Global Gradient Estimate for a Divergence Problem and Its Application to the Homogenization of a Magnetic Suspension

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1 Introduction

The purpose of this paper is to generalize the results obtained by the authors in [10], where the rigorous analysis of the homogenization of a particulate flow consisting of a non-dilute suspension of a viscous Newtonian fluid with magnetizable particles was developed. Here, the fluid is assumed to be described by the Stokes flow and the particles are either paramagnetic or diamagnetic. The coefficients of the corresponding partial differential equations are locally periodic and a one-way coupling between the fluid domain and the particles is also assumed. Such one-way coupling has been observed in nature, see [11, Chapter 1]. For details and information about the applications and literature on the magnetic suspension, we turn to [10] and the references cited therein; however, the mathematical formulation of the considered problem is given in Sect. 2.2 below. References on the effective viscosity of a suspension without the coupling with magnetic field include [5, 6, 12–16, 18, 21, 23, 25, 31, 33].

In [10], a restrictive assumption about the magnetic permeability of the suspension, denoted by $\mathbf{a}$, was made. Here, the function $\mathbf{a}(\cdot)$ is locally periodic and elliptic, where the latter means that $\lambda\mathbf{I} \le \mathbf{a}(x)$ and $\|\mathbf{a}\|_{L^\infty} \le \Lambda$, for all $x \in \Omega$, with the

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suspension domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, including both the ambient fluid and the particles, and $\lambda, \Lambda > 0$ given in (A2)–(A3) below. The assumption on the function $\mathbf{a}$ made in [10] is as follows: for a given $s \in (4, 6]$, there exists a small number $\delta = \delta(\Lambda, d, \Omega) > 0$, for which the magnetic permeability $\mathbf{a}$ satisfies the following condition:

$$\operatorname{ess\,sup} \mathbf{a} - \operatorname{ess\,inf} \mathbf{a} \leq \delta. \tag{1}$$

As a consequence of (1), in [10] we obtained the following gradient estimate for the magnetic potential φ^ε:

$$\int_\Omega \left|\nabla \varphi^\varepsilon\right|^s \mathrm{d}x \leq C \int_\Omega |\mathbf{k}|^s \mathrm{d}x, \tag{2}$$

where the constant $C > 0$ is independent of ε, φ^ε and $\mathbf{k}$; with $0 < \varepsilon \ll 1$ the scale of the microstructure, $\mathbf{k} \in H^1(\Omega, \mathbb{R}^d)$ divergence-free, satisfying the compatibility condition $\int_{\partial\Omega} \mathbf{k} \cdot \mathbf{n}_{\partial\Omega} \, \mathrm{d}s = 0$, and appearing in the Neumann boundary condition on $\partial\Omega$, the boundary of the domain Ω, given by:

$$\left(\mathbf{a}\nabla \varphi^\varepsilon\right) \cdot \mathbf{n}_{\partial\Omega} = \mathbf{k} \cdot \mathbf{n}_{\partial\Omega}, \tag{3}$$

where $\mathbf{n}_{\partial\Omega}$ is the outward-pointing unit normal vector to $\partial\Omega$. The regularity result (2) was then used in the derivation of the effective (or homogenized) response of the given suspension that was rigorously justified in [10].

The main goal of this paper is to relax the assumption (1) on the magnetic permeability $\mathbf{a}$. To achieve this, we consider the Dirichlet boundary condition given in (4b) below, rather than one given in (3), and obtain the Lipschitz estimate (5), instead of (2), for the gradient of the magnetic potential φ^ε, see Theorem 1 below. In this paper, we are able to remove the condition (1) and have $\mathbf{a}$ only required to be *piecewise Hölder continuous*. Such relaxation will be based on the following observations:

- The De Giorgi-Nash-Moser estimate [20, Theorem 8.24] states that the solutions of scalar equations are Hölder continuous.
- If $1 > \varepsilon \geq \varepsilon_0$, for some $\varepsilon_0 > 0$, the uniform gradient bound (5) can be obtained by the result of Li and Vogelius [29]. The case when $\varepsilon_0 > \varepsilon > 0$ is resolved by the compactness method, which is discussed below.
- Provided $\mathbf{a}$ is also symmetric (this assumption is only necessary for the corrector results in Theorem 2), the gradients of the solutions of the cell problems are in $L^\infty(Y)$.

The main tools used in the proof of this theorem are (i) the regularity results of Li and Vogelius [29], and (ii) the celebrated *compactness method*, which was first used in homogenization in the seminal works of Avellaneda and Lin [3, 4]. Its machinery and applications in homogenization are carefully explained in [34]. In the context of homogenization, this method utilizes compactness in order to gain an improved regularity from a limiting equation via a proof by contradiction. This improvement of regularity is iterated and then used in a blow-up argument. Usually, the implementation of this method follows three steps, coined by Avellaneda and Lin [3, 4] as *(i)* "improvement," *(ii)* "iteration," and *(iii)* "blowup."

The main contribution of this improved regularity result is that it will allow us to significantly widen the range of applicability of the results obtained in [10].

The outline of the paper is as follows. In Sect. 2, the main notations are introduced and the formulation of the fine-scale problem is discussed. Theorem 1, which provides an improved gradient estimate for the magnetic potential, is stated and discussed in Sect. 3. In Sect. 4, we obtain the interior Lipschitz and Hölder estimates, which provide the foundation for the boundary and corrector estimates discussed in Sect. 5. With all the results at hand, we then present the proof of our main theorem, also in Sect. 5. In Sect. 6, the homogenization results are obtained and summarized in Theorem 2. The conclusions are given in Sect. 7. The classical Schauder estimate is recalled in Appendix.

2 Formulation

2.1 Notation

For a measurable set A and a measurable function $f : A \to \mathbb{R}$, we define by $|A|$ the measure of A and $\fint_A f(x)\,\mathrm{d}x := \frac{1}{|A|}\int_A f(x)\,\mathrm{d}x$.

Throughout this paper, the scalar-valued functions, such as the pressure p, are written in usual typefaces, while vector-valued or tensor-valued functions, such as the velocity $\mathbf{u}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$, are written in bold. Sequences are indexed by numeric superscripts (ϕ^i), while elements of vectors or tensors are indexed by numeric subscripts (x_i). Finally, the Einstein summation convention is used whenever applicable.

2.2 Setup of the Problem

Consider $\Omega \subset \mathbb{R}^d$, for $d \geq 2$, a simply connected and bounded domain of class $C^{1,\alpha}$, $0 < \alpha < 1$, and let $Y := (0, 1)^d$ be the unit cell in $\mathbb{R}^d$. The unit cell Y is decomposed into:

$$Y = Y_s \cup Y_f \cup \Gamma,$$

where Y_s, representing the magnetic inclusion, and Y_f, representing the fluid domain, are open sets in $\mathbb{R}^d$, and Γ is the closed $C^{1,\alpha}$ interface that separates them.

Let $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$ be a vector of indices and $\{\mathrm{e}^1, \dots, \mathrm{e}^d\}$ be the canonical basis of $\mathbb{R}^d$. For a fixed small $\varepsilon > 0$, we define the dilated sets:

$$Y_i^\varepsilon := \varepsilon(Y + i), \quad Y_{i,s}^\varepsilon := \varepsilon(Y_s + i), \quad Y_{i,f}^\varepsilon := \varepsilon(Y_f + i), \quad \Gamma_i^\varepsilon := \partial Y_{i,s}^\varepsilon.$$

Typically, in homogenization theory, the positive number $\varepsilon \ll 1$ is referred to as the *size of the microstructure*. The effective (or *homogenized*) response of the given suspension corresponds to the case $\varepsilon = 0$.

We denote by $\mathbf{n}_i$, $\mathbf{n}_\Gamma$, and $\mathbf{n}_{\partial\Omega}$ the unit normal vectors to Γ_i^ε pointing outward $Y_{i,s}^\varepsilon$, to Γ pointing outward Y_s, and to $\partial\Omega$ pointing outward, respectively; and also, we denote by $\mathrm{d}\mathcal{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff measure. In addition, we define the sets:

$$I^\varepsilon := \{i \in \mathbb{Z}^d : Y_i^\varepsilon \subset \Omega\}, \quad \Omega_s^\varepsilon := \bigcup_{i \in I^\varepsilon} Y_{i,s}^\varepsilon, \quad \Omega_f^\varepsilon := \Omega \setminus \Omega_s^\varepsilon, \quad \Gamma^\varepsilon := \bigcup_{i \in I^\varepsilon} \Gamma_i^\varepsilon,$$

see Fig. 1.

The magnetic permeability $\mathbf{a}$ is a $d \times d$ matrix satisfying the following conditions:

(A1) Y-periodicity: for all $z \in \mathbb{R}^d$, for all $m \in \mathbb{Z}$, and for all $k \in \{1, \dots, d\}$ we have:

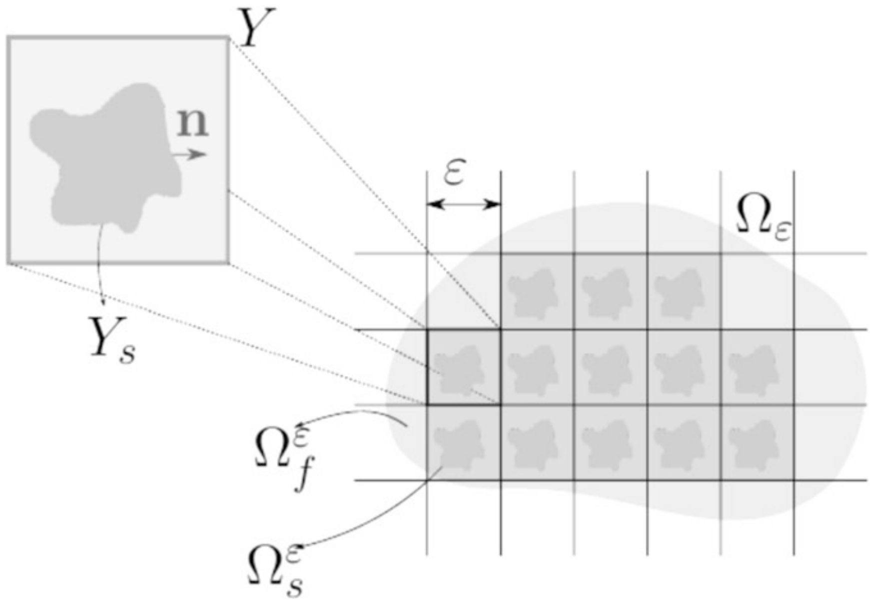

Fig. 1 Reference cell Y and domain Ω

$$\mathbf{a}(z + me^k) = \mathbf{a}(z).$$

(A2) Boundedness and measurability: there exists $\Lambda > 0$ such that:

$$\|\mathbf{a}\|_{L^\infty\left(\mathbb{R}^d\right)} \leq \Lambda.$$

(A3) Ellipticity: there exists $\lambda > 0$ such that for all $\xi \in \mathbb{R}^d$, for all $x \in \mathbb{R}^d$, we have:

$$\mathbf{a}(x)\xi \cdot \xi \geq \lambda\, |\xi|^2 .$$

Denote by $\mathfrak{M}(\lambda, \Lambda)$ the set of matrices that satisfy (A2)–(A3) and $\mathfrak{M}_{\text{per}}(\lambda, \Lambda)$ the subset of matrices in $\mathfrak{M}(\lambda, \Lambda)$ that also satisfy (A1).

3 Statement and Discussion of the Main Result

The main result of this paper is summarized in the following theorem:

Theorem 1 (Global Lipschitz Estimate) *Let Ω be a bounded $C^{1,\alpha}$ domain, $g \in C^{1,\alpha'}(\partial\Omega)$, and $f \in L^\infty(\Omega)$, where $0 < \alpha' < \alpha < 1$. Suppose that $\mathbf{a} \in \mathfrak{M}_{\text{per}}(\lambda, \Lambda)$ is piecewise C^α-continuous. There exists $C = C(\alpha, \alpha', \lambda, \Lambda, d, \Omega) > 0$ such that, for all $\varepsilon > 0$, the (unique) solution φ^ε of:*

$$-\operatorname{div}\left[\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla\varphi^\varepsilon\right] = f, \qquad \text{in } \Omega \tag{4a}$$

$$\varphi^\varepsilon = g, \qquad \text{on } \partial\Omega \tag{4b}$$

satisfies:

$$\left\|\nabla\varphi^\varepsilon\right\|_{L^\infty(\Omega)} \leq C\left(\|g\|_{C^{1,\alpha'}(\partial\Omega)} + \|f\|_{L^\infty(\Omega)}\right). \tag{5}$$

Remark 1 For each $\varepsilon > 0$, let N_ε be the number of subdomains inside Ω such that in each of them the function $\mathbf{a}$ is C^α-continuous. Denote those subdomains by $D_m,\ 1 \leq m \leq N_\varepsilon$. Then, for $0 < \alpha' < \min\left\{\alpha, \frac{\alpha}{(\alpha+1)d}\right\}$, by Li and Vogelius [29, Corollary 1.3], one has:

$$\left\|\nabla\varphi^\varepsilon\right\|_{L^\infty(\Omega)} \leq C\left(\|g\|_{C^{1,\alpha'}(\partial\Omega)} + \|f\|_{L^\infty(\Omega)}\right), \tag{6}$$

where C depends on $\Omega, d, \alpha, \alpha', \lambda, \Lambda, \|\mathbf{a}\|_{C^{\alpha'}(\overline{D_m}, \mathbb{R}^{d\times d})}$, and the $C^{1,\alpha}$-modulus of $\cup_{m=1}^{N_\varepsilon} \partial D_m$ (defined in page 92 [29]). As $\varepsilon \to 0$, the number N_ε increases, while the sizes of the subdomains decrease, which leads to the blowup of the $C^{1,\alpha}$-modulus. Therefore, estimate (6) is not uniform in ε.

However, if $\varepsilon_0 \leq \varepsilon \leq 1$ for some constant $\varepsilon_0 > 0$, then one can control the number, size, distance, and $C^{1,\alpha}$-modulus of the subdomains in Ω, uniformly with respect to ε. Note that the upper and lower bounds of those quantities are positive and independent of ε. Thus, by Li and Vogelius [29, Corollary 1.3], there exists C independent of ε such that (5) holds when $\varepsilon_0 \leq \varepsilon \leq 1$. Therefore, Theorem 1 will be proven, if one can specify a constant $\varepsilon_0 > 0$ such that (5) holds for $0 < \varepsilon < \varepsilon_0$.

Our proof of Theorem 1 follows the classical steps in regularity theory: (i) derive an interior Lipschitz estimate, (ii) derive a boundary Lipschitz estimate, and finally (iii) combine the estimates in (i) and (ii) to obtain the global Lipschitz estimate. Step (i) is obtained via the *compactness method*. For step (ii), we additionally need to establish the following preliminary results:

- Interior and boundary Hölder estimates, see Propositions 1–3.
- Estimates for the Green's function, which are obtained using the Hölder bounds above, see Proposition 5. The existence of the Green's function for scalar uniformly elliptic equations is established in [22, 26, 30].
- Estimates for the Dirichlet boundary corrector, see Proposition 6.

If the coefficient $\mathbf{a}$ is not Hölder continuous, then the classical results in [3, 34, 37] cannot be applied directly. Nevertheless, some of their proofs can be adapted for the case at hand. In those situations, we will explicitly point out what needs to be modified in their proofs, in order to relax the continuity assumption on the coefficient matrix $\mathbf{a}$.

4 Interior Estimates

We start with an estimate for homogenized equations, i.e., the equations with constant coefficients, which are limits of some fine-scale problems.

Lemma 1 *Let $\lambda > 0$, $\Lambda > 0$, $\gamma > 0$ and $0 < \mu < \frac{1}{2}$ be fixed. For each constant matrix $\mathbf{b} \in \mathfrak{M}(\lambda, \Lambda)$, $h \in L^{d+\gamma}(B(x_0, 1))$, with $\|h\|_{L^{d+\gamma}(B(x_0,1))} \leq 1$, there exists $\theta = \theta(\gamma, \mu, \lambda, \Lambda, d) > 0$ such that if $\phi \in H^1(B(x_0, 1))$ satisfies:*

$$-\operatorname{div}(\mathbf{b}\nabla\phi) = h \text{ in } B(x_0, 1),$$

then the following estimate holds:

$$\sup_{|x-x_0|<\theta} \left| \phi(x) - \phi(x_0) - (x - x_0) \cdot \fint_{B(x_0,\theta)} \nabla\phi(z)\, \mathrm{d}z \right| < \theta^{1+3\mu/4}. \tag{7}$$

Proof By the classical Schauder estimate for the scalar equation with constant coefficients (Theorem 3), we have $\phi \in C^{1,\mu}(B(x_0, 1/4))$ and:

$$\begin{aligned}\|\phi\|_{C^{1,\mu}(B(x_0,1/4))} &\le C(\gamma,\mu,\lambda,\Lambda,d)\left(\|h\|_{L^{d+\gamma}(B(x_0,3/8))} + \|\phi\|_{H^1(B(x_0,3/8))}\right)\\ &\le C(\gamma,\mu,\lambda,\Lambda,d)\,\|h\|_{L^{d+\gamma}(B(x_0,1))}\\ &\le C(\gamma,\mu,\lambda,\Lambda,d). \end{aligned} \tag{8}$$

For $0 < \theta < \frac{1}{4}$ and $|x - x_0| < \theta$, there exist z_x such that:

$$\begin{aligned}&\left|\phi(x) - \phi(x_0) - (x - x_0)\cdot \fint_{B(x_0,\theta)} \nabla\phi(z)\,\mathrm{d}z\right|\\ &= \left|\frac{x - x_0}{|B(x_0,\theta)|}\cdot\int_{B(x_0,\theta)}(\nabla\phi(z_x) - \nabla\phi(y))\,\mathrm{d}y\right|\\ &\le C(\gamma,\mu,\lambda,\Lambda,d)\,|x - x_0|^{1+\mu}\\ &\le C(\gamma,\mu,\lambda,\Lambda,d)\theta^{1+\mu}.\end{aligned}$$

Choosing θ small enough so that $C(\gamma,\mu,\lambda,\Lambda,d)\theta^{1+\mu} < \theta^{1+3\mu/4}$, we obtain (7). □

The fact that θ does not depend on the matrix $\mathbf{b}$ or the source term h is crucial for the contradiction argument in Proposition 1 below. We now state the interior Lipschitz estimate. Note that, here, $\mathbf{a}$ is not necessarily Hölder continuous in the domain Ω.

Proposition 1 (Interior Lipschitz Estimate I) *Suppose that* $\mathbf{a} \in \mathfrak{M}_{\text{per}}(\lambda,\Lambda)$ *and* $f \in L^\infty(\Omega)$. *Fix* $x_0 \in \Omega$ *and* $R > 0$, *so that* $B(x_0, R) \subset \Omega$. *There exist* $\varepsilon_0 = \varepsilon_0(\lambda,\Lambda,R,d) > 0$ *and* $C = C(\lambda,\Lambda,R,d) > 0$ *such that, for all* $0 < \varepsilon < \varepsilon_0$ *and for every weak solution* $\varphi^\varepsilon \in H^1(B(x_0,R))$ *of the equation* $-\operatorname{div}\left[\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla\varphi^\varepsilon\right] = f$ *in* $B(x_0,R)$, *the following estimate holds:*

$$\left\|\nabla\varphi^\varepsilon\right\|_{L^\infty(B(x_0,R/2))} \le C\left(\left\|\varphi^\varepsilon\right\|_{L^\infty(B(x_0,R))} + \|f\|_{L^\infty(B(x_0,R))}\right). \tag{9}$$

Proof By dilation, we may assume that $R = 1$. Fix $0 < \mu < \frac{1}{2}$. We prove, by the compactness method, that there exists $\varepsilon_0 = \varepsilon_0(\lambda,\Lambda,d,\mu)$ so that (9) holds for all $0 < \varepsilon < \varepsilon_0$. To do this, we only need to show that there exists $C > 0$, independent of ε, such that:

$$\max\left\{\left\|\varphi^\varepsilon\right\|_{L^\infty(B(x_0,1))}, \|f\|_{L^\infty(B(x_0,1))}\right\} \le 1 \text{ implies } \left\|\nabla\varphi^\varepsilon\right\|_{L^\infty(B(x_0,1/2))} \le C. \tag{10}$$

Let $\boldsymbol{\omega} := (\omega^1, \ldots, \omega^d)$, where $\omega^i \in H^1_{\text{per}}(Y)/\mathbb{R},\ 1 \le i \le d$, is the solution of the *cell problem*:

$$-\operatorname{div}_y \left[\mathbf{a}(y)\left(\mathrm{e}^i + \nabla_y \omega^i(y)\right)\right] = 0 \text{ in } Y. \tag{11}$$

1. *Improvement.* In this step, we prove by contradiction that:
   For fixed $0 < \mu < \frac{1}{2}$, there exist θ and ε^*, with $0 < \theta < \frac{1}{4},\ 0 < \varepsilon^* < 1$, depending on λ, Λ, d, and μ, such that if $\mathbf{a} \in \mathfrak{M}_{\text{per}}(\lambda, L),\ f \in L^\infty(B(x_0, 1))$, $\varphi^\varepsilon \in H^1(B(x_0, 1))$ satisfy:

$$-\operatorname{div}\left[\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla\varphi^\varepsilon\right] = f \text{ in } B(x_0, 1), \tag{12a}$$

$$\max\left\{\|\varphi^\varepsilon\|_{L^\infty(B(x_0,1))}, \|f\|_{L^\infty(B(x_0,1))}\right\} \le 1, \tag{12b}$$

then, for all $0 < \varepsilon < \varepsilon^*$, we have:

$$\sup_{|x-x_0|<\theta}\left|\varphi^\varepsilon(x) - \varphi^\varepsilon(x_0) - \left[x - x_0 + \varepsilon\boldsymbol{\omega}\left(\frac{x}{\varepsilon}\right)\right]\cdot \fint_{B(x_0,\theta)} \nabla\varphi^\varepsilon(z)\,\mathrm{d}z\right| \le \theta^{1+\mu/2}, \tag{13}$$

where $\boldsymbol{\omega}$ solves (11).

Take θ as in (7) of Lemma 1. By contradiction, suppose there exist sequences:

$$\varepsilon_n \to 0, \quad \mathbf{a}_n \in \mathfrak{M}_{\text{per}}(\lambda, L), \quad f_n \in L^\infty\left(B(x_0, 1)\right), \text{ and } \varphi_n \in H^1(B(x_0, 1))$$

satisfying:

$$-\operatorname{div}\left[\mathbf{a}_n\left(\frac{x}{\varepsilon_n}\right)\nabla\varphi_n\right] = f_n \text{ in } B(x_0, 1), \tag{14}$$

$$\max\left\{\|\varphi_n\|_{L^\infty(B(x_0,1))}, \|f_n\|_{L^\infty(B(x_0,1))}\right\} \le 1, \tag{15}$$

such that:

$$\sup_{|x-x_0|<\theta}\left|\varphi_n(x) - \varphi_n(x_0) - \left[x - x_0 + \varepsilon_n\boldsymbol{\omega}\left(\frac{x}{\varepsilon_n}\right)\right]\cdot \fint_{B(x_0,\theta)} \nabla\varphi_n(z)\,\mathrm{d}z\right| > \theta^{1+\mu/2}. \tag{16}$$

Let $\mathcal{A}_n \in \mathfrak{M}(\lambda, \Lambda)$ denote the effective matrix corresponding to $\mathbf{a}_n$. By the Banach–Alaoglu theorem, the Caccioppoli inequality, the Rellich–Kondrachov theorem, and the Schauder theorem (see, e.g., [19, Theorem 4.4] and [7, Theorem 3.16, 6.4 and 9.16]), there exist functions $\varphi_0 \in L^2(B(x_0, 1)),\ f_0 \in L^\infty(B(x_0, 1))$ and a constant matrix $\mathbf{a}_0 \in \mathfrak{M}(\lambda, \Lambda)$ such that, up to subsequences, we have:

$$\varphi_n \rightharpoonup \varphi_0 \text{ in } L^2(B(x_0, 1))$$
$$f_n \overset{*}{\rightharpoonup} f_0 \text{ in } L^\infty(B(x_0, 1))$$
$$\varphi_n \rightharpoonup \varphi_0 \text{ in } H^1(B(x_0, 1/2))$$
$$f_n \to f_0 \text{ in } H^{-1}(B(x_0, 1))$$
$$\mathcal{A}_n \to \mathbf{a}_0.$$

By [9, Theorem 13.4 (iii)] or [37, Theorem 2.3.2], we have φ_0 is the solution of:

$$-\operatorname{div}(\mathbf{a}_0 \nabla \varphi_0) = f_0 \text{ in } B(x_0, 1/2). \tag{17}$$

Fix $x \in B(x_0, 1)$ and let $U \subset B(x_0, 1)$ be an open neighborhood of x. By the De Giorgi-Nash-Moser Theorem [20, Theorem 8.24], there exists $0 < \beta = \beta(d, \lambda/\Lambda) < 1$ such that:

$$\|\varphi_n\|_{C^\beta(\overline{U})} \leq C \left(\|\varphi_n\|_{L^\infty(B(x_0,1))} + \|f_n\|_{L^\infty(B(x_0,1))} \right) \leq 2C.$$

By the Arzela–Ascoli Theorem, up to a subsequence, φ_n uniformly converges to φ^* in $C(U)$ for some φ^*. Since $\varphi_n \rightharpoonup \varphi_0$ in $L^2(B(x_0, 1))$, we conclude that $\varphi^* = \varphi_0$ a.e. in U. Therefore, $\lim_{n\to\infty} \varphi_n(x) = \varphi_0(x)$ a.e. in $B(x_0, 1)$. Letting $n \to \infty$ in (15), the argument above and [7, Theorem 3.13] yield:

$$\max \left\{ \|\varphi_0\|_{L^\infty(B(x_0,1))}, \|f_0\|_{L^\infty(B(x_0,1))} \right\} \leq 1,$$

which, together with (17), implies that (7) still holds for $\phi = \varphi_0$ (observe that, from (8), shrinking the domain from $B(x_0, 1)$ to $B(x_0, 1/2)$ does not affect estimates), that is:

$$\sup_{|x-x_0|<\theta} \left| \varphi_0(x) - \varphi_0(x_0) - (x - x_0) \cdot \fint_{B(x_0,\theta)} \nabla \varphi_0(z)\, \mathrm{d}z \right| < \theta^{1+3\mu/4}. \tag{18}$$

On the other hand, letting $n \to \infty$ in (16) and since $\|\boldsymbol{\omega}\|_{L^\infty(Y)} < \infty$, we obtain:

$$\sup_{|x-x_0|<\theta} \left| \varphi_0(x) - \varphi_0(x_0) - (x - x_0) \cdot \fint_{B(x_0,\theta)} \nabla \varphi_0(z)\, \mathrm{d}z \right| \geq \theta^{1+\mu/2},$$

which contradicts (18), since $0 < \theta < \frac{1}{4}$.

2. *Iteration* Let $0 < \varepsilon < \varepsilon^*$. Direct evaluation yields that:

$$P^\varepsilon(x) := \frac{1}{\theta^{1+\mu/2}} \left\{ \varphi^\varepsilon(\theta x) - \varphi^\varepsilon(\theta x_0) \right.$$

$$-\left[\theta(x-x_0)+\varepsilon\boldsymbol{\omega}\left(\frac{\theta x}{\varepsilon}\right)\right]\cdot \fint_{B(x_0,\theta)} \nabla\varphi^\varepsilon(z)\,\mathrm{d}z\Bigg\}$$

solves the following equation:

$$-\operatorname{div}\left[\mathbf{a}\left(\frac{\theta x}{\varepsilon}\right)\nabla P^\varepsilon(x)\right]=\tilde{f} \text{ in } B(x_0,1), \tag{19}$$

where $\tilde{f} := \theta^{1-\mu/2} f(\theta x)$. Moreover, by (13) and (12b), we have:

$$\left\| P^\varepsilon \right\|_{L^\infty(B(x_0,1))} \le 1 \text{ and } \left\| \tilde{f} \right\|_{L^\infty(B(x_0,1))} \le 1,$$

so by using (13) again, we obtain:

$$\sup_{|x-x_0|<\theta} \left| P^\varepsilon(x) - P^\varepsilon(x_0) - \left[x - x_0 + \frac{\varepsilon}{\theta}\boldsymbol{\omega}\left(\frac{\theta x}{\varepsilon}\right)\right]\cdot \fint_{B(x_0,\theta)} \nabla P^\varepsilon \right| (z)\,\mathrm{d}z \le \theta^{1+\mu/2}. \tag{20}$$

From (20) and scaling down, we have:

$$\sup_{|x-x_0|<\theta^2} \left|\varphi^\varepsilon(x) - \varphi^\varepsilon(x_0) - (x-x_0)\cdot a_2^\varepsilon + \varepsilon b_2^\varepsilon\right| \le \theta^{2(1+\mu/2)},$$

where

$$\begin{aligned} a_2^\varepsilon &:= \fint_{B(x_0,\theta)} \nabla\varphi^\varepsilon(z)\,\mathrm{d}z + \theta^{\mu/2} \fint_{B(x_0,\theta)} \nabla P^\varepsilon(z)\,\mathrm{d}z, \\ b_2^\varepsilon(y) &:= \boldsymbol{\omega}(y)\cdot\left(\fint_{B(x_0,\theta)} \nabla\varphi^\varepsilon(z)\,\mathrm{d}z + \theta^{\mu/2}\fint_{B(x_0,\theta)} \nabla P^\varepsilon(z)\,\mathrm{d}z\right), \text{ for } y := \frac{x}{\varepsilon} \in Y. \end{aligned} \tag{21}$$

By the De Giorgi-Nash-Moser estimate and Caccioppoli inequality [2, Lemma C.2], there exists a constant C, depending only on λ, Λ, and d, such that:

$$\|\boldsymbol{\omega}\|_{L^\infty(Y)} \le C, \qquad \left|\fint_{B(x_0,\theta)} \nabla\varphi^\varepsilon(z)\,\mathrm{d}z\right| \le C/\theta, \qquad \left|\fint_{B(x_0,\theta)} \nabla P^\varepsilon(z)\,\mathrm{d}z\right| \le C/\theta.$$

Therefore, (21) implies that:

$$\begin{aligned} \left|a_2^\varepsilon\right| &\le (C/\theta)\left(1+\theta^{\mu/2}\right), \\ \left\|b_2^\varepsilon\right\|_{L^\infty(Y)} &\le (C/\theta)\left(1+\theta^{\mu/2}\right). \end{aligned}$$

Reiterating this process, we obtain that there exists $C = C(\gamma, \lambda, \Lambda, d, \mu) > 0$ such that:

$$\begin{aligned} \left|a_k^\varepsilon\right| &\le (C/\theta)\left(1+\theta^{\mu/2}+\cdots+\theta^{(k-1)\mu/2}\right), \\ \left\|b_k^\varepsilon\right\|_{L^\infty(Y)} &\le (C/\theta)\left(1+\theta^{\mu/2}+\cdots+\theta^{(k-1)\mu/2}\right), \end{aligned} \tag{22}$$

and:

$$\sup_{|x-x_0|<\theta^k} \left|\varphi^\varepsilon(x)-\varphi^\varepsilon(x_0)-(x-x_0)\cdot a_k^\varepsilon+\varepsilon b_k^\varepsilon\right| \le \theta^{k(1+\mu/2)}. \tag{23}$$

3. *Blowup* Let $\varepsilon_0 := \min\left\{\varepsilon^*, \frac{1}{5\sqrt{d}}\right\}$ and $0 < \varepsilon < \varepsilon_0$.

Choose k such that $\theta^{k+1} \le 4\varepsilon\sqrt{d} < \theta^k$. Then from (23), there exists $C = C(\theta, d) > 0$ so that:

$$\sup_{|x-x_0|<4\varepsilon\sqrt{d}} \left|\varphi^\varepsilon(x)-\varphi^\varepsilon(x_0)-(x-x_0)\cdot a_k^\varepsilon+\varepsilon b_k^\varepsilon\right| \le C\varepsilon^{1+\mu/2}, \tag{24}$$

which, together with (22), leads to:

$$\left\|\varphi^\varepsilon-\varphi^\varepsilon(x_0)\right\|_{L^\infty(B(x_0,4\varepsilon\sqrt{d}))} \le C\varepsilon. \tag{25}$$

Denote by z_0^ε the center of the cell Y_i^ε containing x_0, and define:

$$v^\varepsilon(x) := \frac{1}{\varepsilon}\left[\varphi^\varepsilon(\varepsilon x+z_0^\varepsilon)-\varphi^\varepsilon(\varepsilon x_0+z_0^\varepsilon)\right], \quad x \in \Omega.$$

Then, $\nabla v^\varepsilon(x) = \nabla\varphi^\varepsilon\left(\varepsilon x+z_0^\varepsilon\right)$ and, moreover, v^ε solves:

$$-\operatorname{div}\left[\mathbf{a}\left(x+\frac{z_0^\varepsilon}{\varepsilon}\right)\nabla v^\varepsilon(x)\right] = \varepsilon f(\varepsilon x+z_0^\varepsilon) \quad \text{in } B\left(0, 3\sqrt{d}\right). \tag{26}$$

Observe that:

$$\begin{aligned} \frac{1}{\varepsilon}\left(B\left(x_0, \varepsilon\sqrt{d}\right)-z_0^\varepsilon\right) &\subset B\left(0, 2\sqrt{d}\right) \\ &\subset B\left(0, 3\sqrt{d}\right) \subset \frac{1}{\varepsilon}\left(B\left(x_0, 4\varepsilon\sqrt{d}\right)-z_0^\varepsilon\right). \end{aligned} \tag{27}$$

Applying [29, Theorem 1.1] to (26), we obtain that there exists a constant $C > 0$, independent of ε and x_0, such that:

$$
\begin{aligned}
&\left\|\nabla v^{\varepsilon}\right\|_{L^{\infty}\left(\frac{1}{\varepsilon}\left(B\left(x_0,\varepsilon\sqrt{d}\right)-z_0^{\varepsilon}\right)\right)} \\
&\leq \left\|\nabla v^{\varepsilon}\right\|_{L^{\infty}\left(B\left(0,2\sqrt{d}\right)\right)} \\
&\leq C\left[\left\|v^{\varepsilon}\right\|_{L^{\infty}\left(B\left(0,3\sqrt{d}\right)\right)}+\sup_{x\in B\left(0,3\sqrt{d}\right)}\left|\varepsilon f(\varepsilon x+z_0^{\varepsilon})\right|\right] \qquad (28)\\
&\leq C\left[\left\|v^{\varepsilon}\right\|_{L^{\infty}\left(\frac{1}{\varepsilon}\left(B\left(x_0,4\varepsilon\sqrt{d}\right)-z_0^{\varepsilon}\right)\right)}+\|f\|_{L^{\infty}\left(\frac{1}{\varepsilon}\left(B\left(x_0,4\varepsilon\sqrt{d}\right)-z_0^{\varepsilon}\right)\right)}\right].
\end{aligned}
$$

Scaling down (28) and using (25), we obtain:

$$
\begin{aligned}
&\left\|\nabla \varphi^{\varepsilon}\right\|_{L^{\infty}\left(B\left(x_0,\varepsilon\sqrt{d}\right)\right)} \\
&\leq C\left[\frac{1}{\varepsilon}\left\|\varphi^{\varepsilon}-\varphi^{\varepsilon}(x_0)\right\|_{L^{\infty}\left(\left(B\left(x_0,4\varepsilon\sqrt{d}\right)\right)\right)}+\|f\|_{L^{\infty}\left(B\left(x_0,4\varepsilon\sqrt{d}\right)\right)}\right] \leq C,
\end{aligned}
$$

where $C > 0$ is independent of x_0 and ε.

Remark 2 Under stronger smoothness assumptions on the coefficient **a**, similar estimates to (6) are proved in the literature. In particular, if **a** is in VMO($\mathbb{R}^d$), the real-variable method of L. Caffarelli and I. Peral [8] yields an uniform $W^{1,p}-$estimate; on the other hand, if **a** is Hölder continuous, then one has the uniform Lipschitz estimate. Those results hold also for elliptic systems and even for Neumann boundary condition. We refer the reader to [3, 4, 27, 28, 34–37] and the references cited therein.

However, in this paper, we focus on the case when **a** is only piecewise Hölder continuous. A similar argument as in the papers cited above together with the regularity theorem of Li and Vogelius [29, Theorem 1.1] yield the interior Lipschitz estimate as showed in Proposition 1. Moreover, some additional care is needed to ensure that the constant C in (28) is independent of *both* ε and x_0. In the Blow-up step of the proof above, one may try to let

$$
s^{\varepsilon}(x) := \frac{1}{\varepsilon}\varphi^{\varepsilon}(\varepsilon x + x_0) \qquad (29)
$$

so that

$$
-\operatorname{div}\left[\mathbf{a}(x)\nabla s^{\varepsilon}(x)\right] = \varepsilon f(\varepsilon x + x_0), \qquad (30)
$$

then by applying [29, Theorem 1.1], one obtains

$$
\left\|\nabla s^{\varepsilon}\right\|_{L^{\infty}\left(B\left(0,\frac{1}{2\varepsilon_0}\right)\right)} \leq C'\left[\left\|s^{\varepsilon}\right\|_{L^{\infty}\left(B\left(0,\frac{1}{\varepsilon_0}\right)\right)}+\|f\|_{L^{\infty}\left(B\left(0,\frac{1}{\varepsilon_0}\right)\right)}\right].
$$

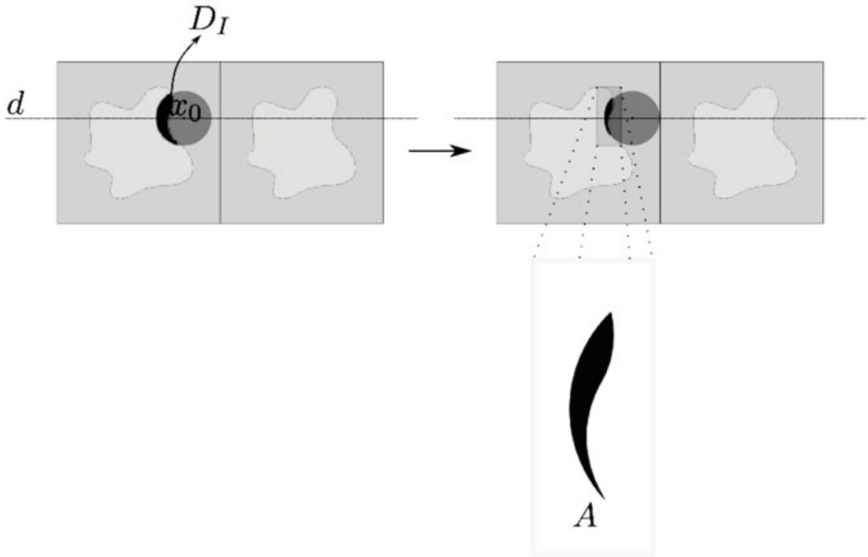

Fig. 2 As the center x_0 of the ball $B\left(x_0, \frac{1}{\varepsilon_0}\right)$ slides on the line d to the right, the subdomain D_I shrinks to 0, which makes the $C^{1,\alpha}$ modulus to become unbounded [29, page 93]. Moreover, in some cases, it is possible that a cusp also appears at some points (point A on the zoomed in figure above)

However, C' *indeed depends on* x_0. The reason is that, when one shifts x_0 in the scaling (29), one also changes the $C^{1,\alpha}$-modulus of the subdomains, where the latter, in the context of our problem, are generated by taking intersections of the ball centered at x_0 and the heterogenous domain. In short, we do not have the uniform control of the subdomains when using the scaling (29) for arbitrary x_0, see Fig. 2. In order to circumvent the dependence on x_0, we use a different scaling and combine with a geometric argument, as demonstrated in the proof of Proposition 1.

The following result follows from Proposition 1, the De Giorgi-Nash-Moser estimate and a change of variable.

Proposition 2 (Interior Lipschitz Estimate II) *Suppose that* $\mathbf{a} \in \mathfrak{M}_{\text{per}}(\lambda, \Lambda)$ *and* $f \in L^\infty(\Omega)$. *Fix* $x_0 \in \Omega$ *and* $R > 0$ *so that* $B(x_0, R) \subset \Omega$. *There exist* $\varepsilon_0 = \varepsilon_0(\lambda, \Lambda, d) > 0$ *and* $C = C(\lambda, \Lambda, d) > 0$ *such that, for all* $0 < \varepsilon < \varepsilon_0$, *the weak solution* $\varphi^\varepsilon \in H^1(B(x_0, R))$ *of the equation* $-\operatorname{div}\left[\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla\varphi^\varepsilon\right] = f$ *in* $B(x_0, R)$ *satisfies:*

$$\left|\nabla\varphi^\varepsilon(x_0)\right| \leq C'\left[\left(\fint_{B(x_0,R)}\left|\nabla\varphi^\varepsilon(z)\right|^2 \mathrm{d}z\right)^{\frac{1}{2}} + R\sup_{x\in B(x_0,R)}|f(x)|\right]. \tag{31}$$

Proof Without loss of generality, we assume $x_0 = 0$. By Proposition 1, with $R = 1$ and considering $\varphi^\varepsilon - \fint_{B(0,1)} \varphi^\varepsilon(z)\,\mathrm{d}z$, which solves $-\operatorname{div}\left[\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla\varphi^\varepsilon\right] = f$ in $B(0,1)$, we have that there exist $\varepsilon_0 > 0$ and $C' > 0$, depending only on λ, Λ, d, such that:

$$\begin{aligned}
\left|\nabla\varphi^\varepsilon(0)\right| &\leq \left\|\nabla\varphi^\varepsilon\right\|_{L^\infty(B(0,1/4))} \\
&\leq C'\left(\left\|\varphi^\varepsilon - \fint_{B(0,1)} \varphi^\varepsilon(z)\,\mathrm{d}z\right\|_{L^\infty(B(0,1/2))} + \|f\|_{L^\infty(B(0,1/2))}\right) \\
&\leq C'\left(\left\|\varphi^\varepsilon - \fint_{B(0,1)} \varphi^\varepsilon(z)\,\mathrm{d}z\right\|_{L^2(B(0,1))} + \|f\|_{L^\infty(B(0,1))}\right) \qquad (32) \\
&\leq C'\left(\left\|\nabla\varphi^\varepsilon\right\|_{L^2(B(0,1))} + \|f\|_{L^\infty(B(0,1))}\right) \\
&\leq C'\left[\left(\fint_{B(0,1)} \left|\nabla\varphi^\varepsilon(z)\right|^2 \mathrm{d}z\right)^{\frac{1}{2}} + \sup_{x\in B(0,1)} |f(x)|\right],
\end{aligned}$$

where we have used the De Giorgi-Nash-Moser estimate and Poincaré's inequality.

For $R > 0$ and $x \in B(0,1)$, let $v^\varepsilon(x) := R^{-1}\varphi^\varepsilon(Rx)$, then $\nabla v^\varepsilon(x) = \nabla\varphi^\varepsilon(Rx)$ and:

$$-\operatorname{div}\left[\mathbf{b}\left(\frac{x}{\varepsilon}\right)\nabla v^\varepsilon(x)\right] = Rf(Rx),$$

where $\mathbf{b}(z) := \mathbf{a}(Rz)$. We have $\mathbf{b} \in \mathfrak{M}(\lambda,\Lambda)$ is $R^{-1}Y-$periodic. Note that the proof of Proposition 1 does not depend on the period, hence, (32) holds for v^ε in particular:

$$\begin{aligned}
\left|\nabla\varphi^\varepsilon(0)\right| &= \left|\nabla v^\varepsilon(0)\right| \\
&\leq C'\left[\left(\fint_{B(0,1)} \left|\nabla v^\varepsilon(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} + R\sup_{x\in B(0,1)} |f(Rx)|\right] \qquad (33) \\
&= C'\left[\left(\fint_{B(0,1)} \left|\nabla\varphi^\varepsilon(Rx)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} + R\sup_{x\in B(0,1)} |f(Rx)|\right].
\end{aligned}$$

By a change of variable in (33), we obtain (31). □

We recall the interior Hölder estimate, adapted from [34, Proposition 1] (or [3, Lemma 9]) that will be used to obtain the boundary Hölder estimate in the next section.

Proposition 3 (Interior Hölder Estimate) *Suppose that* $\mathbf{a} \in \mathfrak{M}_{\text{per}}(\lambda,\Lambda)$ *and* $f \in L^{d+\gamma}(\Omega)$*, for some* $\gamma > 0$*. Fix* $x_0 \in \Omega$ *and* $R > 0$ *such that* $B(x_0, R) \subset \Omega$*. Let*

$0 < \mu := \frac{\gamma}{d+\gamma} < 1$. *There exists* $C = C(\gamma, \lambda, \Lambda, p, R, d) > 0$ *such that, for all* $\varepsilon > 0$*, the weak solution* $\varphi^\varepsilon \in H^1(B(x_0, R))$ *of the equation* $-\operatorname{div}\left[\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla\varphi^\varepsilon\right] = f$ *in* $B(x_0, R)$ *satisfies:*

$$\left[\varphi^\varepsilon\right]_{C^{0,\mu}(B(x_0,R/2))} \leq C\left(\left\|\varphi^\varepsilon\right\|_{L^2(B(x_0,R))} + \|f\|_{L^{d+\gamma}(B(x_0,R))}\right), \tag{34}$$

where $[h]_{C^{0,\mu}(A)} := \sup_{x\neq y\in A} \frac{|h(x)-h(y)|}{|x-y|^\mu}$.

The proof of Proposition 3 is similar to [34, Proposition 1]. Indeed, a closer look at the proof of [34, Proposition 1] reveals that the Hölder continuity assumption on **a** is needed only for the classical Schauder estimate for elliptic systems to hold. However, this paper is devoted to the scalar case, and we use the De Giorgi-Nash-Moser Theorem [20, Theorem 8.24], for which the assumption **a** is bounded is sufficient.

Remark 3 For the case of elliptic systems, the De Giorgi-Nash-Moser Theorem does not hold, see counterexamples by De Giorgi, Giusti and Miranda, and others, cf. [19, Section 9.1], and the references cited therein. Because of that, this paper is concerned with the scalar case only.

5 Boundary Estimates, Green Functions, Dirichlet Correctors, and Proof of Main Theorem

The following result is adapted from [3, Section 2.3] and [37, Section 5.2].

Proposition 4 (Boundary Hölder Estimate) *Suppose that* $\mathbf{a} \in \mathfrak{M}_{\mathrm{per}}(\lambda, \Lambda)$*, and* Ω *is a* C^1*-domain. Fix* $x_0 \in \partial\Omega$*,* $0 < r < \operatorname{diam}(\Omega)$*, and* $0 < \mu < 1$*. Let* $g \in C^{0,1}\left(B(x_0, r) \cap \partial\Omega\right)$*. There exist* $\varepsilon_0 = \varepsilon_0(\mu, \lambda, \Lambda, d) > 0$ *and* $C = c(\mu, \lambda, \Lambda, d) > 0$ *such that, for all* $0 < \varepsilon < \varepsilon_0$*, every weak solution* $\varphi^\varepsilon \in H^1(B(x_0, r))$ *of the equation:*

$$-\operatorname{div}\left[\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla\varphi^\varepsilon\right] = 0 \text{ in } B(x_0, r) \cap \Omega,$$
$$\varphi^\varepsilon = g \text{ on } B(x_0, r) \cap \partial\Omega$$

satisfies:

$$\left[\varphi^\varepsilon\right]_{C^{0,\mu}(B(x_0,r/2)\cap\Omega)} \leq Cr^{-\mu}\left[\left(\fint_{B(x_0,r)\cap\Omega} \left|\varphi^\varepsilon(z)\right|^2 \mathrm{d}z\right)^{\frac{1}{2}} + |g(x_0)| + r\,\|g\|_{C^{0,1}(B(x_0,r)\cap\partial\Omega)}\right]. \tag{35}$$

The proof of Proposition 4 follows the proof of [37, Theorem 5.2.1] with minor modifications. In [37, Theorem 5.2.1], the assumption that $\mathbf{a} \in \mathrm{VMO}(\mathbb{R}^d)$ is used only in two places: (1) when $\varepsilon \geq \varepsilon_0$, which is beyond the scope of this particular theorem, and (2) to obtain the interior Hölder estimate, which we already relaxed in Proposition 3.

Thanks to Propositions 3 and 4, we can drop the assumption that $\mathbf{a} \in \mathrm{VMO}(\mathbb{R}^d)$ of Theorem 5.4.1–2 and Lemma 5.4.5 in [37]. The results are summarized in the following proposition.

Proposition 5 (Green's Functions) *Suppose that* $\mathbf{a} \in \mathfrak{M}_{\mathrm{per}}(\lambda, \Lambda)$ *and* Ω *is a* C^1*-domain. Fix* $0 < \mu, \sigma, \sigma_1 < 1$ *and let* $\delta(x) := \mathrm{dist}(x, \partial\Omega)$. *Then, there exist* $\varepsilon_0 = \varepsilon_0(\mu, \lambda, \Lambda, d) > 0$ *and* $C = C(\lambda, \Lambda, \sigma, \sigma_1, \Omega) > 0$ *such that, for all* $0 < \varepsilon < \varepsilon_0$, *the Green's functions* $G^\varepsilon(x, y)$ *exist and satisfy the following:*

$$\left|G^\varepsilon(x, y)\right| \leq \begin{cases} C\frac{1}{|x-y|^{d-2}} & \text{if } d \geq 3, \\ C\left[1 + \ln\left(\frac{r_0}{|x-y|}\right)\right] & \text{if } d = 2. \end{cases} \tag{36a}$$

$$\left|G^\varepsilon(x, y)\right| \leq \begin{cases} \frac{C\delta(x)^\sigma}{|x-y|^{d-2+\sigma}} & \text{if } \delta(x) < \frac{1}{2}|x-y|, \\ \frac{C\delta(y)^{\sigma_1}}{|x-y|^{d-2+\sigma_1}} & \text{if } \delta(y) < \frac{1}{2}|x-y|, \\ \frac{C\delta(x)^\sigma\delta(y)^{\sigma_1}}{|x-y|^{d-2+\sigma+\sigma_1}} & \text{if } \delta(x) < \frac{1}{2}|x-y| \text{ or } \delta(y) < \frac{1}{2}|x-y|, \end{cases} \tag{36b}$$

$$\int_\Omega \left|\nabla_y G^\varepsilon(x, y)\right| \delta(y)^{\sigma-1}\,\mathrm{d}y \leq C\delta(x)^\sigma, \tag{36c}$$

where $x, y \in \Omega$, $x \neq y$ *and* $r_0 := \mathrm{diam}(\Omega)$.

As a consequence, for $0 < c < 1$ *and* $g \in C^{0,1}(\Omega)$, *there exists* $C = C(\lambda, \Lambda, d) > 0$ *such that, for any* $x_0 \in \partial\Omega$, *for any* ε *satisfying* $c\varepsilon \leq \min\{c\varepsilon_0, r\} \leq r < r_0 := \mathrm{diam}(\Omega)$, *and for any solution* φ^ε *of the Dirichlet problem* $-\operatorname{div}\left[\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla\varphi^\varepsilon\right] = 0$ *in* Ω, $\varphi^\varepsilon = g$ *on* $\partial\Omega$, *the following estimate holds:*

$$\left(\fint_{B(x_0,r)\cap\Omega} \left|\nabla\varphi^\varepsilon\right|^2\right)^{\frac{1}{2}} \leq C\left[\|\nabla g\|_{L^\infty(\Omega)} + \varepsilon^{-1}\|g\|_{L^\infty(\Omega)}\right]. \tag{37}$$

We now define the boundary Dirichlet corrector: For $1 \leq i \leq d$, let $\Phi^{i,\varepsilon} \in H^1(\Omega)$ be the solution of the problem:

$$\begin{aligned} -\operatorname{div}\left[\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla\Phi^{i,\varepsilon}(x)\right] &= 0 \text{ for } x \in \Omega, \\ \Phi^{i,\varepsilon}(z) &= z_i \text{ for } z \in \partial\Omega. \end{aligned} \tag{38}$$

The following proposition provides a bound on the boundary Dirichlet corrector.

Proposition 6 *Let Ω be a bounded $C^{1,\alpha}$-domain. Suppose that $\mathbf{a} \in \mathfrak{M}_{\text{per}}(\lambda, L)$ is piecewise C^α-continuous. Then, for all $\varepsilon > 0$, the solution $\Phi^{i,\varepsilon}$ of (38) satisfies:*

$$\left\| \nabla \Phi^{i,\varepsilon} \right\|_{L^\infty(\Omega)} \leq C, \tag{39}$$

where constant C depends only on λ, Λ and Ω.

The proof is similar to [37, Theorem 5.4.4]. One only needs to use three observations:

- The case $c\varepsilon \geq \min\{c\varepsilon_0, r\}$ follows from [29, Theorem 1.2], by the same argument used in Remark 1.
- Let $\boldsymbol{\omega} = (\omega^1, \omega^2, \ldots, \omega^d)$ be the solutions of the cell problems (11). Given only that $\mathbf{a}$ is piecewise C^α-continuous, then $\nabla\boldsymbol{\omega}$ is bounded in $L^\infty(Y, \mathbb{R}^{d\times d})$, see the first paragraph in the proof of [17, Theorem 3.2] or [38, Corollary 3.5].
- The interior Lipschitz estimate in Proposition 2 only requires $\mathbf{a}$ is piecewise Hölder continuous.

We now combine Propositions 6, 2, and [29, Theorem 1.2] to obtain a discontinuous coefficient-version of [37, Theorem 5.5.1].

Proposition 7 (Boundary Lipschitz Estimate) *Suppose that $\mathbf{a} \in \mathfrak{M}_{\text{per}}(\lambda, L)$ is piecewise C^α-continuous and Ω is a $C^{1,\alpha}$-domain. Fix $x_0 \in \partial\Omega$, $0 < r <$ $\operatorname{diam}(\Omega)$ and $0 < \mu < 1$. Let $g \in C^{1,\alpha}\left(B(x_0, r) \cap \partial\Omega\right)$. There exist $\varepsilon_0 = \varepsilon_0(\mu, \lambda, \Lambda, d, \Omega) > 0$ and $C = C(\mu, \lambda, \Lambda, d, \Omega) > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, the weak solution $\varphi^\varepsilon \in H^1(B(x_0, r))$ of the equation:*

$$-\operatorname{div}\left[\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla\varphi^\varepsilon\right] = 0 \text{ in } B(x_0, r) \cap \Omega,$$
$$\varphi^\varepsilon = g \text{ on } B(x_0, r) \cap \partial\Omega$$

satisfies:

$$\begin{aligned} &\left\|\nabla\varphi^\varepsilon\right\|_{L^\infty(B(x_0,r/2)\cap\partial\Omega)} \\ &\leq C\left[r^{-1}\left(\fint_{B(x_0,r)\cap\Omega}\left|\varphi^\varepsilon\right|^2\right)^{\frac{1}{2}} + r^\alpha \left\|\nabla_{\tan} g\right\|_{C^{0,\alpha}(B(x_0,r)\cap\partial\Omega)}\right. \\ &\qquad \left. + \left\|\nabla_{\tan} g\right\|_{L^\infty(B(x_0,r)\cap\partial\Omega)} + r^{-1}\left\|g\right\|_{L^\infty(B(x_0,r)\cap\partial\Omega)}\right]. \end{aligned} \tag{40}$$

The estimate (5) of Theorem 1 is a consequence of Propositions 2 and 7, by an argument similar to [37, Theorem 5.6.2].

6 Application to Magnetic Suspensions

In this section, we apply the regularity results obtained above to the rigorous homogenization procedure discussed in [10]. For that, we first recap the formulation of the fine-scale problem and the homogenization result itself. We begin by introducing the definition of two-scale convergence, which will be used below.

Definition 1 A sequence $\{v^\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega)$ is said to *two-scale converge* to $v = v(x, y)$, with $v \in L^2(\Omega \times Y)$, if and only if:

$$\lim_{\varepsilon\to 0} \int_\Omega v^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \frac{1}{|Y|} \int_\Omega \int_Y v(x, y)\psi(x, y)\, \mathrm{d}y\, \mathrm{d}x,$$

for any test function $\psi = \psi(x, y)$, with $\psi \in \mathcal{D}(\Omega, C^\infty_{\text{per}}(Y))$, see [1, 9, 32]. In this case, we write $v^\varepsilon \xrightarrow{2} v$.

Let the kinematic viscosity be denoted by $\nu = \frac{\eta}{\rho_f}$, where $\eta > 0$ and $\rho_f > 0$ are the fluid viscosity and the fluid density, respectively. The dimensionless quantities that appear in this problem are the (hydrodynamic) *Reynolds number* $R_e = UL/\nu$, the *Froude number* $F_r = U/\sqrt{FL}$, and the *coupling parameter* $S = \frac{B^2}{\rho_f \Lambda U^2}$, where L, U, B, and F are the characteristic scales corresponding to length, fluid velocity, magnetic field, and body density force, respectively. Moreover, $\Lambda > 0$ is defined in (A2).

From now on, we suppose Ω is $C^{3,\alpha}$, which is needed for the corrector result below. Suppose further that $\mathbf{g} \in H^1(\Omega, \mathbb{R}^d)$, $k \in C^{1,\alpha}(\partial\Omega)$, and $f \in L^\infty(\Omega)$. Let $\mathbf{u}^\varepsilon$ and p^ε be the fluid velocity and the fluid pressure, respectively. Also, in a space free of current, the magnetic field strength is given by $\mathbf{H}^\varepsilon = \nabla\varphi^\varepsilon$, for some magnetic potential $\varphi^\varepsilon(x)$. Let $\mathbf{u}^\varepsilon \in H^1_0(\Omega, \mathbb{R}^d)$, $p^\varepsilon \in L^2(\Omega)/\mathbb{R}$, and $\varphi^\varepsilon \in H^1(\Omega)$ be the solution of the following boundary value problem:

$$-\operatorname{div}\left[\boldsymbol{\sigma}(\mathbf{u}^\varepsilon, p^\varepsilon) + \boldsymbol{\tau}(\varphi^\varepsilon)\right] = \frac{1}{F_r^2}\mathbf{g}, \qquad \text{in } \Omega^\varepsilon_f \tag{41a}$$

$$\operatorname{div} \mathbf{u}^\varepsilon = 0, \qquad \text{in } \Omega^\varepsilon_f \tag{41b}$$

$$\mathbb{D}(\mathbf{u}^\varepsilon) = 0, \qquad \text{in } \Omega^\varepsilon_s \tag{41c}$$

$$-\operatorname{div}\left[\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla\varphi^\varepsilon\right] = f \qquad \text{in } \Omega, \tag{41d}$$

together with the balance equations:

$$\int_{\Gamma^\varepsilon_i} \left[\boldsymbol{\sigma}(\mathbf{u}^\varepsilon, p^\varepsilon) + \boldsymbol{\tau}(\varphi^\varepsilon)\right] \mathbf{n}_i \, \mathrm{d}\mathcal{H}^{d-1} = 0, \tag{42a}$$

$$\int_{\Gamma_i^\varepsilon} \left(\left[\boldsymbol{\sigma}(\mathbf{u}^\varepsilon, p^\varepsilon) + \boldsymbol{\tau}(\varphi^\varepsilon)\right]\mathbf{n}_i\right) \times \mathbf{n}_i \, \mathrm{d}\mathcal{H}^{d-1} = 0, \tag{42b}$$

and boundary conditions:

$$\mathbf{u}^\varepsilon = 0, \text{ on } \partial\Omega, \tag{43a}$$

$$\varphi^\varepsilon = k, \text{ on } \partial\Omega, \tag{43b}$$

where

$$\boldsymbol{\sigma}(\mathbf{u}^\varepsilon, p^\varepsilon) := \frac{2}{R_e}\mathbb{D}(\mathbf{u}^\varepsilon) - p^\varepsilon \mathbf{I}, \tag{44a}$$

$$\mathbb{D}(\mathbf{u}^\varepsilon) := \frac{\nabla \mathbf{u}^\varepsilon + \nabla^\top \mathbf{u}^\varepsilon}{2}, \tag{44b}$$

$$\boldsymbol{\tau}(\varphi^\varepsilon) := S\mathbf{a}\left(\frac{x}{\varepsilon}\right)\left(\nabla\varphi^\varepsilon \otimes \nabla\varphi^\varepsilon - \frac{1}{2}\left|\nabla\varphi^\varepsilon\right|^2 \mathbf{I}\right) \tag{44c}$$

are the *rate of strain*, the *Cauchy stress*, and the *Maxwell stress* tensors, respectively. For the detailed derivation and the physical meaning of the equations above, we refer the readers to [10] and the references therein. Observe that, in the context of this paper, we consider the Dirichlet boundary condition (43b), instead of a Neumann boundary condition (3) in [10], to relax the regularity assumption on the magnetic permeability needed in [10]. Then, the weak formulation for (41d) and (43b) is given by:

$$\begin{aligned} &\int_\Omega \mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla\left(\varphi^\varepsilon - k\right)\cdot\nabla\xi \, \mathrm{d}x \\ &\qquad = -\int_\Omega \mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla k\cdot\nabla\xi\,\mathrm{d}x + \int_\Omega f\xi\,\mathrm{d}x, \quad \forall \xi \in H_0^1(\Omega). \end{aligned} \tag{45}$$

One immediately has that $\|\varphi^\varepsilon\|_{H^1(\Omega)} \leq C\left(\|k\|_{H^{1/2}(\partial\Omega)} + \|f\|_{L^q(\Omega)}\right)$, which implies that φ^ε is two-scale convergent (up to a subsequence). Choosing a test function as in [10, Lemma 3.7], we obtain the cell problem (46) and the first two effective equations defined in (50) below.

Moreover, Theorem 1 ensures that $\nabla\varphi^\varepsilon$ is uniformly bounded in $L^\infty(\Omega, \mathbb{R}^d)$, with respect to $\varepsilon \in (0, \varepsilon_0)$. Therefore, we obtain the existence, uniqueness and a priori bounds for $\mathbf{u}^\varepsilon$ and p^ε as in [10, Corollary 3.11]. Here, we have relaxed the restrictive assumption (1) made in [10] and we can use our results in the case when the constant magnetic permeability is anisotropic, namely when $\mathbf{a}$ is a matrix.

To carry on with the homogenization formulation, for $1 \le i, j \le d$, denote by $\mathbf{U}^{ij}$ the vector defined by $\mathbf{U}^{ij}_k := y_j \delta_{ik}$. Consider $\omega^i \in H^1_{\text{per}}(Y)/\mathbb{R}$, the solution of:

$$-\operatorname{div}_y \left[\mathbf{a}(y)\left(\mathrm{e}^i + \nabla_y \omega^i(y)\right)\right] = 0 \text{ in } Y. \tag{46}$$

Also, consider $\boldsymbol{\chi}^{ij} \in H^1_{\text{per}}(Y, \mathbb{R}^d)/\mathbb{R}$ and $q^{ij} \in L^2(Y)/\mathbb{R}$, solving:

$$\begin{aligned}
\operatorname{div}_y \left[\mathbb{D}_y\left(\mathbf{U}^{ij} - \boldsymbol{\chi}^{ij}\right) + q^{ij}\mathbf{I}\right] &= 0 \text{ in } Y_f,\\
\operatorname{div}_y \boldsymbol{\chi}^{ij} &= 0 \text{ in } Y,\\
\mathbb{D}_y\left(\mathbf{U}^{ij} - \boldsymbol{\chi}^{ij}\right) &= 0 \text{ in } Y_s,\\
\int_\Gamma \left[\mathbb{D}_y\left(\mathbf{U}^{ij} - \boldsymbol{\chi}^{ij}\right) - q^{ij}\mathbf{I}\right]\mathbf{n}_\Gamma \, \mathrm{d}\mathcal{H}^{d-1} &= 0,\\
\int_\Gamma \left[\mathbb{D}_y\left(\mathbf{U}^{ij} - \boldsymbol{\chi}^{ij}\right) - q^{ij}\mathbf{I}\right]\mathbf{n}_\Gamma \times \mathbf{n}_\Gamma \, \mathrm{d}\mathcal{H}^{d-1} &= 0,
\end{aligned} \tag{47}$$

and consider $\boldsymbol{\xi}^{ij} \in H^1_{\text{per}}(Y, \mathbb{R}^d)/\mathbb{R}$ and $r^{ij} \in L^2(Y)/\mathbb{R}$, solving:

$$\begin{aligned}
\operatorname{div}_y \left[\mathbb{D}_y\left(\boldsymbol{\xi}^{ij}\right) + r^{ij}\mathbf{I} + \boldsymbol{\tau}^{ij}\right] &= 0 \text{ in } Y_f,\\
\operatorname{div}_y \boldsymbol{\xi}^{ij} &= 0 \text{ in } Y,\\
\mathbb{D}_y\left(\boldsymbol{\xi}^{ij}\right) &= 0 \text{ in } Y_s,\\
\int_\Gamma \left[\mathbb{D}_y\left(\boldsymbol{\xi}^{ij}\right) + r^{ij}\mathbf{I} + \boldsymbol{\tau}^{ij}\right]\mathbf{n}_\Gamma \, \mathrm{d}\mathcal{H}^{d-1} &= 0,\\
\int_\Gamma \left[\mathbb{D}_y\left(\boldsymbol{\xi}^{ij}\right) + r^{ij}\mathbf{I} + \boldsymbol{\tau}^{ij}\right]\mathbf{n}_\Gamma \times \mathbf{n}_\Gamma \, \mathrm{d}\mathcal{H}^{d-1} &= 0.
\end{aligned} \tag{48}$$

We also define:

$$\begin{aligned}
\mathcal{A}_{jk} &:= \frac{1}{|Y|}\int_Y \mathbf{a}(y)(\mathrm{e}^k + \nabla\omega^k(y)) \cdot (\mathrm{e}^j + \nabla\omega^j(y))\,\mathrm{d}y,\\
\mathcal{N}^{ij}_{mn} &:= \frac{1}{|Y|}\int_Y \mathbb{D}_y(\mathbf{U}^{ij} - \boldsymbol{\chi}^{ij}) : \mathbb{D}_y(\mathbf{U}^{mn} - \boldsymbol{\chi}^{mn})\,\mathrm{d}y,\\
\boldsymbol{\tau}^{ij}_{\text{ref}} &:= \mathbf{a}(y)\left[(\mathrm{e}^i + \nabla_y\omega^i) \otimes (\mathrm{e}^j + \nabla_y\omega^j) - \frac{1}{2}(\mathrm{e}^i + \nabla_y\omega^i)\cdot(\mathrm{e}^j + \nabla_y\omega^j)\mathbf{I}\right],\ y \in Y,\\
\mathcal{B}^{ij} &:= \frac{1}{|Y|}\int_Y \left(\mathbb{D}_y(\boldsymbol{\xi}^{ij}) + \boldsymbol{\tau}^{ij}\right)\mathrm{d}y,
\end{aligned} \tag{49}$$

where $\mathcal{A}$ is the *effective magnetic permeability*, which is symmetric and elliptic. The tensor $\mathcal{N} := \left\{\mathcal{N}_{mn}^{ij}\right\}_{1\le i,j,m,n\le d}$ is the *effective viscosity*, and it is a fourth rank tensor. Moreover, $\mathcal{N}$ is symmetric, i.e., $\mathcal{N}_{mn}^{ij} = \mathcal{N}_{ij}^{mn} = \mathcal{N}_{mn}^{ji} = \mathcal{N}_{nm}^{ij}$, and it satisfies the Legendre-Hadamard condition (or strong ellipticity condition), i.e., there exist $\beta > 0$ such that, for all $\zeta, \eta \in \mathbb{R}^d$, one has $\mathcal{N}_{mn}^{ij}\zeta_i\zeta_m\eta_j\eta_n \ge \beta\,|\zeta|^2\,|\eta|^2$. The matrix $\boldsymbol{\tau}_{\text{ref}}$ is the *Maxwell stress tensor* on Y, and $\mathcal{B}$ is the *effective coupling matrix*.

By the same argument as in Theorem 3.5, Lemma 3.9, and Lemma 3.14 of [10], the following result holds:

Theorem 2 *Let* $(\varphi^\varepsilon, \mathbf{u}^\varepsilon, p^\varepsilon) \in H^1(\Omega) \times H_0^1(\Omega, \mathbb{R}^d) \times L_0^2(\Omega)$ *be the solution of* (41). *Then*

$$\varphi^\varepsilon \rightharpoonup \varphi^0 \text{ in } H^1(\Omega),$$
$$\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u}^0 \text{ in } H_0^1(\Omega, \mathbb{R}^d),$$
$$p^\varepsilon \rightharpoonup \pi^0 \text{ in } L_0^2(\Omega),$$

where $\varphi^0, \mathbf{u}^0$*, and* π^0 *are solutions of:*

$$\begin{aligned} -\operatorname{div}\left(\mathcal{A}\nabla\varphi^0\right) &= f && \text{in } \Omega,\\ \varphi^0 &= k && \text{on } \partial\Omega,\\ \operatorname{div}\left[\frac{2}{R_e}\mathcal{N}^{ij}\mathbb{D}\left(\mathbf{u}^0\right)_{ij} - \pi^0 + S\mathcal{B}^{ij}\frac{\partial\varphi^0}{\partial x_i}\frac{\partial\varphi^0}{\partial x_j}\right] &= \frac{1}{F_r^2}\mathbf{g} && \text{in } \Omega,\\ \operatorname{div}\mathbf{u}^0 &= 0 && \text{in } \Omega\\ \mathbf{u}^0 &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{50}$$

with $\mathcal{A}, \mathcal{N}^{ij}, \mathcal{B}^{ij}$, $1 \le i, j \le d$*, defined in* (49). *Moreover, the first-order correctors satisfy:*

$$\lim_{\varepsilon\to 0}\left\|\nabla\varphi^\varepsilon(\cdot) - \nabla\varphi^0(\cdot) - \nabla_y\varphi^1\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega,\mathbb{R}^d)} = 0,$$
$$\lim_{\varepsilon\to 0}\left\|\mathbb{D}(\mathbf{u}^\varepsilon)(\cdot) - \mathbb{D}(\mathbf{u}^0)(\cdot) - \mathbb{D}_y(\mathbf{u}^1)\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega,\mathbb{R}^{d\times d})} = 0,$$

where

$$\varphi^1(x, y) := \omega^i(y)\frac{\partial\varphi^0}{\partial x_i}(x),$$
$$\mathbf{u}^1(x, y) := -\mathbb{D}\left(\mathbf{u}^0(x)\right)_{ij}\boldsymbol{\chi}^{ij}(y) + S\frac{\partial\varphi^0}{\partial x_i}(x)\frac{\partial\varphi^0}{\partial x_j}(x)\boldsymbol{\xi}^{ij}(y).$$

7 Conclusions

This paper concerns a homogenized description of a non-dilute suspension of magnetic particles in a viscous flow. The results demonstrated in this paper generalize the ones obtained by the authors in [10], where a more restrictive assumption on the magnetic permeability (1) was used and a Neumann boundary condition (3) was imposed instead of the Dirichlet condition (4b). Theorem 2 above demonstrates the *effective response* of a viscous fluid with a locally periodic array of paramagnetic/diamagnetic particles suspended in it, given by the system of equations (41). The effective equations are described by (50), with the effective coefficients given in (49). These effective quantities depend only on the instantaneous position of the particles, their geometry, and the magnetic and flow properties of the original suspension described by (41). Using the tools introduced in [29] and the compactness method, an improved regularity estimate for the gradient of the magnetic potential of the original fine-scale problem (41) was obtained, see Theorem 1. This theorem allows us to drop the restrictive assumption (1) mentioned above. Comparing to the classical results on regularity of this type, we do require the coefficient matrix belongs to a VMO-space, see, e.g., [3, 34, 37]. Recently, in [17, Proposition 3.1], the authors obtained an L^q-bound of the gradient of the solution of the scalar divergence equation, uniform with respect to ε, for $q < \infty$. Our result, in Theorem 1, shows that the gradient bound actually holds for the case $q = \infty$.

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Appendix

Theorem 3 (Interior Schauder Estimates [20, 24]) *Let* $\mathbf{b} \in \mathfrak{M}(\lambda, \Lambda)$ *be a constant matrix and* $w \in H^1(\Omega)$ *be a weak solution of:*

$$\mathbf{b}_{ij} D_i D_j w = f + \sum_{i=1}^{d} D_i f_i .$$

For every $\alpha \in (0, 1)$*, there exists a uniform constant* $C = C(\alpha, d, \lambda, \Lambda)$ *such that if* $\Omega' \subset\subset \Omega$*, with* $\delta = \operatorname{dist}(\Omega', \partial\Omega)$*, then the following estimates hold:*

(i) If $f \in L^p(\Omega)$*,* $f_i \in L^q(\Omega)$ *and* $\alpha = 1 - \frac{d}{q} = 2 - \frac{d}{p} \in (0,1)$*, then* $w \in C^\alpha(\Omega')$ *and:*

$$\|w\|_{C^\alpha(\Omega')} \leq C\delta^{-\frac{d}{2}+1-\alpha}\left(\|f\|_{L^p(\Omega)} + \sum_{i=1}^{d}\|f_j\|_{L^q(\Omega)} + \|w\|_{H^1(\Omega)}\right).$$

(ii) If $f \in L^p(\Omega)$, $\alpha = 1 - \frac{d}{p} \in (0,1)$ *and* $f_i \in C^\alpha(\Omega)$, *then* $\nabla w \in C^\alpha(\Omega')$ *and:*

$$\|\nabla w\|_{C^\alpha(\Omega')} \leq C\delta^{-\frac{d}{2}-\alpha}\left(\|f\|_{L^p(\Omega)} + \sum_{i=1}^{d}\|f_i\|_{C^\alpha(\Omega)} + \|w\|_{H^1(\Omega)}\right).$$

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