Analysis and Numerical Solution of a Modular Convex Nash Equilibrium Problem*

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Abstract. We investigate a modular convex Nash equilibrium problem involving nonsmooth functions acting on linear mixtures of strategies, as well as smooth coupling functions. An asynchronous block-iterative decomposition method is proposed to solve it.

1 Introduction

We consider a noncooperative game with p players indexed by $I = \{1, \ldots, p\}$, in which the strategy x_i of player $i \in I$ lies in a real Hilbert space $\mathcal{H}_i$. A strategy profile is a point $\boldsymbol{x} = (x_i)_{i\in I}$ in the Hilbert direct sum $\boldsymbol{\mathcal{H}} = \bigoplus_{i\in I}\mathcal{H}_i$, and the associated profile of the players other than $i \in I$ is the vector $\boldsymbol{x}_{\smallsetminus i} = (x_j)_{j\in I\smallsetminus\{i\}}$ in $\boldsymbol{\mathcal{H}}_{\smallsetminus i} = \bigoplus_{j\in I\smallsetminus\{i\}}\mathcal{H}_j$. For every $i \in I$ and every $(x_i, \boldsymbol{y}) \in \mathcal{H}_i \times \boldsymbol{\mathcal{H}}$, we set $(x_i; \boldsymbol{y}_{\smallsetminus i}) = (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_p)$. Given a real Hilbert space $\mathcal{H}$, we denote by $\Gamma_0(\mathcal{H})$ the class of lower semicontinuous convex functions $\varphi\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ which are proper in the sense that $\operatorname{dom}\varphi = \big\{x \in \mathcal{H} \mid \varphi(x) < +\infty\big\} \neq \varnothing$.

A fundamental equilibrium notion was introduced by Nash in [30, 31] to describe a state in which the loss of each player cannot be reduced by unilateral deviation. A general formulation of the Nash equilibrium problem is

$$\text{find}\;\; \boldsymbol{x} \in \boldsymbol{\mathcal{H}}\;\; \text{such that}\;\; (\forall i \in I)\;\; x_i \in \operatorname{Argmin} \boldsymbol{\ell}_i(\cdot\,; \boldsymbol{x}_{\smallsetminus i}), \tag{1.1}$$

where $\boldsymbol{\ell}_i\colon \boldsymbol{\mathcal{H}} \to \left]-\infty, +\infty\right]$ is the global loss function of player $i \in I$. We make the following assumption: for every $i \in I$ and every $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$, the function $\boldsymbol{\ell}_i(\cdot\,; \boldsymbol{x}_{\smallsetminus i})$ is convex. Such convex Nash equilibrium problems have been studied since the early 1970s [7]; see [4, 8, 9, 13, 17, 20, 21, 24, 25, 28, 37] for further work. We consider the following modular formulation of (1.1), wherein the functions $(\boldsymbol{\ell}_i)_{i\in I}$ are decomposed into elementary components. This decomposition will provide more modeling flexibility and lead to efficient solution methods.

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Problem 1.1 Let $(\mathcal{H}_i)_{i\in I}$ and $(\mathcal{G}_k)_{k\in K}$ be finite families of real Hilbert spaces, and set $\boldsymbol{\mathcal{H}} = \bigoplus_{i\in I} \mathcal{H}_i$ and $\boldsymbol{\mathcal{G}} = \bigoplus_{k\in K} \mathcal{G}_k$. Suppose that the following are satisfied:

[a] For every $i \in I$, $\varphi_i \in \Gamma_0(\mathcal{H}_i)$.

[b] For every $i \in I$, $\boldsymbol{f}_i \colon \boldsymbol{\mathcal{H}} \to \mathbb{R}$ is such that, for every $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$, $\boldsymbol{f}_i(\cdot\,;\boldsymbol{x}_{\smallsetminus i}) \colon \mathcal{H}_i \to \mathbb{R}$ is convex and differentiable, and we denote its gradient at x_i by $\nabla_i \boldsymbol{f}_i(\boldsymbol{x})$. Further, the operator $\boldsymbol{G} \colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}} \colon \boldsymbol{x} \mapsto (\nabla_i \boldsymbol{f}_i(\boldsymbol{x}))_{i\in I}$ is monotone and Lipschitzian.

[c] For every $k \in K$, $g_k \in \Gamma_0(\mathcal{G}_k)$ and $\boldsymbol{L}_k \colon \boldsymbol{\mathcal{H}} \to \mathcal{G}_k$ is linear and bounded.

The goal is to

$$\text{find } \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \text{ such that } (\forall i \in I)\;\; x_i \in \operatorname{Argmin} \varphi_i + \boldsymbol{f}_i(\cdot\,;\boldsymbol{x}_{\smallsetminus i}) + \sum_{k\in K} (g_k \circ \boldsymbol{L}_k)(\cdot\,;\boldsymbol{x}_{\smallsetminus i}). \tag{1.2}$$

In Problem 1.1, the individual loss of player $i \in I$ is a nondifferentiable function φ_i, while his joint loss is decomposed into a differentiable function $\boldsymbol{f}_i$ and a sum of nonsmooth functions $(g_k)_{k\in K}$ acting on linear mixtures of the strategies. To the best of our knowledge, such a general formulation of a convex Nash equilibrium has not been considered in the literature. As will be seen in Section 3, it constitutes a flexible framework that subsumes a variety of existing equilibrium models. In Section 4, we embed Problem 1.1 in an inclusion problem in the bigger space $\boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$, and we employ the new problem to provide conditions for the existence of solutions to (1.2). This embedding is also exploited in Section 5 to devise an asynchronous block-iterative algorithm to solve Problem 1.1. The proposed method features several innovations that are particularly relevant in large-scale problems: first, each function and each linear operator in (1.2) is activated separately; second, only a subgroup of functions needs to be activated at any iteration; third, the computations are asynchronous in the sense that the result of calculations initiated at earlier iterations can be incorporated at the current one.

2 Notation

General background on monotone operators and related notions can be found in [6]. Let $\mathcal{H}$ be a real Hilbert space. We denote by $2^{\mathcal{H}}$ the power set of $\mathcal{H}$ and by Id the identity operator on $\mathcal{H}$. Let $A \colon \mathcal{H} \to 2^{\mathcal{H}}$. The domain of A is $\operatorname{dom} A = \{x \in \mathcal{H} \mid Ax \neq \varnothing\}$, the range of A is $\operatorname{ran} A = \bigcup_{x\in \operatorname{dom} A} Ax$, the graph of A is $\operatorname{gra} A = \{(x, x^*) \in \mathcal{H} \times \mathcal{H} \mid x^* \in Ax\}$, the set of zeros of A is $\operatorname{zer} A = \{x \in \mathcal{H} \mid 0 \in Ax\}$, the inverse of A is $A^{-1} \colon \mathcal{H} \to 2^{\mathcal{H}} \colon x^* \mapsto \{x \in \mathcal{H} \mid x^* \in Ax\}$, and the resolvent of A is $J_A = (\mathrm{Id} + A)^{-1}$. Now suppose that A is monotone, that is,

$$\big(\forall (x, x^*) \in \operatorname{gra} A\big)\big(\forall (y, y^*) \in \operatorname{gra} A\big) \quad \langle x - y \mid x^* - y^* \rangle \geqslant 0. \tag{2.1}$$

Then A is maximally monotone if, for every monotone operator $\widetilde{A} \colon \mathcal{H} \to 2^{\mathcal{H}}$, $\operatorname{gra} A \subset \operatorname{gra} \widetilde{A} \Rightarrow A = \widetilde{A}$; A is strongly monotone with constant $\alpha \in \left]0, +\infty\right[$ if $A - \alpha\mathrm{Id}$ is monotone; and A is 3^* monotone if

$$(\forall x \in \operatorname{dom} A)(\forall x^* \in \operatorname{ran} A) \quad \sup_{(y,y^*)\in \operatorname{gra} A} \langle x - y \mid y^* - x^* \rangle < +\infty. \tag{2.2}$$

Let $\varphi \in \Gamma_0(\mathcal{H})$. Then φ is supercoercive if $\lim_{\|x\|\to+\infty} \varphi(x)/\|x\| = +\infty$ and uniformly convex if there exists an increasing function $\phi \colon \left[0, +\infty\right[\to [0, +\infty]$ that vanishes only at 0 such that

$$(\forall x \in \operatorname{dom}\varphi)(\forall y \in \operatorname{dom}\varphi)(\forall \alpha \in]0,1[)$$
$$\varphi\big(\alpha x + (1-\alpha)y\big) + \alpha(1-\alpha)\phi\big(\|x-y\|\big) \leqslant \alpha\varphi(x) + (1-\alpha)\varphi(y). \quad (2.3)$$

For every $x \in \mathcal{H}$, $\operatorname{prox}_{\varphi} x$ denotes the unique minimizer of $\varphi + (1/2)\|\cdot - x\|^2$. The subdifferential of φ is the maximally monotone operator

$$\partial\varphi\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{x^* \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \ \langle y - x \mid x^*\rangle + \varphi(x) \leqslant \varphi(y)\big\}. \quad (2.4)$$

Finally, given a nonempty convex subset C of $\mathcal{H}$, the indicator function of C is

$$\iota_C\colon \mathcal{H} \to [0,+\infty] : x \mapsto \begin{cases} 0, & \text{if } \ x \in C;\\ +\infty, & \text{otherwise,}\end{cases} \quad (2.5)$$

and the strong relative interior of C is

$$\operatorname{sri} C = \left\{ x \in C \;\middle|\; \bigcup_{\lambda \in]0,+\infty[} \lambda(C - x) \ \text{ is a closed vector subspace of } \ \mathcal{H} \right\}. \quad (2.6)$$

3 Instantiations of Problem 1.1

Throughout this section, $\mathcal{H}$ is a real Hilbert space. We illustrate the wide span of Problem 1.1 by showing that common formulations encountered in various fields can be recast as special cases of it.

Example 3.1 (quadratic coupling) Let I be a nonempty finite set. For every $i \in I$, let $\varphi_i \in \Gamma_0(\mathcal{H})$, let Λ_i be a nonempty finite set, let $(\omega_{i,\ell,j})_{\ell \in \Lambda_i, j \in I \smallsetminus \{i\}}$ be in $[0,+\infty[$, and let $(\kappa_{i,\ell})_{\ell \in \Lambda_i}$ be in $]0,+\infty[$. Additionally, set $\boldsymbol{\mathcal{H}} = \bigoplus_{i \in I} \mathcal{H}$. The problem is to

$$\text{find } \ \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \ \text{ such that } \ (\forall i \in I) \ \ x_i \in \operatorname{Argmin} \varphi_i + \sum_{\ell \in \Lambda_i} \frac{\kappa_{i,\ell}}{2} \Bigg\| \cdot - \sum_{j \in I \smallsetminus \{i\}} \omega_{i,\ell,j} x_j \Bigg\|^2. \quad (3.1)$$

It is assumed that

$$(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{H}})(\forall \boldsymbol{y} \in \boldsymbol{\mathcal{H}}) \quad \sum_{i \in I} \sum_{\ell \in \Lambda_i} \kappa_{i,\ell} \left\langle x_i - y_i \;\middle|\; x_i - y_i - \sum_{j \in I \smallsetminus \{i\}} \omega_{i,\ell,j}(x_j - y_j) \right\rangle \geqslant 0. \quad (3.2)$$

Define

$$(\forall i \in I) \quad \boldsymbol{f}_i\colon \boldsymbol{\mathcal{H}} \to \mathbb{R}\colon \boldsymbol{x} \mapsto \sum_{\ell \in \Lambda_i} \frac{\kappa_{i,\ell}}{2} \Bigg\| x_i - \sum_{j \in I \smallsetminus \{i\}} \omega_{i,\ell,j} x_j \Bigg\|^2. \quad (3.3)$$

Then, for every $i \in I$ and every $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$, $\boldsymbol{f}_i(\cdot; \boldsymbol{x}_{\smallsetminus i})$ is convex and differentiable with

$$\nabla_i \boldsymbol{f}_i(\boldsymbol{x}) = \sum_{\ell \in \Lambda_i} \kappa_{i,\ell} \Bigg(x_i - \sum_{j \in I \smallsetminus \{i\}} \omega_{i,\ell,j} x_j \Bigg). \quad (3.4)$$

Hence, in view of (3.2), the operator $\boldsymbol{G}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto (\nabla_i \boldsymbol{f}_i(\boldsymbol{x}))_{i \in I}$ is monotone and Lipschitzian. Thus, (3.1) is a special case of (1.2) with $K = \varnothing$ and $(\forall i \in I)\ \mathcal{H}_i = \mathcal{H}$. This scenario unifies models found in [1, 2, 20].

Example 3.2 In (3.1), suppose that, for every $i \in I$, C_i is a nonempty closed convex subset of $\mathcal{H}_i$, $\varphi_i = \iota_{C_i}$, $\Lambda_i = \{1\}$, and $\kappa_{i,1} = 1$. Then (3.1) becomes

$$\text{find } \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \text{ such that } (\forall i \in I) \;\; x_i \in \operatorname{Argmin}_{C_i} \Bigg\| \cdot - \sum_{j \in I \smallsetminus \{i\}} \omega_{i,1,j} x_j \Bigg\|^2. \tag{3.5}$$

In addition, (3.2) is satisfied when

$$\begin{cases} (\forall i \in I) \;\; \sum_{j \in I \smallsetminus \{i\}} \omega_{i,1,j} \leqslant 1 \\ (\forall j \in I) \;\; \sum_{i \in I \smallsetminus \{j\}} \omega_{i,1,j} \leqslant 1, \end{cases} \tag{3.6}$$

which places us in the setting of Example 3.1. The formulation (3.5)–(3.6) unifies models found in [5].

Example 3.3 (minimax) Let I be a finite set and suppose that $\varnothing \neq J \subset I$. Let $(\mathcal{H}_i)_{i \in I}$ be real Hilbert spaces, and set $\boldsymbol{\mathcal{U}} = \bigoplus_{i \in I \smallsetminus J} \mathcal{H}_i$ and $\boldsymbol{\mathcal{V}} = \bigoplus_{j \in J} \mathcal{H}_j$. For every $i \in I$, let $\varphi_i \in \Gamma_0(\mathcal{H}_i)$. Further, let $\boldsymbol{\mathcal{L}} \colon \boldsymbol{\mathcal{U}} \oplus \boldsymbol{\mathcal{V}} \to \mathbb{R}$ be differentiable with a Lipschitzian gradient and such that, for every $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}$ and every $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}$, the functions $-\boldsymbol{\mathcal{L}}(\boldsymbol{u}, \cdot)$ and $\boldsymbol{\mathcal{L}}(\cdot, \boldsymbol{v})$ are convex. Consider the multivariate minimax problem

$$\underset{\boldsymbol{u} \in \boldsymbol{\mathcal{U}}}{\text{minimize}} \;\; \underset{\boldsymbol{v} \in \boldsymbol{\mathcal{V}}}{\text{maximize}} \;\; \sum_{i \in I \smallsetminus J} \varphi_i(u_i) + \boldsymbol{\mathcal{L}}(\boldsymbol{u}, \boldsymbol{v}) - \sum_{j \in J} \varphi_j(v_j). \tag{3.7}$$

Now set $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{U}} \oplus \boldsymbol{\mathcal{V}}$ and define

$$(\forall i \in I) \quad \boldsymbol{f}_i \colon \boldsymbol{\mathcal{H}} \to \mathbb{R} \colon (\boldsymbol{u}, \boldsymbol{v}) \mapsto \begin{cases} \boldsymbol{\mathcal{L}}(\boldsymbol{u}, \boldsymbol{v}), & \text{if } i \in I \smallsetminus J; \\ -\boldsymbol{\mathcal{L}}(\boldsymbol{u}, \boldsymbol{v}), & \text{if } i \in J. \end{cases} \tag{3.8}$$

Then $\boldsymbol{\mathcal{H}} = \bigoplus_{i \in I} \mathcal{H}_i$ and (3.7) can be put in the form

$$\text{find } \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \text{ such that } (\forall i \in I) \;\; x_i \in \operatorname{Argmin} \varphi_i + \boldsymbol{f}_i(\cdot; \boldsymbol{x}_{\smallsetminus i}). \tag{3.9}$$

Let us verify Problem 1.1[b]. On the one hand, we have

$$(\forall i \in I)(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{H}}) \quad \nabla_i \boldsymbol{f}_i(\boldsymbol{x}) = \begin{cases} \nabla_i \boldsymbol{\mathcal{L}}(\boldsymbol{x}), & \text{if } i \in I \smallsetminus J; \\ -\nabla_i \boldsymbol{\mathcal{L}}(\boldsymbol{x}), & \text{if } i \in J. \end{cases} \tag{3.10}$$

Hence, the operator

$$\boldsymbol{G} \colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}} \colon \boldsymbol{x} \mapsto \big(\nabla_i \boldsymbol{f}_i(\boldsymbol{x})\big)_{i \in I} = \Big(\big(\nabla_i \boldsymbol{\mathcal{L}}(\boldsymbol{x})\big)_{i \in I \smallsetminus J}, \big(-\nabla_j \boldsymbol{\mathcal{L}}(\boldsymbol{x})\big)_{j \in J} \Big) \tag{3.11}$$

is monotone [33, 34] and Lipschitzian. Consequently, (3.7) is an instantiation of (1.2). Special cases of (3.7) under the above assumptions can be found in [20, 27, 32, 35, 38, 39].

Example 3.4 ("generalized" Nash equilibria) Consider the setting of Problem 1.1 where [a] and [c] are respectively specialized to

[a'] For every $i \in I$, $\varphi_i = \iota_{C_i}$, where C_i is a nonempty closed convex subset of $\mathcal{H}_i$.

[c'] $K = \{1\}$ and $g_1 = \iota_{D_1}$, where D_1 is a nonempty closed convex subset of $\mathcal{G}_1$.

Then (1.2) reduces to

$$\text{find } \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \text{ such that } (\forall i \in I) \;\; x_i \in \text{Argmin}_{C_i} \, \boldsymbol{f}_i(\cdot\,; \boldsymbol{x}_{\smallsetminus i}) + (\iota_{D_1} \circ \boldsymbol{L}_1)(\cdot\,; \boldsymbol{x}_{\smallsetminus i}). \tag{3.12}$$

This formulation is often referred to as a generalized Nash equilibrium; see, e.g., [24, 28, 29]. However, as noted in [36], it is really a standard Nash equilibrium in the sense of (1.1) since functions are allowed to take the value $+\infty$.

Example 3.5 (PDE model) Let Ω be a nonempty open bounded subset of $\mathbb{R}^N$. In Example 3.4, suppose that, for every $i \in I$, $\mathcal{H}_i = L^2(\Omega)$. Let $z \in L^2(\Omega)$, let $(\Omega_i)_{i\in I}$ be nonempty open subsets of Ω with characteristic functions $(1_{\Omega_i})_{i\in I}$, and, for every $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$, let $\boldsymbol{S}\boldsymbol{x}$ be the unique weak solution in $H_0^1(\Omega)$ of the Dirichlet boundary value problem [23, Chapter IV.2.1]

$$\begin{cases} -\Delta y = z + \displaystyle\sum_{i\in I} 1_{\Omega_i} x_i, & \text{on } \Omega; \\ y = 0, & \text{on } \operatorname{bdry} \Omega. \end{cases} \tag{3.13}$$

For every $i \in I$, let $r_i \in \mathcal{H}_i$, let $\alpha_i \in \left]0, +\infty\right[$, and suppose that

$$\boldsymbol{f}_i \colon \boldsymbol{x} \mapsto \frac{\alpha_i}{2}\|x_i\|^2_{\mathcal{H}_i} + \frac{1}{2}\|\boldsymbol{S}\boldsymbol{x} - r_i\|^2_{\mathcal{H}_i}. \tag{3.14}$$

In addition, suppose that $\mathcal{G}_1 = H_0^1(\Omega)$ and $\boldsymbol{L}_1 = \boldsymbol{S} - \boldsymbol{S}\boldsymbol{0}$. Then we recover frameworks investigated in [9, 29].

Example 3.6 (multivariate minimization) Consider the setting of Problem 1.1 where [b] and [c] are respectively specialized to

[b'] For every $i \in I$, $\boldsymbol{f}_i = \boldsymbol{f}$, where $\boldsymbol{f} \colon \boldsymbol{\mathcal{H}} \to \mathbb{R}$ is a differentiable convex function such that $\boldsymbol{G} = \nabla \boldsymbol{f}$ is Lipschitzian.

[c'] For every $k \in K$, $g_k \colon \mathcal{G}_k \to \mathbb{R}$ is convex and Gâteaux differentiable, and $\boldsymbol{L}_k \colon \boldsymbol{\mathcal{H}} \to \mathcal{G}_k \colon \boldsymbol{x} \mapsto \sum_{j\in I} L_{k,j} x_j$ where, for every $j \in I$, $L_{k,j} \colon \mathcal{H}_j \to \mathcal{G}_k$ is linear and bounded.

Then (1.2) reduces to the multivariate minimization problem

$$\underset{\boldsymbol{x}\in\boldsymbol{\mathcal{H}}}{\text{minimize}} \;\; \sum_{i\in I} \varphi_i(x_i) + \boldsymbol{f}(\boldsymbol{x}) + \sum_{k\in K} g_k\Bigg(\sum_{j\in I} L_{k,j} x_j\Bigg). \tag{3.15}$$

Instances of this problem are found in [3, 4, 11, 12, 14, 22, 26].

4 Existence of solutions

Our first existence result revolves around an embedding of Problem 1.1 in a larger inclusion problem in the space $\boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$.

Proposition 4.1 *Consider the setting of Problem 1.1 and set* $(\forall i \in I)$ $\Pi_i \colon \boldsymbol{\mathcal{H}} \to \mathcal{H}_i \colon \boldsymbol{x} \mapsto x_i$. *Suppose that* $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{v}}^*) \in \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$ *satisfies*

$$\begin{cases} (\forall i \in I) \quad -\nabla_i \boldsymbol{f}_i(\overline{\boldsymbol{x}}) - \sum_{k \in K} \Pi_i(\boldsymbol{L}_k^* \overline{v}_k^*) \in \partial\varphi_i(\overline{x}_i) \\ (\forall k \in K) \;\; \boldsymbol{L}_k \overline{\boldsymbol{x}} \in \partial g_k^*(\overline{v}_k^*). \end{cases} \tag{4.1}$$

Then $\overline{\boldsymbol{x}}$ *solves* (1.2).

Proof. Take $i \in I$ and set

$$f_i = \boldsymbol{f}_i(\cdot\,; \overline{\boldsymbol{x}}_{\smallsetminus i}), \quad \overline{\boldsymbol{s}}_i = (0; \overline{\boldsymbol{x}}_{\smallsetminus i}), \quad \text{and} \quad (\forall k \in K) \;\; \widetilde{g}_k = (g_k \circ \boldsymbol{L}_k)(\cdot\,; \overline{\boldsymbol{x}}_{\smallsetminus i}). \tag{4.2}$$

Then, by Problem 1.1[b], $f_i \colon \mathcal{H}_i \to \mathbb{R}$ is convex and Gâteaux differentiable, and $\nabla f_i(\overline{x}_i) = \nabla_i \boldsymbol{f}_i(\overline{\boldsymbol{x}})$. At the same time,

$$(\forall k \in K)(\forall x_i \in \mathcal{H}_i) \quad \widetilde{g}_k(x_i) = (g_k \circ \boldsymbol{L}_k)(\Pi_i^* x_i + \overline{\boldsymbol{s}}_i) = g_k(\boldsymbol{L}_k(\Pi_i^* x_i) + \boldsymbol{L}_k \overline{\boldsymbol{s}}_i) \tag{4.3}$$

and it thus results from [6, Proposition 16.6(ii)] that

$$(\forall k \in K)(\forall x_i \in \mathcal{H}_i) \quad (\Pi_i \circ \boldsymbol{L}_k^*)\big(\partial g_k(\boldsymbol{L}_k(\Pi_i^* x_i) + \boldsymbol{L}_k \overline{\boldsymbol{s}}_i)\big) \subset \partial \widetilde{g}_k(x_i). \tag{4.4}$$

In particular,

$$(\forall k \in K) \quad (\Pi_i \circ \boldsymbol{L}_k^*)\big(\partial g_k(\boldsymbol{L}_k \overline{\boldsymbol{x}})\big) = (\Pi_i \circ \boldsymbol{L}_k^*)\Big(\partial g_k\big(\boldsymbol{L}_k(\Pi_i^* \overline{x}_i) + \boldsymbol{L}_k \overline{\boldsymbol{s}}_i\big)\Big) \subset \partial \widetilde{g}_k(\overline{x}_i). \tag{4.5}$$

Hence, we deduce from (4.1) and [6, Proposition 16.6(ii)] that

$$\begin{aligned} 0 &\in \partial\varphi_i(\overline{x}_i) + \nabla_i \boldsymbol{f}_i(\overline{\boldsymbol{x}}) + \sum_{k \in K} \Pi_i(\boldsymbol{L}_k^* \overline{v}_k^*) \\ &\subset \partial\varphi_i(\overline{x}_i) + \nabla f_i(\overline{x}_i) + \sum_{k \in K} (\Pi_i \circ \boldsymbol{L}_k^*)\big(\partial g_k(\boldsymbol{L}_k \overline{\boldsymbol{x}})\big) \\ &\subset \partial\varphi_i(\overline{x}_i) + \nabla f_i(\overline{x}_i) + \sum_{k \in K} \partial \widetilde{g}_k(\overline{x}_i) \\ &\subset \partial\Bigg(\varphi_i + f_i + \sum_{k \in K} \widetilde{g}_k\Bigg)(\overline{x}_i). \end{aligned} \tag{4.6}$$

Consequently, appealing to Fermat's rule [6, Theorem 16.3] and (4.2), we arrive at

$$\overline{x}_i \in \operatorname{Argmin} \varphi_i + \boldsymbol{f}_i(\cdot\,; \overline{\boldsymbol{x}}_{\smallsetminus i}) + \sum_{k \in K} (g_k \circ \boldsymbol{L}_k)(\cdot\,; \overline{\boldsymbol{x}}_{\smallsetminus i}), \tag{4.7}$$

which completes the proof. ☐

We are now in a position to provide specific existence conditions.

Proposition 4.2 *Consider the setting of Problem 1.1, set*

$$\boldsymbol{C} = \big\{(\boldsymbol{L}_k \boldsymbol{x} - y_k)_{k \in K} \mid (\forall i \in I) \;\; x_i \in \operatorname{dom} \varphi_i \;\; \text{and} \;\; (\forall k \in K) \;\; y_k \in \operatorname{dom} g_k\big\}, \tag{4.8}$$

and let $\boldsymbol{Z} \subset \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$ *be the set of solutions to* (4.1). *Suppose that* $\boldsymbol{0} \in \operatorname{sri} \boldsymbol{C}$ *and that one of the following is satisfied:*

(i) *For every $i \in I$, one of the following holds:*

1/ $\partial\varphi_i$ *is surjective.*

2/ φ_i *is supercoercive.*

3/ $\operatorname{dom}\varphi_i$ *is bounded.*

4/ φ_i *is uniformly convex.*

(ii) $\boldsymbol{G}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto (\nabla_i \boldsymbol{f}_i(\boldsymbol{x}))_{i\in I}$ *is 3^* monotone and surjective.*

Then $\boldsymbol{Z} \neq \varnothing$ and Problem 1.1 has a solution.

Proof. Define

$$\begin{cases} \boldsymbol{A}\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}\colon \boldsymbol{x} \mapsto \times_{i\in I} \partial\varphi_i(x_i) \\ \boldsymbol{B}\colon \boldsymbol{\mathcal{G}} \to 2^{\boldsymbol{\mathcal{G}}}\colon \boldsymbol{y} \mapsto \times_{k\in K} \partial g_k(y_k) \\ \boldsymbol{L}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{G}}\colon \boldsymbol{x} \mapsto (\boldsymbol{L}_k\boldsymbol{x})_{k\in K} \end{cases} \tag{4.9}$$

and

$$\boldsymbol{T}\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}\colon \boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x} + \boldsymbol{L}^*\big(\boldsymbol{B}(\boldsymbol{L}\boldsymbol{x})\big) + \boldsymbol{G}\boldsymbol{x}. \tag{4.10}$$

Note that the adjoint of $\boldsymbol{L}$ is

$$\boldsymbol{L}^*\colon \boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{v}^* \mapsto \sum_{k\in K} \boldsymbol{L}_k^* v_k^*. \tag{4.11}$$

Now suppose that $\overline{\boldsymbol{x}} \in \operatorname{zer}\boldsymbol{T}$. Then there exists $\overline{\boldsymbol{v}}^* \in \boldsymbol{B}(\boldsymbol{L}\overline{\boldsymbol{x}})$ such that $-\boldsymbol{G}\overline{\boldsymbol{x}} - \boldsymbol{L}^*\overline{\boldsymbol{v}}^* \in \boldsymbol{A}\overline{\boldsymbol{x}}$ or, equivalently, by Problem 1.1[b] and (4.11), $(\forall i \in I)\ -\nabla_i \boldsymbol{f}_i(\overline{\boldsymbol{x}}) - \sum_{k\in K} \Pi_i(\boldsymbol{L}_k^*\overline{v}_k^*) \in \partial\varphi_i(\overline{x}_i)$. Further, (4.9) yields $\overline{v}_k^* \in \partial g_k(\boldsymbol{L}_k\overline{\boldsymbol{x}})$. Altogether, in view of (4.1) and Proposition 4.1, we have established the implications

$$\operatorname{zer}\boldsymbol{T} \neq \varnothing \quad\Rightarrow\quad \boldsymbol{Z} \neq \varnothing \quad\Rightarrow\quad \text{Problem 1.1 has a solution.} \tag{4.12}$$

Therefore, it suffices to show that $\operatorname{zer}\boldsymbol{T} \neq \varnothing$. To do so, define

$$\begin{cases} \boldsymbol{\varphi}\colon \boldsymbol{\mathcal{H}} \to \left]-\infty,+\infty\right]\colon \boldsymbol{x} \mapsto \sum_{i\in I} \varphi_i(x_i) \\ \boldsymbol{g}\colon \boldsymbol{\mathcal{G}} \to \left]-\infty,+\infty\right]\colon \boldsymbol{y} \mapsto \sum_{k\in K} g_k(y_k) \\ \boldsymbol{Q} = \boldsymbol{A} + \boldsymbol{L}^* \circ \boldsymbol{B} \circ \boldsymbol{L}. \end{cases} \tag{4.13}$$

Then, by (4.9) and [6, Proposition 16.9], $\boldsymbol{A} = \partial\boldsymbol{\varphi}$ and $\boldsymbol{B} = \partial\boldsymbol{g}$. In turn, since (4.8) and (4.9) imply that $\boldsymbol{0} \in \operatorname{sri}\boldsymbol{C} = \operatorname{sri}(\boldsymbol{L}(\operatorname{dom}\boldsymbol{\varphi}) - \operatorname{dom}\boldsymbol{g})$, we derive from [6, Theorem 16.47(i)] that $\boldsymbol{Q} = \partial(\boldsymbol{\varphi} + \boldsymbol{g}\circ\boldsymbol{L})$. Therefore, in view of [6, Theorem 20.25 and Example 25.13],

$$\boldsymbol{A},\ \boldsymbol{B},\ \text{and}\ \boldsymbol{Q}\ \text{are maximally monotone and}\ 3^*\ \text{monotone.} \tag{4.14}$$

(i): Fix temporarily $i \in I$. By [6, Theorem 20.25], $\partial\varphi_i$ is maximally monotone. First, if (i)2/ holds, then [6, Corollary 16.30, and Propositions 14.15 and 16.27] entail that $\operatorname{ran}\partial\varphi_i = \operatorname{dom}\partial\varphi_i^* = \mathcal{H}_i$ and, hence, (i)1/ holds. Second, if (i)3/ holds, then $\operatorname{dom}\partial\varphi_i \subset \operatorname{dom}\varphi_i$ is bounded and, therefore, it follows from [6, Corollary 21.25] that (i)1/ holds. Finally, if (i)4/ holds, then [6, Proposition 17.26(ii)] implies that (i)2/ holds and, in turn, that (i)1/ holds. Altogether, it is enough to show that

$$\big[\,(\forall i \in I)\;\; \partial\varphi_i \;\text{ is surjective}\,\big] \quad \Rightarrow \quad \operatorname{zer} \boldsymbol{T} \neq \varnothing. \tag{4.15}$$

Assume that the operators $(\partial\varphi_i)_{i\in I}$ are surjective and set

$$\boldsymbol{P} = -\boldsymbol{Q}^{-1} \circ (-\mathbf{Id}) + \boldsymbol{G}^{-1}. \tag{4.16}$$

Then we derive from (4.9) that $\boldsymbol{A}$ is surjective. On the other hand, [10, Proposition 6] asserts that $\boldsymbol{L}^* \circ \boldsymbol{B} \circ \boldsymbol{L}$ is 3^* monotone. Hence, (4.14) and [6, Corollary 25.27(i)] yields

$$\operatorname{dom} \boldsymbol{Q}^{-1} = \operatorname{ran} \boldsymbol{Q} = \boldsymbol{\mathcal{H}}. \tag{4.17}$$

In turn, since $\boldsymbol{Q}^{-1}$ and $\boldsymbol{G}^{-1}$ are maximally monotone, [6, Theorem 25.3] implies that $\boldsymbol{P}$ is likewise. Furthermore, we observe that

$$\operatorname{dom} \boldsymbol{G}^{-1} \subset \boldsymbol{\mathcal{H}} = \operatorname{dom}\big(-\boldsymbol{Q}^{-1} \circ (-\mathbf{Id})\big) \tag{4.18}$$

and, by virtue of (4.14) and [6, Proposition 25.19(i)], that $-\boldsymbol{Q}^{-1} \circ (-\mathbf{Id})$ is 3^* monotone. Therefore, since $\operatorname{ran} \boldsymbol{G}^{-1} = \operatorname{dom} \boldsymbol{G} = \boldsymbol{\mathcal{H}}$, [6, Corollary 25.27(ii)] entails that $\boldsymbol{P}$ is surjective and, in turn, that $\operatorname{zer} \boldsymbol{P} \neq \varnothing$. Consequently, [6, Proposition 26.33(iii)] asserts that $\operatorname{zer} \boldsymbol{T} \neq \varnothing$.

(ii): Since $\boldsymbol{G}$ is maximally monotone and $\operatorname{dom} \boldsymbol{G} = \boldsymbol{\mathcal{H}}$, it results from (4.14) and [6, Theorem 25.3] that $\boldsymbol{T} = \boldsymbol{Q} + \boldsymbol{G}$ is maximally monotone. Hence, since $\boldsymbol{G}$ is surjective, we derive from (4.14) and [6, Corollary 25.27(i)] that $\boldsymbol{T}$ is surjective and, therefore, that $\operatorname{zer} \boldsymbol{T} \neq \varnothing$. ☐

Remark 4.3 Sufficient conditions for $\boldsymbol{0} \in \operatorname{sri} \boldsymbol{C}$ to hold in Proposition 4.2 can be found in [18, Proposition 5.3].

5 Algorithm

The main result of this section is the following theorem, where we introduce an asynchronous block-iterative algorithm to solve Problem 1.1 and prove its convergence.

Theorem 5.1 *Consider the setting of Problem 1.1 and set* $(\forall i \in I)\ \Pi_i \colon \boldsymbol{\mathcal{H}} \to \mathcal{H}_i \colon \boldsymbol{x} \mapsto x_i$. *Let* $(\chi_i)_{i\in I}$ *be a family in* $[0,+\infty[$ *such that*

$$(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{H}})(\forall \boldsymbol{y} \in \boldsymbol{\mathcal{H}}) \quad \langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{G}\boldsymbol{x} - \boldsymbol{G}\boldsymbol{y} \rangle \leqslant \sum_{i\in I} \chi_i \|x_i - y_i\|^2, \tag{5.1}$$

let $\alpha \in\,]0,+\infty[$ *and* $\varepsilon \in\,]0,1[$ *be such that* $1/\varepsilon > \alpha + \max_{i\in I} \chi_i$, *let* $(\lambda_n)_{n\in\mathbb{N}}$ *be in* $[\varepsilon, 2-\varepsilon]$, *and let* $D \in \mathbb{N}$. *Suppose that the following are satisfied:*

[a] *There exists* $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{v}}^*) \in \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$ *such that (4.1) holds.*

[b] *For every* $i \in I$, $x_{i,0} \in \mathcal{H}_i$ *and, for every* $n \in \mathbb{N}$, $\gamma_{i,n} \in [\varepsilon, 1/(\chi_i + \alpha)]$ *and* $c_i(n) \in \mathbb{N}$ *satisfies* $n - D \leqslant c_i(n) \leqslant n$.

[c] *For every* $k \in K$, $v_{k,0}^* \in \mathcal{G}_k$ *and, for every* $n \in \mathbb{N}$, $\mu_{k,n} \in [\alpha, 1/\varepsilon]$ *and* $d_k(n) \in \mathbb{N}$ *satisfies* $n - D \leqslant d_k(n) \leqslant n$.

[d] $(I_n)_{n\in\mathbb{N}}$ *are nonempty subsets of* I *and* $(K_n)_{n\in\mathbb{N}}$ *are nonempty subsets of* K *such that*

$$I_0 = I, \quad K_0 = K, \quad \textit{and} \quad (\exists\, m \in \mathbb{N})(\forall n \in \mathbb{N}) \ \bigcup_{j=n}^{n+m} I_j = I \ \textit{and} \ \bigcup_{j=n}^{n+m} K_j = K. \tag{5.2}$$

Further, set $\boldsymbol{L} \colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{G}} \colon \boldsymbol{x} \mapsto (\boldsymbol{L}_k\boldsymbol{x})_{k\in K}$. *Iterate*

$$\begin{array}{l}
\textit{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\textit{for every } i \in I_n\\
\left\lfloor
\begin{array}{l}
x_{i,n}^* = x_{i,c_i(n)} - \gamma_{i,c_i(n)}\Big(\nabla_i \boldsymbol{f}_i(\boldsymbol{x}_{c_i(n)}) + \sum_{k\in K}\Pi_i\big(\boldsymbol{L}_k^* v_{k,c_i(n)}^*\big)\Big)\\
a_{i,n} = \operatorname{prox}_{\gamma_{i,c_i(n)}\varphi_i} x_{i,n}^*\\
a_{i,n}^* = \gamma_{i,c_i(n)}^{-1}(x_{i,n}^* - a_{i,n})
\end{array}
\right.\\
\textit{for every } i \in I \smallsetminus I_n\\
\left\lfloor\ (a_{i,n}, a_{i,n}^*) = (a_{i,n-1}, a_{i,n-1}^*)\right.\\
\textit{for every } k \in K_n\\
\left\lfloor
\begin{array}{l}
y_{k,n}^* = \mu_{k,d_k(n)} v_{k,d_k(n)}^* + \boldsymbol{L}_k \boldsymbol{x}_{d_k(n)}\\
b_{k,n} = \operatorname{prox}_{\mu_{k,d_k(n)} g_k} y_{k,n}^*\\
b_{k,n}^* = \mu_{k,d_k(n)}^{-1}(y_{k,n}^* - b_{k,n})
\end{array}
\right.\\
\textit{for every } k \in K \smallsetminus K_n\\
\left\lfloor\ (b_{k,n}, b_{k,n}^*) = (b_{k,n-1}, b_{k,n-1}^*)\right.\\
\boldsymbol{t}_n^* = \boldsymbol{a}_n^* + \boldsymbol{G}\boldsymbol{a}_n + \boldsymbol{L}^*\boldsymbol{b}_n^*\\
\boldsymbol{t}_n = \boldsymbol{b}_n - \boldsymbol{L}\boldsymbol{a}_n\\
\pi_n = \langle \boldsymbol{a}_n - \boldsymbol{x}_n \mid \boldsymbol{t}_n^*\rangle + \langle \boldsymbol{t}_n \mid \boldsymbol{b}_n^* - \boldsymbol{v}_n^*\rangle\\
\textit{if } \pi_n < 0\\
\left\lfloor
\begin{array}{l}
\alpha_n = \lambda_n \pi_n / \big(\|\boldsymbol{t}_n\|^2 + \|\boldsymbol{t}_n^*\|^2\big)\\
\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \alpha_n \boldsymbol{t}_n^*\\
\boldsymbol{v}_{n+1}^* = \boldsymbol{v}_n^* + \alpha_n \boldsymbol{t}_n
\end{array}
\right.\\
\textit{else}\\
\left\lfloor\ \big(\boldsymbol{x}_{n+1}, \boldsymbol{v}_{n+1}^*\big) = (\boldsymbol{x}_n, \boldsymbol{v}_n^*).\right.
\end{array}
\right.
\end{array} \tag{5.3}$$

Then $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ *converges weakly to a solution to Problem* 1.1.

The salient features of the proposed algorithm are the following:

- **Decomposition:** In (5.3), the functions $(\varphi_i)_{i\in I}$ and $(g_k)_{k\in K}$ are activated separately via their proximity operators.
- **Block-iterative implementation:** At iteration n, we require that only the subfamilies of functions $(\varphi_i)_{i\in I_n}$ and $(g_k)_{k\in K_n}$ be activated, as opposed to all of them as in standard splitting methods. To guarantee convergence, we ask in condition [d] of Theorem 5.1 that each of these functions be activated frequently enough.
- **Asynchronous implementation:** Given $i \in I$ and $k \in K$, the asynchronous character of the algorithm is materialized by the variables $c_i(n)$ and $d_k(n)$ which signal when the underlying computations incorporated at iteration n were initiated. Conditions [b] and [c] of Theorem 5.1 ask that the lag between the initiation and the incorporation of such computations do not exceed

D iterations. The synchronous implementation is obtained when $c_i(n) = n$ and $d_k(n) = n$ in (5.3). The introduction of asynchronous and block-iterative techniques in monotone operator splitting were initiated in [19].

In order to prove Theorem 5.1, we need to establish some preliminary properties.

Proposition 5.2 *Let $(\mathcal{X}_i)_{i\in\mathbb{I}}$ be a finite family of real Hilbert spaces with Hilbert direct sum $\boldsymbol{\mathcal{X}} = \bigoplus_{i\in\mathbb{I}} \mathcal{X}_i$. For every $i \in \mathbb{I}$, let $P_i\colon \mathcal{X}_i \to 2^{\mathcal{X}_i}$ be maximally monotone and let $Q_i\colon \boldsymbol{\mathcal{X}} \to \mathcal{X}_i$. It is assumed that $\boldsymbol{Q}\colon \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}\colon \boldsymbol{x} \mapsto (Q_i\boldsymbol{x})_{i\in\mathbb{I}}$ is monotone and Lipschitzian, and that the problem*

$$\text{find } \boldsymbol{x} \in \boldsymbol{\mathcal{X}} \text{ such that } (\forall i \in \mathbb{I})\ \ 0 \in P_i x_i + Q_i \boldsymbol{x} \tag{5.4}$$

has a solution. Let $(\chi_i)_{i\in\mathbb{I}}$ be a family in $[0, +\infty[$ such that

$$(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}})(\forall \boldsymbol{y} \in \boldsymbol{\mathcal{X}}) \quad \langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{Q}\boldsymbol{x} - \boldsymbol{Q}\boldsymbol{y}\rangle \leqslant \sum_{i\in\mathbb{I}} \chi_i \|x_i - y_i\|^2, \tag{5.5}$$

let $\alpha \in \left]0, +\infty\right[$, let $\varepsilon \in \left]0, 1\right[$ be such that $1/\varepsilon > \alpha + \max_{i\in\mathbb{I}} \chi_i$, and let $D \in \mathbb{N}$. For every $i \in \mathbb{I}$, let $x_{i,0} \in \mathcal{X}_i$ and, for every $n \in \mathbb{N}$, let $\gamma_{i,n} \in [\varepsilon, 1/(\chi_i + \alpha)]$, let $\lambda_n \in [\varepsilon, 2 - \varepsilon]$, and let $d_i(n) \in \mathbb{N}$ be such that

$$n - D \leqslant d_i(n) \leqslant n. \tag{5.6}$$

In addition, let $(\mathbb{I}_n)_{n\in\mathbb{N}}$ be nonempty subsets of $\mathbb{I}$ such that

$$\mathbb{I}_0 = \mathbb{I} \quad \text{and} \quad (\exists\, m \in \mathbb{N})(\forall n \in \mathbb{N}) \ \bigcup_{j=n}^{n+m} \mathbb{I}_j = \mathbb{I}. \tag{5.7}$$

Iterate

$$\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\left\lfloor \begin{array}{l}
\text{for every } i \in \mathbb{I}_n \\
\left\lfloor \begin{array}{l}
x^*_{i,n} = x_{i,d_i(n)} - \gamma_{i,d_i(n)} Q_i \boldsymbol{x}_{d_i(n)} \\
p_{i,n} = J_{\gamma_{i,d_i(n)} P_i} x^*_{i,n} \\
p^*_{i,n} = \gamma^{-1}_{i,d_i(n)} (x^*_{i,n} - p_{i,n})
\end{array} \right. \\
\text{for every } i \in \mathbb{I} \smallsetminus \mathbb{I}_n \\
\left\lfloor \ (p_{i,n}, p^*_{i,n}) = (p_{i,n-1}, p^*_{i,n-1}) \right. \\
\boldsymbol{s}^*_n = \boldsymbol{p}^*_n + \boldsymbol{Q}\boldsymbol{p}_n \\
\pi_n = \langle \boldsymbol{p}_n - \boldsymbol{x}_n \mid \boldsymbol{s}^*_n \rangle \\
\text{if } \pi_n < 0 \\
\left\lfloor \begin{array}{l}
\alpha_n = \lambda_n \pi_n / \|\boldsymbol{s}^*_n\|^2 \\
\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \alpha_n \boldsymbol{s}^*_n
\end{array} \right. \\
\text{else} \\
\left\lfloor \ \boldsymbol{x}_{n+1} = \boldsymbol{x}_n. \right.
\end{array} \right.
\end{array} \tag{5.8}$$

Then the following hold:

(i) $(\forall i \in \mathbb{I})\ x_{i,n} - p_{i,n} \to 0$.

(ii) $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ *converges weakly to a solution to* (5.4).

Proof. Define

$$\boldsymbol{M}\colon \boldsymbol{\mathcal{X}} \to 2^{\boldsymbol{\mathcal{X}}}\colon \boldsymbol{x} \mapsto \boldsymbol{Q}\boldsymbol{x} + \bigtimes_{i\in\mathbb{I}} P_i x_i. \tag{5.9}$$

It follows from [6, Proposition 20.23] that the operator $\boldsymbol{x} \mapsto \bigtimes_{i\in\mathbb{I}} P_i x_i$ is maximally monotone. Thus, since $\boldsymbol{Q}$ is maximally monotone by [6, Corollary 20.28], we deduce from [6, Corollary 25.5(i)] that $\boldsymbol{M}$ is maximally monotone. Further, since (5.4) has a solution, zer $\boldsymbol{M} \neq \varnothing$. Set

$$(\forall i \in \mathbb{I})(\forall n \in \mathbb{N}) \quad \overline{\delta}_i(n) = \max\big\{j \in \mathbb{N} \mid j \leqslant n \;\text{ and }\; i \in \mathbb{I}_j\big\} \quad\text{and}\quad \delta_i(n) = d_i\big(\overline{\delta}_i(n)\big), \tag{5.10}$$

and define

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{K}_n\colon \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}\colon \boldsymbol{x} \mapsto \big(\gamma_{i,\delta_i(n)}^{-1} x_i\big)_{i\in\mathbb{I}} - \boldsymbol{Q}\boldsymbol{x}. \tag{5.11}$$

In addition, let χ be a Lipschitz constant of $\boldsymbol{Q}$. Then, the operators $(\boldsymbol{K}_n)_{n\in\mathbb{N}}$ are Lipschitzian with constant $\beta = \sqrt{2(\varepsilon^{-2} + \chi^2)}$. At the same time, for every $n \in \mathbb{N}$, we derive from (5.11) and (5.5) that

$$\begin{aligned}
(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}})(\forall \boldsymbol{y} \in \boldsymbol{\mathcal{X}}) \quad \langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{K}_n\boldsymbol{x} - \boldsymbol{K}_n\boldsymbol{y}\rangle &= \sum_{i\in\mathbb{I}} \gamma_{i,\delta_i(n)}^{-1}\|x_i - y_i\|^2 - \langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{Q}\boldsymbol{x} - \boldsymbol{Q}\boldsymbol{y}\rangle \\
&\geqslant \sum_{i\in\mathbb{I}} (\chi_i + \alpha)\|x_i - y_i\|^2 - \sum_{i\in\mathbb{I}} \chi_i\|x_i - y_i\|^2 \\
&= \alpha\|\boldsymbol{x} - \boldsymbol{y}\|^2
\end{aligned} \tag{5.12}$$

and, in turn, that $\boldsymbol{K}_n$ is α-strongly monotone and maximally monotone [6, Corollary 20.28]. Hence, for every $n \in \mathbb{N}$, [6, Proposition 22.11(ii)] implies that there exists $\widetilde{\boldsymbol{x}}_n \in \boldsymbol{\mathcal{X}}$ such that

$$\big(\gamma_{i,\delta_i(n)}^{-1} x_{i,\overline{\delta}_i(n)}^*\big)_{i\in\mathbb{I}} = \boldsymbol{K}_n\widetilde{\boldsymbol{x}}_n. \tag{5.13}$$

Therefore, we infer from (5.8), (5.10), (5.9), and (5.11) that

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{p}_n = \big(p_{i,\overline{\delta}_i(n)}\big)_{i\in\mathbb{I}} \tag{5.14}$$

$$\begin{aligned}
&= \big(J_{\gamma_{i,\delta_i(n)}P_i} x_{i,\overline{\delta}_i(n)}^*\big)_{i\in\mathbb{I}} \\
&= (\boldsymbol{K}_n + \boldsymbol{M})^{-1}\big(\gamma_{i,\delta_i(n)}^{-1} x_{i,\overline{\delta}_i(n)}^*\big)_{i\in\mathbb{I}} \\
&= (\boldsymbol{K}_n + \boldsymbol{M})^{-1}\big(\boldsymbol{K}_n\widetilde{\boldsymbol{x}}_n\big).
\end{aligned} \tag{5.15}$$

On the other hand, by (5.8), (5.10), (5.13), (5.14), and (5.11),

$$\begin{aligned}
(\forall n \in \mathbb{N}) \quad \boldsymbol{s}_n^* &= \boldsymbol{p}_n^* + \boldsymbol{Q}\boldsymbol{p}_n \\
&= \big(p_{i,\overline{\delta}_i(n)}^*\big)_{i\in\mathbb{I}} + \boldsymbol{Q}\boldsymbol{p}_n \\
&= \big(\gamma_{i,\delta_i(n)}^{-1}\big(x_{i,\overline{\delta}_i(n)}^* - p_{i,\overline{\delta}_i(n)}\big)\big)_{i\in\mathbb{I}} + \boldsymbol{Q}\boldsymbol{p}_n \\
&= \big(\gamma_{i,\delta_i(n)}^{-1} x_{i,\overline{\delta}_i(n)}^*\big)_{i\in\mathbb{I}} - \big(\gamma_{i,\delta_i(n)}^{-1} p_{i,\overline{\delta}_i(n)}\big)_{i\in\mathbb{I}} + \boldsymbol{Q}\boldsymbol{p}_n \\
&= \boldsymbol{K}_n\widetilde{\boldsymbol{x}}_n - \boldsymbol{K}_n\boldsymbol{p}_n.
\end{aligned} \tag{5.16}$$

Thus, (5.8) can be recast as

$$
\begin{array}{l}
\text{for } n = 0,1,\ldots \\
\left\lfloor
\begin{array}{l}
\boldsymbol{p}_n = (\boldsymbol{K}_n + \boldsymbol{M})^{-1}(\boldsymbol{K}_n\widetilde{\boldsymbol{x}}_n) \\
\boldsymbol{s}_n^* = \boldsymbol{K}_n\widetilde{\boldsymbol{x}}_n - \boldsymbol{K}_n\boldsymbol{p}_n \\
\text{if } \langle \boldsymbol{p}_n - \boldsymbol{x}_n \mid \boldsymbol{s}_n^* \rangle < 0 \\
\left\lfloor \boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \dfrac{\lambda_n \langle \boldsymbol{p}_n - \boldsymbol{x}_n \mid \boldsymbol{s}_n^* \rangle}{\|\boldsymbol{s}_n^*\|^2}\, \boldsymbol{s}_n^* \right. \\
\text{else} \\
\left\lfloor \boldsymbol{x}_{n+1} = \boldsymbol{x}_n. \right.
\end{array}
\right.
\end{array}
\tag{5.17}
$$

Therefore, [15, Theorem 4.2(i)] yields $\sum_{n\in\mathbb{N}} \|\boldsymbol{x}_{n+1} - \boldsymbol{x}_n\|^2 < +\infty$. On the one hand, in view of [16, Lemma A.3], we deduce from (5.7) and (5.10) that $(\forall i \in \mathbb{I})\ \boldsymbol{x}_{\delta_i(n)} - \boldsymbol{x}_n \to \boldsymbol{0}$. On the other hand, for every $n \in \mathbb{N}$, every $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$, and every $\boldsymbol{y} \in \boldsymbol{\mathcal{X}}$, we deduce from (5.12) and the Cauchy–Schwarz inequality that $\alpha\|\boldsymbol{x} - \boldsymbol{y}\|^2 \leqslant \langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{K}_n\boldsymbol{x} - \boldsymbol{K}_n\boldsymbol{y} \rangle \leqslant \|\boldsymbol{x} - \boldsymbol{y}\|\, \|\boldsymbol{K}_n\boldsymbol{x} - \boldsymbol{K}_n\boldsymbol{y}\|$, from which it follows that

$$
\alpha\|\boldsymbol{x} - \boldsymbol{y}\| \leqslant \|\boldsymbol{K}_n\boldsymbol{x} - \boldsymbol{K}_n\boldsymbol{y}\|. \tag{5.18}
$$

Hence, using (5.13), (5.8), (5.11), and the fact that $\boldsymbol{Q}$ is χ-Lipschitzian, we get

$$
\begin{aligned}
\alpha^2\|\widetilde{\boldsymbol{x}}_n - \boldsymbol{x}_n\|^2 &\leqslant \|\boldsymbol{K}_n\widetilde{\boldsymbol{x}}_n - \boldsymbol{K}_n\boldsymbol{x}_n\|^2 \\
&= \Big\| \big(\gamma_{i,\delta_i(n)}^{-1}\big(x_{i,\delta_i(n)} - \gamma_{i,\delta_i(n)} Q_i\boldsymbol{x}_{\delta_i(n)}\big)\big)_{i\in\mathbb{I}} - \big(\gamma_{i,\delta_i(n)}^{-1} x_{i,n} - Q_i\boldsymbol{x}_n\big)_{i\in\mathbb{I}} \Big\|^2 \\
&= \sum_{i\in\mathbb{I}} \big\| \gamma_{i,\delta_i(n)}^{-1}\big(x_{i,\delta_i(n)} - x_{i,n}\big) + \big(Q_i\boldsymbol{x}_n - Q_i\boldsymbol{x}_{\delta_i(n)}\big) \big\|^2 \\
&\leqslant \sum_{i\in\mathbb{I}} 2\big(\varepsilon^{-2}\|x_{i,\delta_i(n)} - x_{i,n}\|^2 + \|Q_i\boldsymbol{x}_n - Q_i\boldsymbol{x}_{\delta_i(n)}\|^2\big) \\
&\leqslant \sum_{i\in\mathbb{I}} 2(\varepsilon^{-2} + \chi^2)\|\boldsymbol{x}_{\delta_i(n)} - \boldsymbol{x}_n\|^2 \\
&\to 0.
\end{aligned}
\tag{5.19}
$$

Thus, we conclude via [15, Theorem 4.2(ii) and Remark 4.3] that $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ converges weakly to a point in zer $\boldsymbol{M}$, i.e., a solution to (5.4). Further, it is shown in the proof of [15, Theorem 4.2(ii)] that $\boldsymbol{K}_n\widetilde{\boldsymbol{x}}_n - \boldsymbol{K}_n\boldsymbol{p}_n \to \boldsymbol{0}$. Hence, we derive from (5.18) and (5.19) that $\|\boldsymbol{x}_n - \boldsymbol{p}_n\| \leqslant \|\boldsymbol{x}_n - \widetilde{\boldsymbol{x}}_n\| + \|\widetilde{\boldsymbol{x}}_n - \boldsymbol{p}_n\| \leqslant \|\boldsymbol{x}_n - \widetilde{\boldsymbol{x}}_n\| + (1/\alpha)\|\boldsymbol{K}_n\widetilde{\boldsymbol{x}}_n - \boldsymbol{K}_n\boldsymbol{p}_n\| \to 0$. ☐

We are now ready to prove Theorem 5.1.

Proof. Consider the system of monotone inclusions

$$
\text{find } (\boldsymbol{x}, \boldsymbol{v}^*) \in \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}} \text{ such that } \begin{cases} (\forall i \in I)\ \ 0 \in \partial\varphi_i(x_i) + \nabla_i \boldsymbol{f}_i(\boldsymbol{x}) + \sum_{k\in K} \Pi_i(\boldsymbol{L}_k^* v_k^*) \\ (\forall k \in K)\ \ 0 \in \partial g_k^*(v_k^*) - \boldsymbol{L}_k\boldsymbol{x}. \end{cases} \tag{5.20}
$$

We assume, without loss of generality, that I and K are disjoint subsets of $\mathbb{N}$. Then, in view of (4.11), (5.20) is a special case of (5.4) where $\mathbb{I} = I \cup K$ and

$$
\begin{cases} (\forall i \in I)\ \ \mathcal{X}_i = \mathcal{H}_i \text{ and } P_i = \partial\varphi_i \\ (\forall k \in K)\ \ \mathcal{X}_k = \mathcal{G}_k \text{ and } P_k = \partial g_k^* \\ \boldsymbol{Q}\colon (\boldsymbol{x}, \boldsymbol{v}^*) \mapsto (\boldsymbol{G}\boldsymbol{x} + \boldsymbol{L}^*\boldsymbol{v}^*, -\boldsymbol{L}\boldsymbol{x}). \end{cases} \tag{5.21}
$$

Note that $\boldsymbol{Q}$ is Lipschitzian and that, for every $(\boldsymbol{x}, \boldsymbol{v}^*) \in \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$ and every $(\boldsymbol{y}, \boldsymbol{w}^*) \in \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$, it follows from (5.1) that

$$\big\langle (\boldsymbol{x}, \boldsymbol{v}^*) - (\boldsymbol{y}, \boldsymbol{w}^*) \mid \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{v}^*) - \boldsymbol{Q}(\boldsymbol{y}, \boldsymbol{w}^*) \big\rangle = \langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{G}\boldsymbol{x} - \boldsymbol{G}\boldsymbol{y} \rangle \leqslant \sum_{i \in I} \chi_i \|x_i - y_i\|^2. \tag{5.22}$$

In addition, for every $n \in \mathbb{N}$ and every $k \in K_n$, upon setting $z^*_{k,n} = y^*_{k,n}/\mu_{k,d_k(n)}$, we deduce from (5.3) that

$$z^*_{k,n} = v^*_{k,d_k(n)} + \mu^{-1}_{k,d_k(n)} \boldsymbol{L}_k \boldsymbol{x}_{d_k(n)} \tag{5.23}$$

and from [6, Theorem 14.3(ii) and Example 23.3] that

$$b^*_{k,n} = \operatorname{prox}_{\mu^{-1}_{k,d_k(n)} g^*_k} z^*_{k,n} = J_{\mu^{-1}_{k,d_k(n)} P_k} z^*_{k,n} \quad \text{and} \quad b_{k,n} = \mu_{k,d_k(n)} (z^*_{k,n} - b^*_{k,n}). \tag{5.24}$$

Hence, (5.3) is a realization of (5.8) in the context of (5.21) with

$$\big[\, (\forall n \in \mathbb{N}) \;\; \mathbb{I}_n = I_n \cup K_n \,\big] \quad \text{and} \quad \big[\, (\forall k \in K) \;\; \chi_k = 0 \;\text{ and }\; \gamma_{k,n} = \mu^{-1}_{k,n} \,\big]. \tag{5.25}$$

Moreover, we observe that $\varnothing \neq \boldsymbol{Z}$ is the set of solutions to (5.20). Hence, Proposition 5.2(ii) implies that $(\boldsymbol{x}_n, \boldsymbol{v}^*_n)_{n \in \mathbb{N}}$ converges weakly to a point $(\boldsymbol{x}, \boldsymbol{v}^*) \in \boldsymbol{Z}$. By Proposition 4.1, $\boldsymbol{x}$ solves (1.2). ☐

Remark 5.3 By invoking [15, Theorem 4.8] and arguing as in the proof of Proposition 5.2, we obtain a strongly convergent counterpart of Proposition 5.2 which, in turn, yields a strongly convergent version of Theorem 5.1.

Remark 5.4 Consider the proof of Theorem 5.1. We deduce from Proposition 5.2(i) that $\boldsymbol{x}_n - \boldsymbol{a}_n \to \boldsymbol{0}$ and, thus, that $\boldsymbol{a}_n \rightharpoonup \boldsymbol{x}$. Moreover, by (5.3), given $i \in I$, the sequence $(a_{i,n})_{n \in \mathbb{N}}$ lies in $\operatorname{dom} \partial\varphi_i \subset \operatorname{dom} \varphi_i$. In particular, if a constraint on x_i is enforced via $\varphi_i = \iota_{C_i}$, then $(a_{i,n})_{n \in \mathbb{N}}$ converges weakly to the ith component of a solution $\boldsymbol{x}$ while being feasible in the sense that $C_i \ni a_{i,n} \rightharpoonup x_i$.

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