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Proportional Component Order Network Connectivity

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Abstract

We introduce a new measure of network vulnerability related to the order of the largest component. This new connectivity measure considers a network to be operational if there is a component or order at least some fixed proportion, r, of the original order. Thus, the network is in a failure state if all components are sufficiently small. In this paper, we consider the parameters with vertex deletions as well as edge deletions for particular graph classes. We also find the minimum values of the parameter for graphs with a fixed size and order. We end with a discussion and some conjectures for the maximum value of the parameter for graphs with a fixed size and order.

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1 Introduction

The vulnerability of a network is an important parameter to consider when building a network. Different characteristics of the network can become vulnerable over time or attacked from other forces. In a simple setting, we model a network as a graph where the vertices represent our objects and the edges between the vertices represent connections between these objects. These networks could be computer networks, but they could also represent power stations and power lines, people and friendships, or departments in a company and workflow.

Generally, there are two things to consider when discussing the vulnerability of a network. The first is the conditions that need to be satisfied in order for the network to be operational. This is equivalent to considering what properties need to be satisfied in order to render our network inoperable. The second consideration is what objects in our network tend to fail. In certain applications, edges are prone to failure and in other applications vertices are prone to failure.

One of the first examples of a network vulnerability measure is the connectivity of a graph, which measures the minimum number of vertices (or edges) which can be removed to disconnect the graph. This was also generalized to mixed removal where vertices are removed first followed by edges [1]. Network vulnerability measures have recently been generalized to include other graph properties, not just connectivity. Harary outlined the idea of *conditional connectivity*, which provides a general framework for investigating the vulnerability of networks based on different parameters [8]. In [4] the authors consider the *component order vertex connectivity* (COVC) parameter, denoted $\kappa_c^{(k)}(G)$, which is the minimum number of vertices which can be removed so that the order of all components in the resulting graph are less than some predetermined value, k. A similar measure was developed for edge removals (COEC), denoted $\lambda_c^{(k)}(G)$, in [2], and more recently neighbor removals were studied in [9] and [11]. Component order connectivity has many applications including multiprocessor network vulnerability [3].

In this paper we extend this idea. Instead of a network being operational if there is a component of order larger than a fixed value, k, we define the network to be operational if there is a component with order greater than some proportion, $0 < r < 1$, of the original order. So as our networks become larger, we will need a component of larger order if the network is to be in an operating state. This connectivity measure has been studied for planar graphs with $r = \frac{1}{2}$ for the purposes of VLSI circuit design [10], but we explore the measure with $0 < r < 1$ for simple graphs.

2 Definitions

Throughout the paper, we will assume that $G = (V, E)$ is a simple graph with vertex set V and edge set E. We will follow the standard graph theory notation found in [12].

Given a graph G define the following:

- Let $V(G)$ be the set of vertices of G, and $E(G)$ the set of edges of G.
- Let $G - V'$ be the graph obtained from removing the vertices of $V' \subseteq V(G)$ from G. Similarly for edges define $G - E'$ for any set $E' \subseteq E(G)$.

- Let $|G|$ be the order of the graph, that is, the cardinality of the vertex set $V(G)$.

We will assume throughout that $0 < r < 1$ and n is a positive integer.

As indicated previously, we are concerned with what property makes our network operational and what elements of our graph are prone to failure. We will consider a network to be operational if there is a sufficiently large component. This is formalized in the following definition.

Definition 2.1. Given $0 < r < 1$, we say a graph G of order n is in an *operating state* if at least one component of G has order greater than rn. Otherwise, we will say that G is in a *failure state*.

We will consider vertex and edge removals separately. For vertex removal, if a graph G is in an operating state, we want to find subsets of vertices whose removal renders our graph inoperable. These subsets of vertices will be called *vertex failure sets*.

Definition 2.2. Given a graph $G = (V, E)$ and $0 < r < 1$, a subset of vertices $V' \subseteq V$ is a *vertex failure set* if $G - V'$ is in a failure state; that is, every component of $G - V'$ has order at most rn, where $n = |V|$.

The set of all vertex failure sets of G will be denoted $\mathrm{COPVF}(G)$. To measure the vulnerability of the network, we need to find the smallest set of vertices which, when removed, render our network inoperable. This will allow us to quantify the vulnerability of our network.

Definition 2.3. Given a graph $G = (V, E)$ and $0 < r < 1$, the *Component Order Proportion Vertex Connectivity* of G, denoted $CO_v^r(G)$, is cardinality of the smallest set $V' \in \mathrm{COPVF}(G)$. If $V' \in \mathrm{COPVF}(G)$ is of minimum cardinality, we will call V' a *minimum vertex failure set*.

Now we can consider a similar vulnerability measure for networks where the edges are prone to failure rather than the vertices.

Definition 2.4. Given a graph $G = (V, E)$ and $0 < r < 1$, a subset of edges $E' \subseteq E$ is an *edge failure set* if $G - E'$ is in a failure state; that is, every component of $G - E'$ has order at most rn, where $n = |V|$.

The set of all edge failure sets of G will be denoted $\mathrm{COPEF}(G)$. The cardinality of the smallest set in $\mathrm{COPEF}(G)$ will be the measure of the networks vulnerability.

Definition 2.5. Given a graph $G = (V, E)$ and $0 < r < 1$, the *Component Order Proportion Edge Connectivity* of G, denoted $CO_e^r(G)$, is cardinality of the smallest set $E' \in \mathrm{COPEF}(G)$. If $E' \in \mathrm{COPEF}(G)$ is of minimum cardinality, we will call E' a *minimum edge failure set*.

Note that for the special case when $r = \frac{k-1}{n}$ the k-component order connectivity, $\kappa_c^k(G)$, defined in [4] is equivalent to $CO_v^{\frac{k-1}{n}}(G)$ and the k-component edge connectivity, $\lambda_c^k(G)$, defined in [2] is equivalent to $CO_e^{\frac{k-1}{n}}(G)$.

3 COPVC and COPEC in General Graph Classes

In this section, we outline how CO_v and CO_e work with several general graph classes including paths, cycles, complete graphs, and complete bipartite graphs.

3.1 Vertex

For vertex removal in path graphs, we remove vertices so that the resulting graph will be the disjoint union of the maximum number of large order failure components. This will produce a failure state by removing the fewest vertices.

Theorem 3.1. *If P_n is a path graph of order n and $0 < r < 1$ then*

$$CO_v^r(P_n) = \left\lfloor \frac{n}{\lfloor rn \rfloor + 1} \right\rfloor.$$

Proof. Let $P_n = (V, E)$ be a path graph on n vertices defined in the standard way so that v_1 and v_n have degree 1 and v_i is adjacent to v_{i+1} for all $1 \leq i < n$.

Let $V' = \{v_i \in V : i = j(\lfloor rn \rfloor + 1) \text{ where } j \in \mathbb{Z}^+\}$. Note $|V'| = \left\lfloor \frac{n}{\lfloor rn \rfloor + 1} \right\rfloor$.

So $P_n - V' = G_1 + G_2 + \ldots + G_{|V'|+1}$ where $G_i = P_{\lfloor rn \rfloor}$ except possibly $G_{|V'|+1}$ which will be a subgraph of $P_{\lfloor rn \rfloor}$. So $P_n - V'$ is the disjoint union of paths of length at most $\lfloor rn \rfloor$. Since each graph G_i has order at most $\lfloor rn \rfloor$, we know $V' \in \mathrm{COPVF}(P_n)$.

Next, we show that there is no smaller set in $\mathrm{COPVF}(P_n)$. Assume there is a smaller set $V'' \in \mathrm{COPVF}(P_n)$ so that $|V''| < |V'|$. Since P_n is a tree of maximum degree 2, every vertex deletion creates at most one more component. Therefore, $P_n - V''$ is the disjoint union of graphs $G'_1, ..., G'_{|V''|+1}$ (allowing empty graphs if necessary). Then we have that

$$\begin{aligned} n = |P_n| = |V''| + \sum |G'_i| &\leq |V''| + \lfloor rn \rfloor (|V''| + 1) \\ &< |V'| + \lfloor rn \rfloor |V'| \\ &\leq n. \end{aligned}$$

So $n < n$, a contradiction. □

Note that if $r = \frac{k-1}{n}$, then

$$CO_v^r(P_n) = \left\lfloor \frac{n}{\lfloor ((k-1)/n)n \rfloor + 1} \right\rfloor = \left\lfloor \frac{n}{k} \right\rfloor = \kappa_c^{(k)}(P_n)$$

as shown in Theorem 1 from [4].

Removing a vertex in a cycle of order n results in a path of order $n - 1$. Therefore, we have the following immediate corollary.

Corollary 3.2. *For all $0 < r < 1$,*

$$CO_v^r(C_n) = \left\lfloor \frac{n-1}{\lfloor r(n-1) \rfloor + 1} \right\rfloor + 1.$$

Complete graphs are another class of graphs which are easy to understand since any number of vertex removals will not disconnect our graph.

Theorem 3.3. *For all* $0 < r < 1$,

$$CO_v^r(K_n) = n - \lfloor rn \rfloor.$$

Proof. Removing any j vertices from a complete graph on n vertices gives a complete graph on $n - j$ vertices. So a failure state for K_n will be a complete graph on at most $\lfloor rn \rfloor$ vertices. So for V' to be in COPVF(K_n), it must have at least $n - \lfloor rn \rfloor$ vertices. So $CO_v^r(K_n) = n - \lfloor rn \rfloor$. □

For complete bipartite graphs, we may produce a failure state by removing one of the vertex sets or a portion of a vertex set.

Theorem 3.4. *Assume* $a \leq b$. *For all* $0 < r < 1$,

$$CO_v^r(K_{a,b}) = \min\{a, a + b - \lfloor rn \rfloor\}.$$

Proof. Let $K_{a,b} = (V, E)$ have two parts, $A = \{a_1, a_2, \ldots, a_a\}$ and $B = \{b_1, b_2, \ldots b_b\}$. Let $n = a + b$. Notice that the deletion of any set of vertices will result in a complete bipartite graph or a set of isolated vertices.

Case 1: Suppose $rn \geq b$. Since $rn \geq b$, $a + b - \lfloor rn \rfloor \leq a$. So let $V' \subseteq A$ be a collection of $a+b-\lfloor rn \rfloor$ vertices. Notice $|K_{a,b} - V'| = \lfloor rn \rfloor$ is a failure state, so $V' \in$ COPVF($K_{a,b}$). There are no sets of smaller order in COPVF($K_{a,b}$), since if $V'' \in$ COPVF($K_{a,b}$) and $|V''| < |V'|$ then $K_{a,b} - V''$ is connected (since $|V''| < a$) and $|K_{a,b} - V''| > \lfloor rn \rfloor$. So $V'' \notin$ COPVF($K_{a,b}$) and $CO_v^r(K_{a,b}) = a + b - \lfloor rn \rfloor$ in this case.

Case 2: Suppose $rn < b$. Consider removing A. Then $K_{(a,b)} - A$ is an empty graph with b vertices and is in a failure state. For any subset $V' \subseteq V$ with $|V'| < a$ we know $K_{(a,b)} - V'$ is a complete bipartite graph with order $a + b - |V'|$. However, $a + b - |V'| > b$ and therefore, $K_{(a,b)} - V'$ can not be in a failure state. □

3.2 Edges

We now shift our focus to edge removal for paths, cycles, complete graphs, and complete bipartite graphs.

Theorem 3.5. *For all* $0 < r < 1$, $CO_e^r(P_n) = \left\lfloor \frac{n-1}{\lfloor rn \rfloor} \right\rfloor$.

Proof. Let $P_n = (V, E)$ be a defined as in Theorem 3.1 and label the edge incident to vertices v_i, v_{i+1} as edge e_i. .

Let $E' = \{e_i \in E : i = j\lfloor rn \rfloor$, where $j \in \mathbb{Z}^+\}$. Note $|E'| = \left\lfloor \frac{n-1}{\lfloor rn \rfloor} \right\rfloor$.

So, $P_n - E' = G_1 + G_2 + \ldots + G_{|E'|+1}$ where $G_i = P_{\lfloor rn \rfloor}$ except possibly $G_{|E'|+1}$ which will be a subgraph of $P_{\lfloor rn \rfloor}$. So $P_n - E'$ is the disjoint union of paths of length at most $\lfloor rn \rfloor$. Since each graph G_i has order at most $\lfloor rn \rfloor$, we know $E' \in$ COPEF(G).

Next, we show that there is no smaller set in COPEF(G). Assume there is a smaller set $E'' \in$ COPEF(G) so that $|E''| < |E'|$. Since P_n is a tree, every edge deletion creates at

most one more component. Therefore, $P_n - E''$ is the disjoint union of graphs $G'_1, ..., G'_{|E''|+1}$ (allowing empty graphs if necessary). Then we have that

$$\begin{aligned} n = |E(P_n)| + 1 = \sum |E(G'_i)| + |E''| + 1 &\leq (\lfloor rn \rfloor - 1)(|E''| + 1) + |E''| + 1 \\ &< \lfloor rn \rfloor |E'| + 1 \\ &< n. \end{aligned}$$

So $n < n$, a contradiction. □

As in the vertex removal case, if $r = \frac{k-1}{n}$, then

$$CO_e^r(P_n) = \left\lfloor \frac{n-1}{\lfloor ((k-1)/n)n \rfloor} \right\rfloor = \left\lfloor \frac{n-1}{k-1} \right\rfloor = \lambda_c^{(k)}(P_n)$$

as shown in Example 4.2.2 from [2].

Again, since any edge removed from C_n will produce P_n, we have the following:

Corollary 3.6. *For all* $0 < r < 1$,

$$CO_e^r(C_n) = \left\lfloor \frac{n-1}{\lfloor rn \rfloor} \right\rfloor + 1.$$

Theorem 3.7. *For any* $0 < r < 1$, *let* $n = p\lfloor rn \rfloor + q$ *with* $0 \leq q < \lfloor rn \rfloor$. *Then*

$$CO_e^r(K_n) = \binom{n}{2} - p\binom{\lfloor rn \rfloor}{2} - \binom{q}{2}.$$

Proof. First, we prove that for optimal $E' \in \text{COPEF}(K_n)$, each component of $K_n - E'$ is a complete graph. Let G be a component of $K_n - E'$ that is not complete. Let e be an edge not present in $E(G)$ whose vertices are in $V(G)$. Since K_n is complete, $e \in E(K_n)$ and so $e \in E'$. Let $E'' = E' - \{e\}$. Then note that the components of $K_n - E''$ are the same as $K_n - E' + e$ since the only difference is the edge e, which is contained within the component G. Since no components have their order changed, $K_n - E''$ is a failure state. So $E'' \in \text{COPEF}(K_n)$ and $|E''| < |E'|$. So a failure set of smaller cardinality can always be found if one of the components of E' is not complete, thus, the removal of a minimum failure set will produce a failure state whose components are all complete graphs.

Let $G_1, G_2, \cdots G_k$ denote the components of $K_n - E'$. Assume G_i, and G_j have order strictly less than $\lfloor rn \rfloor$ and let $|V(G_i)| \leq |V(G_j)|$. Then consider a vertex $v \in V(G_i)$. Let $E'' \subseteq E(K_n)$ be the same as E' on all vertices $v' \neq v$, but contains the edges connecting v to vertices in G_i and has no edges connecting v to G_j. Thus,
$E'' = (E' - \{e \in E' : v \text{ is incident to } e\}) \cup \{(v, v') : v' \in V(G_i)\}$. Now let $G'_1, G'_2, \cdots$ be the components of $G - E''$. Then $G'_k = G_k$ for $k \neq i, j$,$G'_i = G_i - v$, and $G'_j = G_j \vee \{v\} \simeq K_{|V(G_j)|+1}$. Note that $|G'_i| < \lfloor rn \rfloor$ and $|G'_j| \leq \lfloor rn \rfloor$, so $E'' \in \text{COPEF}(K_n)$. Since E'' is derived from E' by adding $|V(G_i)| - 1$ edges from v to the other vertices in G_i and deleting the $|V(G_j)|$ edges from v to each vertex in G_j, we see $|E''| = |E'| - |V(G_j)| + |V(G_i)| - 1$ and so $|E''| < |E'|$. Therefore, if there exist two components in $K_n - E'$ with order less than $\lfloor rn \rfloor$, then there exists $E'' \in \text{COPEF}(K_n)$ with strictly lesser cardinality than E'.

Therefore, if $n = p\lfloor rn \rfloor + q$ with $0 \leq q < \lfloor rn \rfloor$ our failure state will consist of p complete graphs of order $\lfloor rn \rfloor$ and a complete graph of order q. □

The analysis of $CO_e^r(K_{a,b})$ is more involved. In [7] the authors consider the k-component order edge connectivity of $K_{a,b}$, denoted $\lambda_c^k(K_{a,b})$, which is defined to be the minimum number of edges that need to be removed so that all of the remaining components have order at most $k-1$. Therefore, if $r = \frac{k-1}{n}$, then $CO_e^r(K_{a,b}) = \lambda_c^k(K_{a,b})$.

4 $G(n,m)$

We will now turn our attention to the family of graph $G(n,m)$, the set of all graphs with n vertices and m edges.

Definition 4.1. Let $G(n,m) = \{G : |V(G)| = n, |E(G)| = m\}$ be the set of all graphs of order n and size m for $0 \leq m \leq \binom{n}{2}$.

In this section, we will be concerned with finding the maximum and minimum values of $CO_v^r(G)$ and $CO_e^r(G)$ over all graphs $G \in G(n,m)$. Intuitively, a graph which achieves this minimum value would be one of the most vulnerable networks we can construct with n vertices and m edges. Similarly, a graph which achieves this maximum value would be one of the least vulnerable networks we can construct with n vertices and m edges.

Definition 4.2. If $\mathcal{G}$ is a collection of graphs, then $CO_v^{r^-}(\mathcal{G}) = \min_{G\in\mathcal{G}} CO_v^r(G)$.

Definition 4.3. If $\mathcal{G}$ is a collection of graphs, then $CO_v^{r^+}(\mathcal{G}) = \max_{G\in\mathcal{G}} CO_v^r(G)$.

We can define $CO_e^{r^-}(\mathcal{G})$ and $CO_e^{r^+}(\mathcal{G})$ similarly. The following lemma gives bounds on the number of edges that we can have in a failure state. In what follows, we can consider a graph of order n and we define p to be the maximum number of failure components of maximum order. Since a component of maximum order has order $\lfloor rn \rfloor$, $p = \left\lfloor \frac{n}{\lfloor rn \rfloor} \right\rfloor$. We will let q denote the number of vertices which remain after we remove p pairwise disjoint sets of $\lfloor rn \rfloor$ vertices.

Lemma 4.4. *Define nonnegative integers p and q so that $n = \lfloor rn \rfloor p + q$ with $0 \leq q < \lfloor rn \rfloor$. If G is in a failure state then $|E(G)| \leq p\binom{\lfloor rn \rfloor}{2} + \binom{q}{2}$. In particular, there exists a graph that is the maximal size failure state which is the disjoint union of p copies of $K_{\lfloor rn \rfloor}$ with one copy of K_q.*

Proof. Assume $n = \lfloor rn \rfloor p + q$. Let G be a graph with order n so that G is in a failure state and there does not exist a graph G' of order n which is also in a failure state so that $|E(G')| > |E(G)|$. This means that G has the maximum number of edges possible to be in a failure state. If G contains a component which is not a complete graph, then we can add an edge into that component and G will remain in a failure state. Therefore, G must be the union of components which are complete graphs of order at most $\lfloor rn \rfloor$. Assume $G = K_{a_1} \cup K_{a_2} \cup \ldots \cup K_{a_c}$. Then $|E(G)|$ is maximized when $a_1 = a_2 = \ldots = a_{c-1} = \lfloor rn \rfloor$ and $a_c = q$. If this is not the case we have K_a and K_b, $a \leq b < \lfloor rn \rfloor$. However, if this is the case, we note

$$\begin{aligned}\binom{b}{2}+\binom{a}{2} &= \frac{b^2-b}{2}+\frac{a^2-a}{2}\\ &= \frac{b^2+b}{2}+\frac{a^2-a}{2}-b\\ &\leq \binom{b+1}{2}+\frac{a^2-a}{2}-a\\ &< \binom{b+1}{2}+\frac{a^2-3a+2}{2}\\ &= \binom{b+1}{2}+\binom{a-1}{2}\end{aligned}$$

and so $|E(K_{a-1})|+|E(K_{b+1})| > |E(K_a)|+|E(K_b)|$. Therefore, G would not be maximal. □

4.1 $CO_v^{r^-}(G(n,m))$

We will establish some basic properties of $CO_v^{r^-}(G(n,m))$ for a fixed n which will be used to find exact values for $CO_v^{r^-}(G(n,m))$.

Since we will be considering the set $G(n,m+1)$, in the following sections we will assume $m < \binom{n}{2}$ since if $m = \binom{n}{2}$, then $G(n,m+1) = \emptyset$.

Lemma 4.5 (Basic properties of $CO_v^{r^-}(G(n,m))$)**.** *Let $n = p\lfloor rn\rfloor + q$, $0 \leq q < \lfloor rn\rfloor$, and let $g_v^{r^-}(m) = CO_v^{r^-}(G(n,m))$. Then the following hold:*

(a) $g_v^{r^-}(m)$ is non-decreasing

(b) $g_v^{r^-}(m+1) - g_v^{r^-}(m) \leq 1$

(c) $g_v^{r^-}(m)$ takes on the value 0 for $0 \leq m \leq p\binom{\lfloor rn\rfloor}{2} + \binom{q}{2}$

(d) The maximum possible value $g_v^{r^-}(m)$ can take on is $n - \left\lfloor \frac{n}{\lfloor rn\rfloor} \right\rfloor$, and $g_v^{r^-}(m)$ always takes on this value for some m.

Proof.

(a) Consider the graph $G \in G(n,m+1)$ where $CO_v^r(G) = g_v^{r^-}(m+1)$. Then removing one edge, e, from G clearly does not increase the vertex connectivity since any vertex set whose removal creates a failure state in G will also create a failure state in $G-e$. So $g_v^{r^-}(m) \leq g_v^{r^-}(m+1)$.

(b) Consider $G \in G(n,m)$. Let D be a minimal component order proportion vertex connectivity set of G. Now consider some $G' \in G(n,m+1)$ so that $G' = G+e$. Assume $v \in V(G)$ is incident to e. Then $D \cup \{v\}$ is a failure set for G' since $G' - D - v$ is a subgraph of $G - D$. Since $g_v^{r^-}(m)$ is non-decreasing, $0 \leq g_v^{r^-}(m+1) - g_v^{r^-}(m) \leq 1$.

(c) Consider the graph $[\cup_{i=1}^{p} K_{\lfloor rn \rfloor}] \cup K_q$, the maximal size failure state from Lemma 4.4. Then this graph is already in a failure state. So by (a), $g_v^{r^-}(m) = 0$ for $0 \leq m \leq p\binom{\lfloor rn \rfloor}{2} + \binom{q}{2}$.

(d) Since $g_v^{r^-}(m)$ is non-decreasing, $g_v^{r^-}(m)$ achieves the greatest value when $m = \binom{n}{2}$, i.e. when $G = K_n$. So the maximum value of $g_v^{r^-}(m)$ is $CO_v^r(K_n) = n - \left\lfloor \frac{n}{\lfloor rn \rfloor} \right\rfloor$.

□

Now using the above lemma and Lemma 4.4, we can obtain a full characterization of $g_v^{r^-}(m) = CO_v^{r^-}(G(n,m))$ for fixed n and r.

Theorem 4.6. *Fix n and let $0 \leq k \leq n - \lfloor rn \rfloor$. Define*

$$f(k) = k(n-k) + \binom{k}{2} + p'\binom{\lfloor rn \rfloor}{2} + \binom{q'}{2}$$

where $n - k = \lfloor rn \rfloor p' + q'$ with $0 \leq q' < \lfloor rn \rfloor$.

Then

$$CO_v^{r^-}(G(n,m)) = \begin{cases} 0 & \text{if } m \leq f(0) \\ k & \text{if } f(k-1) < m \leq f(k). \end{cases}.$$

Proof. Fix $0 \leq k < n - \lfloor rn \rfloor$. We construct the graph G of order n with maximum size such that $CO_v^r(G) = k$. We partition the vertex set into two sets, A and B, with $|A| = k$ and $|B| = n - k$. Connect every vertex in A to every other vertex in G. Also, make the induced subgraph on B, denoted $G[B]$, to be the maximal size failure state as described in Lemma 4.4. Clearly the removal of A will put G into a failure state since $G - A = G[B]$. If we removed some vertex set V' with $|V'| = k$ from G, so that there is a vertex $b \in B$ in V', then there must be at least one vertex $u \in A$ in $G - V'$. Then note that, since $k < n - \lfloor rn \rfloor$, $|G - V'| > \lfloor rn \rfloor$. But since u is connected to every vertex in G, it is also connected to every vertex in $G - V'$. So the component u resides in is $G - V'$ itself. But since $|G - V'| > \lfloor rn \rfloor$, $G - V'$ is not a failure state. Thus, only removing A from G will put it into a failure state, and $CO_v^r(G) = k$.

For G, we can decompose the edge set of G into $E(G) = E(A) \cup E(B) \cup E(A \leftrightarrow B)$ where $E(A \leftrightarrow B)$ is the set of edges incident to one vertex in A and one vertex in B. Then consider G' with $|E(G')| > |E(G)|$ and $CO_v^r(G') = k$. Let A' be the vertex failure set of G' and $B' = V' - A'$. Then let $E(G') = E(A') \cup E(B') \cup E(A' \leftrightarrow B')$. Since $|E(G')| > |E(G)|$, we have that $|E(A')| + |E(B')| + |E(A' \leftrightarrow B')| > |E(A)| + |E(B)| + |E(A \leftrightarrow B)|$. But since $G[A]$ is a complete graph and every vertex of A is adjacent to every vertex of B, then $E(A') \not\succ E(A)$ and $E(A' \leftrightarrow B') \not\succ E(A \leftrightarrow B)$. So $E(B') > E(B)$. This contradicts the fact that $G[B]$ is the failure state with the largest possible edge set. So no G' of larger size exists that satisfies $CO_v^r(G') = k$

The graph G constructed above will have size

$$f(k) = |E(G)| = k(n-k) + \binom{k}{2} + p'\binom{\lfloor rn \rfloor}{2} + \binom{q'}{2}$$

where $n-k = \lfloor rn \rfloor p' + q'$ with $0 \leq q' < \lfloor rn \rfloor$.

If $m < f(0)$, then by Lemma 4.4 there exists a graph of size m which is already in a failure state.

So if $m > f(k)$, then $CO_v^{r^-}(G(n,m)) > k$ and since $CO_v^{r^-}(G(n,m))$ is non-decreasing and can only differ by 1 as m increases, we know if $m = f(k), CO_v^{r^-}(G(n,m)) = k$. This holds for all k, so if $f(k-1) < m \leq f(k)$ the $CO_v^{r^-}(G(n,m)) = k$.

□

It can be shown that Theorem 4.6 implies the following result, which allows for a more straightforward computation of $CO_v^{r^-}(G(n,m))$.

Theorem 4.7. *Fix n and define $p, q \in \mathbb{Z}$ so that $n = \lfloor rn \rfloor p + q$ with $0 \leq q < \lfloor rn \rfloor$. Then consider $0 \leq m \leq \binom{n}{2}$. Let $A = p\binom{\lfloor rn \rfloor}{2} + \binom{q}{2}$, $B = A + qp\lfloor rn \rfloor$, and $C_{i,j} = \sum_{t=1}^{i-1} [\lfloor rn \rfloor^2 (p-t)] + (j-1)\lfloor rn \rfloor (p-i) + B$. Then label m as follows:*

If $0 \leq m \leq A$, then let $f(m) = m_0$.

If $A + tp\lfloor rn \rfloor < m \leq A + (t+1)p\lfloor rn \rfloor$ for $0 < t \leq q$, then let $f(m) = m_t$.

If $C_{i,j} < m \leq C_{i,(j+1)}$ for $1 \leq i \leq p-1$, $1 \leq j < \lfloor rn \rfloor$, then let $f(m) = m_{i,j}$.

If $C_{i,\lfloor rn \rfloor} < m \leq C_{i+1,1}$ for $1 \leq i \leq p-1$, then let $f(m) = m_{i,\lfloor rn \rfloor}$.

If $C_{p,1} \leq m$, then let $f(m) = m_{p,1}$.

Then we have that

$$CO_v^{r^-}(G(n,m)) = \begin{cases} t & \text{if } f(m) = m_t \\ (i-1)\lfloor rn \rfloor + j + q & \text{if } f(m) = m_{i,j}, i \leq p-1 \\ (p-1)\lfloor rn \rfloor + q & \text{if } f(m) = m_{i,j}, i = p. \end{cases}$$

4.2 $CO_e^{r^-}(G(n,m))$

Now we will find the minimum value of $CO_e^{r^-}(G(n,m))$ which considers edge removals rather than vertex removals. Similar to Lemma 4.5, we will first establish some basic properties of $CO_e^{r^-}(G(n,m))$.

Lemma 4.8. *(Basic properties of $CO_e^{r^-}(G(n,m))$) Define non-negative integers p and q so that $n = \lfloor rn \rfloor p + q$ with $0 \leq q < \lfloor rn \rfloor$. For a fixed n, let $g_e^{r^-}(m) = CO_e^{r^-}(G(n,m))$. Then the following hold:*

(a) $g_e^{r^-}(m)$ is non-decreasing

(b) $g_e^{r^-}(m+1) - g_e^{r^-}(m) \leq 1$

(c) $g_e^{r^-}(m)$ takes on the value 0 for $0 \leq m \leq p\binom{\lfloor rn \rfloor}{2} + \binom{q}{2}$

(d) The maximum possible value of $g_e^{r^-}(m)$ *is* $\binom{n}{2} - p\binom{\lfloor rn \rfloor}{2} - \binom{q}{2}$*, and* $g_e^{r^-}(m)$ *always takes on this value for some* m*.*

Proof.

(a) Consider the graph $G \in G(n, m+1)$ that achieves $g_e^{r^-}(m+1)$. Then removing one edge from G clearly does not increase the edge connectivity since the same edge set that creates a failure state in G creates a failure state in $G - e$. So $g_e^{r^-}(m) \leq g_e^{r^-}(m+1)$.

(b) Consider $G \in G(n, m)$. Then let D be a minimal component order proportion edge connectivity set of G. Now consider some $G' \in G(n, m+1)$ so that $G' = G + e$. Then $D \cup \{e\}$ is a failure set for G' since $G' - D - e = G - D$. Since $g_e^{r^-}(m)$ is non-decreasing, $0 \leq g_e^{r^-}(m+1) - g_e^{r^-}(m) \leq 1$.

(c) Follows directly from Lemma 4.4.

(d) Since g is non-decreasing, K_n achieves the greatest value of $g_e^{r^-}()$. Since the maximum failure state is $[\cup_{i=1}^{p} K_{\lfloor rn \rfloor}] \cup K_q$, we have that the maximum value of $g_e^{r^-}(m)$ is $\binom{n}{2} - p\binom{\lfloor rn \rfloor}{2} - \binom{q}{2}$.

□

We can now find a simple closed form for $CO_e^{r^-}(G(n, m))$.

Theorem 4.9. *Fix* n*, let* $g_e^{r^-}(m) = CO_e^{r^-}(G(n, m))$*, and let* $n = \lfloor rn \rfloor p + q$*, with* $0 \leq q < \lfloor rn \rfloor$*. Then*

$$g_e^{r^-}(m) = \begin{cases} 0 & \text{if } m \leq p\binom{\lfloor rn \rfloor}{2} + \binom{q}{2} \\ m - p\binom{\lfloor rn \rfloor}{2} - \binom{q}{2} & \text{otherwise} \end{cases}$$

Proof. The first case of this piecewise function follows from Lemma 4.8(c). If $m > p\binom{\lfloor rn \rfloor}{2} + \binom{q}{2}$, then we can write $m = p\binom{\lfloor rn \rfloor}{2} + \binom{q}{2} + i$ for $i > 0$. Let G_0 denote the maximal size failure state as described in Lemma 4.4. Now if $|E(G)| = m$, then if we remove $j < i$ edges we are left with a graph $G' = G - E'$ with $|E(G')| > |E(G_0)|$, and so G' cannot be a failure state. So $CO_v^{r^-}(G(n, m)) \geq i$.

Now we construct a sequence of graphs recursively: let $G_i = G_{i-1} + e_{i-1}$ and $G_1 = G_0 + e_0$, where e_{i-1} is an arbitrary edge not in G_{i-1} and G_0 is again the maximal size failure state. Then $CO_e^r(G_1) = 1$ since G_0 is a maximal size failure state. Further, if we assume $CO_e^r(G_{i-1}) = i - 1$, then $CO_e^r(G_i) \leq CO_e^r(G_i - e_{i-1}) + 1 = i$. However we know $CO_e^r(G_i) \geq i$, and thus, $CO_e^r(G_i) = i$ and $CO_e^{r^-}(G(n, m)) = i$ in this case.

□

4.3 $CO_v^{r^+}(G(n, m))$ and $CO_e^{r^+}(G(n, m))$

Working with $CO_v^{r^+}(G(n, m))$ and $CO_e^{r^+}(G(n, m))$ is considerably more difficult than the minimum functions, and full characterizations of the functions were not found. In this section we present our results as well as the routes of attack and conjectures concerning these functions.

4.3.1 $CO_v^{r^+}(G(n,m))$

Similar to the minimum value, $CO_v^{r^-}(G(n,m))$ we have the following properties for our maximum function. These follow a similar argument found in Lemma 4.5.

Lemma 4.10 (Basic properties of $CO_v^{r^+}(G(n,m))$)**.** *Let $n = \lfloor rn \rfloor p + q$, $0 \leq q < \lfloor rn \rfloor$, and let $g_v^{r^+}(m) = CO_v^{r^+}(G(n,m))$. Then the following hold:*

(a) $g_v^{r^+}(m)$ is non-decreasing

(b) $g_v^{r^+}(m+1) - g_v^{r^+}(m) \leq 1$

(c) The maximum possible value $g_v^{r^+}(m)$ can take on is $n - \lfloor rn \rfloor$, and $g_v^{r^+}(m)$ always takes on this value for some m.

It is clear that if $m < \lfloor rn \rfloor$, then we are already in a failure state since the largest connected component could have at most $m+1 \leq \lfloor rn \rfloor$ vertices. Also, if $\binom{n}{2} - \lfloor rn \rfloor < m \leq \binom{n}{2}$ then $CO_v^{r^+}(G(n,m)) = n - \lfloor rn \rfloor$. So we have bounds on the tails of $CO_v^{r^+}(G(n,m))$ as follows:

$$CO_v^{r^+}(G(n,m)) = \begin{cases} 0 & \text{if } m < \lfloor rn \rfloor \\ ? & \text{if } \lfloor rn \rfloor \leq m \leq \binom{n}{2} - \lfloor rn \rfloor \\ n - \lfloor rn \rfloor, & \text{if } \binom{n}{2} - \lfloor rn \rfloor < m \leq \binom{n}{2}. \end{cases}$$

However, the case when $\lfloor rn \rfloor \leq m \leq \binom{n}{2} - \lfloor rn \rfloor$ is not as straight forward because the particular failure state structure is unknown. We do not know if the failure state would consist of a small number of components with a few edges in each component or a single large component with more edges.

Another way to approach the problem may be to consider what the minimum number of edges a graph can have if we want to fix $CO_v^r(G) = k$. This could give us a way to find $CO_v^{r^+}(G(n,m))$, however we were unsuccessful with this approach.

4.3.2 $CO_e^{r^+}(G(n,m))$

As in the preceding section, we once again have the following list of simple properties. The proof is essentially identical to Lemma 4.8.

Lemma 4.11 (Basic properties of $CO_e^{r^+}(G(n,m))$)**.** *Let $n = \lfloor rn \rfloor p + q$, $0 \leq q < \lfloor rn \rfloor$, and let $g_e^{r^+}(m) = CO_e^{r^+}(G(n,m))$. Then the following hold:*

(a) $g_e^{r^+}(m)$ is non-decreasing.

(b) $g_e^{r^+}(m+1) - g_e^{r^+}(m) \leq 1$.

(c) The maximum possible value $g_e^{r^+}(m)$ can take on is $\binom{n}{2} - p\binom{\lfloor rn \rfloor}{2} - \binom{q}{2}$, and $g_e^{r^+}(m)$ always takes on this value for some m.

For edge remove, it is also clear that we can figure out the tails as follows:

$$CO_e^{r^+}(G(n,m)) = \begin{cases} 0 & \text{if } m < \lfloor rn \rfloor \\ ? & \text{if } \lfloor rn \rfloor \leq m < \binom{n}{2} \\ \binom{n}{2} - \left(p\binom{\lfloor rn \rfloor}{2} + \binom{q}{2}\right) & \text{if } m = \binom{n}{2}. \end{cases}$$

If there are enough edges to guarantee that any subgraph of G will contain p disjoint copies of $K_{\lfloor rn \rfloor}$ and a disjoint K_q, then we know $CO_e^{r^+}(G(n,m)) = \binom{n}{2} - \left(p\binom{\lfloor rn \rfloor}{2} + \binom{q}{2}\right)$ which is the maximum value possible for $CO_e^{r^+}(G(n,m))$. However, this may not happen for very many values of m. For example, assume $m = \binom{n}{2} - 2$ and $G = K_n - \{e_1, e_2\}$ where e_1 and e_2 are vertex disjoint edges. Consider the case when $\lfloor rn \rfloor = n-1$. Then there is no subgraph of G which contains $K_{\lfloor rn \rfloor} = K_{n-1}$. Therefore, the failure state of G would be $K_{\lfloor rn \rfloor} - e_1$ and we would have to remove $n-2$ edges, not $\binom{n}{2} - \left(p\binom{\lfloor rn \rfloor}{2} + \binom{q}{2}\right) = \binom{n}{2} - \binom{n-1}{2} = n-1$.

4.3.3 Bounds

Since a general formula for $CO_e^{r^+}(G(n,m))$ was not found, we would like to bound the function. Given a formula for a general class of graphs immediately gives a lower bound. As an example, Theorem 3.5 states that $CO_e^r(P_n) = \left\lfloor \frac{n-1}{\lfloor rn \rfloor} \right\rfloor$ and so $CO_e^{r^+}(G(n,m)) \geq \left\lfloor \frac{n-1}{\lfloor rn \rfloor} \right\rfloor$. However, this method does not give a meaningful upper bound.

In the case of $r = \frac{1}{2}$, we attempt to find an upper bound on the $CO_e^{r^+}(G(n,m))$ value by employing a method that looks at bipartite subgraphs in the complement of a graph. We begin with a conjecture about the failure state of a graph that achieves the maximum edge removal connectivity for its graph class.

Conjecture 4.12. *Let $r = \frac{1}{k}$ and let n be a multiple of k. The there exists $G \in G(n,m)$ so that $CO_e^{1/k}(G) = CO_e^{1/k^+}(G(n,m))$ and the failure state of G can be written as the union of k components of order $\frac{n}{k}$.*

In the case $n = 2$, this conjecture is equivalent to the statement that one of the failure states of G is such that the vertices of the components partitions G into two vertex sets of equal cardinality.

This conjecture is fruitful when coupled with lower bounds for the largest bipartite subgraph of a graph G. Since an edge failure set can be seen as a bipartite graph, minimizing an edge failure set in G is dual to finding the largest bipartite graph in the complement, $\overline{G}$. We can then apply a lower bound to the complement graph class $G(n, \overline{m})$, which gives an upper bound for $G(n,m)$.

Definition 4.13. Let G be a graph. We define the function $b(G)$ to be the size of the largest bipartite subgraph of G. For fixed m, we define the function $b(m) = \min_G b(G)$, where $G \in G(n,m)$.

The functions $b(m)$ and $b(G)$ have been previously studied. Some results are summarized below:

Theorem 4.14 (Bounds on the Bipartite Function)**.** *Let $G \in G(n,m)$. Then*

- $b(m) \geq \left\lceil \frac{1}{2}m + \frac{\sqrt{8m+1}-1}{8} \right\rceil$. *[5]*
- $b(G) \geq \frac{1}{2}(m + \frac{1}{3}n)$ *in the case that G has no isolated points. [6]*
- $b(G) \geq \frac{1}{2}(m + \frac{1}{2}(n-1))$ *in the case that G is a connected graph. [6]*

These bounds could be used to provide an upper bound for $CO_e^{1/2^+}(G(n,m))$ as outlined in the next conjecture.

Conjecture 4.15. *Let n be even. Then $CO_e^{1/2^+}(G(n,m)) \leq \frac{m}{2} + \frac{7n}{12}$.*

Idea: Let $\bar{m} = \binom{n}{2} - m$. Then from Conjecture 4.12, let $G \in G(n,m)$ be such that G has maximal $CO_e^{1/2}$ value and has a failure state of two components of equal order. For simplicity, we will assume that G has no isolated vertices. Let C_1, C_2 be these components. Then consider the subgraph H consisting of the vertices of C_1, C_2 and the edges between the components. Then the number of edges is precisely $CO_e^{1/2}(G)$.

Then since H is a bipartite subgraph of G, $CO_e^{1/2}(G) = |E(H)| \leq b(G)$.

Now, we define $\bar{H}$ to be the bipartite complement of B; that is, subgraph consisting of vertex sets C_1 and C_2, with edges connecting the two sets only when an edge is absent in H.

Since each bipartite component has order $\frac{n}{2}$, for any bipartite subgraph B, $|E(B)| = \frac{n^2}{4} - |E(\bar{B})|$, where $\bar{B}$ is the bipartite complement. Then substituting this for the $|E(B)|$ term in the definition of $b(G)$, we obtain $b(G) = \max_{B \subset G} \left[\frac{n^2}{4} - \left| E(\bar{B}) \right| \right]$. But this is simply $\frac{n^2}{2} - b(\bar{G})$, and $CO_e^{1/2}(G) \leq b(G) = \frac{n^2}{4} - b(\bar{G})$.

We now apply the second Erdös bound from Theorem 4.14 to $b(\bar{G})$. We obtain the final inequality, $\mathrm{CO}_e^{1/2^+}(G(n,m)) \leq \frac{n^2}{4} - \frac{1}{2}(\bar{m} + \frac{1}{3}n) = \frac{m}{2} + \frac{7n}{12}$.

If G has l isolated points, we can repeat the previous argument with the isolated points removed and consider $\tilde{n} = n - l$. The upper bound of $\frac{m}{2} + \frac{7n}{12}$ will still hold, as long as l is even.

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