' to the canonical walk:

$$\Pr[T_i' = T \mid W] = \Pr[V' = T \mid W \text{ and } \mathcal{E}_i(W)]$$

We may now decompose the canonical walk by intersection size:

$$\begin{aligned}
N_k^j(V,T) &= \sum_{i=0}^{j} \sum_{W \in X(k+j)} \Pr[W] \Pr[\mathcal{E}_i(W) \mid W] \Pr[V' = T \mid W \text{ and } \mathcal{E}_i(W)] \\
&= \sum_{i=0}^{j} \sum_{W \in X(k+j)} q^{i^2} \frac{\binom{j}{i}_q \binom{k}{k-i}_q}{\binom{k+j}{k}_q} \mathop{\mathbb{E}}_{W \supset V} [\Pr[V' = T \mid W \text{ and } \mathcal{E}_i(W)]] \\
&= \sum_{i=0}^{j} \sum_{W \in X(k+j)} q^{i^2} \frac{\binom{j}{i}_q \binom{k}{k-i}_q}{\binom{k+j}{k}_q} \mathop{\mathbb{E}}_{W \supset V} [\Pr[T_i' = T \mid W\ ]] \\
&= \sum_{i=0}^{j} \sum_{W \in X(k+j)} q^{i^2} \frac{\binom{j}{i}_q \binom{k}{k-i}_q}{\binom{k+j}{k}_q} \Pr[T_i' = T] \\
&= \sum_{i=0}^{j} \sum_{W \in X(k+j)} q^{i^2} \frac{\binom{j}{i}_q \binom{k}{k-i}_q}{\binom{k+j}{k}_q} S_k^i(V,T).
\end{aligned}$$

From this point, the claim can be obtained by applying a q-binomial inversion theorem (Theorem 2.1 of [45]), see the full version for details. ◀

Proof of Proposition 39. For $W \in X(d-i)$, consider the co-link $\bar{X}_W = \{V \in X(k) : V \subset W\}$. For simplicity, let $S := \bar{X}_W$. The density of S in any j-link X_V is:

$$\alpha_j = \frac{(q^{d-i-j}-1)\dots(q^{d-k+1-i}-1)}{(q^{d-j}-1)\dots(q^{d-k+1}-1)} = q^{-i(k-j)} + o_{q,d}(1).$$

The idea is now to examine the (non)-expansion of the co-link with respect to the lower walk $U_{k-1}D_k$. By direct computation, the probability of returning to $\bar{X}_W$ after moving to a $(k-1)$-dimensional subspace is exactly:

$$\bar{\Phi}(\bar{X}_W) = \frac{q^{d-i} - q^{k-1}}{q^d - q^{k-1}} = q^{-i} \pm q^{-\Omega(d)} \tag{4}$$

By Proposition 18, the approximate eigenvalues of the lower walk are

$$\lambda_j = \frac{q^{k-j}-1}{q^k-1} = q^{-j} - O(q^{-k})$$

Since a dimension-i co-link has no projection onto levels $i+1$ through k, it also holds that:

$$\bar{\Phi}(\bar{X}_W) = \frac{1}{\langle \mathbb{1}_S, \mathbb{1}_S \rangle} \sum_{j=0}^{i} q^{-j} \langle \mathbb{1}_S, \mathbb{1}_{S,j} \rangle - O(q^{-k})$$

for large enough q, d. Combined with Equation (4), there exists a universal constant c' such that for large enough q and d, $\mathbb{1}_{\bar{X}_W}$ cannot have more than a $\frac{c'}{q}$ fraction of its mass on levels 1 through $i-1$. Finally, noticing that $\binom{k}{i}_q \alpha_i = 1 + o_q(1)$, we obtain

$$\frac{\langle \mathbb{1}_S, \mathbb{1}_{S,i} \rangle}{\langle \mathbb{1}_S, \mathbb{1}_S \rangle} \geqslant \frac{q - c'}{q} \geqslant c \binom{k}{i}_q \alpha_i$$

since the latter is strictly bounded away from 1 for large enough q. This completes the result since $\bar{X}_W$ is (α_i, i)-pseudorandom. ◀