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Uniqueness of entire graphs evolving by mean curvature flow

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Abstract. In this paper we study the uniqueness of graphical mean curvature flow with locally Lipschitz initial data. We first prove that rotationally symmetric entire graphs are unique, without any further assumptions. Our methods also give an alternative simple proof of uniqueness in the one-dimensional case. In the general case, we establish the uniqueness of entire proper graphs that satisfy a uniform lower bound on the second fundamental form. The latter result extends to initial conditions that are proper graphs over subdomains of $\mathbb{R}^n$. A consequence of our result is the uniqueness of convex entire graphs, which allow us to prove that Hamilton's Harnack estimate holds for mean curvature flow solutions that are convex entire graphs.

1. Introduction

The evolution under *mean curvature flow* studies a family of immersions

$$F(\cdot, t) : M^n \to \mathbb{R}^{n+1}, \quad t \in (0, T),$$

of n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$ such that

$$\frac{\partial}{\partial t} F(p,t) = H(p,t)\nu(p,t), \quad p \in M^n \tag{1.1}$$

where $H(p,t)$ and $\nu(p,t)$ denote the mean curvature and upward pointing normal of the surface $M_t := F(M^n, t)$ at the point $F(p,t)$.

We will assume in this work that $M_t, t \in (0, T]$ is a *complete non-compact graph* over a domain $\Omega_t \subset \mathbb{R}^n$ (if $\partial\Omega_0 \neq \emptyset$, then $\partial\Omega_t$ will evolve by MCF, hence Ω_t is changing in time). Then the solution M_t can be written as $M_t = \{(x, u(x,t)) : x \in \Omega_t\}$ for a height function $u(x,t)$. In the case where $\Omega = \mathbb{R}^n$ we will say that M_t is an *entire graph*.

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The *height function* u satisfies the following quasilinear parabolic initial value problem:

$$\begin{cases} u_t = \left(\delta^{ij} - \dfrac{D^i u D^j u}{1+|Du|^2}\right) D_{ij} u, & (x,t) \in \Omega_t \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega_0, \end{cases}$$

where $M_0 := \{(x, u_0(x)) : x \in \Omega_0\}$. Here we sum over repeated indices. In what follows, we will refer to this equation as *graphical mean curvature flow.*

Although the mean curvature flow (MCF) has extensively been studied in the compact case from many points of view (such as existence and regularity, weak solutions, singularities, the extension of the flow through the singularities, flow with surgery) not much has been done in the non-compact case beyond the fundamental works by Ecker and Huisken [6,7] which deal with graphs over $\mathbb{R}^n$ and the more recent work by the second author and Schnürer [13] which deals with graphs over domains.

The works by Ecker and Huisken [6,7] establish the existence and local a'priori estimates of the graphical MCF over $\mathbb{R}^n$. Also, in [6] the uniqueness of graphical solutions is addressed in some special cases. The results in [7] show that in some sense the MCF on entire graphs behaves better than the heat equation on $\mathbb{R}^n$, namely *an entire graph solution exists for all times, independently from the growth of the initial surface at infinity*. The initial entire graph is assumed to be locally Lipschitz. Methods of similar spirit as in [7] are used by the second author and Schnürer in [13] to establish the existence of MCF solutions which are complete non-compact graphs over domains $\Omega_t \subset \mathbb{R}^n$. Note that if $\partial\Omega_0 \neq \emptyset$, then $\partial\Omega_t$ will evolve by MCF, that is in general it will change in time.

While the works [6,7] and [13] completely address the existence of classical solutions to the graphical MCF with Lipschitz continuous initial data (on $\mathbb{R}^n$ or domains), the uniqueness question in such generality has remained on open question. While the methods in [6,7] imply that polynomial growth at infinity is preserved by the flow, the question of uniqueness is not addressed in those works. In [2] the authors address uniqueness of graphs in general ambient manifolds and high co-dimension. However, their result requires a uniform bound on the second fundamental form for all times. Our goal in this work is to address the *uniqueness of classical solutions to* (1.2) under *minimal assumptions* on the behavior of the initial data $u_0(x)$ as $|x| \to +\infty$, and under *no assumptions* on the behavior of the solutions at infinity.

We will first describe our results in the case of *entire graphs*, these are Theorems 1.1–1.3. We will then state our result in the case of domains.

For the reader's convenience let us state the following *existence result* for graphical MCF over $\mathbb{R}^n$ that follows from the Ecker and Huisken works [6,7]:

Theorem. *Assume that* $u_0 : \mathbb{R}^n \to \mathbb{R}$ *is a locally Lipschitz continuous function. Then there exists a solution* $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ *of the initial value problem*

$$\begin{cases} u_t = \left(\delta^{ij} - \dfrac{D^i u D^j u}{1+|Du|^2}\right) D_{ij} u, & (x,t) \in \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

with $T = +\infty$ *which is continuous up to* $t = 0$ *and* C^∞*-smooth for* $t > 0$.

The striking feature of the result above is that existence holds for *any locally Lipschitz entire graph initial data* that is *independently from the spatial growth of the initial data* $u_0(x)$, *as* $|x| \to +\infty$. This is in contrast with the *heat equation* on $\mathbb{R}^n$, where existence is guaranteed only for initial data with at most quadratic exponential growth at infinity. The underlying reason for this difference is that the diffusion coefficient in this non-linear problem, i.e.

$$g^{ij} = \delta^{ij} - \frac{D^i u D^j u}{1 + |Du|^2},$$

becomes small in a maximal direction of the gradient when $|Du| \to +\infty$. This behavior can simply be observed in the one-dimensional case of an entire graph $u : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ evolving by curve shortening flow (CSF), where $u(x,t)$ satisfies the equation

$$u_t = \frac{u_{xx}}{1 + u_x^2} \tag{1.3}$$

or in higher dimensions under rotational symmetry, where $x_{n+1} = u(r,t)$, $r = |x|$ evolves by

$$u_t = \frac{u_{rr}}{1 + u_r^2} + \frac{n-1}{r} u_r. \tag{1.4}$$

Note that a similar phenomenon has been observed for quasilinear equations of the form

$$u_t = \Delta u^m \quad \text{on } \mathbb{R}^n \times (0, \infty) \tag{1.5}$$

in the range of exponents $\frac{(n-2)_+}{n} < m < 1$ (see [5, 10] and the references therein). In all cases above the *slow diffusion* at spatial infinity when $|Du| \to +\infty$ in (1.3) and (1.4), or $u \to +\infty$ in (1.5) prevents instant blow-up of solutions with large growing initial data as $|x| \to +\infty$.

We will see that in the *one-dimensional* case of the CSF (equation (1.3)) or the *rotationally symmetric* case of MCF (equation (1.4)) *uniqueness holds for any entire graph solution independently of its growth at infinity.* This is in sharp contrast with the *heat equation* in any dimension. More precisely, we will show the following two results. The first shows the uniqueness of entire graph solutions to CSF:

Theorem 1.1 (Uniqueness of solutions to CSF). *Let*

$$u_1, u_2 : \mathbb{R} \times (0, T] \to \mathbb{R}, \quad T > 0,$$

be two smooth solutions of equation (1.3) *with the same Lipschitz continuous initial data* u_0*, that is*

$$\lim_{t \to 0} u_1(\cdot, t) = \lim_{t \to 0} u_2(\cdot, t) = u_0.$$

Then $u_1 = u_2$ *on* $\mathbb{R} \times (0, T]$.

The second result shows the uniqueness of rotationally symmetric entire graph solutions of MCF:

Theorem 1.2 (Uniqueness of rotationally symmetric MCF solutions). *Let*

$$u_1, u_2 : \mathbb{R}^n \times (0, T] \to \mathbb{R}, \quad T > 0,$$

be two entire graph rotationally symmetric smooth solutions of (1.2) *with the same Lipschitz continuous initial data* $u_0(x)$*, that is*

$$\lim_{t \to 0} u_1(\cdot, t) = \lim_{t \to 0} u_2(\cdot, t) = u_0.$$

Then $u_1 = u_2$ *on* $\mathbb{R}^n \times (0, T]$.

We remark that Theorem 1.1 is already covered by the results in [4]. However, we provide here a simpler and more direct proof in the case of entire one-dimensional graphs, in particular pointing out the similarity with fast-diffusion. Regarding the *general case* of proper entire graphs, we establish the uniqueness under a suitable lower bound on the second fundamental form which prevents large oscillations of the solution in different directions. We then extend this result to proper graphs over subdomains $\Omega \subset \mathbb{R}^n$. We begin by recalling the following definition.

Definition 1.1 (Proper graphs over subdomains $\Omega \subset \mathbb{R}^n$). A graph

$$M := \{(x, u(x)) : x \in \Omega\}$$

over a subdomain $\Omega \subset \mathbb{R}^n$ defined by the height function $u : \Omega \to \mathbb{R}$ is said to be proper if $u(x) \to +\infty$ as $x \to \partial\Omega$ or $|x| \to +\infty$ (the latter is assumed if Ω is unbounded, in particular when $\Omega = \mathbb{R}^n$).

Let $M_t = \{(x, u(x,t)) : x \in \mathbb{R}^n\}$, $t \in (0,T)$, be a proper entire graph solution to mean curvature flow (1.1) starting at M_0, which is defined by the height function $u : \mathbb{R}^n \times (0,T) \to \mathbb{R}$. We denote by $v = \langle e_{n+1}, \nu\rangle^{-1}$ the *gradient function* of M_t, where ν denotes the inward pointing unit normal on M_t. Since M_t, $t \in (0,T)$, is assumed to be an entire graph, it follows that $\langle e_{n+1}, \nu\rangle$ has always the same sign. Furthermore, our assumption that M_t is proper, guarantees that

$$v = \langle e_{n+1}, \nu\rangle^{-1} > 0 \quad \text{on } M_t, t \in [0,T],$$

in which case $v = \sqrt{1 + |Du|}$. In our result below we will further assume that M_t satisfies the lower bound curvature condition

$$v h^i_j \geq -c\delta^i_j \quad \text{on } M_t, t \in (0,T], \tag{1.6}$$

for some uniform constant $c > 0$. Here h^i_j is the second fundamental form and in the particular case of graphs corresponds to

$$h^i_j = \left(\delta^{il} - \frac{D_i u D_l u}{1 + |Du|^2}\right) \frac{D_{lj} u}{\sqrt{1 + |Du|^2}}.$$

Our uniqueness result states as follows:

Theorem 1.3 (General uniqueness result for entire graphs). *Assume that $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function defining a proper entire graph*

$$M_0 = \{(x, u_0(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}.$$

Let $u_1, u_2 : \mathbb{R}^n \times (0,T] \to \mathbb{R}$ be two smooth solutions of (1.2) *defining two entire graph solutions $M^1_t = \{(x, u_1(x,t)) : x \in \mathbb{R}^n\}$ and $M^2_t = \{(x, u_2(x,t)) : x \in \mathbb{R}^n\}$ of MCF* (1.1) *which both satisfy condition* (1.6) *and have the same initial data u_0, that is*

$$\lim_{t \to 0} u_1(\cdot, t) = \lim_{t \to 0} u_2(\cdot, t) = u_0.$$

Then $u_1 = u_2$ on $\mathbb{R}^n \times (0,T]$, that is $M^1_t = M^2_t$ for all $t \in (0,T]$.

Remark 1.1. (i) Theorem 1.3 implies that uniqueness holds under convexity with no other growth conditions on the initial data (see in Section 5). As a consequence Hamilton's differential Harnack inequality holds for convex graphs evolving under mean curvature flow (see Corollary 5.3 in Section 5). A related result was recently discussed in [1] in the context of translating solutions.

(ii) Theorem 1.3 shows that uniqueness holds for initial data $u_0(x)$ which has arbitrarily large growth as $|x| \to +\infty$, as long as the lower curvature bound (1.6) holds.

(iii) Theorem 1.3 only assumes the lower bound (1.6) in comparison with the results in [2] which assume upper and lower bounds on the second fundamental form.

At last we will discuss the uniqueness for *graphs over subdomains* of $\mathbb{R}^n$. In that context, the result in [13] guarantees the existence of smooth solutions:

Theorem. *Let $\Omega_0 \subset \mathbb{R}^{n+1}$ be a bounded open set and $u_0 : \Omega_0 \to \mathbb{R}$ a locally Lipschitz continuous function with $u_0(x) \to \infty$ for $x \to x_0 \in \partial\Omega_0$. Then there exists $(\mathcal{D}, u)$, where $\mathcal{D} \subset \mathbb{R}^{n+1} \times [0,\infty)$ is relatively open, such that u is a solutions of the graphical mean curvature flow*

$$\begin{cases} u_t = \left(\delta^{ij} - \dfrac{D_i u D_j u}{1+|Du|^2}\right) D_{ij} u, & (x,t) \in \mathcal{D} \setminus (\Omega_0 \times \{0\}), \\ u(x,0) = u_0(x), & x \in \Omega_0. \end{cases} \tag{1.7}$$

The function u is smooth for $t > 0$ and continuous up to $t = 0$, $u(\cdot\,,0) = u_0$ in Ω_0 and $u(x,t) \to \infty$ as $(x,t) \to \partial\mathcal{D}$, where $\partial\mathcal{D}$ is the relative boundary of $\mathcal{D}$ in $\mathbb{R}^{n+1} \times [0,\infty)$.

It is relevant to remark that in this theorem the domain of definition for the function u changes in time and it is given by the mean curvature flow evolution of $\partial\Omega_0$ (see the discussion in [13]). More precisely, $u(x,t)$ is a graph over Ω_t, where $\Omega_t \times \{t\} = \mathcal{D} \cap (\mathbb{R}^n \times \{t\})$ and $\partial\Omega_t$ agrees with the evolution by mean curvature flow of $\partial\Omega_0$ at time t, provided that this evolution is smooth. In addition, it is possible to see from the proof in [13] that if $(x_k, t_k) \to (\bar{x}, \bar{t}) \in \partial\mathcal{D}$ and $|\bar{x}| \le R$ for some $R > 0$, then $u(x_k, t_k) \to \infty$.

Our uniqueness result for graphs over subdomains states as follows:

Theorem 1.4 (General uniqueness result for subdomains). *Let $\Omega_0 \subset \mathbb{R}^n$ be an open set such that $\partial\Omega_0$ has a unique smooth evolution by mean curvature flow in $(0,T]$ and let Ω_t be such that $\partial\Omega_t$ agrees with the evolution of $\partial\Omega_0$ at time t. Assume that $u_0 : \Omega_0 \to \mathbb{R}$ is a locally Lipschitz function defining a proper graph $M_0 = \{(x, u_0(x)) : x \in \Omega_0\} \subset \mathbb{R}^{n+1}$. Let $u_1, u_2 : \Omega_t \times (0,T] \to \mathbb{R}$ be two smooth solutions of* (1.7) *defining two proper graph solutions $M_t^1 = \{(x, u_1(x,t)) : x \in \Omega_t\}$ and $M_t^2 = \{(x, u_2(x,t)) : x \in \Omega_t\}$ of MCF* (1.1)*, both satisfying condition* (1.6)*, and having the same initial data u_0, that is*

$$\lim_{t\to 0} u_1(\cdot\,,t) = \lim_{t \to 0} u_2(\cdot\,,t) = u_0.$$

Assume, in addition, that if $(x_k, t_k) \to (\bar{x}, \bar{t}) \in \partial\mathcal{D}$ and $|\bar{x}| \le R$ for some $R > 0$, we have $u_i(x_k, t_k) \to \infty$. Then $u_1 = u_2$ on $\mathcal{D} = \bigcup_{t\in[0,T]} \Omega_t \times \{t\}$, that is $M_t^1 = M_t^2$ for all $t \in (0,T]$.

The organization of this paper is as follows: In Sections 2 and 3 we give the proofs of Theorems 1.1 and 1.2, respectively. Section 4 is devoted to the proofs of Theorems 1.3 and 1.4. Finally, Section 5 is devoted to the proof of Hamilton's differential Harnack inequality.

We conclude this section with the following remarks.

Remark 1.2. In Theorem 1.4, if the evolution of $\partial\Omega_0$ is not unique, it follows from the proof of the result that for each evolution Ω_t there is at most one proper graphical solution satisfying assumption (1.6).

Remark 1.3. Uniqueness for other non-compact flows has been discussed in other works. For instance, uniqueness results for complete Ricci flow are discussed in [3] and [15]. The uniqueness for complete Yamabe flow in hyperbolic space is discussed in [14].

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2. Curve shortening flow – Theorem 1.1

In this section we will show that entire graph smooth solutions to Curve Shortening Flow (that is (1.2) for $n = 1$ and $\Omega = \mathbb{R}$) are unique without any growth assumptions at spatial infinity. This result is in contrast with the case of the heat equation where at most quadratic exponential growth at infinity is required for uniqueness. As mentioned in the introduction Theorem 1.1 is already covered by the results in [4]. We provide here a simpler and more direct proof in the case of entire graphs.

The evolution of a curve $y = u(x,t)$ on the plane is given by

$$u_t = \frac{u_{xx}}{1 + u_x^2}$$

which can be also written in divergence form as

$$u_t = (\arctan(u_x))_x.$$

Differentiating in x we see that $v := u_x$ satisfies the equation

$$v_t = (\arctan v)_{xx}. \tag{2.1}$$

The proof of Theorem 1.1 will be based on the following simple observation which we prove next.

Lemma 2.1. *For any $\gamma \in (0,1]$, the following holds:*

$$(\arctan v_1 - \arctan v_2)_+ \le 2(v_1 - v_2)_+^\gamma \quad \textit{for all } v_1, v_2 \in [0, +\infty).$$

Proof. Fix a number $\gamma \in (0,1]$. We may assume that $v_1 > v_2$ and write

$$(\arctan v_1 - \arctan v_2)_+ = \int_{v_2}^{v_1} \frac{1}{1+s^2}\, ds.$$

Assume first that $v_1 > v_2 \ge 1$. In this case, for any number $\gamma \in (0,1]$ we have

$$v_1 \ge (v_1 - v_2)^{1-\gamma},$$

so that the above gives

$$\begin{aligned}(\arctan v_1 - \arctan v_2)_+ &\le \int_{v_2}^{v_1} \frac{1}{s^2}\,ds = \frac{v_1 - v_2}{v_1 v_2}\\ &\le \frac{v_1 - v_2}{v_1} \le \frac{v_1 - v_2}{(v_1 - v_2)^{1-\gamma}}\\ &\le (v_1 - v_2)_+^{\gamma}.\end{aligned}$$

In the case that $0 < v_2 < 1 < v_1$ we have

$$\begin{aligned}(\arctan v_1 - \arctan v_2)_+ &\le \int_{v_2}^{1} ds + \int_{1}^{v_1} \frac{1}{s^2}\,ds\\ &\le (1 - v_2) + \frac{v_1 - 1}{v_1}\\ &\le 2(v_1 - v_2)_+^{\gamma}\end{aligned}$$

since for any $\gamma \in (0, 1]$ we have

$$1 - v_2 < (1 - v_2)^{\gamma} < (v_1 - v_2)^{\gamma}$$

and

$$\frac{v_1 - 1}{v_1} < \frac{v_1 - v_2}{v_1} < (v_1 - v_2)^{\gamma}.$$

The last inequality follows from $v_1 > (v_1 - v_2)^{1-\gamma}$ which holds in this case. Finally, for $0 < v_2 < v_1 \le 1$, we have

$$(\arctan v_1 - \arctan v_2)_+ \le (v_1 - v_2)_+ \le (v_1 - v_2)_+^{\gamma}. \qquad \square$$

Proof of Theorem 1.1. The proof follows the method by Herrero and Pierre in [10]. Let $v_1 = u_{1x}$ and $v_2 = u_{2x}$. We will first show that $v_1 \equiv v_2$ on $\mathbb{R} \times [0, T)$. To this end, we set $w = (v_1 - v_2)_+$. Since v_1, v_2 satisfy equation (2.1), Kato's inequality implies that w satisfies the differential inequality

$$w_t \le (aw)_{xx} \quad \text{on } \mathbb{R} \times (0, T) \tag{2.2}$$

in the sense of distributions, where

$$a := \frac{(\arctan v_1 - \arctan v_2)_+}{(v_1 - v_2)_+}.$$

Our observation in Lemma 2.1 shows that for any $\gamma \in (0, 1)$ we have

$$0 \le a \le 2w^{-1+\gamma}.$$

We will use that momentarily.

Consider the test function $\varphi(x) = \psi(\frac{x}{R})$, where $\psi(\rho)$ is a smooth cut-off function supported in $(-2, 2)$ such that $0 \le \psi \le 1$, $\psi(\rho) = 1$ for $x \in [-1, 1]$. Integrating the differential inequality (2.2) against φ, we obtain

$$\frac{d}{dt}\int w(\cdot, t)\varphi\,dx \le \int (aw)(\cdot, t)\varphi''\,dx, \quad t \in (0, T).$$

For any number $\gamma \in (0,1)$ (to be fixed at the end of our proof) we use the inequality $0 \le a \le 2w^{-1+\gamma}$ to conclude

$$\frac{d}{dt}\int w\varphi\,dx \le 2\int w^{\gamma}|\varphi''|\,dx \le C\left(\int w\varphi\,dx\right)^{\gamma}\left(\int |\varphi''|^{\frac{1}{1-\gamma}}\varphi^{-\frac{\gamma}{1-\gamma}}\,dx\right)^{1-\gamma}.$$

Since $|\varphi''(x)| \le CR^{-2}|\psi''(\rho)|$, $x = R\rho$, and ψ is supported in the interval $[-2,2]$, we have

$$\int |\varphi''|^{\frac{1}{1-\gamma}}\varphi^{-\frac{\gamma}{1-\gamma}}\,dx \le CR^{1-\frac{2}{1-\gamma}}\int |\psi''|^{\frac{1}{1-\gamma}}\psi^{-\frac{\gamma}{1-\gamma}}\,d\rho.$$

For any $\gamma \in (0,1)$ we can choose cutoff $\psi = \psi_\gamma$ such that

$$\int |\psi''|^{\frac{1}{1-\gamma}}\psi^{-\frac{\gamma}{1-\gamma}}\,d\rho \le C_\gamma.$$

We conclude that $I(t) := \int w(\cdot,t)\varphi\,dx$ satisfies

$$I'(t) \le C_\gamma I(t)^{\gamma} R^{-(1+\gamma)}.$$

Integrating the last inequality on $[0,\bar t]$ for any $\bar t \in (0,T)$ while using that $\lim_{t\to 0} I(t) = 0$ (this follows from the fact that $v_1(\cdot,0) = v_2(\cdot,0)$ a.e.), we obtain

$$I(\bar t)^{1-\gamma} \le C_\gamma \bar t R^{-(1+\gamma)} \implies I(\bar t) \le C_n \bar t^{\frac{1}{1-\gamma}} R^{-\frac{1+\gamma}{1-\gamma}}.$$

Finally, recalling that $\varphi \equiv 1$ on $[-R,R]$, we get

$$\int_{-R}^{R}(v_1 - v_2)_+(x,t)\,dx \le C_n \bar t^{\frac{1}{1-\gamma}} R^{-\frac{1+\gamma}{1-\gamma}}.$$

Letting $R \to +\infty$ and using monotone convergence we conclude that

$$\int_0^{\infty}(v_1 - v_2)_+(x,t)\,dx = 0 \quad \text{for all } t \in [0,T).$$

Therefore, we conclude that $(v_1 - v_2)_+ \equiv 0$ on $[0,\infty)\times[0,t_0]$, i.e. $(u_1)_x(\cdot,t) \le (u_2)_x(\cdot,t)$ in $\mathbb{R}$. Similarly, we get $(u_2)_x(\cdot,t) \le (u_1)_x(\cdot,t)$ in $\mathbb{R}$ implying that for any $t \in [0,T)$, we have $(u_1)_x(\cdot,t) = (u_2)_x(\cdot,t)$ in $\mathbb{R}$. This and the fact that $u_1 = u_2$ at time $t = 0$ easily give us that $u_1 \equiv u_2$, finishing our proof. □

3. Rotationally symmetric solutions – Theorem 1.2

In this section we will consider the uniqueness of rotationally symmetric solutions of the initial value problem (1.2) on $\mathbb{R}^n \times (0,T)$. On a radial solution $u(r,t)$ the evolution equation in (1.2) becomes

$$u_t = \frac{u_{rr}}{1+u_r^2} + \frac{n-1}{r}u_r. \tag{3.1}$$

Differentiating (3.1) with respect to r we find that the derivative $v := u_r$ of any solution u of (1.4) satisfies the equation

$$v_t = (\arctan v)_{rr} + \left(\frac{n-1}{r}v\right)_r. \tag{3.2}$$

Proof of Theorem 1.2. The proof follows the method by Herrero and Pierre in [10] and is a generalization of the one-dimensional case with the necessary adaptations. We simply denote by $u_1(r,t), u_2(r,t)$ the rotational symmetric profiles we let $v_1 = u_{1r}$ and $v_2 = u_{2r}$. Set $w = (v_1 - v_2)_+$. Since v_1 and v_2 both satisfy (3.2), Kato's inequality implies that w satisfies

$$w_t \leq \Delta(aw) - \frac{n-1}{r}(aw)_r + \left(\frac{n-1}{r}w\right)_r \tag{3.3}$$

in the sense of distributions, where

$$a := \frac{(\arctan v_1 - \arctan v_2)_+}{(v_1 - v_2)_+}.$$

Similar to the one-dimensional case, the crucial observation is that for any $\gamma \in (0,1)$ we have $0 \leq a \leq 2w^{-1+\gamma}$.

Consider the test function

$$\varphi_R(r,t) = \psi\left(\frac{r^2 + 2(n-1)t}{R^2}\right),$$

where $\psi(\rho)$ is a smooth cut-off function defined on $[0,+\infty)$ such that $0 \leq \psi \leq 1$, $\psi(\rho) = 1$ for $0 \leq \rho \leq 1$ and $\psi(\rho) \equiv 0$ for $\rho \geq 2$. Then

$$(\varphi_R)_t = \frac{2(n-1)}{R^2}\psi', \quad (\varphi_R)_r = \frac{2r}{R^2}\psi' \implies (\varphi_R)_t = \frac{n-1}{r}(\varphi_R)_r$$

and

$$(\varphi_R)_{rr} = \frac{4r^2}{R^4}\psi'' + \frac{2}{R^2}\psi' \implies \Delta\varphi_R = \frac{4r^2}{R^4}\psi'' + \frac{2n}{R^2}\psi'.$$

Hence, using $(\varphi_R)_t = \frac{n-1}{r}(\varphi_R)_r$, we obtain

$$\begin{aligned}\frac{d}{dt}\int w\varphi_R r^{n-1}\,dr &= \int w_t\varphi_R\, r^{n-1}\,dr + \int w(\varphi_R)_t\, r^{n-1}\,dr \\ &\leq \int aw\Delta\varphi_R r^{n-1}\,dr - \int \frac{n-1}{r}(aw)_r\varphi_R r^{n-1}\,dr \\ &\quad + \int \left(\frac{n-1}{r}w\right)_r \varphi_R r^{n-1}\,dr + \int \frac{n-1}{r}w(\varphi_R)_r r^{n-1}\,dr.\end{aligned}$$

Performing integration by parts on the second and third terms, using that

$$\begin{aligned}\int \frac{n-1}{r}(aw)_r\varphi_R r^{n-1}\,dr = &-\int \frac{n-1}{r}aw(\varphi_R)_r r^{n-1}\,dr \\ &- \int \frac{(n-2)(n-1)}{r^2}aw\varphi_R r^{n-1}\,dr,\end{aligned}$$

we obtain (after cancellations) that

$$\begin{aligned}\frac{d}{dt}\int w\varphi_R\,d\mu \leq &\int aw\Delta\varphi_R r^{n-1}\,dr + \int \frac{n-1}{r}aw(\varphi_R)_r r^{n-1}\,dr \\ &+ \int \frac{(n-2)(n-1)}{r^2}aw\varphi_R r^{n-1}\,dr \\ &- \int \frac{(n-1)^2}{r^2}w\varphi_R r^{n-1}\,dr.\end{aligned} \tag{3.4}$$

Next notice that

$$a := \frac{(\arctan v_1 - \arctan v_2)_+}{(v_1 - v_2)_+} = \frac{1}{1+\bar{v}^2}$$

for some $\bar{v}$ between v_1 and v_2, hence $a \le 1$. It follows that

$$\begin{aligned}\int \frac{(n-2)(n-1)}{r^2} a w \varphi_R r^{n-1}\, dr - \int \frac{(n-1)^2}{r^2} w \varphi_R r^{n-1}\, dr \\ \le - \int \frac{n-1}{r^2} w\varphi_R r^{n-1}\, dr \le 0.\end{aligned}$$

Let $\gamma \in (0,1)$ be any number (to be chosen at the end of our proof) and use the inequality $0 \le a \le 2w^{-1+\gamma}$ shown in Lemma 2.1 to bound the first two terms on the right-hand side of (3.4). We conclude that

$$\begin{aligned}&\frac{d}{dt}\int w\varphi_R r^{n-1}\, dr \\ &\le C\int w^{\gamma}\big(|\Delta\varphi_R| + |(n-1)r^{-1}(\varphi_R)_r|\big) r^{n-1}\, dr \\ &\le C\Big(\int w\varphi_R r^{n-1}\, dr\Big)^{\gamma}\Big(\int \big(|\Delta\varphi_R| + |r^{-1}(\varphi_R)_r|\big)^{\frac{1}{1-\gamma}} \varphi_R^{-\frac{\gamma}{1-\gamma}} r^{n-1}\, dr\Big)^{1-\gamma}.\end{aligned}$$

Observing that for $0 \le t \le t_0$ and $R \gg 1$ large we have

$$|\Delta\varphi_R(r,t)| + |r^{-1}(\varphi_R)_r(r,t)| \le C_n R^{-2}\big(|\psi''(\rho)| + |\psi'(\rho)\big),$$

where $\rho := \frac{r^2+2(n-1)t}{R^2}$, we get

$$\begin{aligned}&\Big\{\int \big(|\Delta\varphi_R(r,t)| + |r^{-1}(\varphi_R)_r(r,t)|\big)^{\frac{1}{1-\gamma}} \varphi_R(r)^{-\frac{\gamma}{1-\gamma}} r^{n-1}\, dr\Big\}^{1-\gamma} \\ &\quad \le R^{-2}\Big\{\int \big(|\psi''(\rho)| + |\psi'(\rho)\big)^{\frac{1}{1-\gamma}} \psi(\rho)^{-\frac{\gamma}{1-\gamma}} r^{n-1}(\rho)\, dr(\rho)\Big\}^{1-\gamma},\end{aligned}$$

where $r^2(\rho) = R^2\rho - 2(n-1)t$, which in particular implies $r\, dr = \frac{R^2}{2}\, d\rho$. Thus,

$$\begin{aligned}&\int \big(|\psi''(\rho) + |\psi'(\rho)|\big)^{\frac{1}{1-\gamma}} \psi(\rho)^{-\frac{\gamma}{1-\gamma}} r^{n-1}(\rho)\, dr(\rho) \\ &= \frac{R^2}{2}\int \big(|\psi''(\rho) + |\psi'(\rho)|\big)^{\frac{1}{1-\gamma}} \psi(\rho)^{-\frac{\gamma}{1-\gamma}} \big(R^2\rho - 2(n-1)t\big)^{\frac{n-2}{2}}\, d\rho \\ &\le C_n R^n \int \big(|\psi''(\rho) + |\psi'(\rho)|\big)^{\frac{1}{1-\gamma}} \psi(\rho)^{-\frac{\gamma}{1-\gamma}}\, d\rho\end{aligned}$$

where we have used the fact that on the support of ψ' and ψ'' where $\rho \le 2$, and for $0 \le t \le t_0$ and $R \gg \max(1, t_0)$, one has

$$\big(R^2\rho - 2(n-1)t\big)^{\frac{n-2}{2}} \le C_n R^{n-2}.$$

For any $\gamma \in (0,1)$ we can choose cutoff $\psi = \psi_\gamma$ for which the support of ψ' and ψ'' lies in $[1,2]$ such that

$$\int_1^2 \big(|\psi''(\rho) + |\psi'(\rho)|\big)^{\frac{1}{1-\gamma}} \psi(\rho)^{-\frac{\gamma}{1-\gamma}}\, d\rho \le C(n,\gamma).$$

We then conclude from the above discussion that $I(t) := \int w\varphi_R r^{n-1}\,dr$ satisfies

$$I'(t) \le C(n,\gamma) I(t)^\gamma R^{-2+n(1-\gamma)}.$$

Since $\gamma \in (0,1)$ can be any number, we may choose $\gamma = \gamma(n) \in (0,1]$ so that $n(1-\gamma) < 2$, and integrating the last inequality on $[0,\bar t]$ for any $\bar t \in (0,T)$ while using that $I(0) = 0$, we obtain

$$I(\bar t)^{1-\gamma} \le C_n \bar t R^{-2+n(1-\gamma)} \implies I(\bar t) \le C_n \bar t^{\frac{1}{1-\gamma}} R^{n-\frac{2}{1-\gamma}}.$$

Finally, recalling that $\varphi_R \equiv 1$ on $[0,R]$, we get

$$\int_0^R (v_1 - v_2)_+(r,t) r^{n-1}\,dr \le R^{n-\frac{2}{1-\gamma}}.$$

Letting $R \to +\infty$, using that $n - \frac{2}{1-\gamma} < 0$, and monotone convergence yields

$$\int_0^R (v_1 - v_2)_+(\cdot,t) r^{n-1}\,dr = 0 \quad \text{for all } t \in [0,T).$$

Therefore, we conclude that $(v_1 - v_2)_+ \equiv 0$ on $\mathbb{R}^n \times [0,T)$, i.e. $(u_1)_r \le (u_2)_r$. Similarly, $(u_2)_r \le (u_1)_r$ a.e. in $\mathbb{R}^n \times [0,T)$ implying that $(u_2)_r \equiv (u_1)_r$. This and the fact that $u_1 \equiv u_2$ at time $t = 0$ easily give us that $u_1 \equiv u_2$ on $\mathbb{R}^n \times [0,t_0]$ for all $t_0 < T$, finishing our proof. □

4. The general case

Our goal in this section is to give the proof of our general uniqueness results, Theorem 1.3 and Theorem 1.4. We will see that the proof of the latter theorem is almost identical to the proof of the former. Hence, we will omit most of the proof of Theorem 1.4, pointing out only the minor differences.

For the sake of completeness we show next that for entire graphs the condition $u_0 \ge C$ is preserved under the flow, which implies that if the initial condition is a proper entire graph, then the solution is proper as well, uniformly in time. Both facts will be used our proofs. Because we are dealing with non-compact solutions, we will use the localization techniques developed in [7].

Lemma 4.1. *Let u be a solution to* (1.2) *on $\mathbb{R}^n \times (0,T)$ and assume that $u_0(x) \ge C$ on $|\mathbf{x} - \mathbf{x}_0| \le R$, $\mathbf{x} = (x, u_0(x))$, for some fixed point $\mathbf{x}_0 \in \mathbb{R}^{n+1}$ and some number $R > 1$. Then we have*

$$u(x,t) \ge C - \frac{10}{R} t$$

on the parabolic ball $|\mathbf{x} - \mathbf{x}_0|^2 + 2nt \le \frac{R^2}{2}$, $\mathbf{x} = (x, u(x,t))$ (provided it is non-empty). In particular, if $u_0 \ge C$ on $\mathbb{R}^n$, then for every $t \in (0,T)$ we have $u(\cdot,t) \ge C$ on $\mathbb{R}^n$.

Proof. We will do all calculations in geometric coordinates, that is we assume that our solutions are given by the embedding $\mathbf{x} = F(p,t)$ as in (1.1) and we define

$$U_R(p,t) := (u - C)\left(1 - \frac{|\mathbf{x} - \mathbf{x}_0|^2 + 2nt}{R^2}\right)_+ + \frac{5}{R} t,$$

where $u := \langle F, e_{n+1}\rangle$ and $\mathbf{x} = F(p,t)$. Our assumption $u_0 \geq C$ in $B_R(x_0)$ gives $U_R \geq 0$ at $t = 0$. Furthermore,

$$(U_R)_t - \Delta U_R = 2\nabla u \cdot 2\frac{(\mathbf{x}-\mathbf{x}_0)^T}{R^2} + \frac{5}{R} \geq -\frac{4}{R} + \frac{5}{R} > 0.$$

The maximum principle implies that U_R does not have any interior minima and $U_R \geq 0$. In particular, if $|\mathbf{x}-\mathbf{x}_0|^2 + 2nt \leq \frac{1}{2}R^2$, then

$$0 \leq \frac{u-C}{2} + \frac{5}{R}t,$$

and the first result follows.

In the case where $u_0 \geq C$ globally on $\mathbb{R}^n$, then for any $x_0 \in \mathbb{R}^n$, $t \in (0,T)$, we apply the above result taking $\mathbf{x}_0 = (x_0, u_0(x))$ and choosing $R \gg 1$ so that $|\mathbf{x}-\mathbf{x}_0|^2 + 2nt \leq \frac{1}{2}R^2$ if $\mathbf{x} = (x_0, u(x_0,t))$. We readily conclude that $u(x_0,t) \geq C - \frac{10}{R}t$ and by taking $R \to \infty$, we obtain $u(x_0,t) \geq C$. Since $x_0 \in \mathbb{R}^n$ and $t \in (0,T)$ are arbitrary, the second result follows. □

Corollary 4.2. *Let u be a solution to* (1.2) *on* $\mathbb{R}^n \times (0,T]$ *and assume that*

$$\lim_{|x|\to+\infty} u_0(x) = +\infty.$$

Then we have

$$\lim_{|x|\to+\infty} u(x,t) = +\infty \quad \textit{uniformly in } t \in (0,T].$$

Proof. We begin by observing that our assumption $\lim_{|x|\to+\infty} u_0(x) = +\infty$ implies that $u_0 \geq C$ for some $C \in \mathbb{R}$ and hence by the previous lemma, $u \geq C$ as well.

Now, for every $k \gg 1$ let $R_k > k$ be a sufficiently large number so that $u_0(x) \geq k$ for $|x| \geq R_k$. For any $x_0 \in \mathbb{R}^n$ such that $|x_0| > 4R_k$, let $\mathbf{x}_0 = (x_0, 0)$. Then

$$u_0(x) \geq k \quad \text{on } |\mathbf{x}-\mathbf{x}_0| \leq 2R_k,\ \mathbf{x} = (x, u_0(x))$$

and hence, by the previous lemma, for any $t \in (0,T)$, we have

$$u(x,t) \geq k - \frac{5}{R_k}t \quad \text{on } |\mathbf{x}-\mathbf{x}_0|^2 + 2nt \leq 4R_k^2, \mathbf{x} = (x, u(x,t)).$$

We may choose $k, R_k \gg 1$ so that $2nT < R_k^2$ and $\frac{5}{R_k}T < 1$. Evaluating the above estimate at $\mathbf{x} = (x_0, u(x_0,t))$, for any $t \in (0,T)$, it gives us that

$$u(x_0,t) \geq k-1 \quad \text{provided } |\mathbf{x}-\mathbf{x}_0| = |u(x_0,t)| \leq R_k.$$

We conclude that for any $|x_0| \geq 4R_k$ and $t \in (0,T)$ we either have that $u(x_0,t) \geq k-1$ or $|u(x_0,t)| \geq R_k$. Since $u \geq C$ (be our initial observation) and $R_k \geq k$, we conclude that in either case $u(x_0,t) \geq k-1$, for all $t \in (0,T)$ and all $|x_0| \geq 4R_k$. Since R_k is independent of t, the result readily follows. □

One may ask whether condition (1.6) is preserved in time, namely if $vh_i^j \geq -c\delta_i^j$ at time $t = 0$ implies that $vh_i^j \geq -c\delta_i^j$ for $t > 0$. Although this is easy to verify for the evolution of compact manifolds, in the non-compact setting it becomes challenging. Actually, even the case where $c = 0$ is not known to hold in the general graphical non-compact setting. In the lemma below we show that the condition is preserved under a suitable polynomial growth condition on the solution (which is expected to be preserved by the flow from the results in [6]).

Lemma 4.3. *Assume that $v\, h_{ij} \geq -c g_{ij}$ at time $t = 0$ for some constant $c > 0$ and that for all times we have $(|h_{ij}\xi^i\xi^j|\, v)(x) \leq C|\mathbf{x}|^q$ for any unit vector field ξ and that $\frac{|\nabla v|}{v} \leq C|\mathbf{x}|$. Then condition* (1.6) *holds for every $t \geq 0$.*

Proof. Let $\gamma = |\mathbf{x}|^2 + 2nt + 1$ and $p > q$ (for instance $p = q+1$) and define the tensor

$$f_{ij} = e^{-Kt}\gamma^{-p}(h_{ij}\, v + c g_{ij}),$$

where K is a constant that will be chosen later. From our assumption $f_{ij} \to 0$ as $|x| \to \infty$ (in the sense that $F_{ij}\xi^i\xi^j \to 0$ for every unit vector field ξ). Note in addition that the tensor $f_{ij} \geq 0$ if and only if $f_{ij}\, v + c g_{ij} \geq 0$.

We will use the tensorial maximum principle to prove the results (see [8] for example), hence the equations below are stated in the tensorial sense.

Following [6] and [11], we compute the evolution of f_{ij}. We obtain

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) f_{ij} = -\frac{2}{v}\langle \nabla f_{ij}, \nabla v\rangle - 2H h_{il} g^{lm} f_{ij} \tag{4.1}$$
$$- f_{ij}[K + p(p+1)|x^T|^2\gamma^{-2} + 2v^{-1}\gamma^{-1}\langle x^T, \nabla v\rangle.$$

Let $\epsilon > 0$ and define $\widetilde{f_{ij}} = f_{ij} + \epsilon g_{ij}$. From our assumption at infinity, we have, for every $\epsilon > 0$ and $|x|$ sufficiently large, $\widetilde{f_{ij}}\xi^i\xi^j > \frac{\epsilon}{2}$. In addition, from (4.1) we have

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) \widetilde{f_{ij}} = -\frac{2}{v}\langle \nabla \widetilde{f_{ij}}, \nabla v\rangle - 2H h_{il} g^{lm} \widetilde{f_{ij}} \tag{4.2}$$
$$- f_{ij}[K + p(p+1)|x^T|^2\gamma^{-2} + 2v^{-1}\gamma^{-1}\langle x^T, \nabla v\rangle].$$

Assume that there is a first time $\bar{t}$ such $\inf_{x\in\mathbb{R}^n, \xi\in\mathbb{R}^n} \widetilde{f_{ij}}\xi^i\xi^j = 0$. Form our assumption at infinity, this implies that there is a $\bar{x}$ such that the tensor $\widetilde{f_{ij}}$ has a null-eigenvector, that we denote by ξ. Following [8], we may extend ξ in a neighborhood of $\bar{x}$ such that ξ^m is independent of t, $D_l\xi^m(\bar{x}) = 0$ and $f_{ij}\xi^j = -\epsilon\xi^i$. Then at $\bar{x}$ it holds $0 = \nabla \widetilde{f_{ij}} = (\nabla f_{ij})\xi^i\xi^j$ and $0 \leq \Delta_{M_t}(\widetilde{f_{ij}}\xi_i\xi^j) = (\Delta_{M_t} f_{ij})\xi^i\xi^j$. Note in addition that $f_{ij}\xi^i\xi^j$ (with ξ as above) attains a minimum at $\bar{x}$. Combined with (4.2) we have, at $\bar{x}$, that

$$0 \geq \epsilon[K + p(p+1)|x^T|^2\gamma^{-2} + 2v^{-1}\gamma^{-1}\langle x^T, \nabla v\rangle].$$

From our definition of γ and our growth assumption we have

$$\left|p(p+1)|x^T|^2\gamma^{-2} + 2v^{-1}\gamma^{-1}\langle x^T, \nabla v\rangle\right| \leq p(p+1) + 2C,$$

where C is the constant of our assumption on ∇v. Hence, by choosing K large enough, we get

$$K + p(p+1)|x^T|^2\gamma^{-2} + 2v^{-1}\gamma^{-1}\langle x^T, \nabla v\rangle > 0,$$

which is a contradiction. □

Remark 4.1. Note that $\frac{|\nabla v|}{v} \leq |A|v$. Then the results in [6, 7] imply that if $|A|v \leq |\mathbf{x}|$ holds at $t = 0$, then this is preserved in time and the condition of our lemma is met with $q = 1$.

Remark 4.2. The previous lemma implies that lower bounds on the mean curvature are also preserved under the growth conditions of Lemma 4.3.

4.1. Proof of Theorem 1.3.

Proof. To simplify the notation in this proof, we denote $u = u_1$ and $\bar{u} = u_2$, that is we assume that $u, \bar{u} : \mathbb{R}^n \times (0, T] \to \mathbb{R}$ are the two smooth solutions to (1.2) with initial data u_0 as in the statement of Theorem 1.3. Since u_0 is proper we have $u_0 \geq -C$ for some constant $C > 0$. Hence, by adding on u_0 the constant $C + 1$, we may assume without out loss of generality that $u_0 \geq 1$. Lemma 4.1 implies that

$$u, \bar{u} \geq 1 \quad \text{on } \mathbb{R}^n \times (0, T].$$

To show that $\bar{u} = u$, it is sufficient to prove that $\bar{u} \leq u$, since the same argument will also imply that $u \leq \bar{u}$, thus showing that $u = \bar{u}$.

The solutions $u, \bar{u}$ satisfy the equations

$$u_t = \left(\delta^{ij} - \frac{D_i u D_j u}{1 + |Du|^2}\right) D_{ij} u, \quad \bar{u}_t = \left(\delta^{ij} - \frac{D_i \bar{u} D_j \bar{u}}{1 + |D\bar{u}|^2}\right) D_{ij} \bar{u}.$$

Set

$$a_{ij} := \delta^{ij} - \frac{D_i u D_j u}{1 + |Du|^2}, \quad \bar{a}_{ij} := \delta^{ij} - \frac{D_i \bar{u} D_j \bar{u}}{1 + |D\bar{u}|^2}$$

and define

$$w := u - \bar{u}.$$

Then, subtracting the above equations, we find that the function w satisfies the equation

$$w_t - a_{ij} D_{ij} w = (a_{ij} - \bar{a}_{ij}) D_{ij} \bar{u} \tag{4.3}$$

The main idea in the proof is to introduce the *supersolution*

$$\zeta(x, t) := \epsilon(t + \epsilon) u^2(x, t)$$

for any given $\epsilon > 0$ small. At the end we will let $\epsilon \to 0$. First, we use $u_t - a_{ij} D_{ij} u = 0$ and find that ζ satisfies

$$\zeta_t - a_{ij} D_{ij} \zeta = -2\epsilon(t + \epsilon) a_{ij} D_j u D_i u + \epsilon u^2,$$

where

$$a_{ij} D_j u D_i u = \left(\delta^{ij} - \frac{D_i u D_j u}{1 + |Du|^2}\right) D_i u D_j u = \delta^{ij} D_i u D_j u - \frac{(D_i u)^2 (D_j u)^2}{1 + |Du|^2}$$
$$= |Du|^2 \big(1 - \frac{|Du|^2}{1 + |Du|^2}\big) = \frac{|Du|^2}{1 + |Du|^2}.$$

Combining the above gives

$$\zeta_t - a_{ij} D_{ij} \zeta = -2\epsilon(t + \epsilon) \frac{|Du|^2}{1 + |Du|^2} + \epsilon u^2 \geq \epsilon(u^2 - 2(t + \epsilon)).$$

Since $u \geq 1$, we conclude that for $t \leq \frac{1}{4}$ and $\epsilon < \frac{1}{10}$, we have

$$\zeta_t - a_{ij} D_{ij} \zeta > \frac{\epsilon}{2} u^2. \tag{4.4}$$

Set next

$$W := w - \zeta = u - \bar{u} - \epsilon(t+\epsilon)u^2.$$

By (4.3) and (4.4) we find that W satisfies

$$W_t - a_{ij} D_{ij} W < (a_{ij} - \bar{a}_{ij}) D_{ij} \bar{u} - \frac{\epsilon}{2} u^2. \tag{4.5}$$

Our assumption that $u = \bar{u}$ at $t = 0$ (in the sense that $\lim_{t\to 0}[u(\cdot,t) - \bar{u}(\cdot,t)] = 0$) yields

$$\lim_{t\to 0} W(x,t) = -\epsilon^2 u(x,0) \le -\epsilon^2 < 0 \quad \text{uniformly on any } K \subset \mathbb{R}^n \text{ compact.} \tag{4.6}$$

(The uniform convergence on compact sets follows from the bounds in [7] which give us local bounds on the second fundamental form $|A| \le \frac{C}{\sqrt{t}}$ for both solutions $u, \bar{u}$ where C depends on the initial data.)

Let

$$T^* = \min\left(T, \frac{1}{4}, \frac{1}{10c}\right),$$

where c is the constant in (1.6). We will use (4.5)–(4.6) and the maximum principle to conclude that $W \le 0$ for all $t \in [0, T^*]$. To this end, observe first that $u, \bar{u} \ge 1$ implies that for every fixed $\epsilon > 0$ and for all $t \in (0, T)$,

$$m^* := \sup_{(x,t)\in\mathbb{R}^n\times(0,T^*]} W(x,t) \le \frac{1}{\epsilon^2}. \tag{4.7}$$

Indeed, notice that if there is a point $(x,t) \in \mathbb{R}^n \times (0, T^*]$ where $W(x,t) \ge 0$, then since $\bar{u} \ge 1$, at such a point we have $u \ge \bar{u} + \epsilon(t+\epsilon)u^2 \ge \epsilon^2 u^2$, that is $u(x,t) \le \epsilon^{-2}$. Hence, we obtain $W(x,t) \le u(x,t) \le \epsilon^{-2}$ and the same holds for the supremum m^*.

Claim 4.1. *We have*

$$m^* := \sup_{(x,t)\in\mathbb{R}^n\times(0,T^*]} W(x,t) \le 0$$

provided that ϵ is sufficiently small.

Once this claim is shown, the theorem will follow by simply letting $\epsilon \to 0$ to show that $u \le \bar{u}$ and then switching the roles of u and $\bar{u}$.

Proof of Claim 4.1. To prove the claim, we assume by contradiction that

$$m^* > 0.$$

Since $\lim_{|x|\to+\infty} u(x,t) = +\infty$ uniformly in $[0, T]$ and $\bar{u} \ge 1$, the supremum m^* cannot be attained at infinity. Hence, we have

$$m^* = W(x_{\max}(t_0), t_0)$$

for some point $t_0 \in (0, T^*]$ and $x_{\max}(t_0) \in \mathbb{R}^n$. Then at such point

$$(1 - \epsilon(t_0+\epsilon)u)u = \bar{u} + m^* \quad \text{and} \quad (1 - 2\epsilon(t_0+\epsilon)u) D_i u = D_i \bar{u} \tag{4.8}$$

Note that the first equality, $m^* > 0$ and $u, \bar{u} \geq 1$ imply that $1 - \epsilon(t_0 + \epsilon)u > 0$ at the maximum point, which will be used below. We will now use the second equality in (4.8) to evaluate the right-hand side of (4.5) at the maximum point. First, we have

$$
\begin{aligned}
(4.9) \qquad a_{ij} - \bar{a}_{ij} &= \frac{D_i \bar{u} D_j \bar{u}}{1 + |D\bar{u}|^2} - \frac{D_i u D_j u}{1 + |Du|^2} \\
&= (1 - 2\epsilon(t_0 + \epsilon)u)^2 \frac{D_i u D_j u}{1 + |D\bar{u}|^2} - \frac{D_i u D_j u}{1 + |Du|^2} \\
&= \frac{D_i u D_j u}{(1 + |Du|^2)(1 + |D\bar{u}|^2)} \\
&\qquad \cdot [(1 - 2\epsilon(t_0 + \epsilon)\, u)^2 (1 + |Du|^2) - (1 + |D\bar{u}|^2)] \\
&= -4\epsilon(t_0 + \epsilon)u(1 - \epsilon(t_0 + \epsilon)u) \frac{D_i u D_j u}{(1 + |Du|^2)(1 + |D\bar{u}|^2)}.
\end{aligned}
$$

To derive the last equality we used $(1 - 2\epsilon(t_0 + \epsilon)\, u)^2\, |Du|^2 = |D\bar{u}|^2$ which gave us

$$
\begin{aligned}
(1 - 2\epsilon(t_0 + \epsilon)u)^2 (1 + |Du|^2) - (1 + |D\bar{u}|^2) &= (1 - 2\epsilon(t_0 + \epsilon)u)^2 - 1 \\
&= -4\epsilon(t_0 + \epsilon)u\big(1 - \epsilon(t_0 + \epsilon)u\big).
\end{aligned}
$$

Combining the above with (4.5), we find that at the point $(x_{\max}(t_0), t_0)$ we have

$$
\begin{aligned}
(4.10) \qquad 0 &\leq W_t - a_{ij} D_{ij} W \\
&< -4\epsilon(t_0 + \epsilon)u(1 - \epsilon(t_0 + \epsilon)u) \frac{D_{ij}\bar{u} D_i u D_j u}{(1 + |Du|^2)(1 + |D\bar{u}|^2)} - \frac{\epsilon}{2} u^2.
\end{aligned}
$$

We next use the lower bound on the second fundamental form in (1.6) which implies that

$$
\bar{v} \bar{h}^i_j D_i u D_j u \geq -c|Du|^2.
$$

On the other hand, since

$$
\bar{h}^i_j = \frac{D_{ij}\bar{u}}{\sqrt{1 + |D\bar{u}|^2}} - \frac{D_{lj}\bar{u} D_l \bar{u} D_i \bar{u}}{(1 + |D\bar{u}|^2)^{\frac{3}{2}}},
$$

it follows that at the maximum point $(x_{\max}(t_0), t_0)$ we have

$$
\begin{aligned}
\bar{v} \bar{h}^i_j D_i u D_j u &= \left(D_{ij}\bar{u} - \frac{D_{lj}\bar{u} D_l \bar{u} D_i \bar{u}}{1 + |D\bar{u}|^2} \right) D_i u D_j u \\
&= D_{ij}\bar{u}\; D_i u D_j u - \langle D\bar{u}, Du\rangle \frac{D_{ij}\bar{u}}{1 + |D\bar{u}|^2} D_i \bar{u} D_j u \\
&= \big(1 + |D\bar{u}|^2 - (1 - 2\epsilon(t_0 + \epsilon)\, u)^2 |Du|^2\big) \frac{D_{ij}\bar{u}}{1 + |D\bar{u}|^2} D_i u D_j u \\
&= \frac{D_{ij}\bar{u} D_i u D_j u}{1 + |D\bar{u}|^2}.
\end{aligned}
$$

Combining the last two formula gives

$$
\frac{D_{ij}\bar{u} D_i u D_j u}{1 + |D\bar{u}|^2} = \bar{h}^i_j \bar{v} D_i u D_j u \geq -c|Du|^2.
$$

Inserting this bound in (4.10) implies that at the point $(x_{\max}(t_0), t_0)$ we have

$$0 \le W_t - a_{ij} D_{ij} W < 4\epsilon c(t_0+\epsilon)u(1-\epsilon(t_0+\epsilon)u)\frac{|Du|^2}{1+|Du|^2} - \frac{\epsilon}{2}u^2 \le 4\epsilon c(t_0+\epsilon)u(1-\epsilon(t_0+\epsilon)u) - \frac{\epsilon}{2}u^2. \tag{4.11}$$

We conclude from (4.11) that at the maximum point $(x_{\max}(t_0), t_0)$,

$$4\epsilon c(t_0+\epsilon)u(1-\epsilon(t_0+\epsilon)u) - \frac{\epsilon}{2}u^2 > 0,$$

that is

$$u < 8c(t_0+\epsilon)(1-\epsilon(t_0+\epsilon)u) < 8c(t_0+\epsilon),$$

since $u > 0$. Then $u \ge 1$ yields that $t_0 + \epsilon > \frac{1}{8c}$, where c is the constant from equation (1.6). Since we have assumed that $t_0 \in (0, T^*]$ and $T^* \le \frac{1}{10c}$, we derive a contradiction by choosing ϵ sufficiently small. This shows that, contrary to our assumption, $W^*(t_0) < 0$, finishing the proof of the claim. □

We have just seen that $W := u - \bar{u} - \epsilon(t+\epsilon)u^2 \le 0$ on $\mathbb{R}^n \times (0, T^*]$. Let $\epsilon \to 0$ to obtain that $u \le \bar{u}$ on $\mathbb{R}^n \times (0, T^*]$. Similarly, $\bar{u} \le u$ on the same interval, which means that $u = \bar{u}$. By repeating the same proof starting at $t = T^*$, we conclude after finite many steps that $u \equiv \bar{u}$ on $\mathbb{R}^n \times (0, T)$, finishing the proof of the theorem. □

4.2. Proof of Theorem 1.4.

Proof. The proof of Theorem 1.4 is very similar to that of Theorem 1.3. We briefly outline it in what follows. As before, let $u, \bar{u} : \mathcal{D} := \bigcup_{t\in(0,T]}(\Omega_t \times \{t\}) \to \mathbb{R}$ be the two smooth solutions to (1.7) with initial data u_0 as in the statement of Theorem 1.4 (as above, we simplify the notation by calling $u = u_1$ and $\bar{u} = u_2$). Our assumption that u_0 is proper implies that $u_0 \ge -C$ for some constant $C > 0$ and hence Lemma 4.1 implies that $u, \bar{u} \ge -C$, for $t > 0$ (possibly for a different constant $C > 0$ which is uniform in t for $t < \min(1, T)$, where T is the maximal existence time). By adding on both solutions the constant $C + 1$, we may assume that $u, \bar{u} \ge 1$. As in the proof of Theorem 1.3, we take

$$W := w - \zeta - \epsilon = u - \bar{u} - \epsilon(t+\epsilon)u^2.$$

Let $m^* := \sup_{(x,t)\in\mathcal{D}} W(x,t)$ and assume that $m^* > 0$.

We first remark that Lemma 4.1 and Corollary 4.2 can directly be extended to estimate the infimum of u in $\mathcal{D} \cap B_R(x_0)$ (instead of $\mathbb{R}^n \cap B_R(x_0)$). Hence we have that if u_0 is proper, then $u(x,t) \to \infty$ uniformly in t as $|x| \to \infty$.

Let (x_k, t_k) be a sequence of points in $\mathcal{D}$ such that $W(x_k, t_k) \to m^*$. Note that from our definition and the previous remark we have that if $t_k \to \bar{t}$ and either $x_k \to \partial\Omega_{\bar{t}}$ or $|x_k| \to +\infty$, then $u(x_k, t_k) \to \infty$ and $W \to -\infty$. Hence, we may assume that the supremum of W is attained in the interior of $\Omega_{\bar{t}}$. Now we conclude the desired result by following the proof of Theorem 1.3. □

4.3. Extension of uniqueness for entire graphs (not necessarily proper). In this subsection we provide extensions to our result in Theorem 1.3. We will consider graphical solutions that are not necessarily proper, but their initial height function u_0 and its gradient function v_0 satisfy the following assumption:

$$\text{for every } M \text{ there is a constant } c(M) \text{ such that } \sup_{\{x : u_0(x)) < M\}} v_0 \leq c(M). \tag{4.12}$$

This condition can be understood as excluding oscillatory behavior in the set where the height function u_0 is bounded at the initial time. Then our result states as follows:

Theorem 4.4. *Assume that $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function (not necessarily proper) defining an entire graph hypersurface $M_0 = \{(x, u_0(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ whose height function u_0 is bounded from below and also satisfies condition* (4.12)*. Let*

$$u_1, u_2 : \mathbb{R}^n \times (0, T] \to \mathbb{R}$$

be two smooth solutions of (1.2) *defining two entire graph solutions*

$$M_t^1 = \{(x, u_1(x,t)) : x \in \mathbb{R}^n\} \quad \textit{and} \quad M_t^2 = \{(x, u_2(x,t)) : x \in \mathbb{R}^n\}$$

of MCF (1.1) *satisfying condition* (1.6) *and having the same initial data u_0, that is*

$$\lim_{t \to 0} u_1(\cdot, t) = \lim_{t \to 0} u_2(\cdot, t) = u_0.$$

Then $u_1 = u_2$ on $\mathbb{R}^n \times (0, T]$, that is $M_t^1 = M_t^2$ for all $t \in (0, T]$.

We will first show that condition (4.12) is preserved in time and that implies uniform local bounds for the second fundamental form on the set where $\{u \leq M\}$ (these bounds depend only on M).

Proposition 4.5. *Assume that $u \geq 0$ is a smooth solution of* (1.2) *with initial data u_0 and that* (4.12) *holds. Then:*

(i) *$(M - u)_+^2 v \leq M^2 \, c(M)$ holds for all $t \in (0, T]$.*

(ii) *If we further assume that $|A|^2(x, 0) \leq c(M)$ in the set $\{x : u_0(x) \leq M\}$ (without loss of generality we can take $c(M)$ to be the same as in* (4.12)*), then*

$$|A|^2 (M - u)_+^2 \leq \max\{c(M) M^2, k^{-1}(3 + k^{-1}) M\}, \tag{4.13}$$

where $k = \frac{1}{2 M^2 c(M)}$.

(iii) *Without any assumption on the second fundamental form at the initial time, we have instead*

$$t |A|^2 (M - u)^2 (x, t) \leq k^{-1}(3 + k^{-1} v^{-2}) T (2M + 1)^2 + M^2 \tag{4.14}$$

if $0 \leq t \leq T$, $u(x, t) \leq M$ and $k = \frac{1}{2M^2 c(M)}$.

Proof. (i) Consider the cut-off function (in terms of both u and $\mathbf{x}$) given by

$$\eta_R(x, t) = \left((M - u)_+ \left(1 - \frac{|\mathbf{x}|^2 + 2nt}{R^2} \right)_+ - \frac{4}{R} t \right)_+. \tag{4.15}$$

A direct calculation shows that

$$(\eta_R)_t - \Delta\eta_R = \frac{2}{R^2}\langle\nabla u, \nabla|\mathbf{x}|^2\rangle - \frac{4}{R} \leq 0. \tag{4.16}$$

In the last line we used that $|\nabla|\mathbf{x}|^2| = 2|x^T| \leq 2R$ in the set that $1 - \frac{|\mathbf{x}|^2+2nt}{R^2} \geq 0$ and that $|\nabla u| \leq 1$. Recalling also that

$$v_t - \Delta v = -|A|^2 v - 2\frac{|\nabla v|^2}{v}$$

and defining $V_R = v\eta_R^2$, we have

$$\begin{aligned}(V_R)_t - \Delta V_R &= \eta_R^2\left(-|A|^2 v - 2\frac{|\nabla v|^2}{v}\right) - 2v|\nabla\eta_R|^2 - 4\eta\langle\nabla v, \nabla\eta_R\rangle \\ &\leq \eta_R^2\left(-|A|^2 v - 2\frac{|\nabla v|^2}{v}\right) - 2v|\nabla\eta_R|^2 + 2\eta_R^2\frac{|\nabla v|^2}{v} + 2|\nabla\eta_R|^2 v \\ &= -\eta_R^2|A|^2 v < 0.\end{aligned}$$

A standard application of the maximum principle shows that V_R does not have any interior maximum and hence

$$V_R \leq \max V_R(\cdot, 0) \leq M^2 c(M).$$

The result follows by taking $R \to \infty$.

(ii) We follow the proof of [7, Theorem 3.1] replacing the localization function in that paper by η_R^2 (where η_R is defined by (4.15)). The proof is analogous and we only point out the main steps and differences. Following [7], we define k such that $kv^2 \leq \frac{1}{2}$ in the set that $\eta_R \neq 0$ and define the function

$$g = \frac{v^2|A|^2}{1 - kv^2}.$$

Then

$$g_t - \Delta g \leq -2kg^2 - \frac{2k}{(1-kv^2)^2}|\nabla v|^2 g - 2\frac{v^{-1}}{1-kv^2}\langle\nabla v, \nabla g\rangle.$$

A similar calculation as in [7] where we use (4.16) gives that

$$\begin{aligned}(\eta_R^2 g)_t - \Delta(\eta_R^2 g) \leq -2k\eta_R^2 g^2 - \frac{2k}{(1-kv^2)^2}|\nabla v|^2\eta_R^2 g - 2\eta_R^2\frac{v^{-1}}{1-kv^2}\langle\nabla v, \nabla g\rangle \\ - 2g|\nabla\eta_R|^2 - 4\eta_R\langle\nabla\eta_R, \nabla g\rangle.\end{aligned}$$

Following again [7], we can find a vector function b (that can be explicitly computed, but it is not important) such that

$$(\eta_R^2 g)_t - \Delta(\eta_R^2 g) \leq -2k\eta_R^2 g^2 + (6 + 2k^{-1}v^{-2})g|\nabla\eta_R|^2 + \langle\nabla(g\eta_R^2), b\rangle.$$

Then, observing that for $R >$ it holds $|\nabla\eta_R|^2 \leq (2M+1)^2$, we conclude that if $\eta_R^2 g$ has an interior maximum, then

$$\begin{aligned}0 &\leq -2k\eta_R^2 g^2 + (6 + 2k^{-1}v^{-2})g|\nabla\eta_R|^2 \\ &\leq -2k\eta_R^2 g^2 + (6 + 2k^{-1}v^{-2})g(2M+1)^2\end{aligned}$$

or equivalently,

$$\eta_R^2 g \leq k^{-1}(3 + k^{-1}v^{-2})M.$$

Taking R to infinity (4.13) follows since $v \geq 1$.

(iii) Finally, consider $t\eta_R^2 g$. Then we have

$$(t\eta_R^2 g)_t - \Delta(t\eta_R^2 g) \le -2k\eta_R^2 tg^2 + (6 + 2k^{-1}v^{-2})tg|\nabla\eta_R|^2 + \langle\nabla(tg\eta_R^2), b\rangle + \eta_R^2 g.$$

At a maximum it holds

$$t\eta_R^2 g \le k^{-1}(3 + k^{-1}v^{-2})t(2M+1)^2 + M^2,$$

and we conclude (4.14) by taking $R \to \infty$. □

We will now prove Theorem 4.4:

Proof of Theorem 4.4. As in the proof of Theorem 1.3, we set $u = u_1$, $\bar{u} = u_2$ and assume without loss of generality that $u_0 \ge 1$ in which case $u, \bar{u} \ge 1$ (this follows from $u_0 \ge 1$ and Lemma 4.1). We define as before

$$W := w - \zeta = u - \bar{u} - \epsilon(t+\epsilon)u^2$$

and set

$$T^* = \min\left(T, \frac{1}{4}, \frac{1}{10\bar{c}}\right),$$

where $\bar{c}$ is a uniform constant (to be determined later) and depends on the constant c in (1.6).

We proceed as in the proof of Theorem 1.3, but we need to consider an additional case: *the supremum m^* is attained at infinity.* This means there exist a sequence of points $y_k \in \mathbb{R}^n$ with $|y_k| \to +\infty$ and a sequence of times $s_k \in (0, T^*]$ with $s_k \to t_0$ such that

$$W(y_k, s_k) > \frac{m^*}{2} > 0.$$

Applying the maximum principle, we will deduce that $t_0 > \frac{1}{8c}$ deriving a contradiction to the definition of T^*. Notice that since our initial data is complete non-compact and the convergence of our solutions to the initial data is assumed to be uniform only on compact subsets of $\mathbb{R}^n$, it is not a priori guaranteed that $t_0 > 0$, that is at this point we assume that $s_k \to t_0 \in [0, T^*]$.

To apply the maximum principle, we employ a parabolic version of the Omori–Yau maximum principle (see for example in [12]). We define the functions

$$W_k(x,t) = W(x,t) - t\frac{|x|^2}{C_k^2} \quad \text{for } C_k = \max\{|y_k|^2, k\}$$

and we look at the supremum of W_k in $\mathbb{R}^n \times (0, s_k]$. If this supremum is less than $\frac{m^*}{4}$, then

$$W(y_k, s_k) \le \frac{m^*}{4} + t\frac{|y_k|^2}{C_k^2}$$

and from our choice of C_k we have

$$W(y_k, s_k) \le \frac{3m^*}{8} < \frac{m^*}{2} \quad \text{for } k \gg 1,$$

contradicting our assumption.

We deduce that

$$m_k := \sup_{\mathbb{R}^n \times (0,s_k]} W_k > \frac{m^*}{4} > 0.$$

Since W is uniformly bounded (see (4.7)), this supremum is attained in the interior at a point $(x_k, t_k) \in \mathbb{R}^n \times (0, s_k]$. At this point necessarily we have

$$\begin{aligned} &W(x_k,t_k) \geq t_k \frac{|x_k|^2}{C_k^2} > 0, \qquad W_t(x_k,t_k) = (W_k)_t(x_k,t_k) + \frac{|x_k|^2}{C_k^2} \geq 0, \\ &DW(x_k,t_k) = \frac{2t_k x_k}{C_k^2}, \qquad D_{ij}W(x_k,t_k) \leq \frac{2t_k\delta_{ij}}{C_k^2} \leq \frac{2t_k\delta_{ij}}{k^2}, \end{aligned} \tag{4.17}$$

where the last inequality is understood in the sense of quadratic forms, that is

$$D_{ij}W(x_k,t)\xi_i\xi_j < \frac{2t_k}{k^2}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Furthermore, notice that since (x_k, t_k) is the maximum for W_k on $\mathbb{R}^n \times (0, s_k]$, we have

$$W(x_k,t_k) - t_k \frac{|x_k|^2}{C_k^2} \geq W(0,0),$$

and because $W \leq \epsilon^{-2}$, we have

$$\frac{t_k\,|x_k|^2}{C_k^2} \leq W(x_k,t_k) - W(0,0) \leq \epsilon^{-2} - W(0,0) = \epsilon^{-2} + \epsilon^2 u^2(0,0) =: M_\epsilon.$$

Then

$$|DW(x_k,t_k)| = \frac{2t_k|x_k|}{C_k^2} \leq \frac{2\sqrt{t_k M_\epsilon}}{C_k} \leq \frac{2\sqrt{t_k M_\epsilon}}{k} = \mathcal{O}\left(\frac{\sqrt{t_k}}{k}\right). \tag{4.18}$$

Moreover, since $W_k(x_k,t_k) = m_k^* > \frac{m^*}{4} > 0$, we have

$$W(x_k,t_k) = W_k(x_k,t_k) + t_k \frac{|x_k|^2}{C_k^2} > \frac{m^*}{4} > 0.$$

Combining these with (4.17), we conclude the following:

$$\begin{aligned} &W(x_k,t_k) > \frac{m^*}{4} > 0, \qquad W_t(x_k,t_k) \geq 0, \\ &|DW(x_k,t_k)| \leq \mathcal{O}\left(\frac{\sqrt{t_k}}{k}\right), \quad D_{ij}W(x_k,t_k) \leq \frac{2\delta_{ij}}{k^2}. \end{aligned} \tag{4.19}$$

Hence, we deduce from (4.3), (4.4), (4.19) and the uniform ellipticity of the matrix a_{ij} that

$$-\frac{C}{k^2} \leq W_t - a_{ij}D_{ij}W < (a_{ij} - \bar{a}_{ij})D_{ij}\bar{u} - \frac{\epsilon}{2}u^2 \tag{4.20}$$

holds at each point (x_k, t_k). Furthermore, from $W(x_k, t_k) > 0$ we have

$$(1 - \epsilon(t_k + \epsilon)u(x_k,t_k))u(x_k,t_k) > \bar{u}(x_k,t_k).$$

Next, observe that the fact that $W(x_k, t_k) > 0$ implies that $u(x_k, t_k)$ is bounded (otherwise if $u(x_{k_l}, t_{k_l}) \to +\infty$ for some subsequence, then $\lim_{l\to+\infty} W(x_{k_l}, t_{k_l}) \to -\infty$). Furthermore, $u(x_k, t_k)$ bounded and $u, \bar{u} \geq 1$ imply that $\bar{u}(x_k, t_k)$ is bounded as well. Hence, we may assume without loss of generality that

$$u(x_k, t_k) \to u^* \quad \bar{u}(x_k, t_k) \to \bar{u}^* \quad \text{and} \quad 1 \leq u(x_k, t_k), \bar{u}(x_k, t_k) \leq u^* + 1.$$

Therefore, our assumption that $u, \bar{u}$ satisfy condition (4.12) and the first assertion in Proposition 4.5 applied to $M = u^* + 2$ yield

$$|Du(x_k, t_k)| \leq C(u^*) \quad \text{and} \quad |D\bar{u}(x_k, t_k)| \leq C(u^*).$$

Furthermore, by the third assertion in Proposition 4.5 we have

$$t_k |A|^2(x_k, t_k) \leq C(u^*) \quad \text{and} \quad t_k |\bar{A}|^2(x_k, t_k) \leq C(u^*).$$

It follows that at the points (x_k, t_k) we have for every $i, j \in \{1, \ldots, n\}$ that

$$\sqrt{t_k}\, v |h_i^j|(x_k, t_k) \leq C(u^*) \quad \text{and} \quad \sqrt{t_k}\, v |\bar{h}_i^j|(x_k, t_k) \leq C(u^*)$$

and also

$$\sqrt{t_k} \frac{|D_{ij} u|}{\sqrt{1 + |Du|^2}} \leq C(u^*) \quad \text{and} \quad \sqrt{t_k} \frac{|D_{ij} \bar{u}|}{\sqrt{1 + |D\bar{u}|^2}} \leq C(u^*). \tag{4.21}$$

These bounds will be used momentarily.

We will next analyze the main term on right-hand side of (4.20). From the definition of W we have $D\bar{u}(x_k, t_k) = (1 - 2\epsilon(t_k + \epsilon)\, u) Du - DW$. Then, similarly to (4.9) (the computation here has more terms since $DW \neq 0$), we get

$$\begin{aligned}
a_{ij} - \bar{a}_{ij} &= \frac{D_i \bar{u} D_j \bar{u}}{1 + |D\bar{u}|^2} - \frac{D_i u D_j u}{1 + |Du|^2} \\
&= (1 - 2\epsilon(t_0 + \epsilon) u)^2 \frac{D_i u D_j u}{1 + |D\bar{u}|^2} - \frac{D_i u D_j u}{1 + |Du|^2} \\
&\quad + \frac{D_i W D_j W - (1 - 2\epsilon(t_k + \epsilon)\, u)(D_i u D_j W + D_i W D_j u)}{1 + |D\bar{u}|^2} \\
&= \big(-4\epsilon(t_0 + \epsilon) u (1 - \epsilon(t_0 + \epsilon) u) + \langle DW, b\rangle\big) \frac{D_i u D_j u}{(1 + |Du|^2)(1 + |D\bar{u}|^2)} \\
&\quad + \frac{D_i W D_j W - (1 - 2\epsilon(t_k + \epsilon)\, u)(D_i u D_j W + D_i W D_j u)}{1 + |D\bar{u}|^2},
\end{aligned}$$

where $b = 2(1 - 2\epsilon(t_k + \epsilon)\, u) Du - DW$. Denoting

$$\begin{aligned}
B_{ij} &= \langle DW, b\rangle \frac{D_i u D_j u}{(1 + |Du|^2)(1 + |D\bar{u}|^2)} \\
&\quad + \frac{D_i W D_j W - (1 - 2\epsilon(t_k + \epsilon)\, u)(D_i u D_j W + D_i W D_j u)}{1 + |D\bar{u}|^2},
\end{aligned}$$

we can then express the main term $(a_{ij} - \bar{a}_{ij}) D_{ij} \bar{u}$ on right-hand side of (4.20) as

$$\begin{aligned}
(a_{ij} - \bar{a}_{ij}) D_{ij} \bar{u} = -4\epsilon(t_0 + \epsilon) u (1 - \epsilon(t_0 + \epsilon) u) \frac{D_{ij} \bar{u} D_i u D_j u}{(1 + |Du|^2)(1 + |D\bar{u}|^2)} \\
+ B_{ij} D_{ij} \bar{u}.
\end{aligned} \tag{4.22}$$

Next observe that from (4.18) at (x_k, t_k), we have

$$|B_{ij}| \leq C(u^*)\frac{\sqrt{t_k}}{k},$$

which combined with (4.21) yields

$$|B_{ij} D_{ij}\bar{u}| \leq C\frac{\sqrt{t_k}}{k}(\sqrt{t_k}) = \mathcal{O}\left(\frac{1}{k}\right). \tag{4.23}$$

To bound the first term on the right-hand side of (4.22), we use (1.6) which in particular implies that

$$\bar{v}\bar{h}^i_j D_i u D_j u \geq -c|Du|^2. \tag{4.24}$$

On the other hand,

$$DW = (1-2\epsilon(t_k+\epsilon)\,u)Du - D\bar{u} \quad \text{and} \quad \bar{h}^i_j = \frac{D_{ij}\bar{u}}{\sqrt{1+|D\bar{u}|^2}} - \frac{D_{lj}\bar{u}D_l\bar{u}D_i\bar{u}}{(1+|D\bar{u}|^2)^{\frac{3}{2}}}$$

imply

$$\begin{aligned}
\bar{v}\bar{h}^i_j D_i u D_j u &= \left(D_{ij}\bar{u} - \frac{D_{lj}\bar{u}D_l\bar{u}D_i\bar{u}}{1+|D\bar{u}|^2}\right)D_i u D_j u \\
&= D_{ij}\bar{u}\; D_i u D_j u - \langle D\bar{u}, Du\rangle \frac{D_{lj}\bar{u}}{1+|D\bar{u}|^2} D_l\bar{u} D_j u \\
&= \left(1+|D\bar{u}|^2 - (1-2\epsilon(t_k+\epsilon)\,u)^2|Du|^2\right)\frac{D_{ij}\bar{u}}{1+|D\bar{u}|^2} D_i u D_j u \\
&\qquad + (1-2\epsilon(t_k+\epsilon)\,u)\frac{|Du|^2}{1+|D\bar{u}|^2} D_{ij}u D_j u D_i W \\
&\qquad + (1-2\epsilon(t_k+\epsilon)\,u)\frac{\langle DW, Du\rangle}{1+|D\bar{u}|^2} D_{ij}u D_j u D_i u \\
&\qquad - \frac{\langle DW, Du\rangle}{1+|D\bar{u}|^2} D_{ij}u D_i W D_j u \\
&= \left(1+|D\bar{u}|^2 - (1-2\epsilon(t_k+\epsilon)\,u)^2|Du|^2\right)\frac{D_{ij}\bar{u}}{1+|D\bar{u}|^2} D_i u D_j u + \mathcal{O}\left(\frac{1}{k}\right),
\end{aligned}$$

where to derive the last line we combined (4.18) and (4.21) (following a similar estimate as the one we did for $B_{ij} D_{ij}\bar{u}$).

To further estimate the last line above, we use

$$\begin{aligned}
|D\bar{u}|^2 - (1-2\epsilon(t_k+\epsilon)\,u)^2|Du|^2 &= \langle DW, D\bar{u} + (1-2\epsilon(t_k+\epsilon)\,u)Du\rangle \\
&= \mathcal{O}\left(\frac{\sqrt{t_k}}{k}\right)
\end{aligned}$$

concluding that

$$\bar{v}\bar{h}^i_j D_i u D_j u = \frac{D_{ij}\bar{u}}{1+|D\bar{u}|^2} D_i u D_j u + \mathcal{O}\left(\frac{1}{k}\right)$$

which in turn combined with (4.24) yields

$$\frac{D_{ij}\bar{u}}{1+|D\bar{u}|^2} D_i u D_j u \geq -c|Du|^2 + \mathcal{O}\left(\frac{1}{k}\right). \tag{4.25}$$

Finally, (4.20), (4.22), (4.23) and (4.25) together imply that as $k \to \infty$

$$0 \le 4\bar{c}\epsilon(t_0+\epsilon)u^*(1-\epsilon(t_0+\epsilon)u^*) - \frac{\epsilon}{2}(u^*)^2.$$

We now use the same argument as in the proof of Theorem 1.3 to conclude that this is not possible provided that $t_0 + \epsilon > \frac{1}{8c}$, where c is the constant from (1.6). Since we have assumed that $t_0 \in (0, T^*]$ and $T^* \le \frac{1}{10\bar{c}}$, we derive a contradiction by choosing ϵ sufficiently small. This shows that, contrary to our assumption, $W^*(t_0) < 0$, finishing the proof of the claim. □

5. The convex case and Harnack inequality

In this final section we will state the existence and uniqueness result for convex, proper, non-compact entire graphs mean curvature flow solutions and show that Hamilton's Harnack inequality (proved in [9]) holds.

Theorem 5.1 (Uniqueness of convex entire graph solutions). *Assume that $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a convex function defining a proper entire graph convex hypersurface*

$$M_0 = \{(x, u_0(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}.$$

Let $u_1, u_2 : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ be two solutions of (1.2) *defining two proper smooth convex entire graph solutions $M_t^1 = \{(x, u_1(x,t)) : x \in \mathbb{R}^n\}$ and $M_t^2 = \{(x, u_2(x,t)) : x \in \mathbb{R}^n\}$ of MCF* (1.1) *with the same initial data u_0, that is*

$$\lim_{t\to 0} u_1(\cdot, t) = \lim_{t \to 0} u_2(\cdot, t) = u_0.$$

Then $u_1 = u_2$ on $\mathbb{R}^n \times (0,T)$, that is $M_t^1 = M_t^2$ for all $t \in (0,T)$.

Proof. Since our initial data is a convex proper entire graph over $\mathbb{R}^n$, we may assume that it lies above the $e_{n+1} = 0$ plane, that is $u_0(x) \ge 0$ for all $x \in \mathbb{R}^n$. Furthermore, we have $\lim_{x\to+\infty} u_0(x) = +\infty$ and the same holds for both solutions $u_i(x,t)$, $i = 1, 2$, namely $u_i(\cdot, t) \ge 0$ and $\lim_{x\to+\infty} u_i(x,t) = +\infty$ for all $t > 0$. Then one can apply the maximum principle argument in Theorem 1.3 (actually in the convex case the computation is simpler) to show that for any small number $\epsilon > 0$, one has

$$u_1 - u_2 \le \epsilon t u_1^2 + \epsilon$$

and, similarly,

$$u_2 - u_1 \le \epsilon t u_2^2 + \epsilon$$

for all $t \in (0,T)$. Taking $\epsilon \to 0$ readily gives that $u_1 = u_2$ for all $t \in (0,T)$. □

An immediate consequence of the previous result is that convex graphical MCF solutions can be smoothly approximated by compact ones. For any two compact convex hypersurfaces Σ_1 and Σ_2 we write that $\Sigma_1 \prec \Sigma_2$ if Σ_2 encloses Σ_1 (allowing $\Sigma_1 \cap \Sigma_2 \ne \emptyset$).

Corollary 5.2. *Let $M_t = \{(x, u(x,t)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$, $t \in (0, +\infty)$, be a smooth entire graph mean curvature flow solution with initial data $M_0 = \{(x, u_0(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ which is a proper convex entire graph, normalized in such a way that*

$$u(0) = \min_{x \in \mathbb{R}^n} u_0(x) = 0.$$

Then M_t can be approximated by a sequence M_t^i of compact convex mean curvature flow solutions. More precisely, the surfaces Σ_t^i are reflection symmetric with respect to the hyperplane $\{x_{n+1} = i\}$ and their lower parts $\hat{\Sigma}_t^i$ defined by $\hat{\Sigma}_t^i := \Sigma_t^i \cap \{x_{n+1} < i\}$ converge, as $i \to +\infty$, to M_t, smoothly on compact subsets of $\mathbb{R}^{n+1} \times (0, +\infty)$.

Proof. From our assumptions we have $M_t = \{(x, u(x,t)) : x \in \mathbb{R}^n\}$ for all $t \in (0, +\infty)$ and that $u(\cdot, t) \geq 0$ for all $t \geq 0$, since we have normalized our initial data so that

$$u(0) = \min_{x \in \mathbb{R}^n} u_0(x) = 0.$$

Furthermore, since $u_0(x)$ is assumed to be proper we have

$$\lim_{x \to +\infty} u(x,t) = +\infty \quad \text{for all } t \geq 0.$$

For each integer $i \geq 1$, we define the Lipschitz domains

$$\mathcal{D}_0^i = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : u_0(x) \leq x_{n+1} \leq 2i - u_0(x)\}$$

and we let $\Sigma_0^i = \partial \mathcal{D}_0^i$. Our assumption that $u(0) = 0$ guarantees $\mathcal{D}_0^i \neq \emptyset$ for all $i \geq 1$. Note that $\Sigma_0^i \subset \mathbb{R}^{n+1}$ is just the closed hypersurface that consists of $M_0 \cap \{x_{n+1} \leq i\}$ and its reflection with respect to the hyperplane $x_{n+1} = i$. Furthermore, each Σ_0^i is convex and Lipschitz continuous.

Standard MCF theory shows that for any $i \geq 1$, there exists a unique smooth mean curvature flow Σ_t^i starting at Σ_0^i. The solutions Σ_t^i exists up to times T^i, they satisfy $\Sigma_t^i \prec \Sigma_t^{i+1}$ (Σ_t^{i+1} encloses Σ_t^i), and $\lim_{i \to +\infty} T^i = +\infty$. The strong maximum principle guarantees that each Σ_t^i, $0 < t < T^i$, is strictly convex. Furthermore, Σ_t^i is reflection symmetric with respect to the hyperplane $\{x_{n+1} = i\}$, since Σ_0^i is by construction.

Denote by $\hat{\Sigma}_t^i$ to be the lower half of Σ^i, that is

$$\hat{\Sigma}_t^i := \Sigma_t^i \cap \{x_{n+1} < i\}.$$

Also, for any point $\mathbf{x_0} \in \mathbb{R}^{\mathbf{n+1}}$ let us denote by $B_R^{n+1}(\mathbf{x_0})$ the ball in $\mathbb{R}^{n+1}$ of radius R centered at $\mathbf{x_0}$.

Claim 5.1. *Fix $T > 0$. For any $R > 1$, there exists an integer i_R such as long as $i \geq i_R$, the lower part of $\hat{\Sigma}_t^i \cap B_{2R}^{n+1}(\mathbf{0})$, $t \in [0, T]$, can be written as a graph $\{(x, u^i(x,t)) : |x| \leq R\}$ and satisfies a uniform gradient bound which is independent of i and depends only on R and M_0.*

Proof. Fix $T > 0$ and assume that i is chosen sufficiently large so that $T^i > T$. Given any $R > 1$, we may choose i_R sufficiently large so that $T \ll R$ and if $\mathbf{x_0}^i = (\mathbf{0}, i) \in \mathbb{R}^{n+1}$, then $B_{4R}^{n+1}(\mathbf{x_0}^i) \prec \Sigma_t^i$ for all $i \geq i_R$ and all $t \in [0, T]$. The convexity and symmetry of the solutions Σ_t^i then imply that for any $i \geq i_R$, $\hat{\Sigma}_t^i \cap B_{3R}^{n+1}(\mathbf{0})$, $t \in [0, T]$, can be written as a graph

$$\{(x, u^i(x,t)) : |x| \leq 3R\}.$$

So it remains to show the uniform gradient bound of $\hat{\Sigma}_t^i \cap B_{2R}^{n+1}(\mathbf{0})$, $t \in [0, T]$, for all $i \geq i_R$. This readily follows from the local gradient bound in [7] and the fact that $u^i(x, 0) = u_0(x)$ for all $i \geq i_R$, which implies that $\Sigma_0^i \cap B_{3R}^{n+1}(\mathbf{0})$, $i \geq i_R$, satisfy a uniform gradient bound. □

The results in [7] then imply that $\widehat{\Sigma}_t^i \cap B_R^{n+1}(\mathbf{0})$, $t \in [0, T]$, $i \geq i_R$, have uniformly bounded second fundamental forms. More precisely, there exists a constant $C_{R,T}$ that is independent of i such that the second fundamental form $|A^i|$ of Σ^i satisfies the bound

$$\sup_{\widehat{\Sigma}_t^i \cap B_R^{n+1}(\mathbf{0})} |A^i| \leq C_{R,T}\, t^{-\frac{1}{2}}, \quad t \in (0, T],$$

provided that $i \geq i_R$.

One can then pass to the limit (over a subsequence $i_k \to +\infty$) and obtain a smooth entire graph mean curvature flow solution $\widehat{M}_t$, $t \in (0, T)$ whose second fundamental form satisfies the bound

$$\sup_{\widehat{M}_t \cap B_R^{n+1}(\mathbf{0})} |A| \leq C_{R,T}\, t^{-\frac{1}{2}}, \quad t \in (0, T].$$

Standard arguments then imply that if $\widehat{M}_t = \{(x, \hat{u}(x,t)) : x \in \mathbb{R}^n\}$, then

$$\lim_{t \to 0} \hat{u}(x,t) = u_0(x).$$

Since $x_{n+1} = u_0(x)$ is proper, it follows that $x_{n+1} = \hat{u}(x,t)$ is proper as well. Hence, Theorem 5.1 guarantees that $u = \hat{u}$ on $\mathbb{R}^n \times (0, T)$. Since $T > 0$ was arbitrary, we conclude that $u = \hat{u}$ on $\mathbb{R}^n \times (0, +\infty)$ finishing the proof of the corollary. □

Remark 5.1. Our methods can be applied to study the uniqueness of the (convex) solutions that are analyzed by X.-J. Wang in [16]. More precisely, in that paper, the author studies convex translating solutions to mean curvature flow via a level set method. In the noncompact case, those solutions are obtained via taking limits and our techniques can be used as an alternative proof of the uniqueness of such limits. We leave the details to the interested reader.

An immediate consequence of Corollary 5.2 is that Hamilton's Harnack inequality holds for entire convex graphs.

Corollary 5.3 (Hamilton's Harnack estimate). *Any smooth convex proper entire graph solution M_t, $t \in (0, +\infty)$, of mean curvature flow satisfies Hamilton's Harnack differential inequality, namely for any tangent vector field V,*

$$\frac{\partial H}{\partial t} + 2\langle \nabla H, V \rangle + h(V,V) + \frac{H}{2t} \geq 0. \tag{5.1}$$

Proof. Let Σ_i^t be approximating sequence of compact convex solutions which were constructed in Corollary 5.2. Each of them satisfy the Harnack differential inequality (5.1). Passing to the smooth limit on compact sets, we conclude that (5.1) also holds for our complete non-compact solution M_t, for all $t \in (0, +\infty)$. □

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