SOME POLYCUBES HAVE NO EDGE ZIPPER UNFOLDING*

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Abstract

It is unknown whether every polycube (polyhedron constructed by gluing cubes face-to-face) has an edge unfolding, that is, cuts along edges of the cubes that unfolds the polycube to a single nonoverlapping polygon in the plane. Here we construct polycubes that have no ***edge zipper unfolding*** where the cut edges are further restricted to form a path.

1 Introduction

A ***polycube*** P is an object constructed by gluing cubes whole-face to whole-face, such that its surface is a manifold. Thus the neighborhood of every surface point is a disk; so there are no edge-edge nor vertex-vertex nonmanifold surface touchings. Here we only consider polycubes of genus zero. The ***edges*** of a polycube are all the cube edges on the surface, even when those edges are shared between two coplanar faces. Similarly, the ***vertices*** of a polycube are all the cube vertices on the surface, even when those vertices are ***flat***, incident to 360° total face angle. Such polycube flat vertices have degree 4. It will be useful to distinguish these flat vertices from ***corner vertices***, nonflat vertices with total incident angle $\neq 360°$ (degree 3, 5, or 6). For a polycube P, let its 1-***skeleton***

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graph G_P include every vertex and edge of P, with vertices marked as either corner or flat.

It is an open problem to determine whether every polycube has an ***edge unfolding*** (also called a ***grid unfolding***) — a tree in the 1-skeleton that spans all corner vertices (but need not include flat vertices) which, when cut, unfolds the surface to a ***net***, a planar nonoverlapping polygon [O'R19]. By ***nonoverlapping*** we mean that no two points, each interior to a face, are mapped to the same point in the plane. This definition allows two boundary edges to coincide in the net, so the polygon may be "weakly simple." The intent is that we want to be able to cut out the net and refold to P.

It would be remarkable if every polycube could be edge unfolded, but no counterexample is known. There has been considerable exploration of orthogonal polyhedra, a more general type of object, for which there are examples that cannot be edge-unfolded [BDD$^+$98]. (See [DF18] for citations to earlier work.) But polycubes have more edges in their 1-skeleton graphs for the cut tree to follow than do orthogonal polyhedra, so it is conceivably easier to edge-unfold polycubes.

A restriction of edge unfolding studied in [She75, DDL$^+$10, O'R10, DDU13] is ***edge zipper unfolding*** (also called ***Hamiltonian unfolding***). A ***zipper*** unfolding has a cut tree that is a path (so that the surface could be "unzipped" by a single zipper). It is apparently unknown whether even the highly restricted edge zipper unfolding could unfold every polycube to a net. The result of this note is to settle this question in the negative: polycubes are constructed none of which have an edge zipper unfolding. Two polycubes in particular, shown in Fig. 1, have no such unfolding. Other polycubes with the same property are built upon these two.

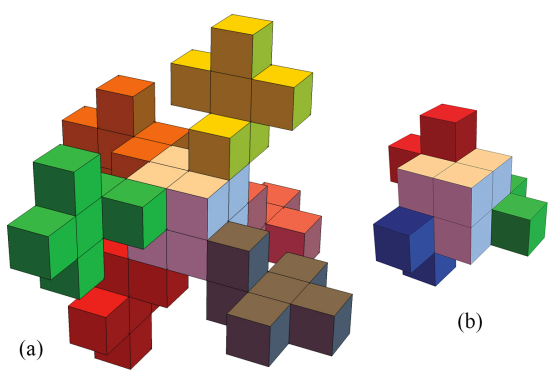

Figure 1: Two polycubes that have no edge zipper unfolding.

2 Hamiltonian Paths

Shephard [She75] introduced Hamiltonian unfoldings of convex polyhedra, what we refer to here as edge zipper unfolding, following the terminology of [DDL+10] Any edge zipper unfolding must cut along a Hamiltonian path of the vertices. It is easy to see that not every convex polyhedron has an edge zipper unfolding, simply because the rhombic dodecahedron has no Hamiltonian path. This counterexample avoids confronting the difficult nonoverlapping condition.

We follow a similar strategy here, constructing a polycube with no Hamiltonian path. But there is a difference in that a polycube edge zipper unfolding need not include flat vertices, and so need not be a Hamiltonian path in G_P. Thus identifying a polycube P that has no Hamiltonian path does not immediately establish that P has no edge zipper unfolding, if P has flat vertices.

So one approach is to construct a polycube P that has no flat vertices—every vertex is a corner vertex. Then, if P has no Hamiltonian path, then it has no edge zipper unfolding. A natural candidate is the polycube object P_6 shown in Fig. 2. However, the 1-skeleton of P_6 does admit Hamiltonian paths, and indeed we found a path that unfolds P_6 to a net.

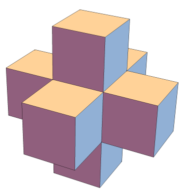

Figure 2: All of P_6's vertices are corner vertices.

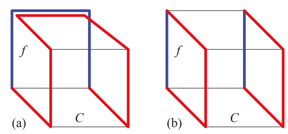

Figure 3: (a) f contains 3 edges of the cycle (blue); (b) f contains 2 edges of the cycle. The cycles are extended to C by replacing the blue with the the red paths.

Let $\overline{G}_P$ be the dual graph of P: each cube is a node, and two nodes are connected if they are glued face-to-face. A ***polycube tree*** is a polycube whose dual graph is a tree. P_6 is a polycube tree. That it has a Hamiltonian path is an instance of a more general claim:

Lemma 1 *The graph G_P for any polycube tree P has a Hamiltonian cycle.*

Proof: It is easy to see by induction that every polycube tree can be built by gluing cubes each of which touches just one face at the time of gluing: never is there a need to glue a cube to more than one face of the previously built object.

A single cube has a Hamiltonian cycle. Now assume that every polycube tree of $\leq n$ cubes has a Hamiltonian cycle. For a tree P of $n+1$ cubes, remove a $\overline{G}_P$ leaf-node cube C, and apply the induction hypothesis. The exposed square face f to which C glues to make P includes either 2 or 3 edges of the Hamiltonian cycle (4 would close the cycle; 1 or 0 would imply the cycle misses some vertices of f). It is then easy to extend the Hamiltonian cycle to include C, as shown in Fig. 3. □

So to prove that a polycube tree has no edge zipper unfolding would require an argument that confronted nonoverlap. This leads to an open question:

Question 1 *Does every polycube tree have an edge zipper unfolding?*

3 Bipartite G_P

To guarantee the nonexistence of Hamiltonian paths, we can exploit the bipartiteness of G_P, using Lemma 3 below.

Lemma 2 *A polycube graph G_P is 2-colorable, and therefore bipartite.*

Proof: Label each lattice point p of $\mathbb{Z}^3$ with the $\{0,1\}$-parity of the sum of the Cartesian coordinates of p. A polycube P's vertices are all lattice points of $\mathbb{Z}^3$. This provides a 2-coloring of G_P; 2-colorable graphs are bipartite. □

The ***parity imbalance*** in a 2-colored (bipartite) graph is the absolute value of the difference in the number of nodes of each color.

Lemma 3 *A bipartite graph G with a parity imbalance >1 has no Hamiltonian path.*[1]

Proof: The nodes in a Hamiltonian path alternate colors $010101\ldots$. Because by definition a Hamiltonian path includes every node, the parity imbalance in a bipartite graph with a Hamiltonian path is either 0 (if of even length) or 1 (if of odd length). □

So if we can construct a polycube P that (a) has no flat vertices, and (b) has parity imbalance >1, then we will have established that P has no Hamiltonian path, and therefore no edge zipper unfolding. We now show that the polycube P_{44}, illustrated in Fig. 4, meets these conditions.

Lemma 4 *The polycube P_{44}'s graph $G_{P_{44}}$ has parity imbalance of* 2.

Proof: Consider first the $2\times2\times2$ cube that is the core of P_{44}; call it P_{222}. The front face F has an extra 0; see Fig. 5. It is clear that the 8 corners of P_{222} are all colored 0. The midpoint vertices of the 12 edges of P_{222} are colored 1. Finally

[1] Stated at http://mathworld.wolfram.com/ HamiltonianPath.html.

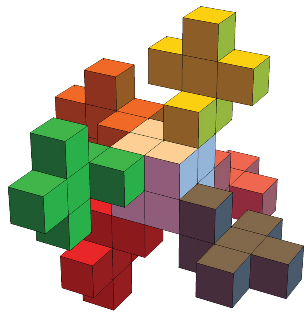

Figure 4: The polycube P_{44}, consisting of 44 cubes, has no Hamiltonian path.

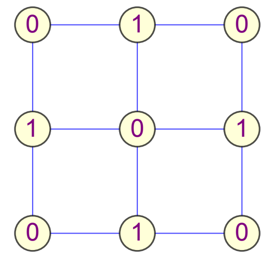

Figure 5: 2-coloring of one face of P_{222}.

the 6 face midpoints are colored 0. So 14 vertices are colored 0 and 12 colored 1.

Next observe that attaching a cube C to exactly one face of any polycube does not change the parity: the receiving face f has colors 0101, and the opposite face of C has colors 1010.

Now, P_{44} can be constructed by attaching six copies of a 6-cube "cross," call it P_+, which in isolation is a polycube tree and so can be built by attaching cubes each to exactly one face. And each P_+ attaches to one corner cube of P_{222}. Therefore P_{44} retains P_{222}'s imbalance of 2. □

The point of the P_+ attachments is to remove the flat vertices of P_{222}. Note that when attached to P_{222}, each P_+ has only corner vertices.

Theorem 1 *Polycube P_{44} has no edge zipper unfolding.*

Proof: Although it takes some scrutiny of Fig. 4 to verify, P_{44} has no (degree-4) flat vertices. Thus an edge zipper unfolding must pass through every vertex, and so be a Hamiltonian path. Lemma 4 says that $G_{P_{44}}$ has imbalance 2, and Lemma 3 says it therefore cannot have a Hamiltonian path. □

4 Construction of P_{14}

It turns out that the smaller polycube P_{14} shown in Fig. 6 also has no edge zipper unfolding, even though it has flat vertices. To establish this, we still need an imbalance > 1, which easily follows just as in Lemma 4:

Lemma 5 *The polycube P_{14}'s graph $G_{P_{14}}$ has parity imbalance of* 2.

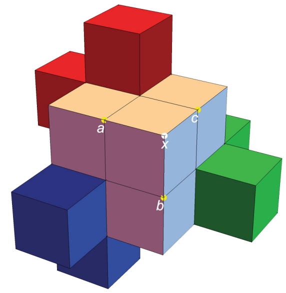

Figure 6: P_{14}: P_{222} with six 1-cube attachments.

But notice that P_{14} has three flat vertices: a, b, and c.

Theorem 2 *Polycube P_{14} has no edge zipper unfolding.*

Proof: An edge zipper unfolding need not pass through the three flat vertices, a, b, and c, but it could pass through one, two, or all three. We show that in all cases, an appropriately modified subgraph of $G_{P_{14}}$ has no Hamiltonian path. Let ρ be a hypothetical edge zipper unfolding cut path. We consider four exhaustive possibilities, and show that each leads to a contradiction.

(0) ρ **includes** a,b,c**.** So ρ is a Hamiltonian path in $G_{P_{14}}$. But Lemma 5 says that $G_{P_{14}}$ has imbalance 2, and Lemma 3 says that no such graph has a Hamiltonian path.

(1) ρ **excludes one flat vertex** a **and includes** b,c**.** (Because of the symmetry of P_{14}, it is no loss of generality to assume that it is a that is excluded.) If ρ excludes a, then it does not travel over any of the four edges incident to a. Thus we can delete a from $G_{P_{14}}$; say that $G_{-a} = G_{P_{14}} \setminus a$. This graph is shown in Fig. 7. Following the coloring in Fig. 5, all corners of P_{222} are colored 0, so each of the edge midpoints a,b,c is colored 1. The parity imbalance of P_{14} is 2 extra 0's. Deleting a maintains bipartiteness and increases the parity imbalance of G_{-a} to 3. Therefore by Lemma 3, G_{-a} has no Hamiltonian path, and such a ρ cannot exist.

(2) ρ **includes just one flat vertex** c**, and excludes** a,b**.** (Again symmetry ensures there is no loss of generality in assuming the one included flat vertex is c.) ρ must include corner x, which is only accessible in $G_{P_{14}}$ through the three flat vertices. If ρ excludes a,b, then it must include the edge cx. Let $G_{-ab} = G_{P_{14}} \setminus \{a,b\}$. In G_{-ab}, x has degree 1, so ρ terminates there. It must be that ρ is a Hamiltonian path in G_{-ab}, but the deletion of a,b increases the parity imbalance to 4, and so again such a Hamiltonian path cannot exist.

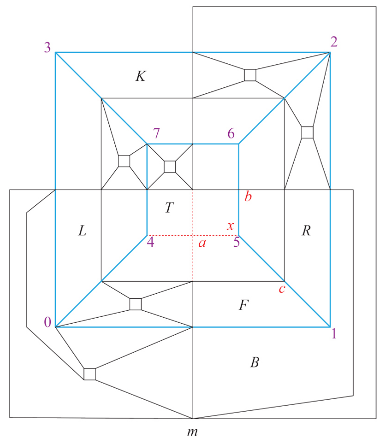

Figure 7: Schlegel diagram of G_{-a}. We follow [DF18] in labeling the faces of a cube as F, K, R, L, T, B for Front, bacK, Right, Left, Top, Bottom respectively. The corners of P_{222} are labeled $0, 1, 2, 3$ around the bottom face B, and $4, 5, 6, 7$ around the top face T. m is the vertex in the middle of B. The edges deleted by removing vertex a are shown dashed.

(3) ρ **excludes** a, b, c**.** Because corner x is only accessible through one of these flat vertices, ρ never reaches x and so cannot be an edge zipper unfolding.

Thus the assumption that there is an edge zipper unfolding cut path ρ for P_{14} reaches a contradiction in all four cases. Therefore, there is no edge zipper unfolding cut path for P_{14}.[2] □

5 Edge Unfoldings of P_{14} and P_{44}

Now that it is known that P_{14} and P_{44} each have no edge zipper unfolding, it is natural to wonder whether either settles the edge-unfolding open problem: can they be edge unfolded? Indeed both can: see Figures 8 and 9. The colors in these layouts are those used by Origami Simulator [GDG18]. Fig. 10 shows a partial folding of P_{44}, and animations are at http://cs.smith.edu/~jorourke/Unf/NoEdgeUnzip.html.

[2] Just to verify this conclusion, we constructed these graphs in Mathematica and `FindHamiltonianPath[]` returned {} for each.

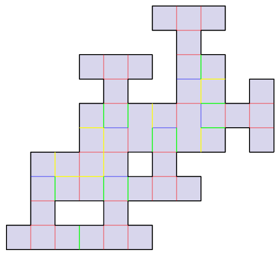

Figure 8: Edge unfolding of P_{14}.

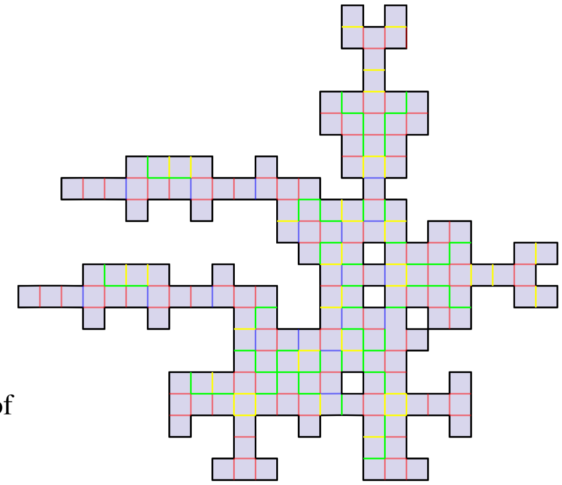

Figure 9: Edge unfolding of P_{44}.

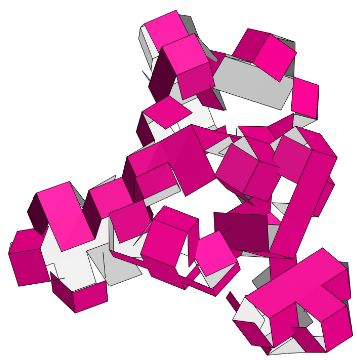

Figure 10: Partial folding of the layout in Fig. 9. Compare with Fig. 4.

6 Many Polycubes with No Edge Zipper Unfolding

As pointed out by Ryuhei Uehara,[3] P_{44} can be extended to an infinite number of polycubes with no edge zipper unfolding. Let P'_6 be the polycube in Fig. 2 with the bottom cube removed. So P'_6 has a '+' sign of five cubes in its base layer. Let B be the bottom face of the cube at the center of the '+' sign. Attach P'_6 to the highest cube of P_{44} in Fig. 1(a) by gluing B to the top face of that top cube. It is easy to verify that all new vertices of this augmented object, call it P'_{44}, are corners. The joining process can be repeated with another copy of P'_6,

[3]Personal communication, June 2020.

producing P''_{44}, and so on. All of these polycubes have no zipper unfolding.

We have not attempted to edge-unfold these larger objects.

7 Open Problem

The most interesting issue remaining in this line of investigation is **Question 1** (Sec. 2): Does every polycube tree have an edge zipper unfolding?

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