Frustration - Analyzing a Card Game with Probability

Introduction

Probability and statistics offer rich opportunities to help mathematics students use their understanding of numbers to make sense of the unpredictable experiences they have everyday. Elements of chance permeate everyday life from daily interactions, to the weather, to sports and games. In grades 6-8, students have opportunities to learn basic tools for interpreting data about random processes and engage in arguments about "what might happen if" something repeats. Unfortunately, many students encounter probability and statistics through inauthentic contexts with an emphasis on calculations instead of exploration of non-routine problems (Sorto 2011). For example, counting different colors of M&Ms is so common that many students encounter it over and over, despite the manufacturer's website showing the percentages since 1997. In this short article, we share a new, classroom-tested lesson designed to engage students in the joy of mathematical inquiry through a game, while building number sense, understanding of uncertainty (c.f., Pratt and Kazak 2018), statistical reasoning, and discourse skills.

Frustration

In 1708, Pierre de Montmort introduced *Jeu de Treize* ("thirteen game"), a perplexing game of pure chance (no strategy) centered around whether a randomly shuffled deck of playing cards will have any cards in their original position. The expected value of the game wasn't solved until 1994 (Doyle et al. 2009), and new players often grow to appreciate the mix of anticipation and apparent low chance of winning that goes with the modern name, *Frustration*. Played as a solitaire (single-player) game, the rules of Frustration are as follows:

1. One at a time, deal a shuffled deck of playing cards face up while calling out the next rank in the canonical sequence: “Ace, two, three, four, five, six, seven, eight, nine, ten, jack, queen, king, ace, two, three, . . .”
2. If a dealt card matches the rank you called out, you lose.
3. If you run out of cards to deal, you win.

The game is traditionally played with a standard deck of 52 playing cards, but it works with any subset of cards, such as the 13 cards in a single suit (e.g., hearts), or, say, a custom 6 card deck using three ranks (ace, five, king) of two suits (hearts, spades) each. The key is to always follow the same order when calling out the ranks in the “deck”. For example, if using the custom 6 card deck, the player could always call out “ace, five, king, ace, five, king.”

Lesson Background

We developed the Frustration lesson through two cycles in which we created lesson materials, taught with them twice, and used video recordings and artifacts of student work to refine the materials. The first version successfully got students engaged in thinking about both theoretical and experimental probability, but the content was new for students and we ran out of class time. From that experience, we built a version focusing on discourse and the Law of Large Numbers. Slides and handouts for both lessons are available as supplements, and we share a vignette about the second version in the next section. Both versions were intentionally designed to incorporate two teaching practices that appear throughout research on increasing students’ conceptual understanding: explicit attention to concepts and students’ opportunity to struggle (Hiebert and Grouws 2007). Explicit attention to concepts focuses on making concepts explicit and public, and emphasizing connections between ideas, solutions, and representations. Students’ opportunity to struggle focuses on sense-making, sustained mental effort, and engaging with meaningful math problems. In our lessons, students engaged in the opportunity to struggle when they justified their predictions, made sense of others' choices, and explained and discussed how their predictions changed over time as the number of experiments increased, and they received explicit attention to concepts through the focus on a main idea in the lesson and how this concept can be used to find probabilities of winning more complex versions of the game.

A Middle School Frustration Lesson

As university teacher educators, we were guest teachers for two class periods in a 7th grade general math classroom in a middle school in rural Idaho. It was spring, and the teacher told us the lesson would be these students' first lesson about probability this year. There were about 20 students in each class, with desks arranged in pairs (for clarity, we sometimes refer to the two classes as Period 1 and Period 2).

Projecting the title slide, we started the lesson on a high note, "We're excited! Because today we're going to be playing a game called 'Frustration'. Which sounds weird, that you'd be excited to play Frustration, but it's really a fun game." We asked students to get a pencil and calculator, and then talked through the learning goals and essential question:

Lesson Goals:

At the end of this lesson, students will:

- Describe probability using numbers between 0 and 1
- Explain how experiments can help us determine probability
- Look for patterns and make conjectures

Essential Question:

- What is the chance that an unpredictable event will happen?

In talking about the goals, we asked students what probability sounded like to them, and students brought up the words “probably” and “maybe.” We agreed, saying probability shows up when “something might happen or it might not.” After talking about playing cards (including the terms “deck”, “suit”, and “rank”), we introduced the rules of Frustration. Instead of the full 52 card version, we began with a simplified 4-card game with unique ranks. Students randomly drew cards to form our custom deck, which we wrote on the whiteboard: 2, Q, 3, 8. We then modeled what students would do on their own later: we shuffled the cards, wrote down the new sequence of cards below the ranks we had listed previously, and recorded the result of the game (Figure 1). To check students’ understanding, we showed an animated version of the game in our slides.

	2	Q	3	8	Result
Game 1	3	2	Q	8	😣 (Lose!)

Figure 1: Model of Frustration Game Results

During this practice round in Period 2, the following conversation came up:

[After drawing a 3 and 2 from a 4-card deck that has 2, Queen, 3, 8]

Teacher: “What rank am I going to say now?”

Students: “3.”

T: “3. And I’m hoping that this [card] is not a 3. Could it be?”

Ss: “No!”

T: “Ha! I think I’m going to be good. How do I know that?”

Ss: “Because it already came up” “You already had it”

T: “Yeah, it already came up, so I’m safe. Okay, let’s draw. It’s a queen!”

[Some murmuring happens amongst students, some laugh, and some say “Wait ohh.”]

T: “Oh no. Am I going to be joyful at the end of this game?”

Ss: "No!"

As students noticed the last card would cause us to lose the game, they were noticing a lack of independence in the game play. This came up naturally in Period 2, but in Period 1, we lost the game on card two. Fortunately, the pre-planned animation of the game in our slides gave us the opportunity to have a similar conversation.

Next, we asked students to use their intuition to estimate the chance of winning the four-card game. We asked, "Just based on learning about the game, how likely do you think it is that when you play, you're gonna win?" and we drew a segmented interval with qualitative labels ("Will not happen", "Unlikely," "Maybe," "Likely", "Will Happen"), as well as numerical anchors (0, 50%, and 100%). Students wrote both a qualitative and numerical estimate on sticky notes (e.g., "unlikely, 15%"), turned and talked (see Chapin et al. 2009) with their partner about what they wrote down and *why,* and again placed sticky notes on the whiteboard (Figure 2).

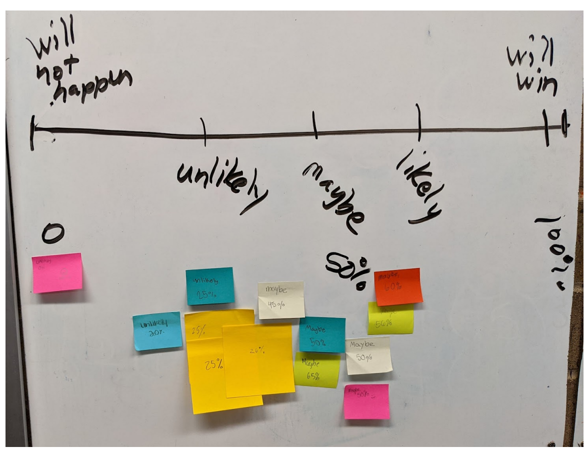

Figure 2a: Students' Predictions Prior to Playing in Period 1

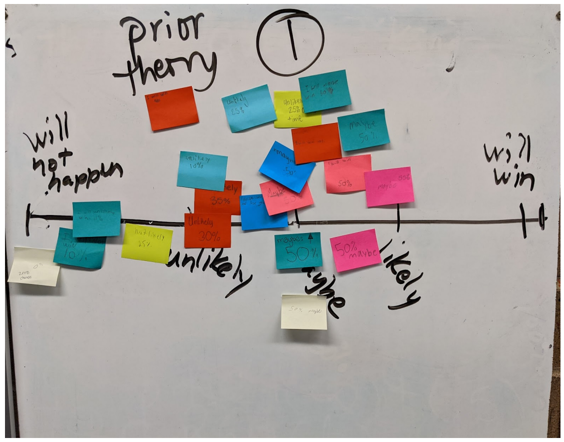

Figure 2b: Students' Predictions Prior to Playing in Period 2

After students placed their sticky notes, we noted the range of predictions (e.g., "we went from 0 to 55") and then passed out cards and the handout to each pair of students. Each pair of students played the Frustration game 10 times, recording their data on the handout. Using the slides, we made sure to point out that this part of the activity was about collecting *experimental* data, and that by playing the game many times and keeping track of wins and losses, we were obtaining an experimental probability value. In contrast, theoretical probability involves analyzing the set of possible outcomes.

Since the game was new, students needed help at first, and we often had to remind players that the ranks needed to be different and they needed to say the ranks in the same order every time they played. Nonetheless, recording the data was quick (about 9 minutes). By having everyone play the game, a lot of small independent samples were being produced and we saw a wide range of outcomes amongst the students (some students won 7 out of 10 games, others won 1 out of 10 games). You could hear the Frustration game living up to its name, as most students were losing most games, and there were audible moments of disappointment or anger at losses, as well as excitement from winners. After completing the 10 trials, students wrote their total number of wins and corresponding percentage on their data sheet.

Next, we had students share their results in groups of 6-8 students each, and make a new prediction about the chance of winning the game that took into account both their results and those of their peers. It was an essential part of the lesson, allowing students to engage in an important practice of statistics, aggregating data to improve predictions. To make students' thinking public, and to help generate discourse, we repeated the sticky note task (Figure 3).

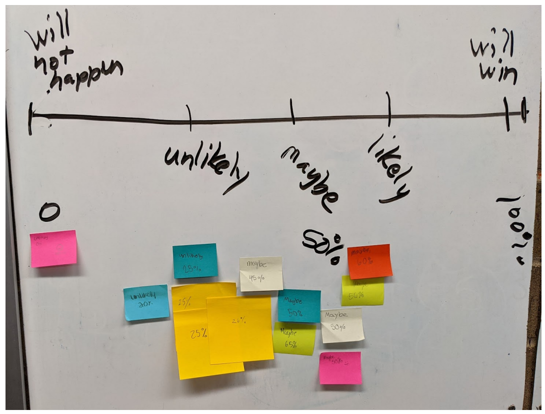

Figure 3a: Students' Predictions Before Playing in Period 1

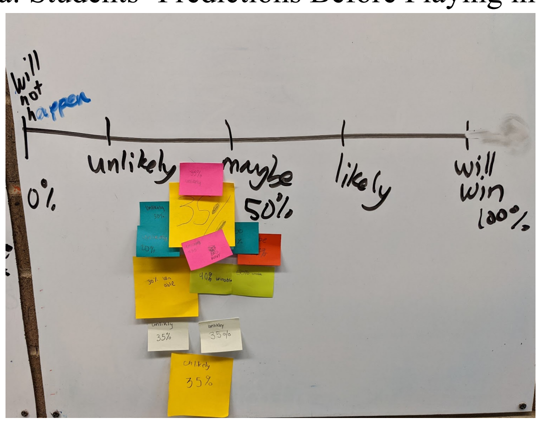

Figure 3b: Students' Predictions After Playing in Period 1

With students' original and revised estimates on the board, we said, "look at the data in the two charts, compare them, and tell me what you're noticing about the difference between the two charts." Before talking as a whole group, we had the students share with their partner. In Period 1, students were quick to note changes to the center of the distribution of estimates:

S1: "Everyone thought that they were going to do amazing and then their dreams were crushed."

T: “And you know that because?”

S1: “Well the first round our guesses were more on the likely, but now they’re not.”

T: “I agree. What else did someone notice about our two graphs?”

S2: “That there’s a huge change… it’s more spread out and on the other one it’s more in like one area.”

After this conversation we asked students to talk to each other about what they think caused the sticky notes to go from more spread out to more stacked. Students discussed how originally they were guessing, and then after playing they were able to see what really happened.

We wrapped up our discussion of the two charts by asking students, “How could we be more sure of the probability of winning the game?” Some students talked about trying to write down all the ways to shuffle cards (a challenging problem), others suggested averaging everyone’s results, and still others suggested playing the game many more times. Once a few ideas had been shared, we showed students the results of a computer simulation of 1,000,000 games (Figure 4, also included in the slides), and helped them to interpret the 37.45% experimental probability as the computer winning about 375,000 times out of a million games.

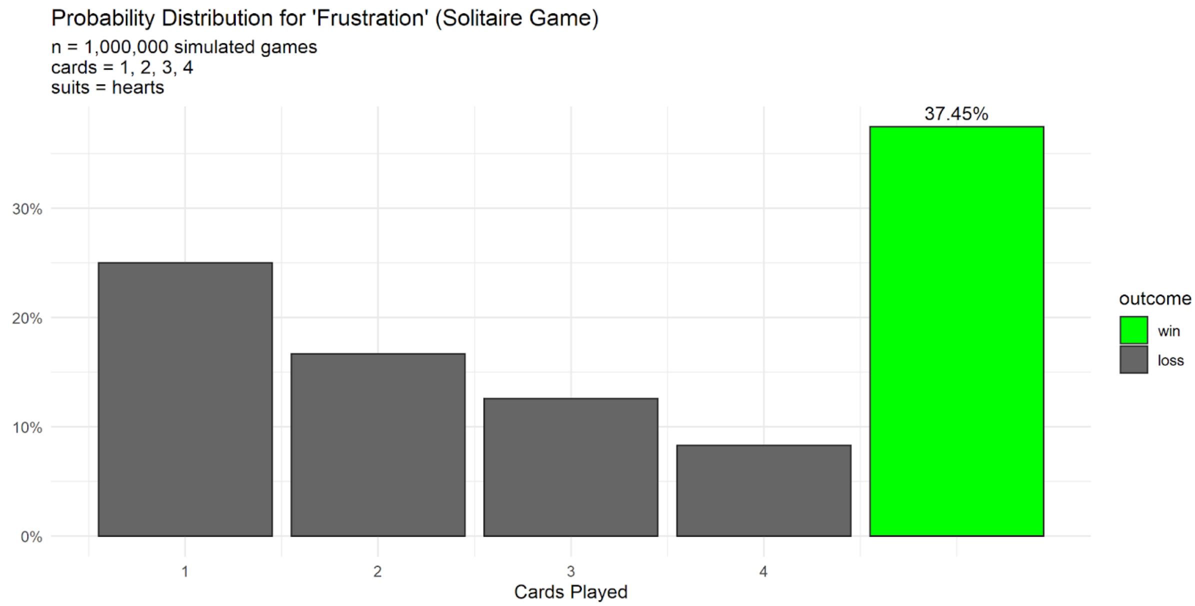

Figure 4: Simulation Results for a 4 Card Version of Frustration.

After this experience, we wanted to have one more discussion about the essential question. We asked students, “If we changed the game so we had 13 different ranks instead of just 4, how likely do you think winning would be?,” and we created one more sticky note distribution of their probability estimates (Figure 5).

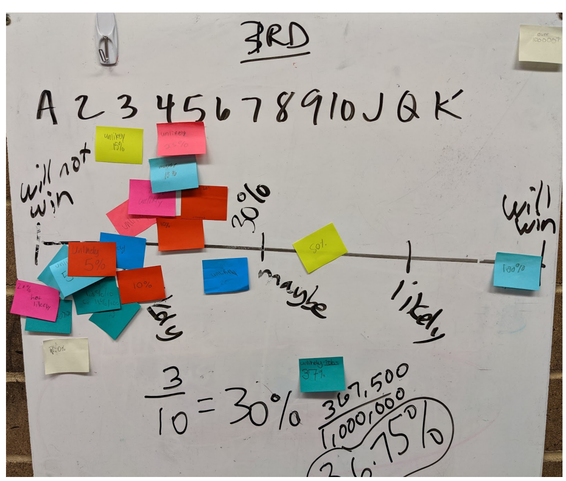

Figure 5: Students’ Third Prediction in Period 1

We talked as a class about the estimates, focusing on where we noticed groups of sticky notes. Students thought it would be less likely to win the game with a larger deck, and most sticky notes fell between 0 and 15%. To allow students to improve their predictions, we played as a whole class using a version of the game we built in Desmos (https://tinyurl.com/DesmosFrustration). The class shouted out the ranks in order as we dealt cards virtually and kept track of wins and losses. After playing 10 times, we calculated the percentage that we won (30%). We pointed out that almost all students' estimates were less than 30%, and asked students whether they would change their estimate. As before, we shared the simulation of results of playing the 13 card version of the game 1 million times, this time with a 36.73% experimental probability value. We explained how the Law of Large Numbers tells us that as the number of repeated trials gets very large, we can expect the winning percentage to be close to the "real" (theoretical) probability of winning the game. We mentioned that this is what the students were doing; each time they played more games, they were using the idea of the Law of Large numbers to hopefully get a better estimate of the theoretical probability. We shared how likely it is to win the game if we play with all 52 cards in the deck, which is MUCH harder to win with a probability of less than 2%. Finally, we closed with a formative assessment, asking students to complete an exit ticket (e.g., Figure 6) where they describe their takeaways from the lesson regarding probability.

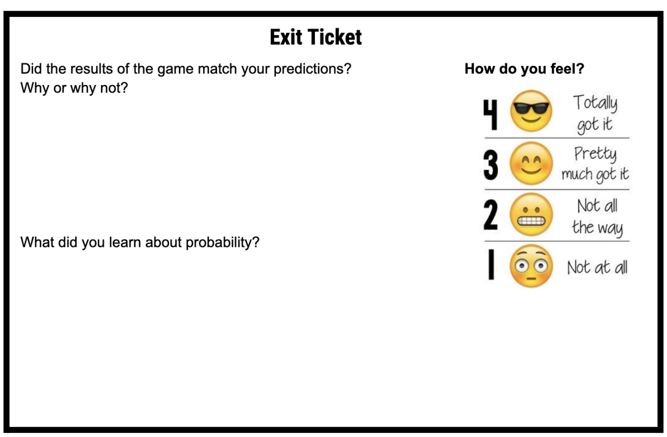

Figure 6: Sample Exit Ticket

Students' Reasoning about Theoretical Probability

Though our vignette described our second version of this lesson (focused on experimental probability), readers may also find our first lesson (focused on theoretical probability) useful too. The theoretical probability lesson has a similar format, but helped students to think more about conditional probability and permutations. It includes a task in which students tried to write all possible ways to deal 4 cards, which surfaced three great strategies: random lists, organized lists, or tree diagrams / bar models.

Random lists were the most common way for the students to think about the possible orders of the small deck of cards (see Figure 7). Some students listed permutations that they got from playing the game, others physically rearranged their cards to generate possibilities, and some simply wrote possible permutations (with no clear organization). As teachers, we tried to question students in a way that might encourage them to organize their lists. For example, "Do you have every possible way? How could you be sure?" or "What are all the possible cards that could come up first? What might come up after that?" If students were convinced they already had all of the permutations listed even after questions like these (but hadn't yet actually found them all), we would point out a specific missing example with comments like, "What if it started with an ace, then a three?" From here, students often developed more organized lists.

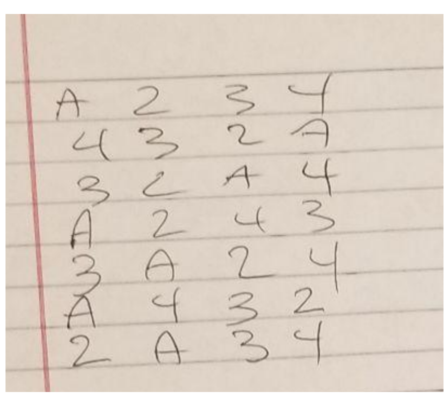

Figure 7: Example of an Unorganized List Strategy

The next most common student approach was an organized list strategy. That is, students considered the permutations case-by-case, often starting by looking at all cases that have the same first card, then moving on to a different first card (see Figure 8).

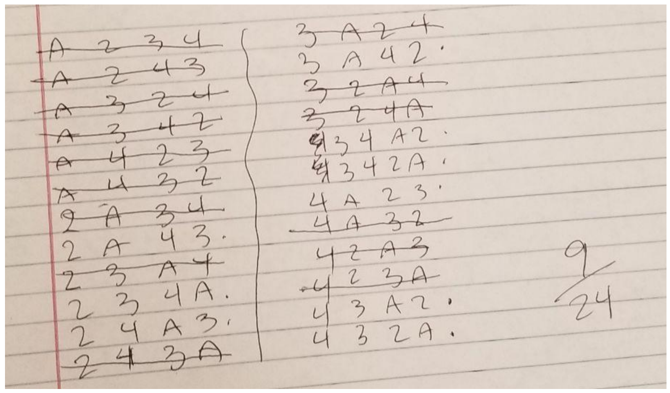

Figure 8: Example of an Organized List Strategy

A few students used a tree diagram strategy to organize their attempt to generate the sample space. That is, students listed what card could be dealt first, then, attached to each of those first cards, what possible cards could come next, and so on. An interesting presentation of this was a bar-like display (see Figure 9). Using this same logic, students divided the space up into bars of paths instead of drawing out lines as in a typical tree diagram.

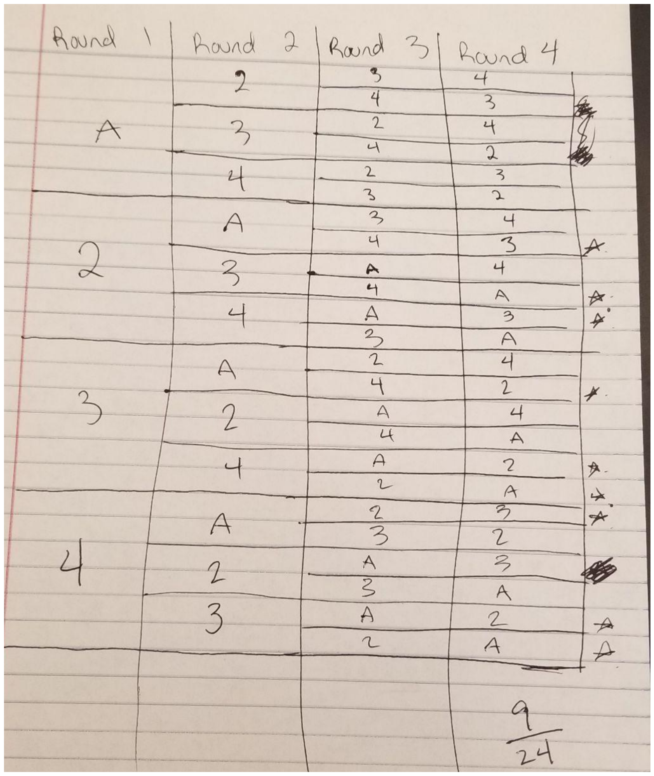

Figure 9: Example of a Tree Diagram / Bar Model Listing Strategy

Standards for Mathematics in the Lesson

Our Frustration lessons address the Common Core State Standards for Mathematics (CCSSM) through standards 7.SP.C.5, understanding probability as a number between 0 and 1, 7.SP.C.6, approximating probability by collecting data and observing relative frequency, and 7.SP.C.8 finding probabilities of compound events using organized lists, tables, tree diagrams, and simulation. The lessons give explicit attention to the difference between experimental probability and theoretical probability, and how they are tied together through the Law of Large Numbers. A key concept that came up naturally from the sticky note activities was center, shape, and spread of the distributions. Along with the three content standards, the lesson foregrounds CCSSM's Standards for Mathematical Practice (SMP), especially SMP3: "Construct viable arguments and critique the reasoning of others." Students discuss their probability estimates and compare ideas with students around them, and (in the lesson that focuses on theoretical probability) discuss their approaches for finding sample space and justifying claims about their listed permutations. Students also engage in SMP8, “Look for and express regularity in repeated reasoning,” when they reason about repeating their experiments and combining results in order to get a more accurate view of the probability of winning.

Challenges for Teaching the Lesson

We learned a lot from teaching these lessons, and we think you will too. First, students often surprised us with their brilliance and creative ways of thinking about the probability of winning the game. We think there’s something about the unpredictability of the game and students’ prior experience with card games that piqued their interest. In every one of our attempts to teach the lessons, students came up with new and unexpected ways to reason about the probability of winning. We set high expectations for our students, but the lessons’ focus on discourse and small-group work gave students the freedom to be creative, and they consistently impressed us with their ideas, strategies, and conversations.

Still, implementing activities like this in the classroom is not without challenges. When we asked students to play the game, they often played slower than expected (sometimes as a result of strategic thinking and problem solving, but often just from moving slowly). We found it helpful to encourage students to play the game quickly. So, one idea might be to challenge students within small groups to see who could get through 10 rounds of the game the fastest.

Additionally, because playing the game is experimental, it is possible the results of students’ ten game plays will indicate a very different chance of winning than the theoretical probability (for example, some students might win 7 out of 10 times). If this happens to several students, this might skew students’ perceptions and cause students’ predictions to become *less* accurate after playing the game. This is important to be aware of so as not to be caught off-guard. Additionally, even if students are starting to build a more accurate prediction of the theoretical probability, they tend to personalize their experiences, and may say that *they* are lucky so they are more likely to win than most people.

We found it essential to plan in advance for facilitating classroom discourse, especially questioning, and selecting student strategies. When students were stuck or questioning their next steps, we tried to encourage them to problem solve on their own without telling them how we would solve the problem. However, it can be helpful to ask students questions that are designed to help them focus on a productive path. We recommend keeping the focus on students' strategies. As students are working on the theoretical probability, for example, you might ask them to explain how they are approaching the problem, or ask them what they are thinking about when they make the list. Then try to ask questions that build on their thinking.

When selecting and sharing student work, it's important to find examples you are prepared to make connections between and that move students towards the lesson goals. For example, you might select a student who was using a random listing strategy and a student who was using an organized listing strategy and highlight that both students were using listing strategies and that they both could solve the problem this way, but if we want to be *sure* that we've listed all the outcomes, the organized list may be more useful.

Conclusion

Our purpose in developing and sharing these lessons is to highlight how games of chance can offer rich opportunities to teach probability and statistics for conceptual understanding in the middle grades. The lessons we described allow students to explore probability with non-routine inquiry and discourse, and we intentionally included ways to use Explicit Attention to Concepts and Student Opportunities to Struggle to further encourage students' understanding. We know these are difficult topics to fit into already packed curricula, so we hope the multiple versions of the lesson can be useful to a wide range of teachers.

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