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EXISTENCE RESULTS FOR FRACTIONAL ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN THE SENSE OF THE DEFORMABLE DERIVATIVE

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ABSTRACT. In this article, we discuss the existence and uniqueness of solutions for initial value problems of fractional order functional and neutral functional differential equations with infinite delay. We use the deformable derivative introduced in 2017 by Zulfeqarr et. al (see [21]). Our results are obtained using the Banach fixed point theorem and the nonlinear alternative Leray–Schauder type theorem. We provide an example as an illustration of the main results.

1. Introduction

The concept of deformable derivative was recently introduced by F. Zulfeqarr, A. Ujlayan, and P. Ahuja [21]. Using this relatively new fractional derivative, we established the existence and uniqueness of solutions to evolution equations with nonlocal and local conditions (See [16, 17]). In [8], we studied solutions to the impulsive fractional differential equation :

$$D^\alpha y(t) = f(t, y),\ t \in J = [0, T],\ t \neq t_k, \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)) \tag{1.2}$$

$$y(0) = y_0, \tag{1.3}$$

where $k = 1, \dots, m, 0 < \alpha \leq 1, D^\alpha$ is the deformable fractional derivative of y. In this paper, we study the existence solutions for initial value problems of fractional order functional differential equations with infinite delay. We consider the initial value problem

$$D^\alpha y(t) = f(t, y_t), for\ t \in J = [0, b], \alpha \in (0, 1), \tag{1.4}$$

$$y(t) = \phi(t), t \in (-\infty, 0], \tag{1.5}$$

where D^α is the deformable derivative, $f : J \times B \to \mathbb{R}$ is a given function,
$\phi \in B$ (a phase space) , $\phi(0) = 0$, and $y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0], t \in J$, y is defined on $(-\infty, b]$, and y_t is the element of B defined here. Here $y_t(\cdot)$ represents the history of the state from time $-\infty$ up to the present time t.
Then we study fractional neutral functional differential equation :

$$D^\alpha[y(t) - g(t, y_t)] = f(t, y_t), \quad t \in J,$$
$$y(t) = \phi(t), \quad t \in (-\infty, 0],$$

where $g : J \times B \to \mathbb{R}$ is a given function such that $g(0, \phi) = 0$.

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2. Preliminaries

In this section, $C(J,\mathbb{R})$ stands for the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

where $|\cdot|$ denotes a suitable complete norm on $\mathbb{R}$.

Definition 2.1. ([21]) Let f be a real valued function on $[a,b]$, $\alpha \in [0,1]$. The deformable derivative of f of order α at $t \in (a,b)$ is defined as:

$$D^\alpha f(t) = \lim_{\epsilon \to 0} \frac{(1+\epsilon\beta)f(t+\epsilon\alpha) - f(t)}{\epsilon},$$

where $\alpha + \beta = 1$. If the limit exists, we say that f is α-differentiable at t.

Remark 2.1. If $\alpha = 1$, then $\beta = 0$, we recover the usual derivative. This shows that the deformable derivative is more general than the usual derivative.

Definition 2.2. ([21]) For $\alpha \in (0,1]$, the α-integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ is defined by

$$I_a^\alpha h(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_a^t e^{\frac{\beta}{\alpha}x} f(x)dx, \quad t \in [a,b],$$

where $\alpha + \beta = 1$. When $a = 0$ we use the notation

$$I^\alpha h(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}x} h(x)dx.$$

Remark 2.2. If $\alpha = 1$, then $\beta = 0$, we recover the usual Riemann integral. This also shows that the α-integral is more general than the usual Riemann integral.

In what follows, we assume that the state space $(B, \|\cdot\|_B)$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$ satisfying the following fundamental set of axioms from Hale and Kato in [11].

(A) If $y : (-\infty, b] \to \mathbb{R}$, and $y_0 \in B$, then for every $t \in [0,b]$ the following conditions hold:

(1) y_t is in B.
(2) $\|y_t\|_B \le K(t)\sup\{|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_B$,
(3) $|y(t)| \le H\|y_t\|_B$, where $H \ge 0$ is a constant, $K : [0,b] \to [0,\infty)$ is continuous, $M : [0,\infty) \to [0,\infty)$ is locally bounded and H, K, M are independent of $y(\cdot)$.

(A-1) For the function $y(\cdot)$ in (A), y_t is a B-valued continuous function on $[0,b]$.
(A-2) The space B is complete.

Theorem 2.1. ([21]) A differentiable function h at a point $t \in (a,b)$ is always α-differentiable at that point for any α. Moreover, we have

$$D^\alpha h(t) = \beta h(t) + \alpha Dh(t).$$

Corollary 2.2. *([21]) An α - differentiable function f defined in (a,b) is differentiable as well.*

Theorem 2.3. ([16],[21]) *The operators D^α and I_a^α possess the following properties: Let $\alpha, \alpha_1, \alpha_2 \in (0,1]$ such that $\alpha + \beta = 1$, $\alpha_i + \beta_i = 1$ for $i = 1, 2$.*

(1) *Let f be differentiable at a point t for some α. Then it is continuous there.*

(2) *Suppose f and g are α-differentiable. Then*

$$D^\alpha(f\circ g)(t) = \beta(f\circ g)(t) + \alpha D(f\circ g)(t)$$
$$= \beta(f\circ g)(t) + \alpha f'(g(t))g'(t).$$

(3) *Let f be continuous on [a,b]. Then $I_a^\alpha f$ is α- differentiable in (a,b), and we have*

$$D^\alpha(I_a^\alpha f(t)) = f(t), and$$
$$I_a^\alpha(D^\alpha f(t) = f(t) - e^{\frac{\beta}{\alpha}(a-t)}f(a).$$

(4) $D^\alpha\left(\dfrac{f}{g}\right) = \dfrac{gD^\alpha(f) - \alpha f}{g^2}.$
(5) *Linearity :* $D^\alpha(af+bg) = aD^\alpha f + bD^\alpha g.$
(6) *Commutativity :* $D^{\alpha_1}\cdot D^{\alpha_2} = D^{\alpha_2}\cdot D^{\alpha_1}.$
(7) *For a constant c,* $D^\alpha(c) = \beta c.$
(8) $D^\alpha(fg) = (D^\alpha f)g + \alpha f Dg.$
(9) *Linearity :* $I_a^\alpha(bf+cg) = bI_a^\alpha f + cI_a^\alpha g.$
(10) *Commutativity :* $I_a^{\alpha_1}I_a^{\alpha_2} = I_a^{\alpha_2}I_a^{\alpha_1}.$

3. Existence of solutions

Let $\Omega = \{y : (-\infty, b] \to \mathbb{R} : y|_{[0,b]}$ is continuous $\}$

Definition 3.1. A function $y \in \Omega$ is said to be a solution of (4)–(5) if y satisfies the equation $D^\alpha y(t) = f(t, y_t)$ on J, and the condition $y(t) = \phi(t)$ on $(-\infty, 0]$.

Lemma 3.1. [8] *Let $0 < \alpha < 1$ and let $h : (0, b] \to \mathbb{R}$ be continuous and $\lim_{t\to 0^+} h(t) = h(0^+) \in \mathbb{R}$. Then y is a solution of the integral equation*

$$e^{\frac{-\beta}{\alpha}t}y_0 + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}h(s)ds,$$

if and only if y is a solution of the initial value problem for the fractional differential equation

$$D^\alpha y(t) = h(t), \quad t \in (0, b]$$
$$y(0) = 0.$$

Our existence result for the IVP (4)-(5) is based on the Banach contraction principle.

Theorem 3.2. *Let $f : J \times B \to \mathbb{R}$. Assume*
(H) There exists $l > 0$ such that $|f(t,u) - f(t,v)| \le l\|u - v\|_B$, for $t \in J$ and every $u, v \in B$

If $\frac{k_b l}{\beta} < 1$, where $k_b = sup\{|k(t)| : t \in [0, b]\}$, then there exists a unique solution for the IVP(4)-(5) on the interval $(-\infty, b]$.

Proof. Transform the problem (4)-(5) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$Ny(t) = \begin{cases} \phi(t), & t \in [-\infty, 0], \\ \dfrac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\displaystyle\int_0^t e^{\frac{\beta}{\alpha}s}f(s, y_s)ds, & t \in (0, b]. \end{cases} \tag{3.1}$$

Let $x(\cdot) : (-\infty, b] \to \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} 0, & if\ t \in [0, b], \\ \phi(t), & if\ t \in [-\infty, 0]. \end{cases} \tag{3.2}$$

Then $x_0 = \phi$, for each $z \in \mathcal{C}([0,b],\mathbb{R})$ with $z(0) = 0$. We denote by $\overline{z}$ the function defined by

$$\overline{z} = \begin{cases} z(t), & if\ t \in [0,b], \\ 0, & if\ t \in (-\infty, 0]. \end{cases} \tag{3.3}$$

If $y(\cdot)$ satisfies the integral equation

$$y(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} f(s, y_s) ds,$$

we can decompose $y(\cdot)$ as $y(t) = \overline{z} + x(t), 0 \le t \le b$, which implies $y_t = \overline{z}_t + x_t$ for every $0 \le t \le b$ and the function $z(\cdot)$ satisfies

$$z(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} f(s, \overline{z}_s + x_s) ds.$$

Set

$$C_0 = \{z \in C([0,b],\mathbb{R}) : z_0\} = 0$$

and let $\|\cdot\|_b$ be the seminorm in C_0 defined by

$$\|z\|_b = \|z_0\|_B + sup\{|z(t)| : 0 \le t \le b,\} = sup\{z(t) : 0 \le t \le b\}\ z \in C_0.$$

C_0 is a Banach space with norm $\|\cdot\|_b$. Let the operator $P : C_0 \to C_0$ be defined by

$$(pz)(t) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} f(s, \overline{z}_s + x_s) ds, \quad t \in [0,b].$$

That the operator N has a fixed point is equivalent to P has a fixed point, and so we turn to proving that$P$ has a fixed point. We shall show that $P : C_0 \to c_0$ is a contraction map. Indeed, consider $z, z^* \in C_0$. Then we have for each $t \in [0,b]$

$$\begin{aligned} |P(z)(t) - P(z^*)(t)| &\le \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} |f(s, \overline{z}_s + x_s) - f(s, \overline{z}^*{}_s + x_s)| ds \\ &\le \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} l ||\overline{z}_s - \overline{z}^*_s||_B ds \\ &\le \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} l K_b \sup_{s \in [0,t]} \|z(s) - z^*(s)\| ds \\ &\le \frac{K_b}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} l\ ds \|z - z^*\|_b. \end{aligned}$$

Therefore,

$$\|p(z) - p(z^*)\| \le \frac{K_b l \|z - z^*\|_b}{\beta},$$

hence P is a contraction. Therefore, P has a unique fixed point by Banach's fixed point principle.

Now we give an existence result based on the nonlinear alternative of Leray–Schauder type. For this, we state the following standard Gronwall's inequality:

Lemma 3.3. [6] *If*

$$x(t) \le h(t) + \int_{t_0}^t k(s) x(s) ds, \quad t \in [t_0, T),$$

where all functions involved are continuous on $[t_0, T), T \leq +\infty$, *and* $k(t) \geq 0$, *then* $x(t)$ *satisfies*

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s)\ exp\left[\int_{s}^{t} k(u)d(u)\right] ds,\ t \in [t_0, T).$$

Theorem 3.4. *Assume that the following hypotheses hold:*
(H1) f is a continuous function
(H2) there exist $p, q \in C(J, R^{+})$ *such that*

$$|f(t,u)| \leq p(t) + q(t)\|u\|_B$$

for $t \in J$ *and each*$u \in B$, *and* $\|I^{\alpha}p\|_{\infty} < \infty$.
Then the IVP (1)–(2) has at least one solution on$(-\infty, b]$

□

Proof. Let$P : C_0 \to C_0$ be defined as in the proof of Theorem 3.2. We shall show that the operator P is continuous and completely continuous.

Step 1. P is continuous
Let $\{z_n\}$ be a sequence such that $z_n \to z \in C_0$. Then

$$|(pz_n)(t) - (pz)(t)| \leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} |f(s, \overline{z}_{n_s} + x_s) - f(s, \overline{z}_s + x_s)| ds$$

Since f is a continuous function, we have

$$\|(pz_n) - (pz)\|_b \leq \frac{\|f(., \overline{z}_{n(.)} + x_{(.)}) - f(., \overline{z}_{(.)} + x_{(.)})\|_{\infty}}{\beta} \to 0$$

as $n \to \infty$.
Step 2. P maps bounded sets into bounded sets in C_0.
Indeed, it is enough to show that for any $\eta > 0$, there exists a positive constant l such that for each $z \in B_{\eta} = \{z \in C_0 : \|z\|_b \leq \eta\}$ one has $\|P(z)\|_{\infty} \leq l$. Let $z \in B_{\eta}$. Since f is continuous function, we have for each $t \in [0, b]$

$$\begin{aligned} |(P(z)(t)| &\leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^b e^{\frac{\beta}{\alpha}s} f(s, \overline{z}_s + x_s) ds \\ &\leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^b e^{\frac{\beta}{\alpha}s} [p(s) + q(s)||\overline{z}_s + x_s||_B] ds \\ &\leq \frac{\|p\|_{\infty}}{\beta} + \frac{\|q\|_{\infty}}{\beta}\eta_* =: l, \end{aligned}$$

where

$$\|\overline{z}_s + x_s\|_B \leq \|\overline{z}_s\|_B + \|x_s\|_B \leq K_b\eta + M_b\|\phi\|_B := \eta^*,$$

and
$M_b = sup\{|M(t)| : t \in [0, b]\}$.
Hence$\|P(z)\|_{\infty} \leq l$.
Step 3. P maps bounded sets into equicontinuous sets of C_0.

Let $t_1, t_2 \in [0,b], t_1 < t_2$, and let B_η be a bounded set of C_0 as in step 2. Let $z \in B_\eta$.Then for each $t \in [0,b]$, we have

$$\begin{aligned}
&|(Pz)(t_2) - (Pz)(t_1)| \\
&= \frac{1}{\alpha}| \int_0^{t_1} e^{\frac{\beta}{\alpha}s}(e^{\frac{-\beta}{\alpha}t_2} - e^{\frac{-\beta}{\alpha}t_1}) f(s, \overline{z}_s + x_s) ds \\
&\quad + \int_{t_1}^{t_2} e^{\frac{\beta}{\alpha}s} e^{\frac{-\beta}{\alpha}t_2} f(s, \overline{z}_s + x_s) ds| \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\alpha} \int_0^{t_1} e^{\frac{\beta}{\alpha}s}(e^{\frac{-\beta}{\alpha}t_1} - e^{\frac{-\beta}{\alpha}t_2}) ds \\
&\quad + \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\alpha} \int_{t_1}^{t_2} e^{\frac{\beta}{\alpha}s} e^{\frac{-\beta}{\alpha}t_2} ds \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\alpha} (e^{\frac{-\beta}{\alpha}t_1} - e^{\frac{-\beta}{\alpha}t_2}) \frac{\alpha}{\beta}(e^{\frac{-\beta}{\alpha}t_1} - 1) \\
&\quad + \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\alpha} (e^{\frac{-\beta}{\alpha}t_2}) \frac{\alpha}{\beta}(e^{\frac{\beta}{\alpha}t_2} - e^{\frac{\beta}{\alpha}t_1}) \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\beta} (1 - e^{\frac{\beta}{\alpha}(t_1 - t_2)} - e^{\frac{-\beta}{\alpha}t_1} + e^{\frac{-\beta}{\alpha}t_2}) \\
&\quad + \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\beta} (1 - e^{\frac{\beta}{\alpha}(t_1 - t_2)}) \\
&\leq \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\beta} (2 - 2e^{\frac{\beta}{\alpha}(t_1 - t_2)} - e^{\frac{-\beta}{\alpha}t_1} + e^{\frac{-\beta}{\alpha}t_2}) \\
&\leq 2\frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\beta} (1 - e^{\frac{\beta}{\alpha}(t_1 - t_2)}).
\end{aligned}$$

As $t_1 \to t_2, |(pz)(t_2) - (pz)(t_1)| \to 0$. As a consequence of steps 1-3, together with Arzela-Ascoli theorem, we can conclude that $P : C_0 \to C_0$ is continuous and completely continuous.
Step 4 (A priori bounds). We now show that there exists an open set $U \subseteq C_0$ with $z \neq \lambda p(z)$ for $\lambda \in (0,1)$ and $z \in \partial U$.
Let $z \in C_0$ and $z = \lambda P(z)$ for some $0 < \lambda < 1$. Then for each $t \in [0,b]$ we have

$$z(t) = \lambda \left[\frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} f(s, \overline{z}_s + x_s) ds \right].$$

This implies by (H2)

$$|z(t)| \leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} q(s) \|\overline{z}_s + x_s\|_B \, ds + \frac{\|p\|_\infty}{\beta}, \quad t \in [0,b].$$

But

$$\begin{aligned}
\|\overline{z}_s + x_s\|_B \leq & |\overline{z}\|_B \leq K(t) \sup\{|z(s)| : 0 \leq s \leq t\} \\
& + M(t)\|z_0\|_B + K(t) \sup\{|x(s)| : 0 \leq s \leq t\} + M(t)\|x_0\|_B \\
\leq & K_b \sup\{|z(s)| : 0 \leq s \leq t\} + M_b \|\phi\|_B.
\end{aligned}$$

If we denote by $w(t)$ the right side of (6), then we have $\|\overline{z}_s + x_s\|_B \leq w(t)$, and therefore

$$z(t) \leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} q(s) w(s) ds + \frac{\|p\|_\infty}{\beta}, t \in [0,b].$$

Using the above inequality and the definition of w we have that

$$w(t) \leq M_b\|\phi\|_B + \frac{K_b\|p\|_\infty}{\beta} + \frac{K_b\|q\|_\infty}{\alpha}\int_0^t e^{\frac{-\beta}{\alpha}t}e^{\frac{\beta}{\alpha}s}w(s)ds, \quad t \in [0,b].$$

Then from Lemma 3.4, we have

$$\begin{aligned}
|w(t)| &\leq M_b\|\phi\|_B + \frac{K_b\|p\|_\infty}{\beta} + \frac{K_b\|q\|_\infty}{\alpha}\int_0^t e^{\frac{-\beta}{\alpha}t}e^{\frac{\beta}{\alpha}s}R\ exp\left[\int_s^t e^{\frac{-\beta}{\alpha}t}e^{\frac{\beta}{\alpha}u}d(u)\right]ds\\
&\leq M_b\|\phi\|_B + \frac{K_b\|p\|_\infty}{\beta} + \frac{K_b\|q\|_\infty}{\alpha}\int_0^t e^{\frac{-\beta}{\alpha}t}e^{\frac{\beta}{\alpha}s}R\ exp\left[\frac{\alpha}{\beta}e^{\frac{-\beta}{\alpha}t}(e^{\frac{\beta}{\alpha}t} - e^{\frac{\beta}{\alpha}s})\right]ds\\
&\leq M_b\|\phi\|_B + \frac{K_b\|p\|_\infty}{\beta} + \frac{K_b\|q\|_\infty}{\alpha}\int_0^t e^{\frac{-\beta}{\alpha}t}e^{\frac{\beta}{\alpha}s}R\ exp\left[\frac{\alpha}{\beta}\right]ds\\
&\leq M_b\|\phi\|_B + \frac{K_b\|p\|_\infty}{\beta} + exp\left[\frac{\alpha}{\beta}\right]\frac{K_b\|q\|_\infty}{\alpha}\int_0^t e^{\frac{-\beta}{\alpha}t}e^{\frac{\beta}{\alpha}s}R\ ds\\
&\leq M_b\|\phi\|_B + \frac{K_b\|p\|_\infty}{\beta} + K(\alpha)\frac{K_b\|q\|_\infty}{\alpha}\int_0^t e^{\frac{-\beta}{\alpha}t}e^{\frac{\beta}{\alpha}s}R\ ds,
\end{aligned}$$

where

$$R = M_b\|\phi\|_B + \frac{K_b\|p\|_\infty}{\beta}\ , \quad K(\alpha) = exp\left[\frac{\alpha}{\beta}\right].$$

Assume that

$$\frac{K_b\|q\|_\infty}{\alpha} \geq 1,$$

hence

$$\|w\|_\infty \leq R + \frac{RK(\alpha)K_b}{\beta} := \tilde{M}.$$

Then

$$\|z\|_\infty \leq \tilde{M}\|I^\alpha q\|_\infty + \frac{\|p\|_\infty}{\beta} := M^*.$$

Set

$$U = \{z \in C_0 : \|z\|_b < M^* + 1\}.$$

$P : \overline{U} \to C_0$ is continuous and completely continuous. From the choice of U,there is no $z \in \partial U$ such that $z = \lambda p(z)$, for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [8], we deduce that p has a fixed point z in U. □

4. NFDEs of fractional order

In this section we give existence result for the IVP (3)-(4)

Definition 4.1. A function $y \in \Omega$ is said to be a solution of (3)-(4) if y satisfies the equation $D^\alpha[y(t) - g(t,y_t)] = f(t,y_t)$ on J, and $y(t) = \phi(t)$ on $(-\infty, 0]$.

Our first existence result for the IVP (3)–(4) is also based on the Banach contraction principle.

Theorem 4.1. *Assume that (H) holds and moreover*
(A) there exists a nonnegative constant c_1 *such that*
$|g(t,u) - g(t,v)| \leq c_1\|u - v\|_B$, *for every* $u, v \in B$.
If $K_b[c_1 + \frac{l}{\beta}] < 1$ *then there exists a unique solution for the IVP (3)-(4) on the interval* $(-\infty, b]$.

Proof. Consider the operator $N_1 : \Omega \to \Omega$ defined by

$$N_1y(t) = \begin{cases} \phi(t), & t \in [-\infty, 0], \\ g(t, y_t) + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} f(s, y_s) ds, & t \in [0, b]. \end{cases} \tag{4.1}$$

In analogy to Theorem 3.5, we consider the operator $P_1 : C_0 \to C_0$ defined by

$$(P_1 z)(t) = \begin{cases} 0, & t \leq 0 \\ g(t, \overline{z}_t + x_t) + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} f(s, \overline{z}_s + x_s) ds, & t \in (0, b]. \end{cases} \tag{4.2}$$

We shall now show that the operator P_1 is a contraction.
Let $z, z_* \in \Omega$. Then following the steps of Theorem 3.3, we have

$$\begin{aligned} |P_1(z)(t) - P_1(z_*)(t)| \leq & |g(t, \overline{z}_t + x_t) - g(t, \overline{z_*}_t + x_t)| \\ & + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t |e^{\frac{\beta}{\alpha} s} f(s, \overline{z}_s + x_s) - f(s, \overline{z_*}_s + x_s)| ds \\ \leq & c_1 \|\overline{z}_t - \overline{z}_*\|_B + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s}\ l \|\overline{z}_s - \overline{z}_{*s}\|_B \ ds \\ \leq & c_1 k_b \sup\{|z(s) - z_*(s)| : s \in [0, t]\} \\ & + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s}\ l\ k_b \sup\{|z(s) - z_*(s)| : s \in [0, t]\} ds \end{aligned}$$

Consequently,

$$|P_1(z)(t) - P_1(z_*)(t)|_b \leq k_b [c_1 + \frac{l}{\beta}] \|z - z_*\|_b,$$

which implies that P_1 is a contraction. Hence, P_1 has a unique fixed point by Banach's contraction principle.

□

Our second existence result for the IVP (3)-(4)is based on the nonlinear alternative of Leray Schauder.

Theorem 4.2. *Assume (H1)-(H2) and the following condition:*
(H3) The function g is consinuous and completely continuous, and for any bounded set $B \in \Omega$, the set $\{t \to g(t, y_t) : y \in B\}$ is equicontinuous in $C([0, b], \mathbb{R})$ and there exist constants $0 \leq k_b d_1 < 1, d_2 \geq 0$ such that $|g(t, u)| \leq d_1 \|u\|_B + d_2,\ t \in [0, b], u \in B$.
Then the IVP (3)-(4) has at least one solution on $(-\infty, b]$.

Proof. Let $P_1 : C_0 \to C_0$ be defined as in Theorem 4.2. We shall show that the operator P_1 is continuous and completely continuous.
Using (H3) it suffices to show that the operator $P_2 : C_0 \to C_0$ defined by

$$P_2(z)(t) = g(t, \overline{z}_s + x_s) + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} f(s, \overline{z}_s + x_s) ds. \ \ t \in [0, b]$$

is continuous and completely continuous. This was proved in Theorem 3.5.
We now show that there exists an open set $U \subseteq C_0$ with $z \neq \lambda P_1 z$ for $\lambda \in (0, 1)$ and $z \in \partial U$.
Let $z \in C_0$ and $z = \lambda N_1(z)$ for some $0 < \lambda < 1$. Then

$$(z)(t) = \lambda \left[g(t, \overline{z}_t + x_t) + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} f(s, \overline{z}_s + x_s) ds \right]. \ \ t \in [0, b]$$

and

$$|z(t)| \le d_1\|\overline{z}_t + x_t\|_B + d_2 + \frac{\|p\|_\infty}{\beta} + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}q(s)\|\overline{z}_s + x_s\|_B ds, \quad t \in [0,b]$$

Thus,

$$w(t) \le \frac{1}{1-k_bd_1}\left[2k_bd_2 + k_b\frac{\|p\|_\infty}{\beta} + k_b\frac{\|q^*\|_\infty}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}w(s)ds\right], \quad t \in [0,b]$$

and consequently,

$$\|w\|_\infty \le R_1 + \frac{R_1K_b\|q^*\|_\infty}{(1-k_bd_1)\beta} := L,$$

where

$$\|q^*\|_\infty = \frac{\|q\|_\infty}{1-k_bd_1}, and\ R_1 = \frac{1}{1-k_bd_1}[2k_bd_2 + k_b\frac{\|p\|_\infty}{\beta}].$$

Then

$$\|z\|_\infty \le d_1\|\phi\|_B + 2d_2 + Ld_1 + \frac{\|p\|_\infty}{\beta} + L\|I^\alpha q\|_\infty := L^*$$

Set

$$U_1 = \{y \in C_0 : \|y\|_b < L^* + 1\}$$

From the choice of U, there is no $y \in \partial U_1$ such that $y = \lambda p_2(y) for \lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that p_2 has a fixed point z in U_1. Then N_1 has a fixed point which is a solution of the IVP (3)-(4). □

5. An Example

We provide an illustrative example as follows.

$$D^\alpha y(t) = \frac{C\left(\frac{1}{x^2+1}\right)^{-\omega}\frac{1}{x^2+1}\|y_t\|}{\left(\frac{1}{x^2+1} + (x^2+1)\right)(1+\|y_t\|)}, \quad t \in J := [0,b],\ \alpha \in (0,1) \tag{5.1}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0]. \tag{5.2}$$

Let $\omega > 0$ and $B_\omega = \{y \in C((-\infty,0],\mathbb{R}) : \lim_{\theta\to-\infty}\left(\frac{1}{\theta^2+1}\right)^\omega y(\theta)$ exists in $\mathbb{R}\}$.
The norm of B_ω is given by $\|y\|_\omega = \sup_{-\infty<\theta\le 0}\left(\frac{1}{\theta^2+1}\right)^\omega |y(\theta)|$.
Let $y : (-\infty, b] \to \mathbb{R}$ be such that $y_0 \in B_\omega$. Then

$$\begin{aligned}\lim_{\theta\to-\infty}\left(\frac{1}{\theta^2+1}\right)^\omega y_t(\theta) &= \lim_{\theta\to-\infty}\left(\frac{1}{\theta^2+1}\right)^\omega y(t+\theta) = \lim_{\theta\to-\infty}\left(\frac{1}{(\theta-t)^2+1}\right)^\omega y(\theta)\\ &\le \lim_{\theta\to-\infty}\left(\frac{1}{(\theta-t)^2+1}\right)^\omega y_0(\theta) = \lim_{\theta\to-\infty}\left(\frac{1}{\theta^2+1}\right)^\omega y_0(\theta) < \infty.\end{aligned}$$

Hence, $y_t \in B_\omega$. Finally we prove that
$\|y_t\|_\omega \le K(t)\sup\{|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_\omega$,
where $K = M = 1$ and $H = 1$. We have $|y_t(\theta)| = |y(t+\theta)|$.
If $\theta + t \le 0$, we get
$|y_t(\theta)| \le \sup\{|y(s)| : -\infty < s \le 0\}$
For $t + \theta \ge 0$, we have
$|y_t(\theta)| \le \sup\{|y(s)| : 0 < s \le t\}$.
Thus for all $t + \theta \in [0,b]$, we get
$|y_t(\theta)| \le \sup\{|y(s)| : -\infty < s \le 0\} + \sup\{|y(s)| : 0 \le s \le t\}$.

It is clear that $(B_\omega, \|\cdot\|_\omega)$ is a Banach space. We can conclude that B_ω is a phase space. Set

$$f(t,x) = \frac{\left(\frac{1}{x^2+1}\right)^{-\omega} \frac{1}{x^2+1} x}{\left(\frac{1}{x^2+1} + (x^2+1)\right)(1+x)}, \qquad (t,x) \in [0,b] \times B_\omega.$$

Let $x, y \in B_\omega$. Then we have

$$|f(t,x) - f(t,y)| = \frac{\left(\frac{1}{x^2+1}\right)^{-\omega} \frac{1}{x^2+1}}{C\left(\frac{1}{x^2+1} + (x^2+1)\right)} \left| \frac{x}{1+x} - \frac{y}{1+y} \right| = \frac{\left(\frac{1}{x^2+1}\right)^{1-\omega} |x-y|}{C\left(\frac{1}{x^2+1} + (x^2+1)\right)(1+x)(1+y)}$$

$$\leq \frac{\left(\frac{1}{x^2+1}\right) |x-y|_{B_\omega}}{C\left(\frac{1}{x^2+1} + (x^2+1)\right)} \leq \frac{1}{C} \|x-y\|_{B_\omega}$$

Hence the condition H holds. Assume that $\frac{1}{\beta C} < 1$. Since $K = 1$, then $\frac{1}{\beta C} < 1$..
Then by theorem 3.3 the problem 11-12 has a unique solution on $(-\infty, b]$.

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