Characterizing Quantitative Structures Students Establish for Real-World Scenarios

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In this report, we characterize a spectrum of mathematical structures of real-world situations. Using data from teaching experiments with undergraduate STEM majors and theories from quantitative reasoning, covariational reasoning, and multi variational reasoning, we build second order accounts of modelers' reasoning with and about conceived quantities. Through these accounts we illustrate four different kinds of structures as means for describing key aspects of how modelers develop their models.

Keywords: Mathematical Modeling, Quantitative Reasoning, Covariational Reasoning, Multivariational Reasoning

Typically, mathematical modeling involves translating real world scenarios into mathematical representations. The mathematical representations can take the form of mathematical expressions, tables and graphs depicting how variables vary with one another (or not), and even diagrams depicting the dynamics of the scenario (e.g. stock-flow diagrams). While tables, graphs, and figures are useful for representing the real-world scenario, the ultimate goal is producing a mathematical expression that is consistent with their previous representations and reasonings about the scenario *and also* representative of the real-world scenario. Quantitative relations govern the mathematical model of a real-world scenario. That is, a mathematical representation of a real-world scenario can be understood as an expression of the relationships among conceived quantities. Therefore, it would make sense to view mathematical modeling through the lens of quantities and relations among quantities (Thompson, 2011; Larsen, 2013; Czocher & Hardison, 2019) in order to find ways to help guide students towards a mathematical expression. However, reasoning with and about quantities doesn't necessarily yield a mathematical expression consistent with the modeler's reasoning *and* representative of the scenario as an end result (Czocher & Hardison, 2019). In milieu of this, we ask: what is the nature of the quantitative relations students establish of real-world scenarios.

Theoretical Orientation

Our research lies within the cognitive perspective of mathematical modeling (Kaiser, 2017). In this perspective, mathematical modeling is considered to be the cognitive processes involved in constructing a mathematical model of real-world scenarios. We define a mathematical model to be the external representation of the relations among the quantities a modeler conceived as relevant to model a real-world scenario. We define mathematical modeling activity as the mental activities involved in creating a mathematical representation of a real-world situation.

Thompson (2011) defines quantification as the "process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship its unit" (p. 37). In that sense, a quantity is a mental construct of a measurable attribute of an object. Quantitative reasoning involves conceiving and reasoning about conceived quantities. Reasoning about conceived quantity can entail operating on conceived quantities and reasoning about how the quantities can vary. Thompson (1994) defines quantitative operation as the "mental operation by which one conceives a new quantity in relation to one or more already-conceived quantities" (p.10). As a result of a quantitative operation a relationship is created: the quantities operated upon along with the quantitative operation in relation to the result of operating (Thompson, 1994). Examples of quantitative operations include combining two quantities additively or multiplicatively and comparing two quantities additively or multiplicatively. Scholars address the following ways individuals can reason about how quantities vary: variational reasoning (Thompson & Carlson, 2017). Co-variational reasoning (Carlson et al, 2002; Thompson & Carlson,2017), and multivariational reasoning (Jones, 2018; Jones & Jeppson, 2020, Panorkou & Germia, 2020).

While variational reasoning involves reasoning about varying quantities independently (Thompson & Carlson, 2017), co-variational reasoning involves "coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al, 2002, p.354). Carlson et al (2002) contributed a framework for the mental actions involved in covariational reasoning. Later, Thompson and Carlson (2017) proposed six major mental operations involved in covariational reasoning. These mental operations are: *no coordination of values*, *pre coordination of values* (envisioning asynchronous changes in variables), *gross coordination of values* (envisioning the general increase/decrease in variables' values), *coordination of values* (coordinating the amounts of change of each quantity), *chunky continuous variation* (envisioning change in variables happening simultaneously but in discrete chunks), *smooth continuous variation* (envisioning change happening simultaneously but smoothly).

Scholars have extended the work of covariational reasoning to multivariational reasoning, which is reasoning about more than two quantities changing in conjunction with each other (Jones 2018; Jones & Jeppson,2020). Jones and Jeppson (2020) identified the following mental actions attendant to multivariational reasoning: recognizing dependence/independence, reduce into isolated covariations, covariational reasoning, switch variables/constants, imagining simultaneous changes in inputs, coordinating multiple simultaneous changes, coordinating qualitative amounts of change, coordinating numeric amounts of change, articulate the type of relationship, identifying the order of effect between variables, and recognize a chain of influence.

Borrowing ideas from the aforementioned theoretical underpinnings, we define establishing structure for a real-world scenario to be creating a network of quantitative relations among the quantities the modeler conceives and recognizes as relevant to modeling the scenario. By network of quantitative relations, we mean the system of quantitative relations that was created as a result of reasoning about and operating on conceived quantities. In this paper we address the question: What is the nature of the quantitative structures students establish for real world scenarios?

Methods

We present data from a set of three teaching experiments (Steffe & Thompson, 2000) conducted with undergraduate STEM majors at a large university. The overall goal of the teaching experiment was to investigate the role of quantitative and co/multi-variational reasoning in students' conception of real-world situations as they attempted to model those scenarios. Baxil, Pai, and Szeth, each participated 10 interview sessions; each session was approximately 1 hour long. Baxil and Pai were enrolled in differential equations and Szeth was repeating the course. During the interviews, in addition to asking students the meanings they attributed to each symbol they introduced, participants were also probed to unpack the reasonings behind certain decisions they made during their mathematical modeling activity. In this report we present data from the following sessions: Baxil and the Fruit Ripening Task, Szeth and the Disease transmission Task, and Pai and the CI8 Account task. We focus on these sessions because they illustrate the finding that quantitative structures established by modelers may have differing natures.

- *Fruit Ripening:* There is a surprising effect in nature where a tree or bush will suddenly ripen all of its fruit or vegetables, without any visible signal. This is our first example of a positive biological feedback loop. If we look at an apple tree, with many apples, seemingly overnight they all go from unripe to ripe to overripe. This will begin with the first apple to ripen. Once ripe, it gives off a gas known as ethylene (C_2H_4) through its skin. When exposed to this gas, the apples near to it also ripen. Once ripe, they too produce ethylene, which continues to ripen the rest of the tree in an effect much like a wave. This feedback loop is often used in fruit production, with apples being exposed to manufactured ethylene gas to make them ripen faster. Develop a mathematical model that captures the dynamics of the ripeness of the fruit.
- *Disease Transmission*: Suppose a disease is spread by contact between sick and well members of the community. If members of the community move about freely among each other, develop a mathematical model that informs us about the dynamics of how the disease would spread through the population.
- *The CI8 Account*: The competing Amtrak Trust has introduced a modification to City bank's SI8, which they call the CI8 account. Like the SI8 account, the CI8 earns 8% of the "initial investment". However, at the end of each year Amtrak Trust recalculates the "initial investment" of the CI8 account to include all the interest that the customer has earned up to that point. Create an expression that gives the value of the CI8 account at any time t (Castillo-Garsow, 2010).

We retrospectively analyzed the data via building second-order models (Steffe & Thompson, 2000) of students' reasoning. Since we did not have direct access to participants' mental activities, the second-order models we constructed are inferences made from the students' observable activities such as language, verbal descriptions and discourse, written work, and their mathematically salient gestures. Our retrospective analysis consisted of five rounds of data analysis to arrive at examples that illustrate the different natures of the structures present in students' conception of real-world situations. First, we watched the subset of videos without interruption to observe students' ways of reasoning about conceived quantities. Second, we paid close attention how they transformed these reasonings into a mathematical expression, or not. Third, we distinguished sessions where a normatively acceptable mathematical expressions were created from those where it wasn't the case. Fourth, we sought to distinguish the sessions where acceptable mathematical expressions weren't created according to the level of sophistication in mental actions attendant to co/multivariational reasoning. Fourth, we created annotated transcripts of such scenarios that provided rich description of the modelers mathematical modeling activity. Finally, we built and refined the explanatory models of participants structural conception of real-world scenarios.

Findings

Qualitative Structure – Baxil and Fruit Ripening

Baxil operationalized ripeness of the fruit as "readiness to eat" the fruit. He indicated that the ripeness of the apple is dependent on "rate and time". By rate, Baxil meant the rate at which the apple will become "ready to eat," as illustrated below:

Baxil: What I'm assuming is the rate of the apple are ready to eat because not all the apple will be always ready to eat, so I assume that, and the time will be keep increasing because if it keep too short, it's not ready. If it's right about time, it's ready, but if it's too long, it's not ready, as well, so it will be the time between ready and too ready. I would say it's same thing because it's any... It's changing maybe every minute or hours, but I think it's continuously again.

He conceived of the situation to be that not all apples will be ready to eat at the same time and the apples' readiness to eat depends on time. From above, it is evident that Baxil was able to coordinate the directions of changes of time and "readiness to eat", thus engaging in gross coordination of values. He reasoned that as time goes on the "readiness to eat" of the apple will increase and then after a certain time the "readiness to eat" will start decreasing. For Baxil, the "readiness to eat" will start decreasing because an overripe fruit cannot be consumed. This is evident in the excerpt below. He also envisioned this change happening smoothly and continuously (see Figure 1).

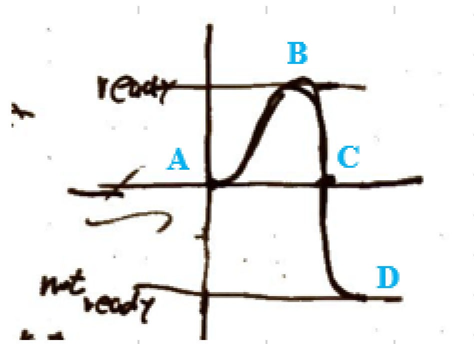


Figure 1. Baxil's graph for "readiness to eat" vs. time

Baxil: It would be negative because it's not ready to eat, so I am assuming it start at 0, and to here is ready to eat [B]. And if I'm here [C], it's not ready to, and that the graph will be from here [C] to here [D], then decreasing, I think, and it will be... because this the time those two... from here to here is ready to eat because one is increasing because giving time to ready to eat, and from here[B] to here is the time you can eat [D], I think. I'm going to assume, and then after you can eat, then you cannot eat anything because it's... I don't know what that word, but it was expired.

Here Baxil explains how the "readiness to eat" of the fruit changes with time. He was mostly involved with the gross coordination of the quantities "readiness to eat" and time and did not produce a mathematical expression that captured the dynamics of the ripening fruit. We call structures of this nature – where no more than the gross coordination of quantities is involved – as *qualitative structures*.

Emerging Quantitative Structure – Pai and Disease Transmission

After reading the Disease Transmission task, Pai immediately reasoned that the account would not change at a constant rate (linear growth) because "the value of what's going to be multiplied by 0.08 changes". This indicates that Pai envisions the amount by which the account grows each period changes. He then wrote down what the values of the account at the end of the first, second, and third year as follows: $S_1 = S_0 + 0.08(S_0)(t)$, $S_2 = S_1 + 0.08(S_1)(t)$, $S_3 = S_2 + 0.08(S_2)(t)$, where S_n is the value of the account at the end of the n^{th} compounding period and $t = 1$ year. Pai continued to reason as:

Pai: Because each year grows 8% times the initial level of the account balance, which is the prior year ending balance. Since the prior year, S_1, is greater than, essentially, S_0. S_2 is going to be taking the S_1 value and adding 8% of that value to it. It will just keep increasing…But if you take the entire account balance over time, it's going to grow at a faster and faster rate.”

As he was reasoning he drew the graph of Account balance vs time (Figure 2). Upon probing to discuss the numerical amounts of change, Pai started with \$100.00 and used his expressions from above to find the accounts’ value at the end of the first and second year as \$108, and \$116.64, respectively. Thus, Pai was coordinating the numeric amounts of change to the account balance with time. He further articulated that the change of the account size during each compounding period would continue to increase. Although Pai was able to coordinate the amounts of change of the value of the account and time, he was uncertain how to write a mathematical expression that would give the value of the account at any time t. We call structures of this nature – where a modeler has coordinated the amounts of change of the conceived quantities but hasn’t translated that into a mathematical expression yet – as *emerging quantitative structures*.

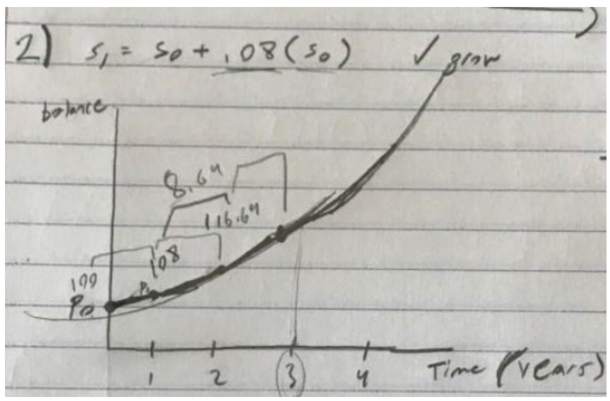


Figure 2. Pai's graph of account balance vs time in years

Quantitative Structure – Szeth and Disease Transmission

Szeth initially wrote down $P(t) = S(t)H(t)$, where $S(t)$ is the sick people, $H(t)$, is the healthy people and $P(t)$ represents the population that becomes sick. After realizing that *P(t)* and $S(t)$ measure the same thing, he changed his expression to be $S'(t) = S(t)H(t)$. When asked why he did so, he reasoned as:

Szeth: Yeah. The big, I guess, driving force was, like I said, these two variables felt like the same thing to me which then the equation doesn't make sense that way. So I was thinking of, well, should I try and change this one [pointing at $P(t)$] or should I try and change this one [pointing at $S(t)$]? So I quickly just look through the wording of the problem, and in the last sentence it says how the disease would spread through the population. So the spreading, that sounds like to me like a rate, how quickly it would spread out, slowly it spreads. So then that led me to change what the equation is equal to. It's equal to the spreading or how quickly people get sick, and then that's based on the interactions between healthy and sick.

As in the except above, Szeth was trying to determine if he should be changing $P(t)$ or $S(t)$ since having them as is doesn’t make any sense”. When re-reading the task, he realized that since he wants to know how the disease spreads, the left-hand side of the equation should be a rate rather than an amount. He then changed $P(t)$ to be $S'(t)$ because the spread of the disease is dependent upon the interactions between healthy and sick people. So, Szeth’s final answer was $S'(t) = S(t)H(t)$. When probed, he indicated that his model assumes that all healthy people who come in contact with the sick people, get sick. When the interviewer asked him how he would modify it to account for only a portion of healthy people who come in contact with the sick people will get sick, Szeth wrote $S'(t) = \alpha S(t)H(t)$ where α is the percentage of interactions that lead to people getting sick. In this scenario, Szeth was able to construct the quantity *the spread of disease* through operating on the quantities $S(t)$, $H(t)$, and α, under two different assumptions. Not only was he able to recognize the dependence among the quantities $S'(t)$, $H(t)$, $S(t)$, *P(t)*, and *α,* but he was also able to translate this dependence into a mathematical expression. We call structures of this nature – where the network of quantitative relations is translated into a mathematical expression – as *quantitative structures*.

Pseudo Quantitative Structure: Baxil and Fruit Ripening

Baxil was asked to draw a graph of the gas produced vs time. Baxil, while drawing his graph (Figure 3), reasoned as "I would say increasing slowly at the beginning, then increasing faster as they are ready to eat because after you're ready to eat, it will produce more instead it didn't ripe yet." The interviewer probed his rationale for why the ethelyne gas production would be faster as the fruit ripens. Baxil explained "When you're not ready to eat, it's just like a little bit amount of the gas, I would think, but after it's ready, it goes faster because everywhere have the gas". Baxil engaged in coordination of three interdependent quantities (amount of ethelyne gas produced, gas production, and time), while maintaining pairwise coordination between amount of gas vs. time and gas production vs. time, and production of gas vs. amount of gas. We can say that Baxil has established a qualitative structure of the situation.

Figure 3. Baxil's graph for ethelyn production vs time

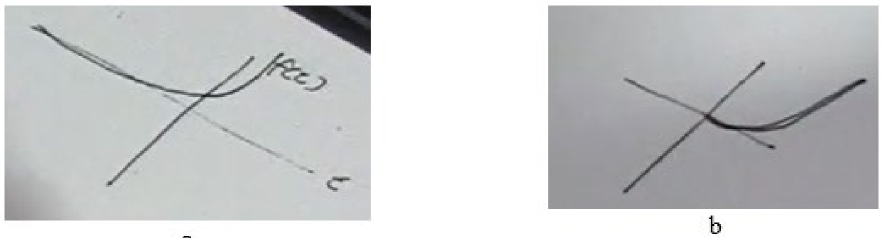
The interviewer then asked Baxil to write an expression for the amount of ethelyne gas produced. He wrote down two expressions and was trying to figure which suited the situation most.

i. Amount of gas produced by the apple which is ready to eat = $e^{rate\ of\ gas * \ time}$
ii. Amount of gas produced by the apple which is ready to eat = $rate^{\ time}$

In expression (i), Baxil conceived of rate of gas to be the "percentage of gas inside the apple". By that, Baxil meant the ripeness to ethylene conversion rate. Whereas in the second expression, he indicated that rate would be "the rate of gas that affect the (ripeness of the apple)". Baxil further indicated that the amount of gas, as represented in the first expression, would be increasing slowly. Whereas in the second expression, the amount of gas would increase quickly. This interpretation was evident in his following explanation:

Baxil: May I make an example like the raw apple there is a little bit of gas like I say 10% of them I guess, so it might be a 20% of them and the next there is something like that and there is a 40% then a 60% it doesn't add to 100% that's the second equation thinking and for the first equation I was thinking if it is 10% the rate won't be changing... I mean not the rate the like the amount then I say like its 10% it might be and depend on the tense it will be increasing by one-tenth, two-tenth, third-tenth, four-tenth... something like that.

The interviewer asked him to draw graphs of the two scenarios, he drew figure a to represent expression 1 and b to represent expression 2. While he attributed the amount of gas in the first one to be increasing slowly, and in the second to be increasing faster, he drew a steeper graph for the first one (Figure 4a).



a b

Figure 4. Baxil's graphs for ethylene production vs time

Here Baxil conceived two distinct measurable attributes of the same object, apple. One was by how much gas is produced by the apple and the other being by how much the gas affects the ripeness of the apple. As a result, he createded two expressions that, despite being mathematically equivalent, behaved different to him in terms of quantities and quantitative operations. He settled on expression (ii) as his final model because, according to him, in the second expression the gas is produced quicker which is most suited for the given situation. Baxil's expression modeling the amount of ethylene gas produced was normatively correct. However, his reasoning evidenced a few kinds of inconsistences. First, Baxil did not justify why he thought the second expression produced ethelyn gas faster than the first. Second, he produced graphs that are inconsistent with the aforementioned reasoning. We call structures of this nature – where the qualitative structure is mapped into an acceptable expression but for incorrect reasons – as a *pseudo quantitative structures*.

Discussion

In this report, we have illustrated four different kinds of structures students may establish for real-world scenarios. For completion, we suggest that it is possible that the modeler does not establish (a quantitative) structure for the real-world scenario. That is, the modeler may have conceived the quantities but has not recognized interdependencies among those quantities, thus explaining their absence from the structural network. We acknowledge that the types of structures reported in this paper are not exhaustive. In addition, the presence of these structures may be limited to students' modeling activities for dynamic systems. These distinct kinds of quantitative structures can be used as a researcher tool to describe the degree of the formality of the network of quantitative relations students established on real-world situations. These descriptions may be used to analyze where the student is in her model developing activity, with quantitative and co/multivariational reasoning as the backbone, complementing existing research on mathematical modeling processes. This in result may provide insights into how educators can guide students into creating a mathematical expression – the favorable outcome of mathematical modeling - as the mathematical representation of a real-world situation.

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