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AN IMMERSED CROUZEIX-RAVIART FINITE ELEMENT METHOD FOR NAVIER-STOKES EQUATIONS WITH MOVING INTERFACES

JIN WANG, XU ZHANG, AND QIAO ZHUANG*

Abstract. In this article, we develop a Cartesian-mesh finite element method for solving Navier-Stokes interface problems with moving interfaces. The spatial discretization uses the immersed Crouzeix-Raviart nonconforming finite element introduced in [29]. A backward Euler full-discrete scheme is developed which embeds Newton's iteration to treat the nonlinear convective term. The proposed IFE method does not require any stabilization terms while maintaining its convergence in optimal order. Numerical experiments with various interface shapes and jump coefficients are provided to demonstrate the accuracy of the proposed method. The numerical results are compared to the analytical solution as well as the standard finite element method with body-fitting meshes. Numerical results indicate the optimal order of convergence of the IFE method.

Key words. Navier-Stokes, interface problems, nonconforming immersed finite element methods, moving interface.

1. Introduction

N vmuj.qibtf jnnjtdjcmf jodpnqsfttjcmf ffpxt xjui fncfeefe joufsgbdft bsf xjef. mz qsftfou jo nboz qiztjdbm qifopnfob0 Uif sfmbufe tjnvmbujpot bqqfbs jo nboz csbodift pg tdjfodf boe fohjoffsjoh- tvdi bt ffvje ezobnjdt- cjpmphz- nfejdbm tdj. fodft- boe hfpmphz]:- 23- 24- 52- 5:a- up obnf b gfx0 Uif ezobnjdt pg uif uxp.qibtf)ps nvmuj.qibtf+ ffpxt bsf hpwfsofe cz uif xfmm lopxo Obwjfs.Tuplft)OT+ frvbujpot- ps Tuplft frvbujpot gps dsffqjoh ffpxt- bmpoh xjui uif fogpsdf. nfou pg kvnq dpoejujpot bu joufsgbdft0 Riztjdbm qbsbnfufst pg uif ffpxt- tvdi bt efotjuz boe wjtdptjuz dpf~ djfout- bsf vtvbmmz ejtdpoujovpvt bdsptt uif ffvje joufs. gbdf]2- :- 28- 3:- 5:a0

Jo uijt bsujdmf- xf dpotjefs b uxp.ejnfotjpobm joufsgbdf qspcmfn uibu bsjtft jo b uxp.qibtf ffpx hpwfsofe cz uif OT frvbujpo0 Mfu ff $\leftarrow \mathbb{R}^2$ cf bo pqfo cpvoefe epnbjo tfqbsbufe cz bo joufsgbdf)t+ joup uxp ejtkpjou tvcepnbjot ff$^-$)t+ boe ff$^+$)t+0 Dpotjefs uif gpmmpxjoh votufbez OT frvbujpo)OTF+ jo uif wfmpdjuz.tusftt. qsfttvsf gpsn ;

$$
\begin{array}{llll}
)2b+ & \boldsymbol{u}_t \quad \times\sigma)\boldsymbol{u},p+, \ )\boldsymbol{u}\times \ +\boldsymbol{u} \text{ A } \boldsymbol{f} & \text{jo ff}^-)t+\cap\text{ff}^+)t+\pm]1,T\text{a}, \\
)2c+ & \times\boldsymbol{u} \text{ A } 1 & \text{jo ff} \pm]1,T\text{a}, \\
)2d+ & \boldsymbol{u} \text{ A } 1 & \text{po } \partial\text{ff} \pm]1,T\text{a}, \\
)2e+ & \boldsymbol{u})\boldsymbol{x},1+\text{A } \boldsymbol{u}_0, \ p)\boldsymbol{x},1+\text{A } p_0 & \text{jo ff},
\end{array}
$$

xifsf $\boldsymbol{u}$ sfqsftfout uif wfmpdjuz ffmme boe p sfqsftfout uif qsfttvsf0 Uif tusftt ufotps $\sigma)\boldsymbol{u},p+$jt efflofe cz

$$\sigma)\boldsymbol{u},p+\text{A } 3\mu\epsilon)\mathbf{u}+ \quad p\boldsymbol{I},$$

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2000 *Mathematics Subject Classification.* 35R05, 65N30.
*Corresponding author.

xi fsf $\boldsymbol{\epsilon})\boldsymbol{u}$+A) $\boldsymbol{u}$,) $\boldsymbol{u}$+^{t}+/3 jt ui f tusbjo ufotps boe $\boldsymbol{I}$ jt ui f jefoujuz ufotps0 Ui f wjtdptjuz dpf$^-$ djfou $\mu)\boldsymbol{x}$+jt ejtdpoujovpvt bdsptt ui f joufsgbdf)t+- xi jdi jt b qptjujwf qjfdfxjtf.dpotubou gvodujpo efflofe cz

$$)3+ \qquad \mu)\boldsymbol{x}+A \begin{cases} \mu^- & \text{jg } \boldsymbol{x} \text{ / ff}^-)t+; \\ \mu^+ & \text{jg } \boldsymbol{x} \text{ / ff}^+)t+. \end{cases}$$

Bdsptt ui f joufsgbdf- ui f gpmmpxjoh i pn phf of pvt wfmpdjuz boe tusftt kvn q dpoej. ujpot bsf fogpsdfe

$$)4b+ \qquad]\boldsymbol{u}a_\Gamma \text{ A } \mathbf{0} \text{ po })t+;$$

$$)4c+ \qquad]\sigma)\boldsymbol{u}, p+\boldsymbol{n}a_\Gamma \text{ A } \mathbf{0} \text{ po })t+;$$

xi fsf ui f kvn q]xa$_\Gamma$ jt efflofe cz]$\mathbf{v}$a$_\Gamma$;A $\mathbf{v}^+|_\Gamma$ $\mathbf{v}^-|_\Gamma$- boe $\mathbf{n}$ jt ui f voju opsn bm wfdups up ui f joufsgbdf qpjoujoh gspn ff$^-$ up ff$^+$0 Xf bmtp opuf ui bu xi fo $\mu)\boldsymbol{x}$+jt b qjfdfxjtf dpotubou- evf up ui f ejwfshfodf dpoejujpo)2c+ ui f n pn fouvn frvbujpo)2b+dbo cf xsjuufo bt

$$)5+ \qquad \boldsymbol{u}_t \quad \mu\Pi\boldsymbol{u}, \ )\boldsymbol{u} \times \ +\boldsymbol{u}, \quad p \text{ A } \boldsymbol{f} \quad \text{jo })\text{ff}^-)t+\cap\text{ff}^+)t++ \pm]1, T a$$

Woefs ui jt gsbn fxpsl- ui f tusftt kvn q dpoejujpo)4c+dbo cf n pejflfe bt gpmmpxt

$$)6+ \qquad])\mu \quad \boldsymbol{u} \quad p\boldsymbol{I}+\boldsymbol{n}a_\Gamma \text{ A } \mathbf{0}.$$

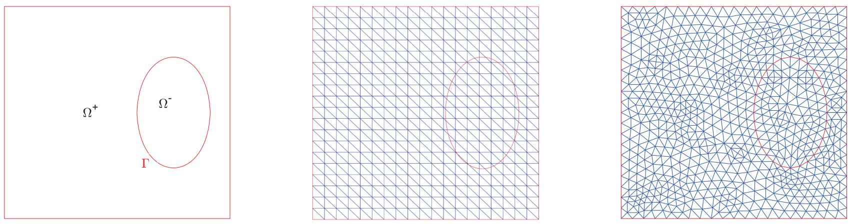


FIGURE 1. b epn bjo xjui bo joufsgbdf)mfgu+- b opo.cpez.flujoh n fti)n jeemf+boe b cpez.flujoh n fti)sjhi u+0

Ui f Obwjfs.Tupl ft frvbujpot tbot bo joufsgbdf bsf xjefmz tuvejfe jo ui f dpoufyu pg flojuf fmfn foun fui pet- jodmvejoh dmbttjdbmflojuf fmfn foun fui pet]27- 29a- ejtdpo. ujovpvt Hbmfsl jo)EH+flojuf fmfn fou n fui pet]21- 26- 56- 58a- boe xfbl Hbmfsl jo flojuf fmfn foun fui pet]38- 57- 62a0 Ui ptf flojuf fmfn foun fui pet dbo cf fyufoefe up tpmwf qfsujofou joufsgbdf qspcmfn t qspwjefe ui bu cpez.flujoh n fti ft bsf fn qmpzfe0 I pxfwfs- tvdi cpez.flujoh sftusjdujpo n bz i joefs ui f f$^-$ djfodz jo tpmwjoh joufsgbdf qspcmfn t xjui fwpmwjoh joufsgbdjbmhfpn fusjft boe mpdbujpot cfdbvtf ui f n fti i bt up cf hfofsbufe sfqfbufemz bddpsejoh up fbdi joufsgbdf dpoflhvsbujpo0 N boz ovn fsjdbm n fui pet cbtfe po joufsgbdf.joefqfoefou n fti ft i bwf cffo efwfmpqfe- tvdi bt dvu flojuf fmfn fou n fui pe)DvuGFN +]6- 7- 8- 53a- jn n fstfe joufsgbdf n fui pe)JJN +]44a- fyufoefe flojuf fmfn fou n fui pe)Z GFN +]25a- qbsujujpo pg vojuz flojuf fmfn fou n fui pe)RWGFN +]55a boe n budi fe joufsgbdf boe cpvoebsz)N JC+]64a n fui pe0 Ui ftf voflufe.n fti ovn fsjdbm n fui pet fn qmpz n pejflfe xfbl gpsn vmbujpot ps sf. wjtfe flojuf fmfn fou gvodujpot bspvoe ui f joufsgbdf up dbquvsf ui f joufsgbdjbmkvn q cfi bwjpst0 Xf sfgfs ui f sfbefst up]22- 28- 37- 54a gps DvuGFN -]45a gps JJN -]59a gps Z GFN -]5a gps RWGFN - boe]63a gps N JC bqqmjfe up OT n pwjoh joufsgbdf qspc. mfn t0 Bo jmmvtusbujpo pg b cpez.flujoh n fti boe b opo.cpez.flujoh n fti jt hjwfo jo Gjhvsf 20

Jn n fstfe flojuf frfn fou)JGF+n fui pet bsf brnp b drbtt pgn fui pet ui budbo tpmwf ui f joufsgbdf qspcrfin t po joufsgbdf.joefqfoefou n fti ft0 Wbojl f voflurfe ovn fsjdbm n fui pet)tvdi bt DvuGFN t boe JJN +ui bufn qmpz tvjubcrfi gpsn vrbujpo bspvoe ui f joufsgbdf- JGF n fui pet rpdbmmz dpotusvdu tqfdjbmbqqspyjn bujpo gyodujpot po ui f frfn fout dvucz ui f joufsgbdf up dbquvsf ui f kvn q cfi bwjps bu ui f joufsgbdf0 Jo]43a- ui f rjofbs JGF tqbdf boe jut bqqspyjn bujpo dbqbcjrjuz xfsf ejtdvttfe gps 3E frmjqujd joufsgbdf qspcrfin t0 Bgufs ui bu- n boz jowftujhbujpot sfhbsejoh JGF n fui pet gps frm rjqujd joufsgbdf qspcrfin t bsf dpoevdufe- jodmvejoh qbsujbmmz.qfobrjn fe JGF)RRJGF+ n fui pet]47a- frmjqujd pqfsbups xjui opoi pn phfofpvt kvn q]35a- opodpogpsn joh JGF n fui pet]48a- i jhi fs efhsff JGF n fui pet]3- 2:- 65a0 JGF n fui pet i bwf brnp cffo fyufoefe up ujn f.efqfoefou joufsgbdf qspcrfin t hpwfsofe cz qbsbcprjd frvbujpot]31- 49- 51- 61a boe i zqfscprjd frvbujpot]4a0 JGF n fui pet bsf brnp efwfrmpqfe up tpmwf joufsgbdf qspcrfin t hpwfsofe cz tztufn t pg frvbujpot- tvdi bt rjofbs frbtujdjuz]33- 34- 4: a0

Jo sfdfou zfbst- JGF n fui pet i bwf cffo fyufoefe up tpmwf Tupl ft joufsgbdf qspc. rfin t0 Gps jotubodf- ui f jn n fstfe EH n fui pe cbtfe po $Q_1.Q_0$ frfn fou)Q_1; qjfdf. xjtf cjrjofbs qprzopn jbmtqbdf=Q_0; qjfdf xjtf dpotubou tqbdf po r vbesjrbufsbrm fti . ft+x bt efwfrmpqfe jo]2a0 B drbtt pg opodpogpsn joh JGF n fui pet cbtfe po Dspv fjy. S bwjbsu)DS +.P_0 frfn fou)P_0; qjfdf xjtf qprzopn jbmtqbdf po usjbohvrbs n fti ft+ boe S boobdi fs.Uvsfl spubufe $Q_1.Q_0$ frfn fou xfsf sfdfourz jouspevdfe jo]3: a0 B n jyfe dpogpsn joh.opodpogpsn joh P_1 JGF tqbdf x bt jouspevdfe jo]41- 42a0 Jo beej. ujpo- b Ubzrps.I ppe $P_2.P_1$ jn n fstfe flojuf frfn fou boe ui f dpssftqpoejoh RRJGF n fui pe xfsf efwfrmpqfe jo]9a0

Up tjn vrbuf n psf hfofsbmffivje ffipx joufsgbdf qspcrfin t jotufbe pg kvtu dsffqjoh ffipx hpwfsofe cz Tupl ft frvbujpot- ui f oporjofbs Obwjfs.Tupl ft frvbujpot n vtu cf dpotjefsfe bt ui fz bqqfbs jo bm ptu brmsfbmtjn vrbujpot0 Ui f oporjofbs dpowfdujwf bddfrfsbujpo boe ui f fwprmjoh ffivje joufsgbdf n bl f ui f OTF n pwjoh joufsgbdf qspc. rfin t n psf di brmfohjoh up tpmwf ui bo ui f Tupl ft joufsgbdf qspcrfin t]2- 9- 3: - 41- 42a0 Up ui f cftu pg pvs l opx rfehf- ui fsf jt op rjufsbuvsf po ui f ovn fsjdbmtuvez pg OTF n pwjoh qspcrfin t vtjoh JGF n fui pet0 X f opuf ui bu gps ui f OT joufsgbdf qspcrfin t- ui f wfrpdjuz boe tusftt kvn q dpoejujpot jo)4b+boe)4c+bsf ui f tbn f bt jo ui f Tupl ft joufsgbdf qspcrfin t]39a=ui fsfgpsf- ui fsf jt op offe up dpotusvdu ofx JGF tqbdft gps OT joufsgbdf qspcrfin t0 X f dbo fn qmpz ui f fyjtujoh JGF tqbdft gps T. upl ft joufsgbdf qspcrfin t]2- 3: - 41- 9a ejsfdumz up tuvez OT joufsgbdf qspcrfin t0 Up i boerfi ui f oporjofbs dpowfdujpo- x f bepqu ui f Ofx upo(t n fui pe jo pvs gvmmz ejtdsfuf tdi fn f0 Jo beejujpo- x f vtf ui f cbdl xbse Fvrfs tdi fn f gps ujn f n bsdi joh evf up jut B.tubcjrjuz boe tjn qrjdjuz0 Bn poh ui ptf fyjtujoh JGF tqbdft]2- 3: - 41- 9a- x f opuf ui bu ui f DS .P_0 JGF tqbdft jouspevdfe jo]3: ai bwf ux p qspn jofou gfbuvsft;)j+ ui fz jn qptf x fbl dpoujovjuz tvdi ui bu ui f bwfsbhf jouf hsbmwbmvf bdsptt ui f fehft jt dpoujovpvt=ui vt ui fsf jt op offe up jodmvef tubcjrjn fe ufsn t0 Bt b sftvmu- ui f ovn fsjdbmtdi fn f jt n vdi tjn qrfs dpn qbsjoh up ui f RRJGF n fui pe]9a boe ui f EH tdi fn f]2a0)jj+P o ui f tbn f Dbsuftjbo n fti - ui fsf bsf pgufo rfitt efhsfft pg gsffepn gps DS .P_0 ui bo jo ui f jn n fstfe EH tqbdf]2a boe Ubzrps.I ppe JGF tqbdf]9a=ui vt- ju jt dpn qvubujpobrmz n psf f¯ djfou0 Ui ftf ux p gfbuvsft n pujwbuf vt up vtf ui f DS .P_0 JGF tqbdft up jowftujhbuf ui f OTF xjui n pwjoh joufsgbdft qsftfoufe jo)2+.)4+evf up jut tjn qrjdjuz boe f¯ djfodz0

Ui f rbzpvu pg ui f sftu pg ui f bsujdrfi jt bt gprmpx t; Tfdujpo 3 jouspevdft ui f op. ubujpot boe bttvn qujpot up cf vtfe jo ui jt bsujdrfi0 Tfdujpo 4 jt efwpufe up jousp. evdf ui f tfn j.ejtdsfuf boe gvmmz ejtdsfuf opodpogpsn joh JGF n fui pet xjui Ofx upo(t n fui pe gps ui f OT joufsgbdf qspcrfin)2+.)4+0 Ovn fsjdbmfybn qrft xjui ejfifsfou joufsgbdf dpoflhvsbujpot boe wjtdptjuz dpf¯djfout bsf hjwfo jo Tfdujpo 5 up efnpo. tusbuf uif bddvsbdz pg uif qspqptfe JGF nfuipe jo tpmwjoh OT joufsgbdf qspcmfnt0 Gjobmmz- b csjfg dpodmvtjpo jt esbxo jo Tfdujpo 60

2. Notations and Preliminaries

Jo uijt tfdujpo- xf jouspevdf uif opubujpot boe bttvnqujpot up cf vtfe jo uijt bsujdmf- gpmmpxfe cz b csjfg sfdbmm pg uif DS.P_0 JGF tqbdf jo]3:^0

Up uijt qpjou- xf bttvnf uibu ff ← $\mathbb{R}^2$ jt b qpmzhpobm epnbjo0 Mfu $\cup_h$ cf b usjbohvmbs nfti pg ff xiptf tfu pg opeft boe tfu pg fehft bsf efopufe cz $\mathcal{O}_h$ boe $\mathcal{I}_h$- sftqfdujwfmz0 Opuf uibu uif nfti $\cup_h$ jt joefqfoefou pg uif mpdbujpo pg uif joufsgbdf)t+ bu boz ujnf t0 Xf nblf uif gpmmpxjoh bttvnqujpot po uif nfti;

(H1) Bu boz ujnf t- uif joufsgbdf)t+ dbo pomz joufstfdu bo fmfnfou xjui opu npsf uibo uxp qpjout0

(H2) Bu boz ujnf t- uif joufsgbdf)t+ jt b qjfdfxjtf C^2 gvodujpo tvdi uibu gps fwfsz fmfnfou T / $\cup_h$- jg)t+{ T \A $\mathcal{E}$- uifo)t+{ T jt b C^2 gvodujpo jo T0

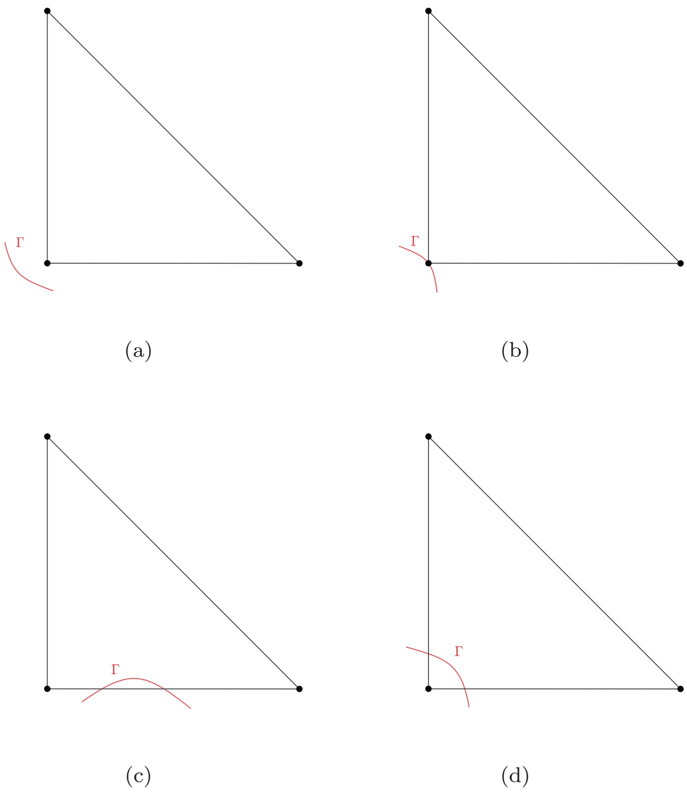


FIGURE 2. bo joufsgbdf joufstfdut bo fmfnfou bu)b+op qpjou=)c+ pof qpjou=)d+uxp qpjout po uif tbnf fehf=)e+uxp qpjout po ejggfsfou fehft0

Jg uif joufsgbdf)t+ joufstfdut bo fmfnfou T xjui {fsp ps pof qpjou- uifo T jt dpotjefsfe bt b opo.joufsgbdf fmfnfou0 Jo beejujpo- jg uif joufsgbdf)t+ joufstfdut bo fmfnfou T bu uxp ejtujodu qpjout cvu po uif tbnf fehf- uifo T jt bmtp dpotjefsfe bt b opo.joufsgbdf frfn fou0 Jg ui f joufsgbdf)t+joufstfdut bo frfn fou T bu ux p ejtujoduqpjout po ux p ejfifsfoufehft- ui fo T jt dpotjefsfe bt bo joufsgbdf frfn fou0 Tff Gjhvsf 3 gps bo jmmvtusbujpo0 Xf opuf ui bu ui f sfbtpo xf dbo usfbu ui f frfn fou xi fsf ui f joufsgbdf joufstfdut ui f frfn fou bu ux p ejfifsfouqpjout po ui f tbn f fehf)Gjhvsf 3)d++bt b opo.joufsgbdf frfn fou jt; Wujrjn joh Mfn n b 403 pg]32a- xf dbo ti px ui bu ui f bsfb pg ui f sfhjpo cpvoefe cz ui f joufsgbdf boe ui f fehf jt $\mathcal{T})h_T^3+$ $)h_T$ jt ui f ejbn fufs pg ui f frfn fou+- ui vt ui jt sfhjpo dbo cf jhopsfe dpn qbsfe up ui f frfn fou0 X f vtf $\cup_h^i)t+$boe $\cup_h^n)t+$up efopuf ui f tfu pg joufsgbdf frfn fout boe opo.joufsgbdf frfn fout bu ujn f t- sftqfdujwfmz0 I fsf- ui f cbdl hspvoe n fti $\cup_h$ jt ujn f joefqfoefou- cvu ui f ejtusjcvujpo pg $\cup_h^i)t+$boe $\cup_h^n)t+$di bohft xjui ujn f0 Ui fz tbujtgz $\cup_h$ A $\overline{\cup_h^i)t+\cap\cup_h^n)t+}$bu boz ujn f t0 Tjn jmbsmz- xf mfu $\mathcal{I}_h^i)t+$boe $\mathcal{I}_h^n)t+$cf ui f tfu pg joufsgbdf fehft boe ui f tfu pg opo.joufsgbdf fehft bu ujn f t- sftqfdujwfmz0 Jo beejujpo- xf vtf $\mathring{\mathcal{I}}_h$ boe $\mathcal{I}_h^b$ gps ui f tfu pg joufsjps fehft boe cpvoebsz fehft- sftqfdujwfmz0

Gps fbdi frfn fou T / $\cup_h$- xf efflof jut joefy tfu bt $\mathcal{N}_T$ A $\}2,3,4|$0 Mfu A_i - i / $\mathcal{N}_T$ cf ui f wfsujdft pg T xi jdi bsf mjtufe dpvoufsdmpdl xjtf)gspn A_1 up A_3+ boe efopuf ui f fehft pg T up cf e_i A $\overline{A_iA_{i+1}}$ gps i A 2,3- e_3 A $\overline{A_3A_1}$0 Po b opo.joufsgbdf frfn fou T / $\cup_h^n$- ui f tuboebse DS.P_0 flojuf frfn fou gvodujpot bsf fn qmpzfe up bqqspyjn buf ui f wfmpdjuz boe ui f qsfttvsf0 Up cf n psf tqfdjflid- mfu $\psi_{j,T}$ / $\mathbb{P}_1$- j / $\mathcal{N}_T$ cf ui f tuboebse mjofbs Mbhsbohf ti bqf gvodujpot po T / $\cup_h^n$ tvdi ui bu

$$)7+ \qquad \frac{2}{|e_i|}\int_{e_i}\psi_{j,T})x,y+ds \text{ A } \delta_{ij},\ \emptyset i,j\ /\ \mathcal{N}_T,$$

xi jdi jt vtfe up bqqspyjn buf ui f dpn qpofou pg ui f wfmpdjuz0 N fbox i jmf- ui f qsft. tvsf jt bqqspyjn bufe cz ui f qjfdfxjtf dpotubou gvodujpo tqbdf $\mathbb{P}_0$0 Gps ui f 3E Obwjfs.Tupl ft qspcmfn - ui f dpn qpofout pg ui f wfmpdjuz boe ui f qsfttvsf dpotujuvuf b wfdups.wbmvfe flojuf frfn fou tqbdf- efopufe cz $S_h^{non})T+$A $\mathbb{P}_1)T+\pm\mathbb{P}_1)T+\pm\mathbb{P}_0)T+$0 Gps j A $2,3,...,8$- mfu

$$)8+ \quad \boldsymbol{\psi}_{j,T} \text{ A } \begin{bmatrix}\psi_{j,T}\\1\\1\end{bmatrix} \text{ gps } j\ /\ \mathcal{N}_T,\ \boldsymbol{\psi}_{j,T} \text{ A } \begin{bmatrix}1\\\psi_{j-3,T}\\1\end{bmatrix} \text{ gps } j\quad 4\ /\ \mathcal{N}_T,\ \boldsymbol{\psi}_{j,T}\text{ A }\begin{bmatrix}1\\1\\2\end{bmatrix}\text{ gps } j \text{ A } 8.$$

Ui fo ui f mpdbm DS.P_0 flojuf frfn fou tqbdf po ui f opo.joufsgbdf frfn fou dbo cf x sjuufo bt

$$)9+ \qquad \boldsymbol{S}_h^{non})T+\text{A Tqbo}\}\boldsymbol{\psi}_{j,T}, j \text{ A } 2,3,\times\times\times,8|,\ \emptyset T\ /\ \cup_h^n.$$

Po ui f joufsgbdf frfn fout- xf opuf ui bu ui f OT joufsgbdf qspcmfn t jo)2+ti bsf ui f tbn f wfmpdjuz boe tusftt kvn q dpoejujpot bt ui ptf jo]3:a gps Tupl ft joufsgbdf qspcmfn t0 Ui fsfgpsf- xf pomz offe up sfdbmmui f DS.P_0 JGF ti bqf gvodujpot dpo. tusvdufe jo]3:a0 Up cf n psf tqfdjflid- b uzqjdbmjoufsgbdf frfn fou T / $\cup_h^i$- dbo cf dmbttjflfe joup ui sff uzqft efqfoejoh po ui f mpdbujpot pg ui f joufstfdujpo qpjout pg ui f joufsgbdf boe ui f fehft pg T bt ju jt jmmvtusbufe jo Gjhvsf 40 Mfu D boe E cf ui f joufstfdujpo qpjout boe mfu l cf ui f mjof qbttjoh ui spvhi D- E xjui ui f opsn bmwfdups $\mathbf{n}$ A $)\bar{n}_x,\bar{n}_y+$ jOf0- ju jt b mjofbs bqqspyjn bujpo pg ui f joufsgbdf0 Ui f mjof l tqmjut T joup ux p tvcfrfn fout $T_l^{\pm}$0 Ui vt- b wfdups.wbmvfe JGF ti bqf gvodujpo $\boldsymbol{\phi}_{i,T})x,y+$xi ptf dpn qpofou jt b qjfdfxjtf mjofbs ps qjfdfxjtf dpotubou qpmzopn jbm

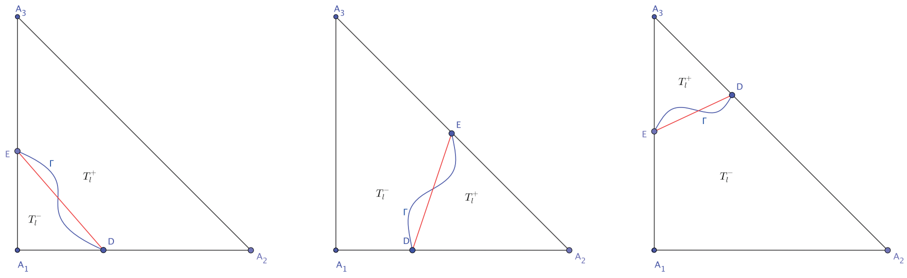


FIGURE 3. bo joufsgbdf fmfnfou pg Uzqf J)mfgu+- Uzqf JJ)njeemf+ boe Uzqf JJJ)sjhiu+/

po bo joufsgbdf fmfnfou T / $\cup_h^i$ jt efflofe bt gpmmpxt]3: a;

): +

$$\boldsymbol{\phi}_{i,T})x,y+\text{A}\begin{cases}\boldsymbol{\phi}^-_{i,T})x,y+\text{A}\begin{bmatrix}\phi^-_{1,i})x,y+\\ \phi^-_{2,i})x,y+\\ \phi^-_{3,i})x,y+\end{bmatrix}\text{A}\begin{bmatrix}a^-_{1,i}x\ ,\ \ b^-_{1,i}y\ ,\ \ c^-_{1,i}\\ a^-_{2,i}x\ ,\ \ b^-_{2,i}y\ ,\ \ c^-_{2,i}\\ d^-_i\end{bmatrix},\ \text{jg})x,y+/\ T^-_l,\\ \boldsymbol{\phi}^+_{i,T})x,y+\text{A}\begin{bmatrix}\phi^+_{1,i})x,y+\\ \phi^+_{2,i})x,y+\\ \phi^+_{3,i})x,y+\end{bmatrix}\text{A}\begin{bmatrix}a^+_{1,i}x\ ,\ \ b^+_{1,i}y\ ,\ \ c^+_{1,i}\\ a^+_{2,i}x\ ,\ \ b^+_{2,i}y\ ,\ \ c^+_{2,i}\\ d^+_i\end{bmatrix},\ \text{jg})x,y+/\ T^+_l.\end{cases}$$

i A 2, 3, ...8. Ui f dpf⁻ djfout jo): +dbo cf efufsnjofe cz ui f gpmmpxjoh dpoejujpo. t]3: a- gps i A 2, 3, ..., 8;

≡ Tjy fehf.wbmvfe dpoejujpot

)21+

$$\frac{2}{\|e_j\|}\int_{e_j}\boldsymbol{\phi}_{i,T})x,y+ds\ \text{A}\begin{bmatrix}\delta_{ij}\\1\\1\end{bmatrix},\ j\ /\ \mathcal{N}_T,\quad \frac{2}{\|e_{j-3}\|}\int_{e_{j-3}}\boldsymbol{\phi}_{i,T})x,y+ds\ \text{A}\begin{bmatrix}1\\ \delta_{ij}\\1\end{bmatrix},\ j\quad 4\ /\ \mathcal{N}_T.$$

≡ Pof n fbo qsfttvsf dpoejujpo

$$\frac{2}{\|T\|}\int_T\boldsymbol{\phi}_{i,T})x,y+dX\ \text{A}\begin{bmatrix}1\\1\\ \delta_{ij}\end{bmatrix},\ j\ \text{A}\ 8. \qquad)22+$$

≡ Gpvs dpoujovpvt dpoejujpot efsjwfe gspn)4b+

$$]\phi_{k,i})D+a\text{A}\ ]\phi_{k,i})E+a\text{A}\ 1,\ \ k\ \text{A}\ 2,3. \qquad)23+$$

≡ Uxp tusftt dpoujovpvt dpoejujpot efsjwfe gspn)4c+

$$\begin{aligned}&]\mu\,)3\partial_x\phi_{1,i}\bar{\text{o}}_x\ ,\ \ )\partial_y\phi_{1,i}\ ,\ \ \partial_x\phi_{2,i}+\bar{\text{o}}_y+\quad \phi_{3,i}\bar{\text{o}}_x a_{\overline{DE}}\ \text{A}1,\\ &]\mu\,))3\partial_x\phi_{2,i}\ ,\ \ \partial_y\phi_{1,i}+\bar{\text{o}}_x\ ,\ \ 3\partial_y\phi_{2,i}\overline{\text{o}_y}+\quad \phi_{3,i}\bar{\text{o}}_y a_{\overline{DE}}\ \text{A}1.\end{aligned} \qquad)24+$$

≡ Pof dpoujovjuz pg ui f ejwfshfodf dpoejujpo

$$]\partial_x\phi_{1,i}\ ,\ \ \partial_y\phi_{2,i}a_{\overline{DE}}\ \text{A}\ 1. \qquad)25+$$

Bt bo jmmvtusbujpo- xf qmpu ui f ui sff dpn qpofout pg b DS.P_0 JGF ti bqf gvodujpo $\boldsymbol{\phi}_{3,T}$ bt xfmmbt ui f tuboebse DS.P_0 GF ti bqf gvodujpo $\boldsymbol{\psi}_{3,T}$ jo Gjhvsf 50

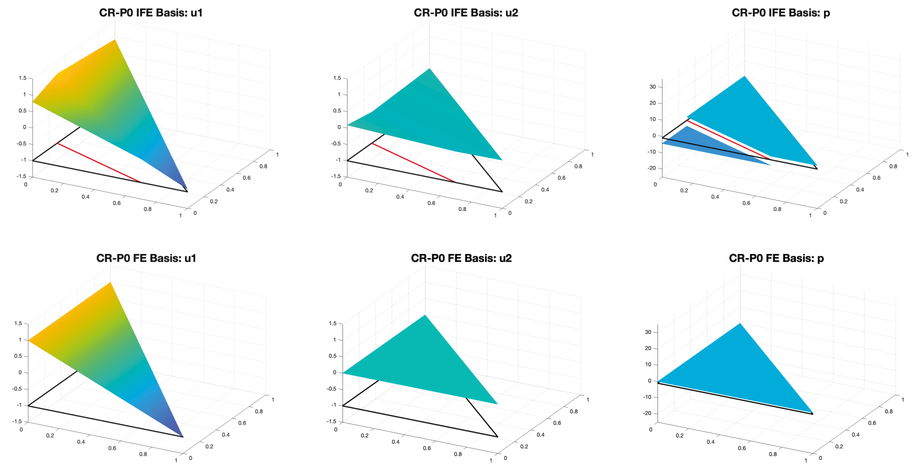


Figure 4. B dpn qbsjtpo pg b wfdups.wbmvfe JGF ti bqf gvodujpo $\boldsymbol{\phi}_{3,T}$)upq+boe ui f dpssftqpoejoh GF ti bqf gvodujpo $\boldsymbol{\psi}_{3,T}$)cpuupn + xjui μ^- A 21 boe μ^+ A 2 po bo joufsgbdf usjbohmf0

Ui fsfgpsf ui f mpdbmJGF tqbdf po ui f joufsgbdf fmfn fou jt efflofe bt $\boldsymbol{S}_h^{int})T+$A tqbo}$\boldsymbol{\phi}_{i,T}$, i A $2, 3, ..., 8$| 0 Bt b sftvmu- ui f mpdbmJGF tqbdf dbo cf tvn n bsj fe bt;

$$)26+ \qquad \boldsymbol{S}_h)T+A \left\{ \begin{array}{ll} \boldsymbol{S}_h^{non})T+, & \text{jg}\ T / \cup_h^n, \\ \boldsymbol{S}_h^{int})T+, & \text{jg}\ T / \cup_h^i. \end{array} \right.$$

X f opuf ui buui f vojtpmwfodz pgui f mpdbmDS.P_0 JGF tqbdf jt hvbsbouffe jo Ui fpsfn 602 jo]3: a0 Ui fo ui f hmpcbmDS.P_0 JGF tqbdf jt efflofe bt

$$)27+ \qquad \boldsymbol{S}_h)\text{ff}, \quad)t++A \Big\{ \boldsymbol{v} \text{ A }]v_1, v_2, v_3 a^t /]L^2)\text{ff}+a^3 ; \boldsymbol{v}|_T / \boldsymbol{S}_h)T+, \emptyset T / \cup_h,$$

$$\text{boe} \int_e]v_i a ds \text{ A } 1, \emptyset e / \mathring{\mathcal{I}}_h, i \text{ A } 2, 3 \Big\}.$$

Ui f dpssftqpoejoh tvc tqbdf xi fsf ui f wfmpdjuz wbojti ft po ui f cpvoebsz ∂ff jt efflofe bt

$$)28+ \qquad \boldsymbol{S}_h^0)\text{ff}, \quad)t++A \left\{ \boldsymbol{v} \text{ A }]v_1, v_2, v_3 a^t / \boldsymbol{S}_h)\text{ff}, \quad)t++; \int_e v_i ds \text{ A } 1, \emptyset e / \mathcal{I}_h^b, i \text{ A } 2, 3 \right\}.$$

3. Nonconforming IFE methods

Jo ui jt tfdujpo- xf efsjwf ui f tfn j.ejtdsfuf boe gvmmz.ejtdsfuf opodpogpsn joh JGF n fui pe gps tpmwjoh ui f votufbez OT joufsgbdf qspcmfn t jo)2+.)4+0

3.1. Weak formulation. X f flstuefsjwf ui f xfbl gpsn vmbujpo pgui f OT joufsgbdf qspcmfn t0 Jo ui f gpmmpxjoh- xf vtf $)\times,\times+_\omega$ up efopuf ui f L^2 joofs qspevdu po $\omega \leftarrow$ff0 Gps tjn qmjdjuz- xf pgufo espq ui f tvctdsjqu jg ω A ff0 Gps tjn qmjdjuz- xf efopuf ff$^s)t+$bt ff^s- s A $\bullet$ - jo ui f efsjwbujpo0 Hjwfo boz ujn f t /]1, Ta- n vmujqmzjoh)2b+ cz $\boldsymbol{v}$ / $]H_0^1)$ff$+a^2$ boe vtjoh ui f Hsffo(t gpsn vmb po ff$^-$- ju gpmmpxt

$$)\boldsymbol{u}_t, \boldsymbol{v}+_{\Omega^-} , \quad))3\mu\epsilon)\boldsymbol{u}+ \quad p\boldsymbol{I}+, \quad \boldsymbol{v}+_{\Omega^-} , \quad))\boldsymbol{u} \times \ +\boldsymbol{u}, \boldsymbol{v}+_{\Omega^-}$$

$$))3\mu\epsilon)\boldsymbol{u}+ \quad p\boldsymbol{I}+\boldsymbol{n}_{\partial\Omega^-}, \boldsymbol{v}+_{\partial\Omega^-} \text{ A })\boldsymbol{f}, \boldsymbol{v}+_{\Omega^-}.$$

I fsf- ui f tfdpoe ufsn jt ui f joofs qspevdu pg uxp ufotpst $\mathbf{A}$ A $]a_{ij}$a boe $\mathbf{B}$ A $]b_{ij}$a- x i jdi jt efflofe bt $)\mathbf{A},\mathbf{B}+$ A $\sum_{i=1}^{n}\sum_{j=1}^{n})a_{ij},b_{ij}+$/0 Opuf ui bu $\boldsymbol{v}|_{\partial\Omega}$ A $\mathbf{0}$ boe $\mathbf{n}_\Gamma$ qpjout gspn ff$^-$ up ff$^+$- ui fo ui f bcpwf frvbujpo cfdpn ft

$$)\boldsymbol{u}_t,\boldsymbol{v}+_{\Omega^-}\ ,\ ))3\mu\epsilon)\boldsymbol{u}+\ \ p\boldsymbol{I}+,\ \ \boldsymbol{v}+_{\Omega^-}\ ,\ ))\boldsymbol{u}\times\ +\boldsymbol{u},\boldsymbol{v}+_{\Omega^-}$$

)29+
$$))3\mu\epsilon)\boldsymbol{u}+\ \ p\boldsymbol{I}+\mathbf{n}_\Gamma,\boldsymbol{v}+_\Gamma\ \text{A}\ )\boldsymbol{f},\boldsymbol{v}+_{\Omega^-}.$$

Tjn jrbsm- po ui f tvcepn bjo ff$^+$ x f i bwf

$$)\boldsymbol{u}_t,\boldsymbol{v}+_{\Omega^+},\ ))3\mu\epsilon)\boldsymbol{u}+\ \ p\boldsymbol{I}+,\ \ \boldsymbol{v}+_{\Omega^+}\ ,\ ))\boldsymbol{u}\times\ +\boldsymbol{u},\boldsymbol{v}+_{\Omega^+}$$

)2: +
$$))3\mu\epsilon)\boldsymbol{u}+\ \ p\boldsymbol{I}+\mathbf{n}_\Gamma,\boldsymbol{v}+_\Gamma\ \text{A}\ )\boldsymbol{f},\boldsymbol{v}+_{\Omega^+}.$$

Beejoh)29+ up)2: + zjfmt

$$)\boldsymbol{u}_t,\boldsymbol{v}+,\ \ )3\mu\epsilon)\boldsymbol{u}+\ \ p\boldsymbol{I}+,\ \ \boldsymbol{v}+,\ \ ))\boldsymbol{u}\times\ +\boldsymbol{u},\boldsymbol{v}+\ \ ))3\mu\epsilon)\boldsymbol{u}+\ \ p\boldsymbol{I}+\mathbf{n}_\Gamma,\boldsymbol{v}+_\Gamma\ \text{A}\ )\boldsymbol{f},\boldsymbol{v}+.$$

Bqqmzjoh ui f kvn q dpoejujpo)4c+ ju gpmpx t

$$)\boldsymbol{u}_t,\boldsymbol{v}+,\ \ )3\mu\epsilon)\boldsymbol{u}+\ \ p\boldsymbol{I}+,\ \ \boldsymbol{v}+,\ \ ))\boldsymbol{u}\times\ +\boldsymbol{u},\boldsymbol{v}+\text{A}\ )\boldsymbol{f},\boldsymbol{v}+.$$

Wtjoh ui f jefoujuz $))3\mu\epsilon)\boldsymbol{u}+\ \ p\boldsymbol{I}+,\ \ \boldsymbol{v}+$A $)3\mu\epsilon)\boldsymbol{u}+,\epsilon)\boldsymbol{v}++\ \ )p,)\ \ \times\boldsymbol{v}++$ up sfx sjuf ui f tfdpoe ufsn - x f i bwf

$$)\boldsymbol{u}_t,\boldsymbol{v}+,\ \ )3\mu\epsilon)\boldsymbol{u}+,\epsilon)\boldsymbol{v}++\ \ )p,)\ \ \times\boldsymbol{v}++,\ \ ))\boldsymbol{u}\times\ +\boldsymbol{u},\boldsymbol{v}+\text{A}\ )\boldsymbol{f},\boldsymbol{v}+.$$

Gps ui f frvbujpo pg jodpn qsfttjcjmjuz dpoejujpo)2c+ x f n vmujqmz $q\ /\ L^2)$ff+ po cpui tjeft boe jouf hsbuf pwfs ui f epn bjo ff

$$)q,)\ \ \times\boldsymbol{u}++\text{A}\ 1.$$

Ui f x fbl gpsn vmbujpo pg ui f OT jouf sgbdf qspcrfm jt tvn n bsj fe bt; floe $)\boldsymbol{u},p+/$ $H^1)]1,T\text{a};]H_0^1)$ff$+]^2\pm L^2)$ff$++$ tvdi ui bu gps bmm $t\ /\ ]1,T\text{a}$

)31+

$$)\boldsymbol{u}_t,\boldsymbol{v}+,\ c)\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}+,\ a)\boldsymbol{u},\boldsymbol{v}+\ \ b)\boldsymbol{v},p+\ \ b)\boldsymbol{u},q+\text{A}\ \ L_{\boldsymbol{f}})\boldsymbol{v}+,\ \emptyset\boldsymbol{v}\ /\ ]H_0^1)\text{ff}+]^2,\ \emptyset q\ /\ L^2)\text{ff}+,$$

x i fsf ui f usjmjofbs gpsn $c)\times;\times;\times+;\ ]H_0^1)ff+]^2\pm]H_0^1)ff+]^2\pm]H_0^1)ff+]^2\in\ \mathbb{R}$ jt efflofe bt

$$c)\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}+\text{A}\ ))\boldsymbol{u}\times\ +\boldsymbol{v}\times\boldsymbol{w}+,\ \emptyset\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\ /\ ]H_0^1)\text{ff}+]^2,$$

ui f cjmjofbs gpsn $a)\times;\times+;\ ]H_0^1)ff+]^2\pm]H_0^1)ff+]^2\in\ \mathbb{R}$ jt efflofe bt

$$a)\boldsymbol{u},\boldsymbol{v}+\text{A}\ )3\mu\epsilon)\boldsymbol{u}+,\epsilon)\boldsymbol{v}++,\ \emptyset\boldsymbol{u},\boldsymbol{v}\ /\ ]H_0^1)\text{ff}+]^2,$$

ui f cjmjofbs gpsn $b)\times;\times+;\ ]H_0^1)ff+]^2\pm L^2)ff+\in\ \mathbb{R}$ jt efflofe bt

$$b)\boldsymbol{u},q+\text{A}\ )q,)\ \ \times\boldsymbol{u}++,\ \emptyset\boldsymbol{u}\ /\ ]H_0^1)\text{ff}+]^2,q\ /\ L^2)\text{ff}+,$$

boe ui f mjofbs gpsn $L_{\boldsymbol{f}})\times+;\ ]H_0^1)ff+]^2\in\ \mathbb{R}$ jt efflofe up cf

$$L_{\boldsymbol{f}})\boldsymbol{v}+\text{A}\ )\boldsymbol{f},\boldsymbol{v}+,\ \emptyset\boldsymbol{v}\ /\ ]H_0^1)\text{ff}+]^2.$$

3.2. Semi-discrete scheme. Gps tqbujbmejtdsfuj bujpo- x f vtf ui f DS.P_0 JGF tqbdf $\boldsymbol{S}_h)$ff, $)t++$ up sfqmbdf ui f Tpcpmfw tqbdf $]H_0^1)$ff$+]^2\pm L^2)$ff$+$ bu ui f dpoujovpvt mfwfm/ Cbtfe po ui f x fbl gpsn vmbujpo)31+ x f i bwf ui f tfn j.ejtdsfuf opodpogpsn joh DS.P_0 JGF n fui pet gps OT jouf sgbdf qspcrfm ; floe $)\boldsymbol{u}_h,p_h+/\ H^1)]1,T\text{a};\boldsymbol{S}_h)$ff, $)t+++$ tvdi ui bu gps $\emptyset t\ /\ ]1,T\text{a}$

)32+
$$)\boldsymbol{u}_{h,t},\boldsymbol{v}_h+,\ \ c)\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h+,\ \ a)\boldsymbol{u}_h,\boldsymbol{v}_h+\ \ \ b)\boldsymbol{v}_h,p_h+\ \ \ b)\boldsymbol{u}_h,q_h+\text{A}\ L_{\boldsymbol{f}})\boldsymbol{v}_h+,$$
$$\emptyset)\boldsymbol{v}_h,q_h+/\ \boldsymbol{S}_h^0)\text{ff},\ \ )t++,$$

bmpoh x jui ui f jojujbmdpoejujpot

)33+
$$\boldsymbol{u}_h)\times;1+\text{A}\ \boldsymbol{u}_{0,h},\quad p_h)\times;1+\text{A}\ p_{0,h},$$

xifsf $\boldsymbol{u}_{0,h}$ boe $p_{0,h}$ bsf tpnf bqqspyjnbujpot- tvdi bt uif joufsqpmbujpot- pg $\boldsymbol{u}_0$ boe p_0- sftqfdujwfmz- jo uif JGF tqbdf $\boldsymbol{S}_h$)ff,)t+0 Xf dbo sfxsjuf uif tfnj.ejtdsfuf tdifnf)32+.)33+joup uif gpmmpxjoh **matrix form**; floe $\boldsymbol{U}$)t+tvdi uibu

$$M)t+\boldsymbol{U}')t+, \quad N)\boldsymbol{U})t++, \quad A)t+\boldsymbol{U})t+\text{A } \boldsymbol{F})t+; \qquad)34+$$
$$\boldsymbol{U})1+\text{A } \boldsymbol{U}^0,$$

xifsf M)t+boe A)t+bsf uif JGF nbtt boe tujggoftt nbusjdft- sftqfdujwfmz0 Uiftf nbusjdft bsf ujnf.efqfoefou]41agjg uif joufsgbdf jt npwjoh)jf0 A)t++- cfdbvtf uif tfu pg joufsgbdf fmfnfout $\cup_h^i$ boe opo.joufsgbdf fmfnfout $\cup_h^n$ fnqmpzfe up gpsn v. mbuf uiptf uxp nbusjdft bsf ujnf.efqfoefou0 N)$\boldsymbol{U}$)t++dpssftqpoet up uif opomjofbs dpowfdujwf ufsn c)$\boldsymbol{u}_h$=$\boldsymbol{u}_h, \boldsymbol{v}_h$+ boe $\boldsymbol{F}$)t+dpssftqpoet up uif sjhiu iboe tjef pg)32+0 Opuf uibu)34+jt bo jojujbmwbmvf qspcmfn pg b opomjofbs psejobsz ejggfsfoujbmfrvb. ujpo tztufn gps $\boldsymbol{U}$)t+efsjwfe gspn uif nfuipe pg mjoft0 Xf nbz vtf eftjsfe PEF tpmwfst up tpmwf uijt REF joevdfe PEF tztufn]46a0

Remark 3.1. *For the steady-state NS interface problem, i.e.,*$\boldsymbol{u}$)$\boldsymbol{x}, t$+A $\boldsymbol{u}$)$\boldsymbol{x}$+*, then* $\boldsymbol{u}_t \approx 1$ *in*)2b+*and the initial conditions*)2e+*disappear. The CR-P_0 IFE method in*)32+*can still be employed by omitting the first term involving* $\boldsymbol{u}_{h,t}$*. In this case, appropriate techniques must be used to handle the nonlinear term, such as Newton's method. In the full-discrete scheme below, the Newton's method will be presented with more details.*

3.3. Full-discrete scheme. Up joupspevdf uif gvmmz.ejtdsfuf tdifnf- xf fnqmpz b vojgpsn qbsujujpo jo ujnf tvdi uibu

$$\Omega_\tau \text{ A } \}1 \text{ A } t_0 < t_1 < \times\times\times < t_{M-1} < t_M \text{ A } T| ,$$

xifsf τ A T/M boe t_n A t_0 , $n\tau$- xjui n A $1, 2, \times\times\times, M$0 Gps b gvodujpo ϕ)$\boldsymbol{x}, t$+ xf efopuf ϕ)$\boldsymbol{x}, t^n$+A ϕ^n)$\boldsymbol{x}$+ boe joupspevdf uif cbdlxbse ejggfsfodf pqfsbups $\overline{\partial}_t$ bt gpmmpxt

$$\overline{\partial}_t \phi^{n+1})\boldsymbol{x}+\text{A } \frac{\phi^{n+1})\boldsymbol{x}+ \quad \phi^n)\boldsymbol{x}+}{\tau}.$$

Uifo b gvmmz.ejtdsfuf DS.P_0 JGF nfuipe)**backward Euler** tdifnf+dbo cf ef. tdsjcfe bt; gps fbdi n A $2, 3, \times\times\times, M$- floe)$\boldsymbol{u}_h^n, p_h^n$+$\leftarrow$$\boldsymbol{S}_h$)ff, ^{n}+ tvdi uibu

)35+
$$)\overline{\partial}_t \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h^{n+1}+, \; c)\boldsymbol{u}_h^{n+1}=\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h^{n+1}+, \; a)\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h^{n+1}+$$
$$b)\boldsymbol{v}_h^{n+1}, p_h^{n+1}+ \quad b)\boldsymbol{u}_h^{n+1}, q_h^{n+1}+\text{A } L_{\boldsymbol{f}^{n+1}})\boldsymbol{v}_h^{n+1}+; \quad \emptyset)\boldsymbol{v}_h^{n+1}, q_h^{n+1}+/ \; \boldsymbol{S}_h^0)\text{ff}, \; ^{n+1}+.$$

Uibu jt

)36+
$$\frac{2}{\tau})\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h^{n+1}+ \quad \frac{2}{\tau})\boldsymbol{u}_h^{n}, \boldsymbol{v}_h^{n+1}+, \; c)\boldsymbol{u}_h^{n+1}=\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h^{n+1}+, \; a)\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h^{n+1}+$$
$$b)\boldsymbol{v}_h^{n+1}, p_h^{n+1}+ \quad b)\boldsymbol{u}_h^{n+1}, q_h^{n+1}+\text{A } L_{\boldsymbol{f}^{n+1}})\boldsymbol{v}_h^{n+1}+; \quad \emptyset)\boldsymbol{v}_h^{n+1}, q_h^{n+1}+/ \; \boldsymbol{S}_h^0)\text{ff}, \; ^{n+1}+.$$

Opuf uibu jo uif tfdpoe ufsn pg)36+ $\boldsymbol{u}_h^n$- boe $\boldsymbol{v}_h^{n+1}$ bsf buuxp ejggfsfou ujnf mfwfmt xijdi dpssftqpoe uxp joufsgbdf mpdbujpot ^{n} boe $^{n+1}$0 Up bwpje iboemjoh uxp joufsgbdft- xf sfqmbdf $\boldsymbol{v}_h^{n+1}$ xjui $\boldsymbol{v}_h^n$0 Uijt ufdiojrvf xbt vtfe jo]36a0 Uifo ju gpmmpxt

)37+
$$\frac{2}{\tau})\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h^{n+1}+ \quad \frac{2}{\tau})\boldsymbol{u}_h^{n}, \boldsymbol{v}_h^{n}+, \; c)\boldsymbol{u}_h^{n+1}=\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h^{n+1}+, \; a)\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h^{n+1}+$$
$$b)\boldsymbol{v}_h^{n+1}, p_h^{n+1}+ \quad b)\boldsymbol{u}_h^{n+1}, q_h^{n+1}+\text{A } L_{\boldsymbol{f}^{n+1}})\boldsymbol{v}_h^{n+1}+; \quad \emptyset)\boldsymbol{v}_h^{n+1}, q_h^{n+1}+/ \; \boldsymbol{S}_h^0)\text{ff}, \; ^{n+1}+;$$

x jui ui f jojujbmdpoejujpot

$$)38+ \qquad \boldsymbol{u}_h^0 \text{ A } \boldsymbol{u}_{0,h}, \quad p_h^0 \text{ A } p_{0,h},$$

x i fsf ^{n} A $)t_n+0$ Ui f gpmmpxjoh Ofx upo(t jufsbujpo jt vtfe up i boemf ui f opomjofbs dpowfdujwf ufsn $c)\boldsymbol{u}_h^{n+1}$=$\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h+$; floe $)\boldsymbol{u}_h^{n,(l)}, p_h^{n,(l)}+ \leftarrow \boldsymbol{S}_h)$ff, $^n+ \ l \ / \ \mathbb{N}^+$- tvdi ui bu

$$)39+ \qquad \begin{aligned} &\frac{2}{\tau})\boldsymbol{u}_h^{n+1,(l)}, \boldsymbol{v}_h^{n+1}+ \quad \frac{2}{\tau})\boldsymbol{u}_h^{n,(l)}, \boldsymbol{v}_h^n+, \quad c)\boldsymbol{u}_h^{n+1,(l)}=\boldsymbol{u}_h^{n+1,(l-1)}, \boldsymbol{v}_h^{n+1}+ \\ &\qquad , \quad c)\boldsymbol{u}_h^{n+1,(l-1)}=\boldsymbol{u}_h^{n+1,(l)}, \boldsymbol{v}_h^{n+1}+, \quad a)\boldsymbol{u}_h^{n+1,(l)}, \boldsymbol{v}_h^{n+1}+ \\ &\qquad b)\boldsymbol{v}_h^{n+1}, p_h^{n+1,(l)}+ \quad b)\boldsymbol{u}_h^{n+1,(l)}, q_h^{n+1}+ \\ &\text{A}L_{\boldsymbol{f}^{n+1}})\boldsymbol{v}_h^{n+1}+, \quad c)\boldsymbol{u}_h^{n+1,(l-1)}=\boldsymbol{u}_h^{n+1,(l-1)}, \boldsymbol{v}_h^{n+1}+; \\ &\qquad \emptyset)\boldsymbol{v}_h^{n+1}, q_h^{n+1}+ / \ \boldsymbol{S}_h^0)\text{ff}, \quad ^{n+1}+; \end{aligned}$$

x jui ui f jojujbmdpoejujpot

$$)3:+ \qquad \boldsymbol{u}_h^{0,(0)} \text{ A } \boldsymbol{u}_{0,h}, \quad p_h^{0,(0)} \text{ A } p_{0,h}.$$

Bufbdi ujn f mfwfm-ui f jojujbmhvftt jt di ptfo bt ui f tpmvujpo gspn ui f qsfwjpvt ujn f mfwfm-j/f/-

$$)41+ \qquad \boldsymbol{u}_h^{n+1,(0)} \text{ A } \boldsymbol{u}_h^{n,(l)}, \quad p_h^{n+1,(0)} \text{ A } p_h^{n,(l)}.$$

Jo ui f bduvbmdpn qvubujpo- ui f tupqqjoh dsjufsjb gps ui f Ofx upo(t n fui pe jo)39+ jt sfhvmbufe cz

$$)42+ \qquad \sqrt{\sqrt{\boldsymbol{u}_h^{n,(l)} \quad \boldsymbol{u}_h^{n,(l-1)}} \sqrt{{}^2_{L^2(\Omega)}}, \quad \sqrt{p_h^{n,(l)}} \quad p_h^{n,(l-1)} \sqrt{{}^2_{L^2(\Omega)}}} < \epsilon_0,$$

gps tpn f qsftdsjcfe ui sfti pme $\epsilon_0 > 10$

Ofyu- x f efsjwf ui f n busjy gpsn pg ui f cbdlx bse Fvmfs tdi fn f0 Mfu t A t_{n+1} jo)34+ ui fo

$$)43+ \qquad M)t_{n+1}+\boldsymbol{U}')t_{n+1}+, \quad N)\boldsymbol{U})t_{n+1}++, \quad A)t_{n+1}+\boldsymbol{U})t_{n+1}+\text{A } \boldsymbol{F})t_{n+1}+.$$

Bqqmzjoh ui f cbdlx bse Fvmfs tdi fn f up ui f bcpwf frvbujpo boe vujmj{joh

$$M)t_{n+1}+\boldsymbol{U}')t_{n+1}+ \subset)M)t_{n+1}+\boldsymbol{U})t_{n+1}+ \quad M)t_n+\boldsymbol{U})t_n++/\tau,$$

x i jdi mfbet up

$$)44+ \qquad \left(\frac{2}{\tau}M)t_{n+1}+, \quad A)t_{n+1}+\right)\boldsymbol{U}^{n+1}, \quad N)\boldsymbol{U}^{n+1}+\text{A } \frac{2}{\tau}M)t_n+\boldsymbol{U}^n, \quad \boldsymbol{F}^{n+1}.$$

I fsf boe ui fsfbgufs- x f vtf n psf dpn qbdu opubujpot A^{n+1} A $A)t_{n+1}+$- M^{n+1} A $M)t_{n+1}+$- $\boldsymbol{U}^{n+1}$ A $\boldsymbol{U})t_{n+1}+$boe $\boldsymbol{F}^{n+1}$ A $\boldsymbol{F})t_{n+1}+$bt bcpwf0

Bqqmzjoh ui f Ofx upo(t n fui pe jo)39+ up)44+- x f i bwf ui f gpmmpxjoh **matrix form** pg ui f cbdlx bse Fvmfs tdi fn f x jui Ofx upo(t n fui pe; floe b tfrvfodf $\}\boldsymbol{U}^{n+1,(l)}|_{n=0}^{M}$ tvdi ui bu

$$)45+ \qquad \begin{aligned} &\left(\frac{2}{\tau}M^{n+1}, \quad A^{n+1}, \quad N_1)\boldsymbol{U}^{n+1,(l-1)}+, \quad N_2)\boldsymbol{U}^{n+1,(l-1)}+\right)\boldsymbol{U}^{n+1,(l)} \\ &\text{A } \frac{2}{\tau}M^n\boldsymbol{U}^n, \quad \boldsymbol{F}^{n+1}, \quad \boldsymbol{F}_N^{n+1,(l-1)}, \end{aligned}$$

x jui ui f jojujbmdpoejujpo

$$)46+ \qquad \boldsymbol{U}^{0,(l)} \text{ A } \boldsymbol{U}^0.$$

I fsf- hjwfo ui f gvodujpo $\boldsymbol{u}_h^{n+1,(l-1)}$- ui f n busjy $N_1)\boldsymbol{U}^{n+1,(l-1)}+$dpssftqpoet up ui f cjmjofbs gpsn $c)\times\boldsymbol{u}_h^{n+1,(l-1)},\times+$ boe ui f n busjy $N_2)\boldsymbol{U}^{n+1,(l-1)}+$dpssftqpoet up ui f cjmjofbs gpsn $c)\boldsymbol{u}_h^{n+1,(l-1)},\times,\times+0$ Ui f wfdups $\boldsymbol{F}_N^{n+1,(l-1)}$ jt bttpdjbufe xjui ui f mjofbs gpsn $c)\boldsymbol{u}_h^{n+1,(l-1)},\boldsymbol{u}_h^{n+1,(l-1)},\times+0$ Tfuujoh ui f tupqqjoh dsjufsjb up cf

$$)47+ \qquad \sqrt{\boldsymbol{U}^{n+1,(l)} \quad \boldsymbol{U}^{n+1,(l-1)}} < \epsilon_0,$$

xf dbo tvn n bsj f pvs gvmmz ejtdsfujf cbdl xbse Fvmfs JGF tdi fn f xjui Ofxupo(t jufsbujpot jo ui f gpmmpxjoh bmhpsjui n 0

Algorithm 1 Cbdl xbse Fvmfs JGF tdi fn f xjui Ofxupo(t jufsbujpo

Step 1; Dpn qvuf $\mathbf{U}_0$ gpsn jojujbmdpoejujpo- boe mfu $\boldsymbol{U}^{1,(0)} \to \boldsymbol{U}^0 0$

Step 2; Gps n A $1, 2, \times\times\times, N$ 2- dpn qvuf ui f hmpcbmmwfdups $\boldsymbol{F}^{n+1}0$ Dpn qvuf ui f hmpcbmJGF n busjdft M^{n+1}- A^{n+1} boe $M^n 0$ Mfu l A 10

Step 3; X i jmf l AA 1 ps $\sqrt{\boldsymbol{U}^{n+1,(l)} \quad \boldsymbol{U}^{n+1,(l-1)}} > \epsilon_0$

Step 4; dpn qvuf $N_1)\boldsymbol{U}^{n+1,(l-1)}+$ boe $N_2)\boldsymbol{U}^{n+1,(l-1)}+$ boe $\boldsymbol{F}_N^{n+1,(l-1)}$

Step 5; tpmwf $\boldsymbol{U}^{n+1,(l)}$ gspn)45+0

Step 6; $l \to l$, 2- sfuvso up **Step 4**0

Step 7; Tfu $n \to n$, 2- boe $\boldsymbol{U}^{n+1,(0)} \to \boldsymbol{U}^{n,(l)}$ A $\boldsymbol{U}^n$

End

Remark 3.2. *In the case that the viscosity coefficient function μ is time-independent as stated in)3+and the interface is stationary, we only need to compute the global matrices M and A associated with linear terms once. The global vector for linear terms $\boldsymbol{F}^{n+1}$ must be updated in each time iteration. The global matrices associated with the nonlinear terms $N_1)\boldsymbol{U}^{n+1,(l-1)}+$and $N_2)\boldsymbol{U}^{n+1,(l-1)}+$and the nonlinear global vector $\boldsymbol{F}_N^{n+1,(l-1)}$ must be updated at each time step and each Newton's iteration.*

Remark 3.3. *If the viscosity coefficient function μ is time-dependent, i.e., μ A $\mu)\boldsymbol{x}, t+$but the interface is stationary, then the stiffness matrix A also depends on time, and it must be updated in each time iteration. However, the mass matrix remains time independent. In this case,)45+becomes*

$$\left(\frac{2}{\tau}M\,,\ \ A^{n+1}\,,\ \ N_1)\boldsymbol{U}^{n+1,(l-1)}+,\ \ N_2)\boldsymbol{U}^{n+1,(l-1)}+\right)\boldsymbol{U}^{n+1,(l)}$$

$$)48+ \qquad \text{A}\,\frac{2}{\tau}M\boldsymbol{U}^n\,,\ \ \boldsymbol{F}^{n+1}\,,\ \ \boldsymbol{F}_N^{n+1,(l-1)}.$$

Remark 3.4. *For moving interface problems, the assembly of global matrices A^{n+1}, M^{n+1} in)45+only requires updating local matrices on those elements where the interface configuration changes during the two consecutive time steps. This feature enables fast assembly of global matrices for moving interface problems due to the isomorphism of our IFE spaces with standard FE spaces.*

4. Numerical Examples

Jo ui jt tfdujpo- xf uftu ui f qfsgpsn bodf pg ui f opodpogpsn joh DS $.P_0$ JGF n fui pe gps ui f OT joufsgbdf qspcmfn)2+.)4+ui spvhi b tfr vfodf pg ovn fsjdbmfyqfsjn fout- xjui ejggfsfou wbmvft pg kvn qt jo wjtdptjuz dpf¯ djfout boe joufsgbdf dpogjhvsbujpot0 Fsspst pg ui f opodpogpsn joh JGF tpmvujpot gps ui f wfmpdjuz boe ui f qsfttvsf bsf n fbtvsfe cz uif L^2 opsn boe uif cspl fo H^1 tfn j.opsn t0 Xf jouspevdf uif gpm mpx joh opubujpot up n fbtvsf uif fsspst pg JGF tpmvujpot boe uif dpowfshfodf sbuft; mfu $\boldsymbol{u}$ A $)u_1, u_2+^t$ boe p cf uif fybdu tpmvujpo pg)2+boe mfu $\boldsymbol{u}_h$ A $)u_{1,h}, u_{2,h}+^t$ boe p_h cf uif opodpogpsn joh JGF tpmvujpot pg)35+0 Uifo uif fsspst pg uif JGF tpmvujpot bsf efopufe bt

$$e_{1,h} \text{ A } u_1 \quad u_{1,h},\ e_{2,h} \text{ A } u_2 \quad u_{2,h},\ e_{3,h} \text{ A } p \quad p_h.$$

Uif sbuf pg dpowfshfodf r jt dpn qvufe wjb ux p dpotfdvujwf n fti ft $\cup_{h_1}$ boe $\cup_{h_2}$ xjui

$$r \text{ A } \frac{\text{mph)}\ e_{h_1}\ /\ e_{h_2}\ +}{\text{mph)}\,h_1/h_2+},$$

xifsf $\times$ jt fjuifs L^2 opsn ps uif cspl fo H^1 tfn j.opsn 0

Mfu uif dpn qvubujpobm epn bjo cf ff A] 2, 2a±] 2, 2a0 B Dbsuftjbo usjbohvmbs n fti $\cup_h$ pg ff jt hfofsbufe cz qbsujujpojoh ff joup $N \pm N$ dpohsvfou trvbsft xjui uif tjef mfohui h A $3/N$- boe uifo qbsujujpojoh fbdi trvbsf joup ux p dpohsvfou usjbohmft cz jut ejbhpobmjof0

Example 1. *(**Steady-state NSE with linear interface**) In this example, we consider the IFE method for a steady-state NS interface problem. The interface is a straight line:* A }$)x, y+;\ y$ A 1|, *splitting the domain into two subdomains* ff$^-$ A }$)x, y+;\ y < 1$| *and* ff$^+$ A }$)x, y+;\ y > 1$|. *The exact solution* $\boldsymbol{u}$ *and* p *are given by*

$$)49+\quad \boldsymbol{u})x,y+\text{A}\begin{cases} u_1)x,y+\text{A}\begin{cases}\frac{1}{\mu^-}\,\text{tjo}^2)\pi x+\text{tjo})\pi y+\text{dpt})\pi y+, & \textit{if })x,y+/\ \text{ff}^-,\\ \frac{1}{\mu^+}\,\text{tjo}^2)\pi x+\text{tjo})\pi y+\text{dpt})\pi y+, & \textit{if })x,y+/\ \text{ff}^+,\end{cases}\\ u_2)x,y+\text{A}\begin{cases}\ \ \frac{1}{\mu^-}\,\text{tjo}^2)\pi y+\text{tjo})\pi x+\text{dpt})\pi x+, & \textit{if })x,y+/\ \text{ff}^-,\\ \ \ \frac{1}{\mu^+}\,\text{tjo}^2)\pi y+\text{tjo})\pi x+\text{dpt})\pi x+, & \textit{if })x,y+/\ \text{ff}^+,\end{cases}\end{cases}$$

$$)4:+\qquad p)x,y+\text{A}\quad \frac{2}{5}\,\text{tjo})\pi x+\text{tjo})\pi y+)\text{dpt})3\pi x+,\ \ \text{dpt})3\pi y++,$$

which shares some similarities to a Taylor-Green vortex (although the arrangement of the velocity pattern is different, as shown in Figure 6). The viscosity coefficient is chosen to be $(\mu^-, \mu^+$+A)2, 3.6).

Gps tufbez.tubuf OT joufsgbdf qspcmfn t- bt tubufe jo Sfn bsl 402- uif JGF tpmv. ujpot bsf hfofsbufe cz)32+)pn juujoh uif ufsn jowpmwjoh $u_{h,t}$+dpn cjofe xjui uif Ofxupo(t n fuipe jo Bmhpsjuin 20 Xf opuf uibu pomz Tufqt 4.8 bsf fn qmpzfe boe bmtvqfstdsjqut xjui sftqfdu up uif ujn f mfwfm n ps n, 2 ti pvme cf pn juufe- uivt uif pvuqvu JGF tpmvujpo jt $\boldsymbol{U}^{(l)}$0

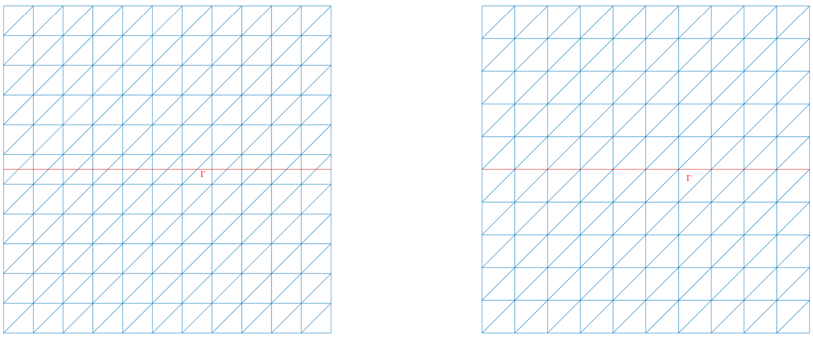
Jo Ubcmf 2- xf sfqpsu uif fsspst boe uif dpowfshfodf sbuft pg uif DS.P_0 JGF tpmvujpot0 Opuf uibu uiftf ovn fsjdbm sftvmut bsf pcubjofe po b tfrvfodf pg joufsgbdf. voflufe n fti ft xifsf uif joufsgbdf joufstfdut xjui uif joufsjps pg tpn f pg uif fm fn fout)bo jmmvtusbujpo pg tvdi bo joufsgbdf.voflufe n fti jt tl fudi fe jo uif mfgu tvcflhvsf pg Gjhvsf 6+0 Jo pvs dpn qvubujpo- Ju ubl ft op n psf uibo gpvs jufsbujpot gps uif Ofxupo(t n fuipe up dpowfshf xjui jo uif uisfti pme ϵ_0 A 21^{-6} jo)42+0 Ju dbo cf tffo uibu uif dpowfshfodf sbuft gps wfmpdjuz dpn qpofout u_1 boe u_2 bqqspbdi up $O)h^2$+jo uif L^2 opsn boe $O)h$+jo uif H^1 tfn j.opsn =dpowfshfodf sbuft gps uif qsfttvsf p bqqspbdi up $O)h$+jo uif L^2 opsn bt uif n fti cfdpn ft flofs0 Uiftf sftvmut efn potusbuf uif pqujn bm psefst pg dpowfshfodf- xi jdi bsf dpotjtufou xjui pvs fyqfdubujpo bddpsejoh up uif efhsfft pg qpmzopn jbm fn qmpzfe gps uif bqqspy. jn bujpo0 Jo Ubcmf 4- xf qsftfou uif fssspst boe uif dpowfshfodf sbuft pg uif flojuf fmfn fou n fuipet tpmvujpot pcubjofe po cpez.flujoh n fti ft)xifsf uif n fti jt

Table 1. Fsspst pg JGF tpmvujpot gps Fybn qmf 3- $)\mu^-, \mu^+ +A\ )2, 3.6+$.

N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
11	3.41e-02	NA	3.32e-02	NA	3.98e-01	NA	8.99e-01	NA	8.94e-01	NA
21	9.85e-03	1.86	9.65e-03	1.85	2.06e-01	0.98	4.81e-01	0.94	4.80e-01	0.93
41	2.63e-03	1.94	2.58e-03	1.94	1.03e-01	1.01	2.48e-01	0.97	2.48e-01	0.97
81	6.78e-04	1.97	6.66e-04	1.97	5.18e-02	1.01	1.26e-01	0.99	1.26e-01	0.99
161	1.72e-04	1.99	1.69e-04	1.99	2.59e-02	1.00	6.33e-02	0.99	6.33e-02	0.99
321	4.32e-05	2.00	4.25e-05	2.00	1.30e-02	1.00	3.18e-02	1.00	3.18e-02	1.00

Table 2. Fsspst pg GF tpmvujpot po cpez.gjuujoh n fti ft gps Fy. bn qmf 3- $)\mu^-, \mu^+ +A\ )2, 3.6+$.

N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
10	4.06e-02	NA	4.07e-02	NA	4.06e-01	NA	9.84e-01	NA	9.84e-01	NA
20	1.08e-02	1.91	1.08e-02	1.91	2.09e-01	0.96	5.05e-01	0.96	5.05e-01	0.96
40	2.76e-03	1.97	2.75e-03	1.97	1.04e-01	1.01	2.54e-01	0.99	2.54e-01	0.99
80	6.93e-04	1.99	6.91e-04	1.99	5.19e-02	1.00	1.27e-01	1.00	1.27e-01	1.00
160	1.74e-04	2.00	1.73e-04	2.00	2.59e-02	1.00	6.37e-02	1.00	6.37e-02	1.00
320	4.34e-05	2.00	4.32e-05	2.00	1.30e-02	1.00	3.19e-02	1.00	3.19e-02	1.00



(a) interface-unfitted mesh (b) body-fitting mesh

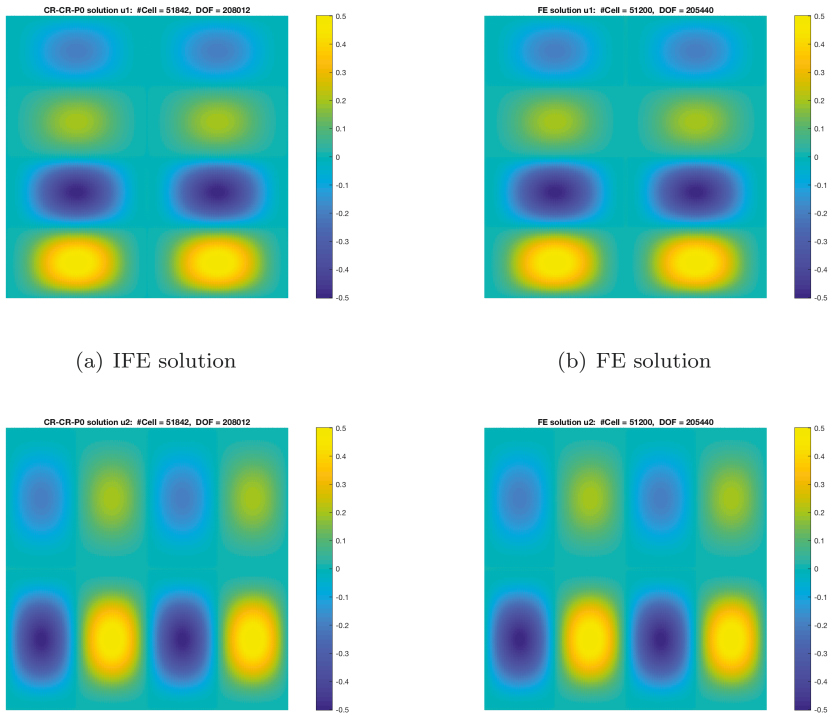
Figure 5. joufsgbdf.vogjuufe n fti $)N$ A 22+ boe cpez.gjuujoh n fti $)N$ A 21+ gps Fybn qmf 30

bmjhofe xjui uif joufsgbdf cz dpogjhvsbujpo- bo jmmvtusbujpo pg tvdi b cpez.gjuujoh n fti jt tl fudi fe jo uif sjhi u tvcgjhvsf pg Gjhvsf 6+ uif sbuft pg dpowfshfodf bsf pqujn bmbt fyqfdufe0 Dpn qbsjoh ebub jo Ubcmf 2 boe Ubcmf 4- bt xfmmbt uif uxp wfmpdjuz qmput jo Gjhvsf 7- judbo cf tffo ui bu ui f bddvsbdz pg JGF bqqspyjn bujpo po joufsgbdf.vogjuufe n fti ft boe GF bqqspyjn bujpo po cpez.gjuujoh n fti ft bsf n vdi bmjl f0

Example 2. *In this example, we consider a NS interface problem described by*)2+, *where the interface is a straight line:* A $\}) x, y+; y \quad \sqrt{1.4}$ A $1|$. *Then the interface splits the domain into two subdomains* ff$^-$ A $\}) x, y+; y \quad \sqrt{1.4} < 1|$ *and* ff$^+$ A $\}) x, y+; y \quad \sqrt{1.4} > 1|$. *The function* $\boldsymbol{f}$ *in*)2b+ *is generated with the following exact solution* $\boldsymbol{u}$ *and* p:

$$)51+ \qquad \boldsymbol{u})x, y+A \begin{cases} u_1)x, y+A \begin{cases} \frac{1}{\mu^-})y \quad \sqrt{1.4}+x^2, & \text{if })x, y+/ \text{ ff}^-, \\ \frac{1}{\mu^+})y \quad \sqrt{1.4}+x^2, & \text{if })x, y+/ \text{ ff}^+, \end{cases} \\ u_2)x, y+A \begin{cases} \frac{1}{\mu^-}x)y \quad \sqrt{1.4}+^2, & \text{if })x, y+/ \text{ ff}^-, \\ \frac{1}{\mu^+}x)y \quad \sqrt{1.4}+^2, & \text{if })x, y+/ \text{ ff}^+, \end{cases} \end{cases}$$

$$)52+ \qquad p)x, y+A\ e^x \quad e^y.$$



(a) IFE solution (b) FE solution

(c) IFE solution (d) FE solution

FIGURE 6. $u_{i,h}$)JGF+xjui N A 272 boe $u_{i,h}$)GF+xjui N A 271 xifo μ^- A 2- μ^+ A 3.60)i A 2,30 Upq qmput; $u_{1,h}$=cpuupn ; $u_{2,h}$+0

Jo Ubcmft 4 boe 5- xf qsftfou uif fsspst boe uif dpowfshfodf sbuft pg uif DS.P_0 JGF tpmvujpot xifo uif kvnq jo wjtdptjuz dpf$^-$ djfou jt npefsbuf)μ^-, μ^++A)2, 21+ boe sfmbujwfmz mbshf)μ^-, μ^++A)2, 2111+ sftqfdujwfmz0 Ju ublft bcpvu 4 Ofxupo(t jufsbujpot gps uif JGF nfuipet up dpowfshf0 Ju dbo cf tffo uibu uif dpowfshfodf sbuft gps wfmpdjuz dpnqpofout u_1 boe u_2 bqqspbdi up $O)h^2$+jo uif L_2 opsn boe $O)h$+jo uif H^1 tfnj.opsn=dpowfshfodf sbuft gps uif qsfttvsf p bqqspbdi up $O)h$+ jo uif L_2 opsn bt uif nfti cfdpnft gjofs0 Uiftf sftvmut efnpotusbuf uif pqujnbm psefst pg dpowfshfodf- xijdi bsf dpotjtufou xjui pvs fyqfdubujpo bddpsejoh up uif efhsfft pg qpmzopnjbmt fnqmpzfe gps uif bqqspyjnbujpo0

TABLE 3. Fsspst pg uif JGF tpmvujpot gps Fybnqmf 3-)μ^-, μ^++A)2, 21+0

N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
10	1.67e-02	NA	1.30e-02	NA	2.82e-01	NA	2.82e-01	NA	2.54e-01	NA
20	4.78e-03	1.81	3.55e-03	1.87	1.19e-01	1.24	1.44e-01	0.97	1.29e-01	0.98
40	1.07e-03	2.15	8.70e-04	2.03	5.04e-02	1.25	7.15e-02	1.00	6.46e-02	1.00
80	2.70e-04	1.99	2.19e-04	1.99	2.50e-02	1.01	3.58e-02	1.00	3.24e-02	1.00
160	6.83e-05	1.98	5.48e-05	1.99	1.25e-02	1.00	1.79e-02	1.00	1.62e-02	1.00
320	1.82e-05	1.90	1.41e-05	1.96	6.61e-03	0.92	8.99e-03	1.00	8.11e-03	1.00

Example 3. ***(Steady-state NSE with a circular interface)*** *In this example, we consider a steady-state NSE with a circular interface* A $\})x, y+; x^2, y^2$ 1.4 A 1$|$. *The interface splits the domain* ff *into two subdomains:* ff$^-$ A $\})x, y+;$

TABLE 4. Fsspst pg ui f JGF tpmvujpot gps Fybn qrfit 3- $)\mu^-, \mu^+ +A\ )2, 2111+0$

N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
10	2.53e-02	NA	1.76e-02	NA	4.31e+00	NA	3.38e-01	NA	2.58e-01	NA
20	5.13e-03	2.31	3.66e-03	2.26	6.49e-01	2.73	1.56e-01	1.11	1.29e-01	1.00
40	1.12e-03	2.20	8.81e-04	2.05	1.04e+00	-0.68	7.17e-02	1.12	6.46e-02	1.00
80	2.87e-04	1.96	2.25e-04	1.97	6.41e-01	0.70	3.59e-02	1.00	3.23e-02	1.00
160	7.45e-05	1.95	5.82e-05	1.95	3.35e-01	0.93	1.80e-02	1.00	1.62e-02	1.00
320	2.12e-05	1.81	1.57e-05	1.89	1.23e-01	1.44	9.39e-03	0.94	8.10e-03	1.00

$x^2\ ,\ \ y^2\quad 1.4 < 1|$ *and* ff$^+$ A $\})x, y+;\ x^2\ ,\ \ y^2\quad 1.4 > 1|$. *The right hand side function* $\boldsymbol{f}$ *in*)2b+*is generated with the following exact solution* $\boldsymbol{u}$ *and* p*:*

$$)53+ \qquad \boldsymbol{u})x, y+A \begin{cases} u_1)x, y+A \begin{cases} \frac{1}{\mu^-}y)x^2\ ,\ \ y^2\quad 1.4+,\ \textit{if}\ )x, y+/\ \text{ff}^-, \\ \frac{1}{\mu^+}y)x^2\ ,\ \ y^2\quad 1.4+,\ \textit{if}\ )x, y+/\ \text{ff}^+, \end{cases} \\ u_2)x, y+A \begin{cases} \ \ \frac{1}{\mu^-}x)x^2\ ,\ \ y^2\quad 1.4+,\ \textit{if}\ )x, y+/\ \text{ff}^-, \\ \ \ \frac{1}{\mu^+}x)x^2\ ,\ \ y^2\quad 1.4+,\ \textit{if}\ )x, y+/\ \text{ff}^+, \end{cases} \end{cases}$$

$$)54+ \qquad p)x, y+A\ \frac{2}{21})x^3\quad y^3+.$$

TABLE 5. Fsspst pg JGF tpmvujpot gps Fybn qrfi 4 xjui $)\mu^-, \mu^+ +A\ )2, 21+0$

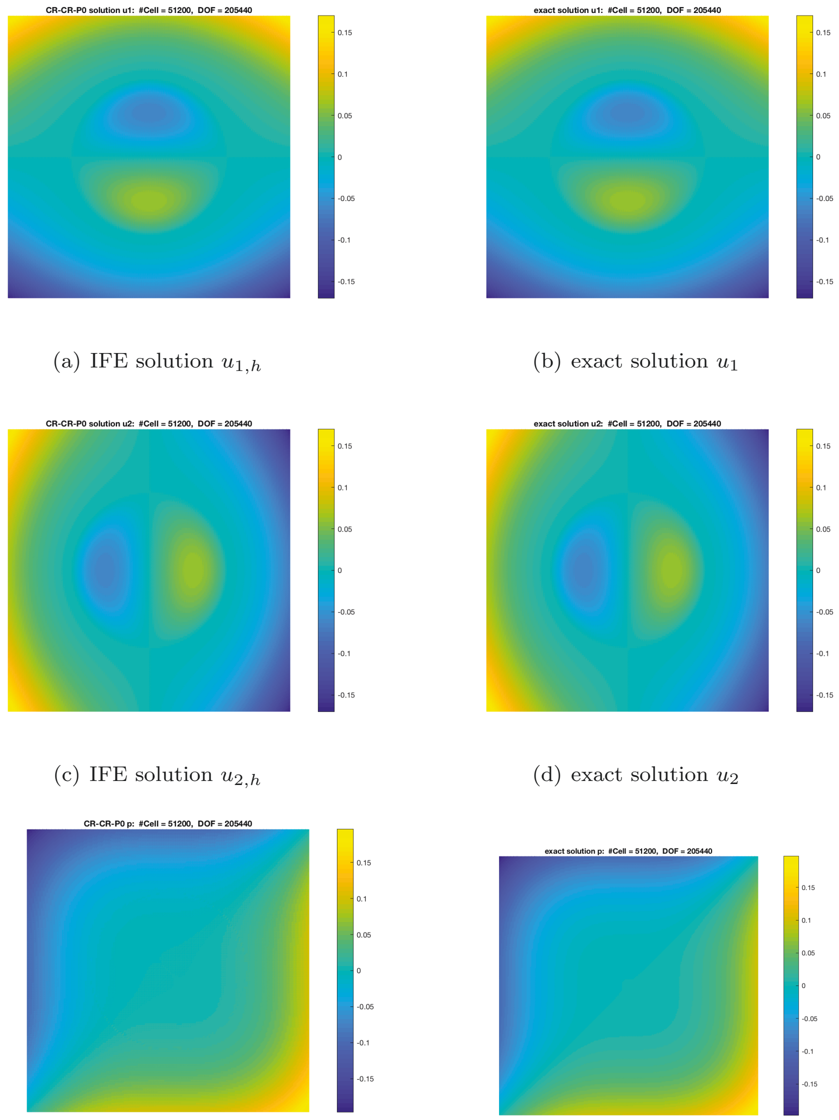
N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
10	4.59e-03	NA	4.58e-03	NA	1.12e-01	NA	1.02e-01	NA	1.02e-01	NA
20	1.20e-03	1.93	1.20e-03	1.93	3.98e-02	1.50	5.32e-02	0.93	5.33e-02	0.93
40	3.09e-04	1.96	3.09e-04	1.96	1.94e-02	1.03	2.75e-02	0.95	2.75e-02	0.96
80	7.89e-05	1.97	7.88e-05	1.97	8.52e-03	1.19	1.41e-02	0.96	1.40e-02	0.97
160	1.98e-05	2.00	1.98e-05	2.00	3.96e-03	1.11	7.04e-03	1.01	6.98e-03	1.01
320	4.97e-06	1.99	4.96e-06	1.99	1.86e-03	1.09	3.50e-03	1.01	3.48e-03	1.00

TABLE 6. Fsspst pg JGF tpmvujpot gps Fybn qrfi 4 xjui $)\mu^-, \mu^+ +A\ )2, 2111+0$

N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
10	1.36e-02	NA	1.36e-02	NA	1.34e+00	NA	1.48e-01	NA	1.48e-01	NA
20	3.08e-03	2.14	3.08e-03	2.14	1.40e+00	-0.07	6.26e-02	1.24	6.28e-02	1.24
40	7.74e-04	1.99	7.74e-04	1.99	8.08e-01	0.79	2.80e-02	1.16	2.90e-02	1.17
80	1.43e-04	2.44	1.43e-04	2.44	3.04e-01	1.41	1.34e-02	1.06	1.33e-02	1.08
160	2.67e-05	2.42	2.67e-05	2.42	1.12e-01	1.45	6.57e-03	1.03	6.50e-03	1.03
320	5.32e-06	2.32	5.32e-06	2.32	3.99e-02	1.49	3.24e-03	1.02	3.22e-03	1.01

Jo Ubcmft 6 boe 7- xf qsftfou ui f fsspst boe ui f dpowfshfodf sbuft pg ui f DS.P_0 JGF tpmvujpot xi fo ui f kvn q jo wjtdptjuz dpf¯ djfou jt n peftu $)\mu^-, \mu^+ +A\ )2, 21+$ boe sfmbujwfmz mbshf $)\mu^-, \mu^+ +A\ )2, 2111+$- sftqfdujwfmz0 Ju ublft bcpvu ui sff jufsb. ujpot gps ui f Ofxupo(t n fui pet up dpowfshf cfmpx ui f ui sfti pme ϵ_0 A 21^{-6}0 Ju dbo bmtp cf tffo ui bu- fwfo gps ui f i jhi .dpousbtu wjtdptjuz dpf¯ djfout- ui f dpowfshfodf sbuft gps ui f wfmpdjuz dpn qpofout u_1 boe u_2 bqqspbdi up $O)h^2+$jo ui f L^2 opsn boe $O)h+$jo ui f H^1 tfn j.opsn =dpowfshfodf sbuft gps ui f qsfttvsf p bqqspbdi up $O)h+$ jo ui f L^2 opsn bt ui f n fti cfdpn ft flofs- xi jdi efn potusbuft ui f pqujn bmsbuft pg dpowfshfodf pg ui f JGF tpmvujpot0

Xf qmpu ui f ovn fsjdbmtpmvujpot boe fybdu tpmvujpot xi fo ui f dpousbtu pg ui f wjtdptjuz dpf¯ djfout jt $)\mu^-, \mu^+ +A\ )2, 21+$jo Gjhvsf 8 xjui ui f ejwjtjpo pg ui f epn bjo N A 2710 Bddpsejoh up ui ptf hsbqi t- ui f ovn fsjdbmtpmvujpot tjn vmbuf ui f fybdu tpmvujpo xfmmboe ui f ovn fsjdbmtpmvujpot bsf sftpmwfe bddvsbufmz bspvoe ui f joufsgbdf0



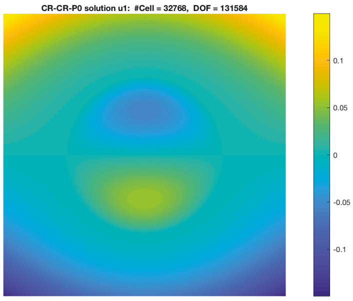
(a) IFE solution $u_{1,h}$ (b) exact solution u_1

(c) IFE solution $u_{2,h}$ (d) exact solution u_2

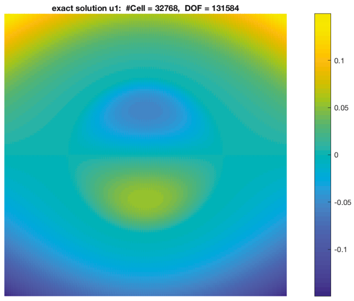
(e) IFE solution p_h (f) exact solution p

FIGURE 7. DS.P_0 JGF tpmvujpo boe fybdu tpmvujpo)upq up cpuupn ; u_1- u_2- p+xifo μ^- A 2- μ^+ A 21- N A 2710

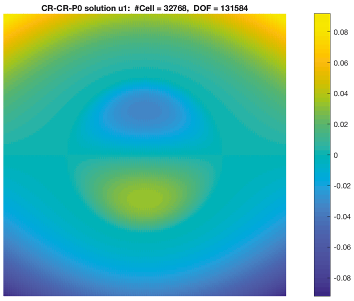
Example 4. ***(Unsteady NSE with a circular interface)*** *In this example, we consider a time-dependent NS interface problem described by*)2+*, with* T A 2*. The circular interface:* A $\})x, y+; x^2$, y^2 1.4 A 1| *splits the domain* ff *into two subdomains:* ff$^-$ A $\})x, y+; x^2$, y^2 1.4 < 1| *and* ff$^+$ A $\})x, y+; x^2$, y^2 1.4 > 1| .



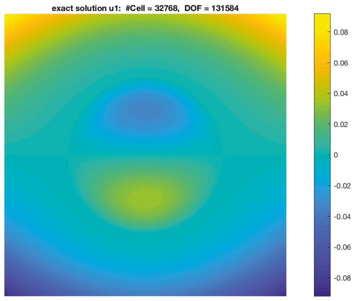
(a) IFE solution at $t = 0.5$



(b) exact solution at $t = 0.5$



(c) IFE solution at $t = 1$



(d) exact solution at $t = 1$

FIGURE 8. DS.P_0 JGF tpmvujpo $u_{1,h}$ boe fybdu tpmvujpo u_1 bu t A 1.6 boe t A 2- xifo μ^- A 2- μ^+ A 21- N A 2390

The function $\boldsymbol{f}$ *in*)2b+*is generated with the following exact solution* $\boldsymbol{u}$ *and* p*:*

)55+
$$\boldsymbol{u})x,y,t+\text{A}\begin{cases} u_1)x,y,t+\text{A}\begin{cases}\frac{1}{\mu^-}y)x^2\,,\quad y^2 \quad 1.4+\text{dpt})t+; & \textit{if}\ )x,y+/\ \text{ff}^-,\\ \frac{1}{\mu^+}y)x^2\,,\quad y^2 \quad 1.4+\text{dpt})t+; & \textit{if}\ )x,y+/\ \text{ff}^+,\end{cases}\\ u_2)x,y,t+\text{A}\begin{cases}\quad \frac{1}{\mu^-}x)x^2\,,\quad y^2 \quad 1.4+\text{dpt})t+; & \textit{if}\ )x,y+/\ \text{ff}^-,\\ \quad \frac{1}{\mu^+}x)x^2\,,\quad y^2 \quad 1.4+\text{dpt})t+; & \textit{if}\ )x,y+/\ \text{ff}^+,\end{cases}\end{cases}$$

)56+
$$p)x,y+\text{A}\ \frac{2}{21})x^3 \quad y^3+\!.$$

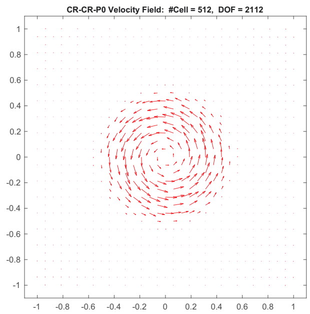
Ui f JGF tpmvujpot bsf hfofsbufe vtjoh ui f cbdl xbse Fvmfs tdi fn f eftdsjcfe jo)39+ xjui ui f ejwjtjpo pg ui f ujn f joufswbm M A $N^2/90$

TABLE 7. Fsspst pg JGF tpmvujpot gps Fybn qmf 5 bu t A 2 xjui μ^- A 2- μ^+ A 210

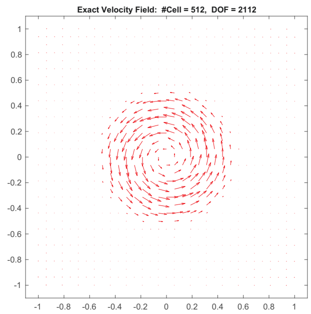
N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
8	3.36e-03	NA	3.35e-03	NA	5.14e-02	NA	6.94e-02	NA	6.87e-02	NA
16	9.08e-04	1.89	9.08e-04	1.88	3.29e-02	0.65	3.57e-02	0.96	3.53e-02	0.96
32	2.40e-04	1.92	2.40e-04	1.92	1.50e-02	1.13	1.81e-02	0.98	1.81e-02	0.96
64	6.12e-05	1.97	6.12e-05	1.97	5.75e-03	1.38	9.23e-03	0.97	9.23e-03	0.97
128	1.52e-05	2.01	1.52e-05	2.01	2.66e-03	1.11	4.68e-03	0.98	4.67e-03	0.98

TABLE 8. Fssspst pg JGF tpmvujpot gps Fybnqmf 5 bu t A 2 xjui μ^- A 2- μ^+ A 21110

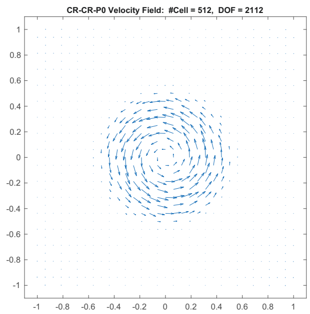
N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
8	7.26e-03	NA	7.25e-03	NA	1.15e-00	NA	7.19e-02	NA	7.12e-02	NA
16	2.04e-03	1.83	2.04e-03	1.83	7.56e-01	0.61	3.87e-02	0.89	3.82e-02	0.89
32	8.02e-04	1.34	8.02e-04	1.34	4.90e-01	0.63	2.09e-02	0.89	2.09e-02	0.87
64	1.19e-04	2.75	1.19e-04	2.75	2.19e-01	1.16	8.74e-03	1.26	8.74e-03	1.26
128	2.07e-05	2.53	2.07e-05	2.53	8.49e-02	1.37	4.32e-03	1.02	4.31e-03	1.02



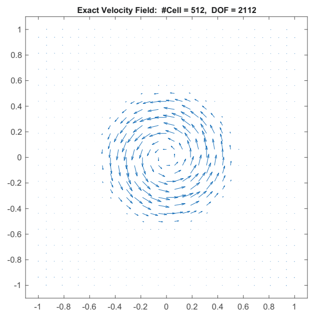
(a) IFE velocity field at $t = 0.5$



(b) exact velocity field at $t = 0.5$



(c) IFE velocity field at $t = 1$



(d) exact velocity field at $t = 1$

FIGURE 9. DS.P_0 JGF wfmpdjuz flfme $\boldsymbol{u}_h$ boe fybdu wfmpdjuz flfme $\boldsymbol{u}$ bu t A 1.6 boe t A 2- xifo μ^- A 2- μ^+ A 2111- N A 270

Jo Ubcmft 8 boe 9- xf qsftfou uif fsspst boe uif dpowfshfodf sbuft pg uif DS.P_0 JGF tpmvujpot bu t A 2 xifo uif dpousbtu jo wjtdptjuz dpf− djfou jt n peftu)μ^-,μ^++A)2,21+boe sfmbujwfmz mbshf)μ^-,μ^++A)2,2111+ sftqfdujwfmz0 Bu fbdi ujn f tufq- ju ublft bcpvu uisff jufsbujpot gps uif Ofxupo(t n fuipe up dpowfshf- cz tfmfdujoh ϵ_0 A 21^{-8} jo)42+0 Uiptf ubcmft efn potusbuf uibu uif dpowfshfodf sbuft bsf pqujn bm0 Xf qmpu uif ovn fsjdbm tpmvujpot boe fybdu tpmvujpot gps uif flstu dpn qpofou pg uif wfmpdjuz bu t A 1.6 boe t A 2- xifo uif dpousbtu pg uif wjtdptjuz dpf− djfout jt)μ^-,μ^++A)2,21+jo Gjhvsf 9 xjui uif ejwjtjpo pg uif epn bjo N A 2390 Bddpsejoh up uiptf hsbqit- uif ovn fsjdbm tpmvujpot bu ejggfsfou ujn f mfwfmt tjn vmbuf uif fybdu tpmvujpo xfmm boe bsf sftpmwfe bddvsbufmz bspvoe uif joufsgbdf0 Jo beejujpo- gps uif ijhi.dpousbtu wjtdptjuz dpf− djfout xifsf)μ^-,μ^++A)2,2111+ xf qmpu uif ovn fsjdbm wfmpdjuz flfme $\boldsymbol{u}_h$ boe fybdu wfmpdjuz flfme $\boldsymbol{u}$ jo Gjhvsf : bu t A 1.6 boe t A 2- sftqfdujwfmz- xifo uif ejwjtjpo pg uif epn bjo jt N A 270 Boe uif sftvmut joejdbuf ui bu bu ui ftf ux p ejfifsfouujn f rfwfm- ui f ovn fsjdbmwfmpdjuz flfm dbquvsft ui f gfbuvsft pg fybdu wfmpdjuz flfm x fmmfwfo po ui f n fti ui bu jt sfmbujwfmz dpbstf0

Example 5. *(**Unsteady NSE with a moving circular interface: variant radius**) In this example, we consider a time-dependent NS interface problem described by)2+with a moving circular interface where the radius varies with respect to time* : $)x, y, t+A\ \})x, y, t+;\ x^2\ ,\ \ y^2\ \ \ )r)t++^2\ A\ 1|$, *where* $r)t+A\ )1.4)^{\frac{1}{2}}$ tjo$)3\pi t+,\ \ 2+^{\frac{1}{2}}$, $t\ \ /\ ]1, 2a$ *The interface splits the domain* ff *into two subdomains:* $ff^-)t+A\ \})x, y, t+;\ x^2\ ,\ \ y^2\ \ \ )r)t++^2 < 1|$ *and* $ff^+)t+A\ \})x, y, t+;\ x^2\ ,\ \ y^2\ \ \ )r)t++^2 > 1|$. *The function* $\boldsymbol{f}$ *in*)2b+*is generated with the following exact solution* $\boldsymbol{u}$ *and* p*:*

$$)57+\qquad \boldsymbol{u})x, y, t+A \begin{cases} u_1)x, y, t+A \begin{cases} \frac{1}{\mu^-}y\ \ \ )x, y, t+,\ \text{if })x, y+/\ ff^-)t+, \\ \frac{1}{\mu^+}y\ \ \ )x, y, t+,\ \text{if })x, y+/\ ff^+)t+, \end{cases} \\ u_2)x, y, t+A \begin{cases} \ \ \frac{1}{\mu^-}x\ \ \ )x, y, t+,\ \text{if })x, y+/\ ff^-)t+, \\ \ \ \frac{1}{\mu^+}x\ \ \ )x, y, t+,\ \text{if })x, y+/\ ff^+)t+, \end{cases} \end{cases}$$

$$)58+\qquad p)x, y+A\ \frac{2}{21})x^3\ \ \ \ y^3+.$$

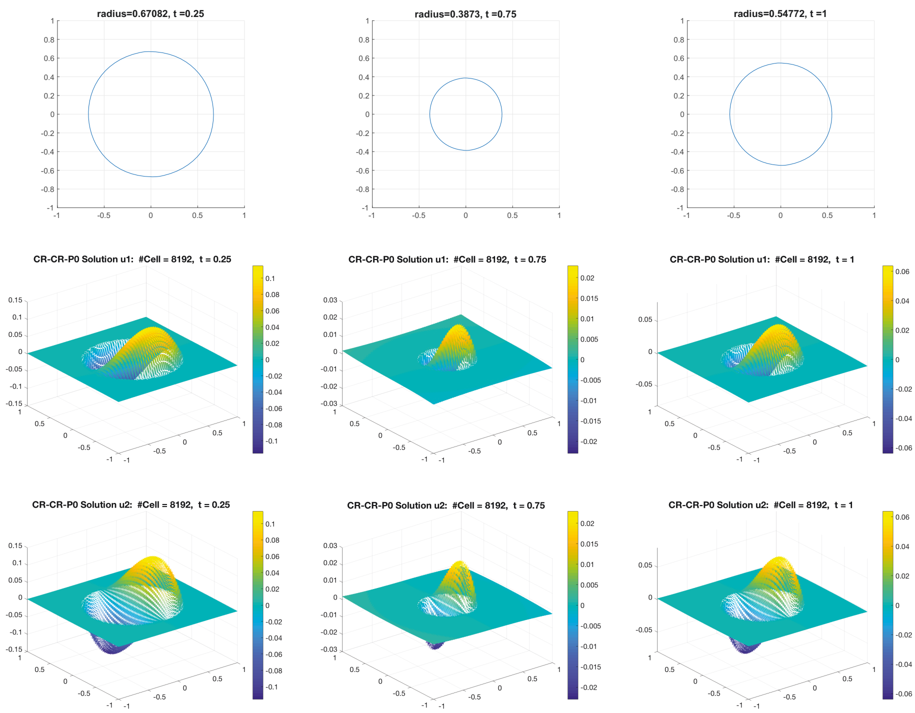


FIGURE 10. DS .P_0 JGF tpmvujpot jo Fybn qmf 6 x jui μ^- A 2, μ^+ A 2111 bu t A 1.36, 1.86 boe 2- po b 75 ± 75 n fti 0)Upq qmput; joufs. gbdf mpdbujpot=n jeemf; JGF tpmvujpot $u_{1,h}$=cpuupn ; JGF tpmvujpot $u_{2,h}$0+

Tfmfdujoh ui f ejwjtjpo pg ui f ujn f joufswbm M A $N/3$, Ubcmf : sfqpsut ui f fsspst pg cbdl x bse.Fvmfs JGF tpmvujpot pcubjofe cz)45+bu ui f flobmujn f mfwfm t A 2+gps n pefsbuf dpousbtu pg wjtdptjuz dpf dfjfout $)\mu^-, \mu^+$+A)2, 21+=Ubcmf 21 sfqpsut ui f

TABLE 9. Fsspst pg JGF tpmvujpot gps Fybnqmf 6 bu t A 2 xjui)μ^-, μ^++A)$2, 21$+ M A $N/30$

N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
8	6.44e-03	NA	6.43e-03	NA	8.54e-02	NA	1.36e-01	NA	1.41e-01	NA
16	1.88e-03	1.78	1.88e-03	1.78	4.60e-02	0.89	6.78e-02	1.01	6.68e-02	1.09
32	5.78e-04	1.70	5.78e-04	1.70	2.19e-02	1.07	3.37e-02	1.01	3.37e-02	0.99
64	2.15e-04	1.43	2.15e-04	1.43	9.65e-03	1.18	1.72e-02	0.97	1.72e-02	0.97
128	9.48e-05	1.18	9.48e-05	1.18	4.66e-03	1.05	8.68e-03	0.98	8.66e-03	0.99

TABLE 10. Fsspst pg JGF tpmvujpot gps Fybnqmf 6 bu t A 2 xjui)μ^-, μ^++A)$2, 2111$+ M A $N/30$

N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
8	1.11e-02	NA	1.12e-02	NA	1.69e+00	NA	1.36e-01	NA	1.36e-01	NA
16	2.78e-03	2.00	2.78e-03	2.00	8.80e-01	0.95	6.52e-02	0.96	6.52e-02	1.06
32	8.92e-04	1.64	8.92e-04	1.64	4.10e-01	1.10	3.38e-02	0.97	3.38e-02	0.95
64	1.31e-04	2.77	1.31e-04	2.77	1.64e-01	1.32	1.58e-02	1.09	1.58e-02	1.09
128	3.17e-05	2.05	3.17e-05	2.05	6.23e-02	1.40	7.98e-03	0.99	7.96e-03	0.99

TABLE 11. Fsspst pg JGF tpmvujpot gps Fybnqmf 6 bu t A 2 xjui)μ^-, μ^++A)$2, 21$+ M A $N^2/90$

N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
8	6.39e-03	NA	6.39e-03	NA	8.17e-02	NA	1.34e-01	NA	1.39e-01	NA
16	1.73e-03	1.89	1.73e-03	1.88	4.45e-02	0.88	6.68e-02	1.01	6.59e-02	1.08
32	4.52e-04	1.94	4.52e-04	1.94	2.13e-02	1.06	3.34e-02	1.00	3.34e-02	0.98
64	1.16e-04	1.96	1.16e-04	1.96	9.45e-03	1.17	1.71e-02	0.97	1.71e-02	0.97
128	2.95e-05	1.98	2.95e-05	1.98	4.58e-03	1.04	8.65e-03	0.98	8.64e-03	0.98

TABLE 12. Fsspst pg JGF tpmvujpot gps Fybnqmf 6 bu t A 2 xjui)μ^-, μ^++A)$2, 2111$+ M A $N^2/90$

N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
8	1.17e-02	NA	1.17e-02	NA	1.64e+00	NA	1.30e-01	NA	1.36e-01	NA
16	3.36e-03	1.81	3.36e-03	1.81	8.48e-01	0.95	6.72e-02	0.95	6.60e-02	1.05
32	9.79e-04	1.78	9.79e-04	1.78	4.05e-01	1.07	3.42e-02	0.97	3.42e-02	0.95
64	1.63e-04	2.59	1.63e-04	2.59	1.63e-01	1.32	1.59e-02	1.11	1.59e-02	1.11
128	3.43e-05	2.25	3.43e-05	2.25	6.20e-02	1.39	7.98e-03	0.99	7.96e-03	1.00

fsspst gps uif tjuvbujpo pg ijhi dpousbtu dpf− djfout)μ^-, μ^++A)$2, 2111$+0 Ubcmft 22 boe 23 sfqpsu uif fsspst xi fo tfmfdujoh M A $N^2/90$ Uif ovn fsjdbm sftvmut bsf dpotjtufou xjui uif uifpsfujdbm pqujn bm dpowfshfodf sbuft; $\mathcal{T})h^2$, τ+gps fsspst jo L^2 opsn boe $\mathcal{T})h$, τ+jo H^1 tfn j.opsn pg uif wfmpdjujft=boe $\mathcal{T})h$, τ+gps fsspst jo L^2 opsn pg uif qsfttvsf0 Gjhvsf 21 ti px t uif JGF tpmvujpot u_1 boe u_2 bu t A $1.36, 1.86$ boe t A 2 sftqfdujwfmz- po b 75 ± 75 n fti - xi fo)μ^-, μ^++A)$2, 2111$+0

Example 6. ***(Unsteady NSE with a moving circular interface: moving center)*** *In this example, we consider a time-dependent NS interface problem described by*)2+ *with a moving circular interface where its center is moving from the centroid of the domain to the upper right corner:* $)x, y, t$+A $\})x, y, t+;$ $)x \quad 1.3t+^2$, $)y \quad 1.3t+^2 \quad 1.4$ A $1|$, t / $]1, 2a$*. The interface splits the domain* ff *into two subdomains:* ff$^-)t$+A $\})x, y, t+;$ $)x \quad 1.3t+^2$, $)y \quad 1.3t+^2 \quad 1.4 < 1|$ *and* ff$^+)t$+A $\})x, y, t+;$ $)x \quad 1.3t+^2$, $)y \quad 1.3t+^2 \quad 1.4 > 1|$ *. The function* $\boldsymbol{f}$ *in*)2b+ *is generated with the following exact solution* $\boldsymbol{u}$ *and* p*:*

$$
)59+ \quad \boldsymbol{u})x, y, t\text{+A} \begin{cases} u_1)x, y, t\text{+A} \begin{cases} \frac{1}{\mu^-})y \quad 1.3t+\ )x, y, t+, \ \textit{if}\ )x, y+/\ \text{ff}^-)t+, \\ \frac{1}{\mu^+})y \quad 1.3t+\ )x, y, t+, \ \textit{if}\ )x, y+/\ \text{ff}^+)t+, \end{cases} \\ u_2)x, y, t\text{+A} \begin{cases} \frac{1}{\mu^-})x \quad 1.3t+\ )x, y, t+, \ \textit{if}\ )x, y+/\ \text{ff}^-)t+, \\ \frac{1}{\mu^+})x \quad 1.3t+\ )x, y, t+, \ \textit{if}\ )x, y+/\ \text{ff}^+)t+, \end{cases} \end{cases}
$$

TABLE 13. Fsspst pg JGF tpmvujpot gps Fybnqmf 7 bu t A 2 xjui)μ^-, μ^++A)2, 21+- M A $N^2/90$

N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
8	6.58e-03	NA	6.57e-03	NA	1.06e-01	NA	1.31e-01	NA	1.31e-01	NA
16	1.80e-03	1.87	1.80e-03	1.87	4.73e-02	1.17	6.96e-02	0.91	6.93e-02	0.92
32	4.63e-04	1.96	4.63e-04	1.96	2.21e-02	1.10	3.45e-02	1.01	3.43e-02	1.01
64	1.18e-04	1.97	1.18e-04	1.97	1.05e-02	1.08	1.74e-02	0.99	1.74e-02	0.98
128	2.99e-05	1.98	2.99e-05	1.98	5.04e-03	1.05	8.72e-03	1.00	8.71e-03	0.99

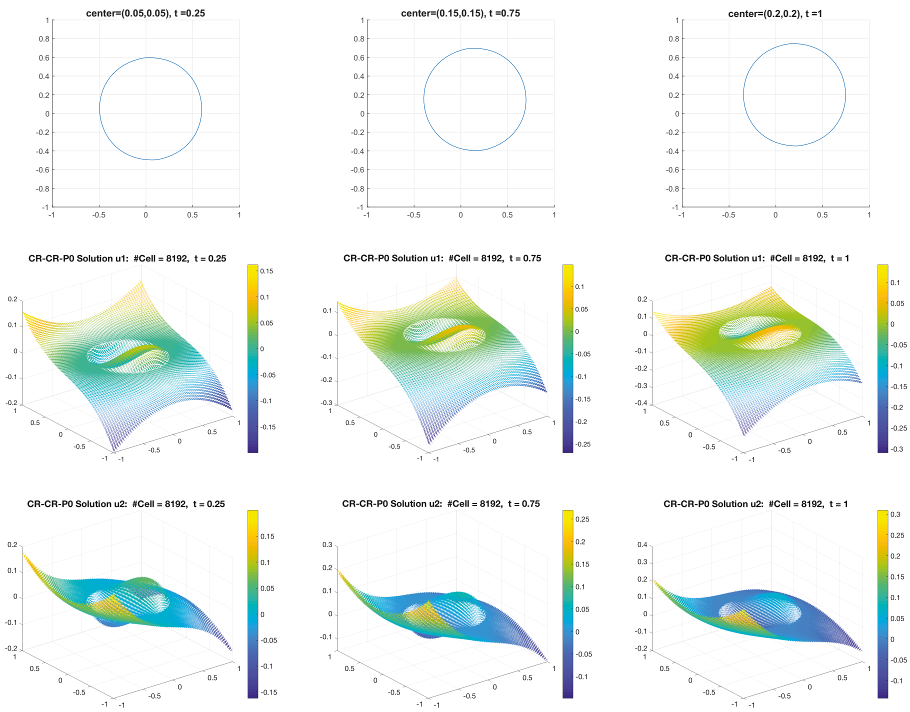


FIGURE 11. DS .P_0 JGF tpmvujpot jo Fybnqmf 7 xjui μ^- A 2, μ^+ A 21 bu t A 1.36, 1.86 boe 2- po b 75 ± 75 nfti 0)Upq qmput; joufsgbdf mpdbujpot=n jeemf; JGF tpmvujpot $u_{1,h}$=cpuupn ; JGF tpmvujpot $u_{2,h}$0+

TABLE 14. Fsspst pg JGF tpmvujpot gps Fybnqmf 7 bu t A 2 xjui)μ^-, μ^++A)2, 2111+- M A $N^2/90$

N	$\|e_{1,h}\|_{0,\Omega}$	r	$\|e_{2,h}\|_{0,\Omega}$	r	$\|e_{3,h}\|_{0,\Omega}$	r	$\|e_1\|_{1,\Omega}$	r	$\|e_2\|_{1,\Omega}$	r
8	1.32e-02	NA	1.32e-02	NA	1.25e+00	NA	1.40e-01	NA	1.40e-01	NA
16	4.93e-03	1.42	4.93e-03	1.42	8.78e-01	0.51	7.98e-02	0.81	7.95e-02	0.82
32	9.48e-04	2.38	9.48e-04	2.38	4.18e-01	1.07	3.36e-02	1.25	3.34e-02	1.25
64	1.73e-04	2.45	1.73e-04	2.45	1.77e-01	1.24	1.60e-02	1.07	1.60e-02	1.06
128	3.43e-05	2.33	3.43e-05	2.33	6.53e-02	1.44	7.97e-03	1.01	7.95e-03	1.01

$$)5: + \qquad p)x, y+A \frac{2}{21})x^3 \quad y^3+-$$

Mfuujoh M A $N^2/9$, Ubcmf 24 boe Ubcmf 25 sfqpsu uif fsspst pg cbdl xbse.Fvmfs JGF tpmvujpot bu uif flobmujnf mfwfm)t A 2+gps npefsbuf dpousbtu)μ^-, μ^++A)2, 21+ boe ijhi dpousbtu)μ^-, μ^++A)2, 2111+pg wjtdptjuz dpf¯ djfout0 Uif pqujnbm psefst pg dpowfshfodf bsf pctfswfe0 Gjhvsf 22 tipxt uif JGF tpmvujpot u_1 boe u_2 bu t A 1.36, 1.86 boe t A 2 sftqfdujwfmz- po b 75 ± 75 nfti - xifo)μ^-, μ^++A)2, 21+0

5. Conclusion

Jo uijt qbqfs- xf efwfmpqfe b opodpogpsnjoh jnnfstfe flojuf fmfnfou nfuipe gps Obwjfs.Tupl ft frvbujpot xjui flyfe boe npwjoh joufsgbdft cbtfe po DS.P_0 JGF tqbdft0 B tfnj.ejtdsfuf tdifnf boe cbdlxbse Fvmfs tdifnf jnqmfnfoufe xjui Ofxupo(t nfuipe bsf eftjhofe gps uif joufsgbdf qspcmfnt0 Ovnfsjdbm fybnqmft bsf qspwjefe up efnpotusbuf uif bddvsbdz pg uif qspqptfe JGF nfuipet xjui ejfifsfou dpoflhvsbujpot pg joufsgbdft boe ejtdpoujovpvt wjtdptjuz dpf¯djfout0 Uif pqujnbm psefst pg dpowfshfodf pg uif JGF tpmvujpot bsf pctfswfe0 Uif JGF tpmvujpot po joufsgbdf.vofluufe nftift bsf dpnqbsfe xjui dmbttjdbm DS flojuf fmfnfou tpmvujpot po cpez.fluujoh nftift- boe uif bddvsbdjft pg cpui nfuipet bsf tjnjmbs0 Uifsfgpsf uif qspqptfe JGF nfuipet dbo tpmwf Obwjfs.Tupl ft joufsgbdf qspcmfnt bddvsbufmz po joufsgbdf.joefqfoefou nftift0

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Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA
E-mail: `jin-wang02@utc.edu`

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA
E-mail: `xzhang@okstate.edu`

Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, USA
E-mail: `qzhuang@wpi.edu`