PLANAR MINIMAL SURFACES WITH POLYNOMIAL GROWTH IN THE $\mathrm{Sp}(4,\mathbb{R})$-SYMMETRIC SPACE

ANDREA TAMBURELLI AND MICHAEL WOLF

Abstract. We study the asymptotic geometry of a family of conformally planar minimal surfaces with polynomial growth in the $\mathrm{Sp}(4,\mathbb{R})$-symmetric space. We describe a homeomorphism between the "Hitchin component" of wild $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles over $\mathbb{CP}^1$ with a single pole at infinity and a component of maximal surfaces with light-like polygonal boundary in $\mathbb{H}^{2,2}$. Moreover, we identify those surfaces with convex embeddings into the Grassmannian of symplectic planes of $\mathbb{R}^4$. We show, in addition, that our planar maximal surfaces are the local limits of equivariant maximal surfaces in $\mathbb{H}^{2,2}$ associated to $\mathrm{Sp}(4,\mathbb{R})$-Hitchin representations along rays of holomorphic quartic differentials.

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Introduction

Let S be a surface of finite type and let G be a real semisimple Lie group. Higher Rank Teichmüller theory is a quickly growing area of research that studies dynamical, geometric and algebraic properties of representations of $\pi_1(S)$ into G ([Wie18]), whose origins can be traced back to the pioneering work of Hitchin on Higgs bundles ([Hit87]), and to the foundational work of Corlette ([Cor88]), Donaldson ([Don87]), and Simpson ([Sim90]). In brief, they developed a theory, generally referred to as nonabelian Hodge correspondence, that provides homeomorphisms between three natural objects: the character variety $\chi(\Gamma, G)$, the de Rahm moduli space of flat connections on principal G-bundles over a Riemann surface $X = (S, J)$; and the

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Dolbeaut moduli space of G-Higgs bundles on X. A fundamental role in this theory is played by equivariant harmonic maps from the universal cover of X to the symmetric space G/K ([Li19a]). In particular, when G is (non-exceptional) real split of rank 2 the theory is very rich and well-understood: by work of Hitchin ([Hit92]) there is a connected component in the character variety that generalizes Teichmüller space; there is a unique preferred choice of conformal structure on S that makes the associated equivariant harmonic maps conformal, and thus (branched) minimal immersions ([Lab17]); and representations in this connected component all arise as holonomy of geometric structures on (bundles over) S ([Bar10], [CTT19], [Lab07], [Lof01], [Mes07], [GW08]).

More recently, the nonabelian Hodge correspondence has been extended to include surfaces with punctures and Higgs bundles with meromorphic Higgs field with tame or irregular singularities at the punctures ([BB04], [Boa14], [Sim90]). The main aim of this paper is to describe harmonic maps arising from Higgs bundles over $\mathbb{CP}^1$ with polynomial Higgs field for the Lie group $\mathrm{Sp}(4,\mathbb{R})$ and the associated geometric structures. In particular, we aim to unite (see Theorem C) two perspectives: the complex values of the polynomial in the plane with the synthetic pseudo-Riemannian geometry of polygons in the boundary of $\mathbb{H}^{2,2}$. The latter naturally reflects some combinatorics of flats in the symmetric space of $\mathrm{Sp}(4,\mathbb{R})$.

In turn, the study of these planar Higgs bundles provides tools for studying families of representations that leave compacta in a character variety. In particular, and in the setting of the Hitchin component of $\mathrm{Sp}(4,\mathbb{R})$ representations of surface groups, consider a family of representations that has associated harmonic maps from a Riemann surface X that are conformal and of energies growing without bound. In that case, we show that the high energy harmonic maps *localize* in the sense that the restriction of global harmonic maps of the surface to a small neighborhood on the surface is well-approximated by the harmonic maps associated to a Higgs bundle over $\mathbb{CP}^1$ with polynomial Higgs field (for the Lie group $\mathrm{Sp}(4,\mathbb{R})$).

Let us introduce some notation and terminology. Let $f:\mathbb{C}\to \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ be a harmonic map. We can interpret the differential df of f as a 1-form with values in the vector space $\mathfrak{m}=\{A\in\mathfrak{sl}(n,\mathbb{R})\mid A=A^t\}$. The harmonicity of f implies that the $(1,0)$-part $\varphi=\partial f$ of its differential is holomorphic. We can thus associate to f holomorphic k-differentials q_k on the complex plane defined by $q_k=\mathrm{tr}(\varphi^k)$. Notice, in particular, that $q_1=0$ and q_2 is the Hopf differential of the harmonic map f. We say that f has polynomial growth if q_k are all polynomials. One naturally asks,

Question A. *Given holomorphic polynomial k-differentials $(q_2,\dots,q_n)$, is there a harmonic map $f:\mathbb{C}\to\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ such that $q_k=\mathrm{tr}((\partial f)^k)$? Moreover, can we describe its image?*

In this generality, the above question is still open. The first work in this direction is due to Han, Tam, Treibergs and Wan ([HTTW95]) who studied harmonic maps

from the complex plane to the hyperbolic plane with polynomial Hopf differential, showing that the quadratic differential determines a harmonic diffeomorphism from $\mathbb{C}$ to an ideal polygon in $\mathbb{H}^2$ with $m+2$ vertices if m is the degree of the polynomial. Recent work of Li ([Li19b]) shows, in addition, that such a harmonic diffeomorphism is unique. A simple dimension count, however, shows that there cannot be a one-to-one correspondence between polynomial quadratic differentials of degree m on the complex plane and ideal polygons in $\mathbb{H}^2$ with $m+2$ vertices. Gupta ([Gup21]) explained this phenomenon in terms of rate in which the harmonic maps take the end of $\mathbb{C}$ into the cusps and obtained a homeomorphism between these moduli spaces by prescribing the principal part at infinity of the Hopf differential. Another interpretation was given by the first author ([Tam19]), who, using anti-de Sitter geometry, showed that there is actually a homeomorphism between polynomial quadratic differentials on the complex plane and pairs of ideal polygons in $\mathbb{H}^2$ with the same number of vertices.

An answer to Question A is also known for $n = 3$ in case of conformal harmonic maps. In joint work with David Dumas ([DW15]), the second author used techniques from affine differential geometry to show that there is a conformal equivariant harmonic map from $\mathbb{C}$ to $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ with prescribed polynomial cubic differential q_3. Moreover, they constructed a homeomorphism between the moduli space of polynomial cubic differentials of degree m and convex polygons in $\mathbb{RP}^3$ with $m+3$ vertices, exploiting the fact that these harmonic maps arise as Gauss maps of hyperbolic affine spheres in $\mathbb{R}^3$, which project to convex sets, in this case polygons, in $\mathbb{RP}^2$. In terms of the geometry of the minimal surface in the symmetric space, this result can be interpreted as the solution of an asymptotic Dirichlet problem for minimal surfaces in $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$: the minimal surfaces found in [DW15] are asymptotic to $2(m+3)$ flats at infinity with the property that each consecutive pair shares three adjacent Weyl chambers at infinity.

In this paper we extend this last result to conformal harmonic maps with polynomial growth into $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$. We prove the following:

Theorem B. *Assume that q_4 is a polynomial holomorphic quartic differential of degree n. Then there exists a conformal harmonic map $f : \mathbb{C} \to \mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ such that $q_4 = \mathrm{tr}((\partial f)^4)$. Moreover, the associated minimal surface $f(\mathbb{C})$ is asymptotic to $2(n+4)$ flats as $|z| \to +\infty$, with the property that any consecutive pair shares four adjacent Weyl chambers at infinity. Such a collection determines the minimal surface and q_4 uniquely.*

Although the general idea of the proof resembles that in [DW15], the techniques used are very different for two main reasons. First of all, we construct the harmonic map using Higgs bundles: we associate to q_4 an irregular Higgs bundle over $\mathbb{CP}^1$ and find the solution to Hitchin's self-duality equations. We then obtain the minimal surface by parallel transport of a unitary frame using the associated flat connection. In particular, the study of the geometry at infinity of the minimal surface requires

precise estimates on the parallel transport as $|z| \to +\infty$; this in turn involves the study of the asymptotic behavior of solutions of a (coupled) system of elliptic PDE on the complex plane (unlike the $\mathrm{SL}(3,\mathbb{R})$ case where the equation can be reduced to a single scalar PDE). These techniques have the advantage that they might be easily adapted to every cyclic Higgs bundle and thus used for the study of the asymptotic geometry of planar minimal surfaces into $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ with only the requirement that q_n does not vanish identically.

The other main difference is that, in order to prove the second part of Theorem B, we use techniques from pseudo-Riemannian geometry. Exploiting the low-dimensional isomorphism $\mathbb{P}\mathrm{Sp}(4,\mathbb{R}) \cong \mathrm{SO}_0(2,3)$, we interpret the harmonic maps found before as Gauss maps of maximal surfaces in the pseudo-hyperbolic space $\mathbb{H}^{2,2}$ bounding a future-directed negative light-like polygon in the Einstein Universe $\mathrm{Ein}^{1,2}$, which we view as the Lorentzian conformal boundary at infinity of $\mathbb{H}^{2,2}$. We show the following:

Theorem C. *There is a homeomorphism between the moduli space of polynomial quartic differentials of degree n on the complex plane and a connected component of the moduli space of future-directed negative light-like polygons with $n+4$ vertices in the Einstein Universe.*

One cannot ignore the theme running through Theorem C, [DW15] and even [HTTW95] (compare [Wol91]). All of these works identify a "Stokes phenomenon" in which certain cyclic Higgs bundles on $\mathbb{CP}^1$ (with wild singularity at ∞) define geometric shapes – ideal polygons in $\mathbb{H}^2$, convex real projective polygons in $\mathbb{RP}^2$, or future-directed negative light-like polygons in $\mathrm{Ein}^{1,2}$ – which arise in a common way. In particular, the associated harmonic maps from $\mathbb{C}$ to the symmetric space which have a constant holomorphic differential, and (hence) map onto a flat, provide asymptotic solutions (for the solutions of the Hitchin equations under study) in a region of the plane defined by the geometry of the quadratic, cubic or (in the present case) quartic differential; passing from one of these regions to another through a Stokes direction in the plane provides that the solutions transition to be asymptotic to a different flat in the symmetric space (typically sharing a collection of Weyl chambers).

However, unlike [HTTW95] or [DW15], we have reasons to believe that in our case this moduli space of geometric structures that cyclic Higgs bundles on $\mathbb{CP}^1$ induce is not connected. Note that in both [HTTW95] and [DW15], the geometric objects under study (harmonic diffeomorphisms onto ideal polygons in $\mathbb{H}^2$ and affine spheres projecting onto convex polygons in $\mathbb{RP}^2$) can arise only from one family of wild ($\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(3,\mathbb{R})$) Higgs bundles on $\mathbb{CP}^1$ with singularity at infinity that are themselves reminiscent of the Higgs bundles in the Hitchin component in the case of surfaces with negative Euler characteristic. In our context, however, already in the classical setting of closed surfaces of genus at least 2, complete maximal surfaces in $\mathbb{H}^{2,2}$ can be obtained from families of Higgs bundles ([CTT19]) belonging to different

connected components in the moduli space. This suggests that there should be a one-to-one correspondence between the connected components of the moduli space of wild $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles over $\mathbb{CP}^1$ and the connected components of future-directed negative light-like polygons in $\mathrm{Ein}^{1,2}$. In support of this idea, we find an explicit parametrization of the moduli space of future-directed negative light-like hexagons in $\mathrm{Ein}^{1,2}$ and show that this has two connected components. Our Theorem C gives a homeomorphism with the component that does not contain the unique (up to the action of $\mathrm{SO}_0(2,3)$) future-directed negative light-like hexagon in $\mathrm{Ein}^{1,1} \subset \mathrm{Ein}^{1,2}$. This is consistent with our conjecture as the family of wild $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles over $\mathbb{CP}^1$ that we consider in this paper are those belonging to the Hitchin section (cfr. [FN17]) that can never be reduced to $\mathrm{SO}_0(2,2)$-Higgs bundles.

We believe that this study is also relevant to developing a harmonic map compactification of the Hitchin component in the spirit of [Wol89]. By work of Labourie ([Lab17]) the $\mathrm{Sp}(4,\mathbb{R})$-Hitchin component of a closed surface S may be parametrized by the bundle $\mathcal{Q}^4$ of quartic differentials over the Teichmüller space of S. A natural question is to understand the asymptotic behavior of the representations ρ_s and of the associated ρ_s-equivariant harmonic maps $g_s : \tilde{S} \to \mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ along a ray $q_s = sq_0$ of quartic differentials. Works of Collier-Li ([CL17]) and Mochizuki ([Moc14]) give a precise picture away from the zeros of the quartic differential q_0 (as well as for general n-differentials). A consequence of Theorem B is an initial study of the asymptotics of the harmonic maps g_s on all neighborhoods of the surface S in this case of quartic differentials. In particular, we imagine rescaling the coordinate chart in a neighborhood so that q_s converges to the polynomial quartic differential $z^k dz^4$ over $\mathbb{C}$. We can then use the solution of Hitchin's equations on the plane, found in Theorem B, to give the following asymptotic estimates, which extend the ones found in [DW22] (see also [LTW22b]) for the Blaschke metrics along rays of cubic differentials to the present $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ setting:

Theorem D. *Let $q_s = sq_0$ be a ray of holomorphic quartic differentials on a closed Riemann surface $X = (S, J)$. Let σ be the conformal hyperbolic metric on X and let $g_s = \mathrm{diag}(g_{1,s}, g_{2,s}^{-1}, g_{1,s}^{-1}, g_{2,s})$ be the harmonic metric on the Higgs bundle $(\mathcal{E}, \varphi_s)$ over X, where*

$$\mathcal{E} = K^{\frac{3}{2}} \oplus K^{-\frac{1}{2}} \oplus K^{-\frac{3}{2}} \oplus K^{\frac{1}{2}} \quad \textit{and} \quad \varphi_s = \begin{pmatrix} 0 & 0 & q_s & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

Let p be a zero of order k for q_0. Then, as $s \to +\infty$, there exists a sequence of radii $r_s \to 0$ such that

$$g_{1,s}^{-1}{}_{|_{B(p,r_s)}} = O(s^{\frac{3}{k+4}}\sigma^{\frac{3}{2}}) \quad \textit{and} \quad g_{2,s}^{-1}{}_{|_{B(p,r_s)}} = O(s^{\frac{1}{k+4}}\sigma^{\frac{1}{2}}) .$$

As a consequence, we deduce (see Corollary 7.8) that the family of maximal surfaces in $\mathbb{H}^{2,2}$ arising from the Higgs bundles described above ([CTT19]) converge in

the pointed Gromov-Hausdorff topology to the planar maximal surfaces of Theorem C with polynomial quartic differential $z^k dz^4$, where k is the vanishing order of q_0 at the chosen base point. This "localization" result mirrors the result in [DW22] that rays of affine spheres in the Labourie-Loftin coordinates converge to affine spheres over regular polygons. We believe also that these estimates should play a role in the study of the asymptotic holonomy along paths that go through some zeros of q_0 (cf. [LTW22b]) and in the description of the (rescaled) limiting harmonic map to a building ([KNPS15]). We leave these aspects to future work.

Finally, we compare the present work to another recent response to [CTT19]. Labourie, Toulisse and the second author [LTW22a] study the case of spacelike maximal surfaces in $\mathbb{H}^{2,n}$ with positive boundary on $\mathrm{Ein}^{1,n}$ and no specification of the conformal type of the maximal surface (instead, their focus is on removing the restriction in [CTT19] to a cocompact group action). In contrast, Theorems B and C in the present work study boundary maps which are polygonal, hence only semi-positive, on planar surfaces.

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1. Background material

1.1. **Lie theory for $\mathrm{Sp}(4,\mathbb{R})$.** We recall briefly the relevant Lie theory for the Lie group $\mathrm{Sp}(4,\mathbb{R})$. In particular, we fix once and for all an identification of $\mathfrak{sp}(4,\mathbb{R})$ as subalgebra of $\mathfrak{sl}(4,\mathbb{C})$.

We consider on $\mathbb{C}^4$ the symplectic form given by

$$\Omega = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} . \tag{1.1}$$

The complex symplectic group $\mathrm{Sp}(4,\mathbb{C})$ consists of all linear transformations g in $\mathrm{GL}(4,\mathbb{C})$ such that $g^t\Omega g = \Omega$. Hence, its Lie algebra is

$$\mathfrak{sp}(4,\mathbb{C}) = \{X \in \mathfrak{gl}(4,\mathbb{C}) \mid X^t\Omega + \Omega X = 0\} .$$

A simple computation shows that $X \in \mathfrak{sp}(4,\mathbb{C})$ if and only if it can be written as

$$X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$

for some $A \in \mathrm{GL}(2,\mathbb{C})$ and $B, C \in \mathrm{Sym}(2,\mathbb{C})$. The anti-linear involution

$$\begin{aligned} \rho : \mathrm{Sp}(4,\mathbb{C}) &\to \mathrm{Sp}(4,\mathbb{C}) \\ g &\mapsto (\overline{g^{-1}})^t \end{aligned}$$

fixes a maximal compact subgroup isomorphic to SU(4).
We identify $\mathrm{Sp}(4,\mathbb{R})$ with the fixed points in the complex group $\mathrm{Sp}(4,\mathbb{C})$ of the anti-linear involution

$$\begin{aligned} \lambda : \mathrm{Sp}(4,\mathbb{C}) &\to \mathrm{Sp}(4,\mathbb{C}) \\ g &\mapsto \begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix} \overline{g} \begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix} . \end{aligned}$$

Remark 1.1. This group is conjugate via

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \end{pmatrix} \in \mathrm{SU}(4)$$

to the standard $\mathrm{Sp}(4,\mathbb{R})$, consisting of matrices with real coefficient preserving the symplectic form Ω.

At the Lie algebra level, this identification of $\mathrm{Sp}(4,\mathbb{R})$ provides for the identification

$$\mathfrak{sp}(4,\mathbb{R}) = \left\{ \begin{pmatrix} C & D \\ \overline{D} & -C^t \end{pmatrix} \mid C \in \mathfrak{u}(2);\ D \in \mathrm{Sym}(2,\mathbb{C}) \right\} .$$

The involutions ρ and λ commute and the composition $\sigma = \lambda \circ \rho$ acts on $\mathfrak{sp}(4,\mathbb{R})$ as

$$\sigma \begin{pmatrix} C & D \\ \overline{D} & -C^t \end{pmatrix} = \begin{pmatrix} C & -D \\ -\overline{D} & -C^t \end{pmatrix} .$$

We deduce that σ is a Cartan involution for $\mathfrak{sp}(4,\mathbb{R})$ and induces the (Cartan) decomposition

$$\mathfrak{sp}(4,\mathbb{R}) = \mathfrak{u}(2) \oplus (\mathrm{Sym}(2,\mathbb{R}) \oplus \mathrm{Sym}(2,\mathbb{R})).$$

By complexifying, we obtain the splitting

$$\mathfrak{sp}(4,\mathbb{C}) = \mathfrak{gl}(2,\mathbb{C}) \oplus (\mathrm{Sym}(2,\mathbb{C}) \oplus \mathrm{Sym}(2,\mathbb{C})).$$

1.2. $\mathrm{Sp}(4,\mathbb{R})$**-Higgs bundles.** We recall here the definition of $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles over closed Riemann surfaces and their connection with harmonic maps in the symmetric space $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$.

Definition 1.2. An $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle on a closed Riemann surface Σ is a triple (V,β,γ), where V is a holomorphic vector bundle of rank 2, and the forms $\beta \in H^0(\Sigma, \mathrm{Sym}(V) \otimes K)$ and $\gamma \in H^0(\Sigma, \mathrm{Sym}(V)^* \otimes K)$, where K is the canonical bundle over Σ.

The associated $\mathrm{SL}(4,\mathbb{C})$-Higgs bundle is given by the holomorphic vector bundle $\mathcal{E} = V \oplus V^*$ on Σ and the Higgs field $\varphi : \mathcal{E} \to \mathcal{E} \otimes K$ represented by the matrix

$$\varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} .$$

This bundle comes equipped with a symplectic form Ω and an orthogonal structure $Q : \mathcal{E} \to \mathcal{E}^*$, which, in the above splitting $\mathcal{E} = V \oplus V^*$, are given by

$$\Omega = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix} .$$

More generally, we will say that a frame for $\mathcal{E}$ is Ω-symplectic and Q-adapted, if the symplectic form and the orthogonal structure are represented by the above matrices. We are interested in Higgs bundles in the $\mathrm{Sp}(4,\mathbb{R})$-Hitchin component. Those are parameterized [Lab17] by a point in Teichmüller space (corresponding to the complex structure on Σ) and a holomorphic quartic differential q, and they are given by the triple

$$V = K^{\frac{3}{2}} \oplus K^{-\frac{1}{2}} \quad \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} .$$

Hitchin's equations look for a (harmonic) hermitian metric H on $\mathcal{E}$ such that the $\mathrm{Sp}(4,\mathbb{R})$-connection

$$\nabla = D_H + \varphi + \varphi^{*H}$$

is flat, where D_H denotes the Chern connection of H. It is well-known that the solution is unique [Sim88] and diagonal [Sim09] of the form $H = \mathrm{diag}(h_1, h_2^{-1}, h_1^{-1}, h_2)$ in the above splitting $\mathcal{E} = V \oplus V^*$. Notice that the hermitian metric is compatible with the symplectic structure Ω and the orthogonal structure Q in the sense that $H^t \Omega H = \Omega$ and $H^t Q H = Q$. The monodromy of the flat connection ∇ defines then a representation $\rho : \pi_1(S) \to \mathrm{Sp}(4,\mathbb{R})$.

Moreover, the metric H induces a ρ-equivariant harmonic map

$$\tilde{f}_\rho : \tilde{\Sigma} \to \mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$$

as follows. Fix a point $\tilde{p}_0 \in \tilde{\Sigma}$ and fix a holomorphic, Q-adapted, Ω-symplectic and H-unitary frame $N(\tilde{p})$ for the bundle $\mathcal{E}$ at every point $\tilde{p} \in \tilde{\Sigma}$. For every $\tilde{p} \in \tilde{\Sigma}$, we denote by $\mathcal{N}(\tilde{p})$ the parallel transport (relative to the connection ∇) of $N(\tilde{p}_0)$ at $\tilde{p}$. Notice that in general (i.e. when $\varphi \neq 0$), the frame $\mathcal{N}(\tilde{p})$ will not be unitary. If we identify the symmetric space $\mathrm{SL}(4,\mathbb{C})/\mathrm{SU}(4)$ with the space of hermitian metrics on $\mathbb{C}^4$, the harmonic map is given by

$$\begin{aligned} \tilde{f}_\rho : \tilde{\Sigma} &\to \mathrm{SL}(4,\mathbb{C})/\mathrm{SU}(4) \\ \tilde{p} &\mapsto H^{\mathcal{N}(\tilde{p})} . \end{aligned}$$

Here $H^{\mathcal{N}(\tilde{p})}$ is the metric H expressed in the frame $\mathcal{N}(\tilde{p})$. We then notice that the image of $\tilde{f}_\rho$ is actually contained in the copy of $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ consisting of hermitian metrics H on $\mathbb{C}^4$ that are Q-symmetric (i.e. $H^t Q H^{-1} = Q$) and Ω-symplectic (i.e.

$H^t\Omega H = \Omega$). In fact, if we denote by $g(\tilde{p}) \in \mathrm{Sp}(4,\mathbb{R})$ the family of matrices such that $\mathcal{N}(\tilde{p})g(\tilde{p}) = N(\tilde{p})$, then

$$\tilde{f}_\rho(\tilde{p}) = \overline{(g(\tilde{p})^{-1})^t} g(\tilde{p})^{-1} \ , \tag{1.2}$$

and an easy computation shows that $g(\tilde{p}) \in \mathrm{Sp}(4,\mathbb{R})$ is equivalent to the hermitian metric $\tilde{f}_\rho(\tilde{p})$ being Ω-symplectic and Q-symmetric. In addition, noting that $\mathrm{tr}(\varphi^2)$ vanishes, we see that the map $\tilde{f}_\rho$ is conformal [Cor88] and thus parameterizes a minimal surface in the symmetric space.

1.3. **Planar minimal surfaces with polynomial growth.** In this paper we are interested in the study of a particular class of minimal surfaces in $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ which are described by conformal, harmonic maps $f : \mathbb{C} \to \mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ with polynomial growth.

Given a map $f : \mathbb{C} \to \mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$, we recall ([Cor88]) that if f is harmonic then, for a lift $\tilde{f} : \mathbb{C} \to \mathrm{Sp}(4,\mathbb{R})$, we have that $\varphi = (\partial\tilde{f})^\perp$ is holomorphic, where $(\partial\tilde{f})^\perp$ denotes the component of the (1,0)-part of the differential of $\tilde{f}$, which is orthogonal to $\mathfrak{u}(2)$ with respect to the Killing form

$$\begin{aligned} B : \mathfrak{sp}(4,\mathbb{R}) \times \mathfrak{sp}(4,\mathbb{R}) &\to \mathbb{C} \\ (X, Y) &\mapsto \mathrm{tr}(XY) \ . \end{aligned}$$

In particular the quadratic differential

$$q_2 = \mathrm{tr}(\varphi^2)$$

and the quartic differential

$$q_4 = \mathrm{tr}(\varphi^4) \tag{1.3}$$

are holomorphic. The Killing form B induces a Riemannian metric g on the symmetric space, and its pull-back via f is

$$f^* g_p(X, Y) = B((\varphi + \varphi^{*H})(X), (\varphi + \varphi^{*H})(Y))$$

where $H = f(p)$. Therefore, q_2 is the Hopf differential of the harmonic map f and the vanishing of q_2 is equivalent to the map f being conformal. In this case the pull-back metric reduces to

$$f^* g_p = \mathrm{tr}(\varphi\varphi^{*H}) \ .$$

Finally, we say that f *has polynomial growth* if the quartic differential $q_4 = q$ is a polynomial over $\mathbb{C}$.

We can actually interpret the harmonic map f as the harmonic metric induced by some Higgs bundle over $\mathbb{CP}^1$ with singularity at infinity. It is sufficient to consider

the holomorphic bundle $\mathcal{E} = \oplus_{i=1}^{4}\mathcal{O}(\alpha_i)$ over $\mathbb{CP}^1$ endowed with the Higgs field

$$\varphi = \begin{pmatrix} 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

The Higgs bundle $(\mathcal{E}, \varphi)$ is the $\mathrm{SL}(4, \mathbb{C})$-Higgs bundle associated to an $\mathrm{Sp}(4, \mathbb{R})$-Higgs bundle with singularity at infinity. Here, we consider $(\mathcal{E}, \varphi)$ as a good filtered Higgs bundle ([Moc14],[FN17]) with weights $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. In particular, for a meromorphic section $s = (s_1, s_2, s_3, s_4)$ of $\mathcal{E}$, we define $v_\infty(s) = \max_{i=1,\dots,4}\{v_\infty(s_i) + \alpha_i\}$, where $v_\infty(s_i) \in \mathbb{Z}$ is the order of singularity at infinity of the section s_i. We then seek a hermitian metric H on $\mathcal{E}$ satisfying the self-duality equations

$$F_H + [\varphi, \varphi^{*H}] = 0 , \tag{1.4}$$

which is compatible with the filtration in the following sense: for every meromorphic section s of $\mathcal{E}$ we require that

$$H(s(z), s(z)) = O(|z|^{2v_\infty(s)}) \quad \text{as} \quad |z| \to +\infty .$$

If such H exists, then the map f coincides with the conformal harmonic map induced by H via the procedure described in the previous subsection.

In the section 2, we will find a solution to Equation (1.4) for the Higgs bundle $(\mathcal{E}, \varphi)$ with weights

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{3n}{8}, -\frac{n}{8}, -\frac{3n}{8}, \frac{n}{8}\right) ,$$

where we assume that q is a polynomial quartic differential of degree n.

1.4. **Moduli space of polynomial quartic differentials.** A polynomial quartic differential is a holomorphic differential on the complex plane of the form $q(z)dz^4$, where $q(z)$ is a polynomial function. We denote by $\mathcal{Q}_n$ the space of polynomial quartic differentials of degree n. The group $\mathrm{Aut}(\mathbb{C})$ of biholomophisms of $\mathbb{C}$ acts on this space by push-forward. Let $\mathcal{MQ}_n$ be the quotient of $\mathcal{Q}_n$ by this action. The geometry of the resulting moduli space is analogous to that described for polynomial cubic differentials in [DW15].

Proposition 1.3. *The moduli space $\mathcal{MQ}_n$ is a complex orbifold of real dimension $2(n-1)$ if $n \geq 1$.*

Proof. Every polynomial quartic differential may be written as

$$q = (a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0)dz^4$$

for some $a_i \in \mathbb{C}$ and $a_n \in \mathbb{C}^*$. An element $T(z) = bz + c \in \mathrm{Aut}(\mathbb{C})$ acts on q via

$$T_* q = (a_n b^{n+4}(z + c/b)^n + a_{n-1} b^{n+3}(z + c/b)^{n-1} + \cdots + b^4 a_0)dz^4.$$

Hence by choosing $b = a_n^{-1/(n+4)}$ we may make T_*q monic (i.e. with leading coefficient equal to 1); then a suitable choice of the translation component c allows us to assume

that T_*q is centered (i.e. with $a_{n-1}=0$). Notice that these choices are unique up to multiplying b by an $(n+4)$-root of unity. Thus we can describe the moduli space $\mathcal{MQ}_n$ as the quotient

$$\mathcal{MQ}_n = \mathcal{TQ}_n/\mathbb{Z}_{n+4}$$

where $\mathcal{TQ}_n$ is the space of monic and centered polynomials of degree n and $\mathbb{Z}_{n+4}$ denotes the cyclic group of order $n+4$ generated by $T(z)=\zeta_{n+4}^{-1}z$ for a primitive $(n+4)$-root of unity ζ_{n+4}. Since $\mathcal{TQ}_n$ is naturally identified with $\mathbb{C}^{n-1}$ by

$$\mathcal{TQ}_n \to \mathbb{C}^{n-1}$$
$$(z^n + a_{n-2}z^{n-2} + \cdots a_0) \mapsto (a_{n-2},\dots,a_0)\ ,$$

it follows that $\mathcal{MQ}_n$ is a complex orbifold of real dimension $2(n-1)$. □

Remark 1.4. If $n=0$, the space $\mathcal{MQ}_0$ consists of only one point, represented by the quartic differential $q=dz^4$.

We put on $\mathcal{MQ}_n$ the topology induced by the identification

$$\mathcal{MQ}_n = \mathcal{TQ}_n/\mathbb{Z}_{n+4}$$

found in Proposition 1.3.

2. Existence

In this section we prove the existence of a conformal harmonic map $f:\mathbb{C}\to \mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ with given polynomial quartic differential $q_4=q$ (cfr. Equation (1.3)). We will provide also precise estimates of the behaviour of the associated harmonic metric H when $|z|\to\infty$.

Theorem 2.1. *Let q be a polynomial quartic differential of degree n. Consider the good filtered $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle $(\mathcal{E},\varphi)$ over $\mathbb{CP}^1$ where*

$$\mathcal{E} = \mathcal{O}\left(\frac{3n}{8}\right)\oplus\mathcal{O}\left(-\frac{n}{8}\right)\oplus\mathcal{O}\left(-\frac{3n}{8}\right)\oplus\mathcal{O}\left(\frac{n}{8}\right)$$

and

$$\varphi = \begin{pmatrix} 0 & 0 & q & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\ .$$

Then there exists a unique diagonal harmonic metric H satisfying Hitchin's self-duality equation $F_H+[\varphi,\varphi^{*H}]=0$.

Inspired by the solution of Hitchin's equations for $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles over closed Riemann surfaces (see Section 1), we look for a diagonal metric of the form $H=\mathrm{diag}(h_1,h_2^{-1},h_1^{-1},h_2)$. Under this assumption, the equation $F_H+[\varphi,\varphi^{*H}]=0$ simplifies into the following coupled system of elliptic PDE

$$\begin{cases}\Delta\log(h_1)+h_1^{-1}h_2-h_1^2|q|^2=0\\ \Delta\log(h_2)+h_2^{-2}-h_1^{-1}h_2=0\ .\end{cases}$$

Note that here we adopt the convention that $\Delta = \partial_z \partial_{\bar{z}}$; while this convention is more common for authors writing on Hitchin equations, it differs from that invoked often by authors writing from a harmonic maps or conformal variational problem viewpoint.

It is convenient to define $u_i = \log(\frac{1}{h_i})$ and study the system in the following form

$$\begin{cases} \Delta u_1 = e^{u_1 - u_2} - e^{-2u_1}|q|^2 \\ \Delta u_2 = e^{2u_2} - e^{u_1 - u_2} \ . \end{cases} \tag{2.1}$$

Namely, if we define

$$F : \mathbb{R}^2 \to \mathbb{R}^2$$
$$F(u_1, u_2) = (e^{u_1 - u_2} - e^{-2u_1}|q|^2, e^{2u_2} - e^{u_1 - u_2}) = (F_1, F_2)$$

the above system may be written as

$$\Delta u = F(u)$$

where $u = (u_1, u_2)$ and the map F satisfies a monotone condition

$$\frac{\partial F_j}{\partial u_i} \leq 0 \qquad \text{for} \quad i \neq j \ .$$

In this setting we can apply a super- and sub-solution method to prove the existence of a smooth solution defined over all $\mathbb{C}$.

Since we did not manage to find a precise reference for this method applied to a system of PDE, we provide a detailed description of its application to Equation (2.1).

Let B_R be the ball of radius R centered at 0. We start by proving the existence of a solution to Equation (2.1) on the domain B_R with some smooth boundary values (w_1, w_2) and for R sufficiently large.

Definition 2.2. We say that u^+ is a super-solution of Equation (2.1) with boundary values (w_1, w_2) on the ball B_R if it is continuous and satisfies

$$\begin{cases} \Delta u_i^+ \leq F_i(u^+) & \text{for} \quad i = 1, 2 \\ u_i^+ \geq w_i & \text{on} \quad \partial B_R \end{cases}$$

in the weak sense. Similarly u^- is a sub-solution if it is continuous and satisfies

$$\begin{cases} \Delta u_i^- \geq F_i(u^-) & \text{for} \quad i = 1, 2 \\ u_i^- \leq w_i & \text{on} \quad \partial B_R \end{cases}$$

in the weak sense.

Our sub- and super-solution for System (2.1) will be slightly modifications of

$$(u_1, u_2) = \left(\frac{3}{4} \log(|q|), \frac{1}{4} \log(|q|) \right) \ ,$$

which is the exact solution of the system if q is a non-zero constant quartic differential (or an exact solution in regions where q does not vanish). We will also choose the boundary values

$$(w_1, w_2) = \left(\frac{3}{4}\log(|q|), \frac{1}{4}\log(|q|)\right) ,$$

which are smooth on ∂B_R as soon as it does not contain any zeros of q.

Lemma 2.3. *The following function $u^- = (u_1^-, u_2^-)$ is a sub-solution of Equation (2.1):*

$$u_1^- = \begin{cases} \log(|q|^{\frac{3}{4}}) & \text{if} \quad |z| > d \\ \max(\log(g_{2d}^{\frac{3}{2}}), \log(|q|^{\frac{3}{4}})) & \text{if} \quad |z| \leq d \end{cases}$$

$$u_2^- = \begin{cases} \log(|q|^{\frac{1}{4}}) & \text{if} \quad |z| > d \\ \max(\log(g_{2d}^{\frac{1}{2}}), \log(|q|^{\frac{1}{4}})) & \text{if} \quad |z| \leq d \end{cases}$$

where g_{2d} denotes the density of the metric with constant curvature -2 on the ball $B(0, 2d)$ centred at the origin with radius $2d$

$$g_{2d} = \frac{1}{2}\left(\frac{4d}{4d^2 - |z|^2}\right)^2$$

and d is a positive real number that depends only on the quartic differential q.

Proof. Let us verify first that u_i^- are continuous. We can choose d sufficiently large such that $\{z \mid |q(z)| \leq 1\} \subset B(0, d)$ and we can suppose that $d > \frac{4}{3}$ in such a way that $\log(g_{2d})$ is negative for $|z| \leq d$. This implies that the functions u_i^- are continuous in a neighbourhood of $|z| = d$. Moreover, they are continuous in a neighbourhood of the zeros of q because $\log(|q|)$ tends to $-\infty$ at the zeros of q, whereas g_{2d} is bounded away from 0. We notice also that the functions u_i^- are piece-wise smooth and thus locally Lipschitz.

Let us now verify that u^- is a sub-solution of the system. Since

$$\max(\log(g_{2d}^{\frac{3}{2}}), \log(|q|^{\frac{3}{4}})) = \log(|q|^{\frac{3}{4}}) \Leftrightarrow \max(\log(g_{2d}^{\frac{1}{2}}), \log(|q|^{\frac{1}{4}})) = \log(|q|^{\frac{1}{4}})$$

it is sufficient to verify that the pairs $(\log(|q|^{\frac{3}{4}}), \log(|q|^{\frac{1}{4}}))$ and $(\log(g_{2d}^{\frac{3}{2}}), \log(g_{2d}^{\frac{1}{2}}))$ are sub-solutions. Away from the zeros of q, the pair $(\log(|q|^{\frac{3}{4}}), \log(|q|^{\frac{1}{4}}))$ is a solution of the system, hence in particular it is a sub-solution. As for the second pair, the density of the metric with constant curvature -2 satisfies the differential equation

$$\Delta \log(g_{2d}) = g_{2d}$$

therefore,

$$F_1(\log(g_{2d}^{\frac{3}{2}}), \log(g_{2d}^{\frac{1}{2}})) = g_{2d} - g_{2d}^{-1}|q|^2 \leq \frac{3}{2}g_{2d} = \Delta \log(g_{2d}^{\frac{3}{2}})$$

$$F_2(\log(g_{2d}^{\frac{3}{2}}), \log(g_{2d}^{\frac{1}{2}})) = 0 \leq \frac{1}{2}g_{2d} = \Delta \log(g_{2d}^{\frac{1}{2}})$$

We deduce that at every point the function u^- is a sub-solution or the maximum of two sub-solutions, hence it is a sub-solution. Notice also that the boundary conditions are satisfied as soon as $R > d$. □

Lemma 2.4. *There exists a constant $C > 1$ such that, for any choice of R and consequent boundary values (w_1, w_2) on ∂B_R, the pair*

$$(u_1^+, u_2^+) = \left(\frac{3}{8}\log(|q|^2 + C), \frac{1}{8}\log(|q|^2 + 3C)\right)$$

is a super-solution of System (2.1) with those boundary values (w_1, w_2).

Proof. Of course, as soon as $C > 0$, we have that $u_i^+ > w_i$, so the boundary conditions for a supersolution are satisfied on ∂B_R for any R sufficiently large. Then to find a constant C for which (u_1^+, u_2^+) is a supersolution of equation (2.1), we begin by noting that a simple computation shows that

$$\Delta u_1^+ = \frac{3}{8}\frac{|q_z|^2 C}{(|q|^2 + C)^2}$$

and

$$\Delta u_2^+ = \frac{3}{8}\frac{|q_z|^2 C}{(|q|^2 + 3C)^2}.$$

Moreover,

$$F_1(u_1^+, u_2^+) = (|q|^2 + C)^{\frac{3}{8}}(|q|^2 + 3C)^{-\frac{1}{8}} - (|q|^2 + C)^{-\frac{3}{4}}|q|^2$$
$$F_2(u_1^+, u_2^+) = (|q|^2 + 3C)^{\frac{1}{4}} - (|q|^2 + C)^{\frac{3}{8}}(|q|^2 + 3C)^{-\frac{1}{8}} .$$

Therefore, we need to show that there exists a constant $C > 0$ such that

$$\begin{cases} (|q|^2 + C)^{\frac{19}{8}}(|q|^2 + 3C)^{-\frac{1}{8}} - (|q|^2 + C)^{\frac{5}{4}}|q|^2 \geq \frac{3}{8}|q_z|^2 C \\ (|q|^2 + 3C)^{\frac{9}{4}} - (|q|^2 + C)^{\frac{3}{8}}(|q|^2 + 3C)^{\frac{15}{8}} \geq \frac{3}{8}|q_z|^2 C \end{cases} .$$

Let us consider the following one-parameter family of functions:

$$f_C(z) = (|q|^2 + C)^{\frac{19}{8}}(|q|^2 + 3C)^{-\frac{1}{8}} - (|q|^2 + C)^{\frac{5}{4}}|q|^2 - \frac{3}{8}|q_z|^2 C$$

$$g_C(z) = (|q|^2 + 3C)^{\frac{9}{4}} - (|q|^2 + C)^{\frac{3}{8}}(|q|^2 + 3C)^{\frac{15}{8}} - \frac{3}{8}|q_z|^2 C.$$

We will show that f_C and g_C diverge uniformly to infinity when $C \to +\infty$.
We first remark that for every $C > 0$ the functions f_C and g_C admit a global minimum. Namely, since $|q| \to +\infty$ when $|z| \to +\infty$ and q is a polynomial, the leading terms of the asymptotic expansions of f_C and g_C for $|z| \to +\infty$ are given by

$$f_C(z) = \frac{3}{4}C(|q|^2)^{\frac{5}{4}} + o(|q|^2)$$
$$g_C(z) = \frac{3}{4}C(|q|^2)^{\frac{5}{4}} + o(|q|^2)$$

and thus, for C fixed, they are unbounded when $|z| \to +\infty$. Let us denote by $z_f(C)$ and $z_g(C)$ the point of global minimum of f_C and g_C, respectively. It is sufficient to

show that $f_C(z_f(C))$ and $g_C(z_g(C))$ tend to infinity when $C \to +\infty$. Since it seems difficult to find an explicit expression for $z_f(C)$ and $z_g(C)$, we give an abstract argument by considering two different cases. We explain the complete argument for the function f_C, the other being analogous.
Suppose first that $z_f(C)$ is uniformly bounded when $C \to +\infty$. In this case, there exists a ball B_r of radius r centred at the origin such that $z_f(C) \in B_r$ for every C. Let us denote $M' = \max_{B_r}(|q_z|^2)$ and $M = \max_{B_r}(|q|^2)$. Then,

$$f_C(z_f(C)) \geq C^{\frac{19}{8}}(M+3C)^{-\frac{1}{8}} - (M+C)^{\frac{3}{8}}M - \frac{3}{8}CM'$$

and it is clear that the right-hand side tends to infinity when $C \to +\infty$.
Let us then suppose that $z_f(C)$ is unbounded. This implies that $|q(z_f(C))|$ is diverging as a function of C. Assume that $|q(z_f(C))|^2$ has super-linear growth as a function of C, i.e. $|q(z_f(C))|^2 = O(C^n)$ for some $n > 1$ as $C \to +\infty$. Since q_z is a polynomial of lower degree than q, we deduce that $|q_z(z_f(C))| = o(|q(z_f(C))|)$ as $C \to +\infty$, hence the asymptotic expansion of $f_C(z_f(C))$ reads

$$f_C(z_f(C)) = O(C \cdot C^{\frac{5}{4}n}) \ .$$

Therefore, we can conclude that f_C diverges to $+\infty$ as C tends to $+\infty$. The argument is similar if $|q(z_f(C))|^2$ has linear or sublinear growth as a function of C. □

Theorem 2.5. *Let $d > 0$ be the constant appearing in Lemma 2.3. For every $R > d$, there exists an analytic solution $u^R = (u_1^R, u_2^R)$ of the following boundary value problem*

$$\begin{cases} \Delta u_1^R = e^{u_1^R - u_2^R} - e^{-2u_1^R}|q|^2 \\ \Delta u_2^R = e^{2u_2^R} - e^{u_1^R - u_2^R} \\ u_1^R = \frac{3}{4}\log(|q|) \qquad on \quad \partial B_R \\ u_2^R = \frac{1}{4}\log(|q|) \qquad on \quad \partial B_R \end{cases}$$

Moreover, $u_i^- \leq u_i^R \leq u_i^+$.

Proof. For this proof we remove the dependence on R in the notation. Let us define the sequence of functions $u^k = (u_1^k, u_2^k)$ by

$$\begin{cases} \Delta u_1^k = -\Omega_1 u_1^{k-2} + F_1(u_1^{k-2}, u_2^{k-2}) + \Omega_1 u_1^k \\ \Delta u_2^k = -\Omega_2 u_2^{k-2} + F_2(u_1^k, u_2^{k-2}) + \Omega_2 u_2^k \\ u_1^k = \frac{3}{4}\log(|q|) \qquad \text{on} \quad \partial B_R \\ u_2^k = \frac{1}{4}\log(|q|) \qquad \text{on} \quad \partial B_R \end{cases} \tag{2.2}$$

where $\Omega_i = \sup\left\{ \left| \frac{\partial F_i}{\partial u_i} \right| \ | \ u \in [u^0, u^{-1}] \right\}$ and u^0, u^{-1} are a sub-solution and a super-solution of Equation (2.1), respectively.
We claim that

$$u^0 \leq u^2 \leq \cdots \leq u^{2k} \leq u^{2k-1} \leq u^{2k-3} \leq \cdots \leq u^1 \leq u^{-1} \tag{2.3}$$

for every $k \geq 1$. Then the result follows from the Schauder fixed point theorem [Ama76, p.660] applied to the differential operator defined by (2.2) on the Banach

space of pairs of Hölder functions on B_R, standard bootstrap arguments and Morrey's regularity theorem [Mor58, p.198]. Moreover, the above inequalities imply that the solution is bounded from above by (any of) the super-solution(s) and from below by (any of) the sub-solution(s).

Let us now prove the claim (2.3). We first show that

$$u_1^0 \leq u_1^1 \leq u_1^{-1} \ .$$

By definition and the monotonicity properties of the function F_1, the following equations hold

$$\begin{aligned}
\Delta u_1^1 &= -\Omega_1 u_1^{-1} + \Omega_1 u_1^1 + F_1(u_1^{-1}, u_2^{-1}) \\
\Delta u_1^0 &\geq F_1(u_1^0, u_2^0) \geq F_1(u_1^0, u_2^{-1}) - \Omega_1 u_1^0 + \Omega_1 u_1^0 \\
\Delta u_1^{-1} &\leq F_1(u_1^{-1}, u_2^{-1}) - \Omega_1 u_1^{-1} + \Omega_1 u_1^{-1}
\end{aligned}$$

and the claim follows from the maximum principle (for Sobolev functions, at this first iteration of the process) applied to differences of the above equations. Namely,

$$\Delta(u_1^1 - u_1^{-1}) \geq \Omega_1(u_1^1 - u_1^{-1})$$

and the maximum principle implies that $u_1 - u_1^{-1} \leq 0$. Similarly,

$$\Delta(u_1^1 - u_1^0) \leq \Omega_1(u_1^1 - u_1^0) + F_1(u_1^{-1}, u_2^{-1}) - F_1(u_1^0, u_2^{-1}) - \Omega_1(u_1^{-1} - u_1^0) \leq \Omega_1(u_1^1 - u_1^0)$$

by definition of Ω_1, and from the maximum principle we deduce that $u_1^1 - u_1^0 \geq 0$.

The reasoning is similar also for the second components. In this case we have

$$\begin{aligned}
\Delta u_2^1 &= -\Omega_2 u_2^{-1} + F_2(u_1^1, u_2^{-1}) + \Omega_2 u_2^1 \\
\Delta u_2^0 &\geq F_2(u_1^0, u_2^0) \geq F_2(u_1^1, u_2^0) - \Omega_2 u_2^0 + \Omega_2 u_2^0 \\
\Delta u_2^{-1} &\leq F_2(u_1^{-1}, u_2^{-1}) \leq F_2(u_1^1, u_2^{-1}) - \Omega_2 u_2^{-1} + \Omega_2 u_2^{-1}
\end{aligned}$$

and the inequalities $u_2^0 \leq u_2^1 \leq u_2^{-1}$ follow from the maximum principle as above. With the same argument, one can show that

$$u^0 \leq u^2 \leq u^1$$

and the chain of inequalities (2.3) follows then by induction. (Note that elliptic regularity implies that the functions u_j^k are increasingly smooth, so that for $k \geq 2$, $u_j^k \in C^2$.) □

We now deduce the existence of an analytic solution $u = (u_1, u_2)$ to Equation (2.1) defined on the whole complex plane, via a limiting argument.

By Theorem 2.5, we obtain a sequence of analytic functions u_i^R defined on the ball B_R for every $R > d$. By using the fact that u_i^R is bounded between the sub-solution and the super-solution for every R, we deduce a uniform bound on Δu_i^R on every compact set, which is independent of R. By elliptic regularity, the functions u_i^R are bounded in the $C^{1,\alpha}$ norm, uniformly on every compact set. By Ascoli-Arzelá, this implies that the sequences u_i^R converge in the C^1 norm on compact sets for

every $i = 1, 2$. In particular, the limit functions u_i are defined over all $\mathbb{C}$ and are weak solutions of the system. By elliptic regularity of Poisson equations (applied to each single equation), we deduce that u_i are smooth and hence are strong solutions of Equation (2.1). By Morrey's results [Mor58], the functions u_i are analytic. Moreover, by construction we have

$$\frac{3}{8}\log(|q|^2) \leq u_1 \leq \frac{3}{8}\log(|q|^2 + C) \tag{2.4}$$

$$\frac{1}{8}\log(|q|^2) \leq u_2 \leq \frac{1}{8}\log(|q|^2 + 3C)\ . \tag{2.5}$$

Of course, here it is important that we found in Lemma 2.4 a single constant C for which the right-hand sides were supersolutions on B_R for all R sufficiently large.

Corollary 2.6. *There exist constants $A, R > 0$ and an exponent $\alpha > 1$ as follows. If the $|q|^{\frac{1}{2}}$-distance of a point $p \in \mathbb{C}$ from the zeros of q is $r > R$, then*

$$0 \leq u_1(p) - \frac{3}{8}\log(|q|^2) \leq Ar^{-\alpha}$$

$$0 \leq u_2(p) - \frac{1}{8}\log(|q|^2) \leq Ar^{-\alpha}\ .$$

Proof. Outside a disc D containing the zeros of q, the polynomial q is comparable to z^n up to multiplicative constants, where n is the degree of q. As a consequence, the $|q|^{\frac{1}{2}}$-distance r of a point $p \notin D$ from a zero of q is bounded from above by a multiple of the $|z|^{\frac{n}{2}}$-distance of p from the origin. We deduce that there exists a constant $c > 0$ such that

$$r < c|p|^{\frac{(n+4)}{4}}\ ,$$

where $|p|$ refers to the norm of the complex number $p \in D$. Since $|q|$ is bounded also from below by a multiple of $|z|^n$, we have

$$|q(p)| > c'|p|^n \geq c''r^{\frac{4n}{(n+4)}}\ .$$

From the previous theorem, we obtain that

$$u_1 - \frac{3}{8}\log(|q|^2) \leq u_1^+ - \frac{3}{8}\log(|q|^2) \leq \frac{M}{|q|^2} \leq \frac{A}{r^{\frac{8n}{n+4}}}$$

$$u_2 - \frac{1}{8}\log(|q|^2) \leq u_2^+ - \frac{1}{8}\log(|q|^2) \leq \frac{M}{|q|^2} \leq \frac{A}{r^{\frac{8n}{n+4}}}\ .$$

By noticing that $\alpha = \frac{8n}{n+4} > 1$ for every $n \geq 1$, the result follows if we fix R big enough such that $r > R$ implies $p \notin D$. □

Remark 2.7 (On uniqueness). By work of Mochizuki ([Moc14]), the solution $H = (h_1, h_2^{-1}, h_1^{-1}, h_2)$ found above is the unique diagonal solution of the self-duality equation

$$F_H + [\varphi, \varphi^{*H}] = 0$$

on the Higgs bundle $(\mathcal{E}, \varphi)$ on $\mathbb{CP}^1$. Moreover, in recent work ([LM20]), Li and Mochizuki applied similar sub- and super-solution techniques to show existence and uniqueness of diagonal solutions of Hitchin's self-duality equation on every cyclic Higgs bundle with wild singularities over non-compact Riemann surfaces.

3. Geometry of the minimal surface

In this section we study the geometry of the minimal surface with polynomial growth induced by the harmonic metric found in Section 2.

In particular, the results in this section will imply Theorem B. Moreover, for S the minimal surface in the symmetric space $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ associated to a monic polynomial q of degree $n \geq 1$, we find in Theorem 3.9 that S is asymptotic to $2(n+4)$ maximal flats in $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$; two consecutive flats that are asymptotic to S share four adjacent Weyl chambers (Proposition 3.10). Intrinsically, by Proposition 3.11, the metric on S induced by this immersion is asymptotically $4|q|^{\frac{1}{2}}$, up to an (additive) error that decays at a rate of $O(|q|^{-2})$.

We organize the argument as follows. After some preliminaries, we display the solution for the case of $q_0 = dz^4$. Then we choose good charts away from a compact set which contains the zeroes that respect the geometry that q imposes on the plane $\mathbb{C}$: each such plane cuts off a region in $\mathbb{C}$ which is roughly a half-plane in the $|q|$ metric, positioned to develop in a controlled manner, with overlaps that also develop in a controlled way. In those charts, we find, roughly, that the minimal surface in the symmetric space $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ may be well-approximated by an isometric image of the flat defined by q_0. Describing those asymptotics carefully, up to some estimates deferred until the end of the section, occupies the first half of this section, and culminates in the proof of Theorem 3.9, Proposition 3.10 and Proposition 3.11. A careful treatment of the error estimates completes the section.

3.1. **Construction of the minimal surface.** In Section 1, we recalled how a solution to Hitchin's equation induces a harmonic map into a symmetric space. The construction goes as follows. Let H denote the associated Hermitian metric on $\mathcal{E}$ (guaranteed by Theorem 2.1). Let $\{N(z)\}_{z\in\mathbb{C}}$ be a holomorphic, Ω-symplectic, Q-adapted and H-unitary frame for the bundle $\mathcal{E}$. The frame $\{N(z)\}_{z\in\mathbb{C}}$ is not parallel for the $\mathrm{Sp}(4,\mathbb{R})$-connection $\nabla = D_H + \varphi + \varphi^{*H}$. Fix a base point z_0. We denote by $\{\mathcal{N}(z)\}_{z\in\mathbb{C}}$ the parallel transport of the frame $N(z_0)$ via the connection ∇. By expressing the metric H in the frame $\{\mathcal{N}(z)\}_{z\in\mathbb{C}}$, we obtain a map

$$\begin{aligned} f : \mathbb{C} &\to \mathrm{SL}(4,\mathbb{C}))/\mathrm{SU}(4) \\ z &\mapsto H^{\mathcal{N}(z)} \ . \end{aligned}$$

We then notice that the image of f is in fact contained in the copy of $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ inside $\mathrm{Sp}(4,\mathbb{C})/\mathrm{SU}(4)$, consisting of Ω-symplectic and Q-symmetric hermitian matrices with determinant 1.

Let us now find an explicit expression for $H^{\mathcal{N}(z)}$ (cfr. Equation (1.2)). Let $\{F(z)\}_{z\in\mathbb{C}}$ be the standard holomorphic frame of $\mathcal{E}$ where $H^{F(z)} = \mathrm{diag}(h_1, h_2^{-1}, h_1^{-1}, h_2)$.

We denote by $\{\mathcal{F}(z)\}_{z\in\mathbb{C}}$ the parallel transport of $F(z_0)$ with respect to ∇. For every $z \in \mathbb{C}$, we can find a matrix $\psi(z)$ such that

$$\mathcal{F}(z)\psi(z) = F(z) \ ,$$

i.e. $\psi(z)$ expresses the change of frame from $\mathcal{F}(z)$ to $F(z)$ at every point. Let $\gamma(s) = z_0 + se^{i\theta}$ be a path connecting the base point z_0 with z. We observe that the one-parameter family of matrices $\psi(s) = \psi(\gamma(s))$ satisfies the ordinary differential equation

$$\begin{cases} \frac{d\psi}{ds}(s) = \psi(s)(e^{i\theta}U + e^{-i\theta}V) \\ \psi(0) = \mathrm{Id} \end{cases} , \tag{3.1}$$

where we denoted by

$$U = D_H + \varphi \quad \text{and} \quad V = \varphi^{*H}$$

the $(1,0)$-part and the $(0,1)$-part of the connection ∇, respectively. To see this, note that equation (3.1) is a direct consequence of the fact that $\{\mathcal{F}(z)\}_{z\in\mathbb{C}}$ is parallel. Namely,

$$\begin{aligned} \mathcal{F}(\gamma(s))\frac{d\psi}{ds}(s) &= \nabla(\mathcal{F}\psi)(\gamma(s)) \\ &= \nabla_{\frac{\partial}{\partial s}} F(\gamma(s)) \\ &= F(\gamma(s))(e^{i\theta}U + e^{-i\theta}V) \\ &= \mathcal{F}(\gamma(s))\psi(\gamma(s))(e^{i\theta}U + e^{-i\theta}V) \ . \end{aligned}$$

Moreover, since the connection ∇ is flat, there exists a constant matrix $P \in \mathrm{SL}(4,\mathbb{C})$ such that $\mathcal{N}(z) = \mathcal{F}(z)P$. In fact, P is the change of frame between $N(z_0)$ and $F(z_0)$, i.e. $N(z_0) = F(z_0)P$. We thus deduce that

$$\begin{aligned} f(z) = H^{\mathcal{N}(z)} = H^{\mathcal{F}(z)P} = H^{F(z)\psi^{-1}(z)P} &= \overline{P^t(\psi(z)^{-1})^t} H^{F(z)}(\psi(z))^{-1}P \\ = \overline{P^t(\psi(z)^{-1})^t}\mathrm{diag}(h_1(z), h_2^{-1}(z), h_1^{-1}(z), h_2(z))(\psi(z))^{-1}P \ . \end{aligned}$$

We notice in particular that the geometry of the minimal surface will depend not only on the functions $h_i(z)$, but also on the solution to the ODE (3.1). This will play a fundamental role in Section 3.3.

3.2. **The case of constant quartic differential.** In the special case, when the quartic differential q is constant, the solution of the ODE (3.1) can be written explicitly and the minimal surface turns out to be a flat in $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$.

Up to biholomorphisms of $\mathbb{C}$, we can suppose that $q = dz^4$. As mentioned in Section 2, the solution to Hitchin's equations in this case is

$$H = \mathrm{diag}(1,1,1,1) \ .$$

As a consequence, the system of ODE (3.1) simplifies into

$$\begin{cases} \frac{d\psi_0}{ds}(s) = \psi_0(s)(e^{i\theta}U_0 + e^{-i\theta}V_0) \\ \psi_0(0) = \mathrm{Id} \end{cases} \tag{3.2}$$

where we are using the notation ψ_0 to indicate that we are dealing with the special case of constant quartic differential. Moreover, the $(1,0)$-part and the $(0,1)$-part of the connection $\nabla_0 = D_H + \varphi + \varphi^{*H}$ are respectively

$$U_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad V_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ,$$

as the Chern connection D_H of H vanishes in these coordinates, since H is constant. Therefore, equations (3.2) become a system of ODE with constant coefficients, that can be explicitly integrated:

$$\psi_0(s) = \exp\left(s \begin{pmatrix} 0 & 0 & e^{i\theta} & e^{-i\theta} \\ 0 & 0 & e^{-i\theta} & e^{i\theta} \\ e^{-i\theta} & e^{i\theta} & 0 & 0 \\ e^{i\theta} & e^{-i\theta} & 0 & 0 \end{pmatrix} \right) .$$

By observing that

$$S^{-1} \begin{pmatrix} 0 & 0 & e^{i\theta} & e^{-i\theta} \\ 0 & 0 & e^{-i\theta} & e^{i\theta} \\ e^{-i\theta} & e^{i\theta} & 0 & 0 \\ e^{i\theta} & e^{-i\theta} & 0 & 0 \end{pmatrix} S = \begin{pmatrix} 2\cos(\theta) & 0 & 0 & 0 \\ 0 & -2\sin(\theta) & 0 & 0 \\ 0 & 0 & -2\cos(\theta) & 0 \\ 0 & 0 & 0 & 2\sin(\theta) \end{pmatrix}$$

for a constant unitary matrix

$$S^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -i & i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i & -i \end{pmatrix} ,$$

we obtain that

$$\psi_0(s) = S \begin{pmatrix} e^{2s\cos(\theta)} & 0 & 0 & 0 \\ 0 & e^{-2s\sin(\theta)} & 0 & 0 \\ 0 & 0 & e^{-2s\cos(\theta)} & 0 \\ 0 & 0 & 0 & e^{2s\sin(\theta)} \end{pmatrix} S^{-1} .$$

If we fix the origin as base point z_0 in the definition of the harmonic map f_0 (see Section 1), and write $z = se^{i\theta}$, then $f_0 : \mathbb{C} \to \mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ is given by

$$\begin{aligned} f_0(z) &= \overline{(\psi_0(z)^{-1})^t}\mathrm{diag}(1,1,1,1)\psi_0(z)^{-1} \\ &= \overline{(S^{-1})^t}\mathrm{diag}(e^{-4\Re(z)}, e^{4\Im(z)}, e^{4\Re(z)}, e^{-4\Im(z)})S^{-1} \\ &= S \cdot \mathrm{diag}(e^{-4\Re(z)}, e^{4\Im(z)}, e^{4\Re(z)}, e^{-4\Im(z)}) , \end{aligned}$$

where we denoted with $\cdot$ the action of an element $g \in \mathrm{SL}(4,\mathbb{C})$. This shows that the image of f_0 is a maximal flat in the symmetric space.

3.3. **The general case.** In order to study the general case, i.e. when the quartic differential q is an arbitrary polynomial $q(z)$ of degree $n \geq 1$, the main idea is to estimate the solution to Equation (3.1) by comparing it with the solution to Equation (3.2). In fact, the complement of a compact set containing the roots of $q(z)$ is covered by $(n+4)$ charts, which are conformal to the upper-half plane, where the quartic differential q is constant. This suggests that in each chart the solution to Equation (3.1) should look like the solution to Equation (3.2), at least when we are far enough from the zeros of the polynomial $q(z)$. We will thus describe the asymptotic geometry of the associated minimal surface and we will focus, in particular, on studying an interesting "Stokes phenomenon", that already occurred for affine spheres with polynomial Pick differential ([DW15]) (and indeed implicitly in [HTTW95]).

3.3.1. *Standard half-planes and rays.* Given a quartic differential q, a natural coordinate w for q is a local coordinate on an open set of $\mathbb{C}$ in which $q = dw^4$. Such a coordinate always exists locally away from the zeros of q, as it is possible to choose a holomorphic fourth root of q and define

$$w(z) = \int_{z_0}^{z} q^{\frac{1}{4}} \ .$$

Notice, in particular, that a natural coordinate is not unique, but every two natural coordinates for q differ by a multiplication by a fourth root of unity and an additive constant.

Definition 3.1. We define a q-half-plane (or a standard half-plane, when the reference to the differential q is obvious) as a pair (U, w), where $U \subset \mathbb{C}$ is open and w is a natural coordinate for q that maps U diffeomorphically to the upper-half plane $\{\Im(w) > 0\}$. Note that U then determines w up to addition of a real constant.

A path in $\mathbb{C}$ whose image in a natural coordinate for q is a Euclidean ray with angle θ is called *a q-ray of angle θ*. (Note that the angle is well-defined mod $\frac{\pi}{2}$.) This means that in a suitable natural coordinate, a q-ray is parameterized by $t \mapsto b + e^{i\theta}t$. Similarly, a *$q$-quasi-ray with angle θ* is a path that can be parameterized so that its image in a natural coordinate w is $t \mapsto e^{i\theta}t + o(t)$.

It turns out that every monic polynomial quartic differential q admits a finite number of q-half-planes that cover the complement in $\mathbb{C}$ of a compact set containing the zeros of q.

Proposition 3.2 ([DW15])**.** *Let q be a monic polynomial quartic differential and let K be a compact subset of $\mathbb{C}$ containing the zeros of q. Suppose q has degree $n \geq 1$.*

Then, there exist a compact subset $K' \supseteq K$ and a collection of $(n+4)$ q-half-planes $\{(U_k, w_k)\}_{k=1,\dots,n+4}$ with the following properties:

i) the complement of $\bigcup_k U_k$ is K';
ii) the ray $\{\arg(z) = \frac{2\pi k}{n+4}\}$ is eventually contained in U_k;
iii) the rays $\{\arg(z) = \frac{2\pi(k\pm1)}{n+4}\}$ are disjoint from U_k;
iv) on $U_k \cap U_{k+1}$ we have $w_{k+1} = iw_k + c$ for some constant c and each w_k, w_{k+1} maps this intersection onto a sector of angle $\frac{\pi}{2}$ based at a real point.
v) any Euclidean ray in $\mathbb{C}$ is a q-quasi ray and is eventually contained in U_k for some k.

Let $r : \mathbb{C} \to \mathbb{R}^+$ be the $|q|^{\frac{1}{2}}$-distance from the zeros of q. We recall the following result that will be used in Section 3.3.5.

Proposition 3.3 ([DW15])**.** *Let q be a monic polynomial quartic differential and let K be a compact set containing the zeros of q. Then there are constant A, a, R_0 with $a > 0$ so that for every point $p \in \mathbb{C}$ with $r(p) > R_0$, there exists a q-half-plane (U, w) with $U \cap K = \emptyset$ such that $\Im(w(p)) \geq r(p) - A$. In addition, on the boundary of this half-plane we have $r(x) \geq a|\Re(w(x))|$, for x large.*

We remark that the monic condition in the above propositions is not restrictive, as every polynomial can be made monic via a biholomorphic change of coordinates on $\mathbb{C}$.

3.3.2. *Comparing ψ_0 and ψ.* Let us fix the origin of $\mathbb{C}$ as base point z_0 in the construction of the harmonic map (see Section 3.1). By Proposition 3.2, any point z far enough from z_0 is connected by a ray $\gamma(s) = se^{i\theta}$, which is definitely contained in a standard half-plane. Therefore, there exists a time $s_0 > 0$, such that the ray $\gamma(s)$ lies in a standard half-plane for every $s \geq s_0$. We can write thus a differential equation satisfied by $\psi\psi_0^{-1}(s)$ for $s \geq s_0$ using Equation (3.1) and Equation (3.2):

$$\begin{cases} \frac{d\psi\psi_0^{-1}}{ds}(s) = \psi\psi_0^{-1}(s)(\psi_0(s)R\psi_0(s)^{-1}) \\ \psi\psi_0^{-1}(s_0) = A_0 \end{cases}, \tag{3.3}$$

for some matrix $A_0 \in \mathrm{Sp}(4, \mathbb{C})$, which represents the difference between ψ_0 and ψ at the point s_0. In Equation (3.3) we have denoted

$$\begin{aligned} R &= \psi^{-1}(s)\frac{d\psi}{ds}(s) - \psi_0^{-1}(s)\frac{d\psi_0}{ds}(s) \\ &= e^{i\theta}(U - U_0) + e^{-i\theta}(V - V_0) + e^{i\theta}D_H \end{aligned}$$

the error between the connection ∇_0 and ∇. Let us denote by $\tilde{u}_j$ the functions

$$\tilde{u}_1 = u_1 - \frac{3}{8}\log(|q|^2) \quad \text{and} \quad \tilde{u}_2 = u_2 - \frac{1}{8}\log(|q|^2)\ ,$$

which represent the error between the solution to Equation (2.1) and the particular solution in the case of constant quartic differential. By Corollary 2.6, the function $\tilde{u}_j$ decays as $|z| \to +\infty$. We can now write the error R in terms of the function $\tilde{u}_j$

in the natural coordinate chart w of the half-plane.

First, since $q = dw^4$, the term $U - U_0$ vanishes. Moreover, since the metric H is diagonal, it is easy to verify that the Chern connection D_H is

$$D_H = H^{-1}\partial H = \begin{pmatrix} -\partial\tilde{u}_1 & 0 & 0 & 0 \\ 0 & \partial\tilde{u}_2 & 0 & 0 \\ 0 & 0 & \partial\tilde{u}_1 & 0 \\ 0 & 0 & 0 & -\partial\tilde{u}_2 \end{pmatrix} ,$$

because in a natural coordinate $\tilde{u}_j = u_j$. Finally, by definition of V and V_0, we have

$$V - V_0 = \begin{pmatrix} 0 & 0 & 0 & e^{\tilde{u}_1-\tilde{u}_2} - 1 \\ 0 & 0 & e^{\tilde{u}_1-\tilde{u}_2} - 1 & 0 \\ e^{-2\tilde{u}_1} - 1 & 0 & 0 & 0 \\ 0 & e^{2\tilde{u}_2} - 1 & 0 & 0 \end{pmatrix} .$$

Let us denote $D = \mathrm{diag}(e^{2s\cos(\theta)}, e^{-2s\sin(\theta)}, e^{-2s\cos(\theta)}, e^{2s\sin(\theta)})$ and $R' = S^{-1}RS$, where S is the unitary matrix introduced in Section 3.2. We can then write the error term as

$$\begin{aligned} \Theta(s) = \psi_0(s)R\psi_0(s)^{-1} &= SDS^{-1}RSD^{-1}S^{-1} = SDR'D^{-1}S^{-1} \\ &= SD(R'_1 + R'_2)D^{-1}S^{-1} \end{aligned}$$

with

$$\begin{aligned} R'_1 &= e^{i\theta}S^{-1}\mathrm{diag}(-\partial\tilde{u}_1, \partial\tilde{u}_2, \partial\tilde{u}_1, -\partial\tilde{u}_2)S \\ &= -\frac{e^{i\theta}}{4}\begin{pmatrix} 0 & (1-i)\partial(\tilde{u}_1+\tilde{u}_2) & 2\partial(\tilde{u}_1-\tilde{u}_2) & (1+i)\partial(\tilde{u}_1+\tilde{u}_2) \\ (1+i)\partial(\tilde{u}_1+\tilde{u}_2) & 0 & (1-i)\partial(\tilde{u}_1+\tilde{u}_2) & 2\partial(\tilde{u}_1-\tilde{u}_2) \\ 2\partial(\tilde{u}_1-\tilde{u}_2) & (1+i)\partial(\tilde{u}_1+\tilde{u}_2) & 0 & (1-i)\partial(\tilde{u}_1+\tilde{u}_2) \\ (1-i)\partial(\tilde{u}_1+\tilde{u}_2) & 2\partial(\tilde{u}_1-\tilde{u}_2) & (1+i)\partial(\tilde{u}_1+\tilde{u}_2) & 0 \end{pmatrix} \end{aligned} \tag{3.4}$$

and

$$R'_2 = e^{-i\theta}S^{-1}(V - V_0)S .$$

If we introduce the notation $u_3 = -u_2$ and $u_4 = -u_1$, an elementary but tedious computation shows that

$$R'_{kl} = -\frac{e^{i\theta}}{4}\sum_{j=0}^{3} i^{(k-l)j}\partial\tilde{u}_{j+1} + \frac{e^{-i\theta}i^{1-k}}{4}\sum_{j\in\mathbb{Z}_4} i^{(k-l)j}e^{\tilde{u}_j-\tilde{u}_{j+1}} \quad \text{for } k \neq l$$

and

$$R'_{kk} = \frac{e^{-i\theta}(-i)^{k-1}}{4}(e^{-2\tilde{u}_1} + 2e^{\tilde{u}_1-\tilde{u}_2} + e^{2\tilde{u}_2} - 4) .$$

In Section 3.3.5 we will prove the following:

Proposition 3.4. *Let r be the distance from the zeros of q. Then for $r \to +\infty$*

$$R'_{kl} = O\left(\frac{e^{-2|1-i^{k-l}|r}}{\sqrt{r}}\right) \quad \text{if } k \neq l ,$$

and

$$R'_{kk} = o\left(\frac{e^{-2\sqrt{2}r}}{\sqrt{r}}\right) .$$

These exponential decays allow us to find a limit of $\psi\psi_0^{-1}(s)$ along rays in any "stable direction", which we now define.

Definition 3.5. Let $\gamma(t) = b + e^{i\theta}t$ be a ray in a standard half-plane. The direction of the ray is the angle $\theta \in [0, \pi]$. We say that the ray is stable if $\theta \notin \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$. Similarly, a quasi-ray is stable, if the direction of the associated ray is stable.

The possible directions of stable rays form four intervals of length $\frac{\pi}{4}$ which we denote by

$$J_{++} = \left(0, \frac{\pi}{4}\right) \quad J_{+} = \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \quad J_{-} = \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \quad J_{--} = \left(\frac{3\pi}{4}, \pi\right) .$$

The stability of rays and quasi-rays is related to the convergence of $\psi\psi_0^{-1}(\gamma(s))$:

Lemma 3.6. *If γ is a stable ray or quasi-ray, then the limit $\lim_{s\to+\infty} \psi\psi_0^{-1}(\gamma(s))$ exists. Furthermore, among all such rays only four limits are seen, i.e. there exist $L_{++}, L_{+}, L_{-}, L_{--} \in \mathrm{Sp}(4, \mathbb{C})$ such that*

$$\lim_{s\to+\infty} \psi\psi_0^{-1}(\gamma(s)) = \begin{cases} L_{++} & \text{if } \theta \in J_{++} \\ L_{+} & \text{if } \theta \in J_{+} \\ L_{-} & \text{if } \theta \in J_{-} \\ L_{--} & \text{if } \theta \in J_{--} \end{cases}$$

Proof. First we consider rays, and at the end of the proof we show that quasi-rays have the same behaviour.

Let γ be a ray and let us write $G(s) = \psi\psi_0^{-1}(\gamma(s))$. It satisfies the ODE

$$\begin{cases} \frac{dG}{ds}(s) = G(s)\Theta(s) \\ G(0) = A_0 \end{cases}$$

for some $A_0 \in \mathrm{Sp}(4, \mathbb{C})$. Recalling the definition of $\Theta(s) = SDR'D^{-1}S^{-1}$, the decay of the error $\Theta(s)$ is determined by comparing the decay of R' and the growth of the diagonal matrix $D = \mathrm{diag}(e^{2s\cos(\theta)}, e^{-2s\sin(\theta)}, e^{-2s\cos(\theta)}, e^{2s\sin(\theta)})$. Conjugating R' by the diagonal matrix $D(s)$ multiplies the entry R'_{kl} by

$$\lambda_{kl} = \exp\left(2s\left(\cos\left(\theta - \frac{(k-1)\pi}{2}\right)\right) - \cos\left(\theta - \frac{(l-1)\pi}{2}\right)\right) .$$

Combining this with Proposition 3.4, we deduce that for any stable ray, we have a definite exponential decay in the equation satisfied by G, i.e.

$$G(s)^{-1}G'(s) = O\left(\frac{e^{-\alpha s}}{\sqrt{s}}\right)$$

for some $\alpha > 0$ (where $\alpha = \alpha(\theta)$ depends on θ). Standard ODE techniques (see [DW15, Appendix B]) then show that the limit $\lim_{s\to+\infty} G(s)$ exists.

Now suppose that γ_1 and γ_2 are stable rays with angles θ_1 and θ_2 that belong to the same interval (J_{++}, J_+, J_-, or J_{--}). We will show that $G_1(s)G_2(s)^{-1} \to \mathrm{Id}$ as $s \to +\infty$, where $G_i(s) = \psi\psi_0^{-1}(\gamma_i(s))$. This means that the limit does not depend on the direction of the ray in a same interval, thus concluding the proof.

For any $s > 0$, let $\eta_s(t) = (1-t)\gamma_1(s)+t\gamma_2(s)$ be the constant speed parameterization of the segment from $\gamma_1(s)$ and $\gamma_2(s)$. Let $g_s(t) = (\psi\psi_0^{-1}(\eta_s(0)))^{-1}\psi\psi_0^{-1}(\eta_s(t))$, which satisfies

$$\begin{cases} g_s^{-1}(t)g_s'(t) = \Theta(\eta_s(t))\eta_s'(t) \\ g_s(0) = \mathrm{Id} \\ g_s(1) = G_1(s)^{-1}G_2(s) \end{cases} .$$

Since $|\eta_s'(t)| = O(s)$, the analysis above shows that $g_s^{-1}g_s'(t) = O(\sqrt{s}e^{-\alpha s})$, for some $\alpha > 0$, because the path $\eta_s(t)$ never crosses an unstable direction. In particular, by making s large enough we can arrange for $g_s(t)^{-1}g_s'(t)$ to be uniformly small for all $t \in [0, 1]$. Once again standard ODE methods ([DW15, Lemma B.1 (i)]) give the desired convergence,

$$G_1(s)^{-1}G_2(s) = g_s(1) \to \mathrm{Id} \quad \text{as} \quad s \to +\infty \ .$$

Finally, suppose that γ_1 is a stable quasi-ray, and γ_2 is the ray that approximates γ_1 with direction θ. We can study as above the homotopy $\eta_s(t) = (1-t)\gamma_1(s)+t\gamma_2(s)$ between the ray and the quasi-ray. Since we have a bound $|\eta_s'(t)| = o(s)$, the previous argument applies, thus $G(s)$ has the same limit along the stable quasi-ray γ_1 and along the associated stable ray γ_2. □

We now investigate how limits along rays in different intervals are related.

Lemma 3.7. *Let L_{++}, L_+, L_-, L_{--} be as in the previous lemma. Then there exist unipotent matrices U_+, U_0, U_-, such that*

$$L_{++}^{-1}L_+ = SU_+S^{-1}, \quad L_+^{-1}L_- = SU_0S^{-1} \quad \textit{and} \quad L_-^{-1}L_{--} = SU_-S^{-1} \ .$$

Proof. We give a detailed proof for $L_+^{-1}L_-$, the other cases being analogous.

Consider the rays $\gamma_+(s) = e^{i\frac{3\pi}{8}}s$ and $\gamma_-(s) = e^{i\frac{5\pi}{8}}s$. By the previous lemma $G_+(s) = \psi\psi_0^{-1}(\gamma_+(s))$ and $G_-(s) = \psi\psi_0^{-1}(\gamma_-(s))$ have respective limits L_+ and L_-. For any $s > 0$, we can join $\gamma_+(s)$ and $\gamma_-(s)$ by a circular arc

$$\eta_s(t) = e^{i(t+\frac{3\pi}{8})}s \quad \text{for} \quad t \in \left[0, \frac{\pi}{4}\right] \ .$$

Let $g_s(t) = (\psi\psi_0^{-1}(\eta_s(0)))^{-1}\psi\psi_0^{-1}(\eta_s(t))$, which satisfies

$$\begin{cases} g_s^{-1}(t)g_s'(t) = \Theta(\eta_s(t))\eta_s'(t) \\ g_s(0) = \mathrm{Id} \\ g_s(\pi/4) = G_+(s)^{-1}G_-(s) \ . \end{cases}$$

Unlike the previous case, however, the coefficient $\Theta(\eta_s(t))$ is not exponentially small in s throughout the interval. At $t = \frac{\pi}{8}$, conjugation by the diagonal matrix D multiplies the $(4,2)$-entry of R' by a factor e^{4s}, exactly matching the exponential decay rate of R' and giving

$$\Theta\left(\eta_s\left(\frac{\pi}{8}\right)\right) = O(\sqrt{s}) \ .$$

However, this potential growth is seen only in the $(4,2)$-entry, because the other entries are scaled by smaller exponential factors. In fact, for $t \in \left[0, \frac{\pi}{4}\right]$, we have

$$\lambda_{42} = \exp\left(4\cos\left(t - \frac{\pi}{8}\right)\right) \leq \exp\left(4 - \left(t - \frac{\pi}{8}\right)^2\right) \ .$$

We can thus separate the unbounded entry in $\Theta(\eta_s(t))$ and write

$$\Theta(\eta_s(t)) = \Theta^0(\eta_s(t)) + \mu_s(t) S E_{42} S^{-1}$$

where $\Theta^0_s(t) = O(e^{-\alpha s})$ for some $\alpha > 0$, E_{42} is the elementary matrix, and

$$\mu_s(t) = O\left(|\eta_s'(t)|\lambda_{42}\frac{e^{-4s}}{\sqrt{s}}\right) = O(\sqrt{s}\exp(-(\pi/8 - t)^2 s)) \ .$$

This upper bound is a Gaussian function on t, renormalized such that its integral over $\mathbb{R}$ is independent of s. Therefore, the function $\mu_s(t)$ is uniformly absolutely integrable over $t \in \left[0, \frac{\pi}{4}\right]$. We can apply [DW15, Lemma B.2] and conclude that

$$\left\| g_s(\pi/4) - S\exp\left(E_{42}\int_0^{\frac{\pi}{4}} \mu_s(t)dt\right) S^{-1} \right\| \to 0 \quad \text{as} \quad s \to +\infty \ .$$

Since $g_s(\pi/4) \to L_+^{-1}L_-$ as $s \to +\infty$, we obtain the desired unipotent difference. □

3.3.3. *Asymptotic behaviour of the minimal surface.* We can now describe the asymptotic geometry of the minimal surface $S = f(\mathbb{C})$ in the general case. We begin with the definition of "asymptotic" minimal surfaces adapted to our setting.

Definition 3.8. We say that two minimal surfaces S_1 and S_2 in a Riemannian manifold Y are asymptotic if there is a domain Riemann surface X and conformal harmonic parametrizations $u_i : X \to Y$ of S_i so that $u_1(x) = A(x)u_2(x)$ for some isometries $A(x)$ of Y such that $A(x) \to \mathrm{Id}$ as x leaves compacta in X.

Theorem 3.9. *Let q be a monic polynomial quartic differential of degree $n \geq 1$. Then the associated minimal surface S is asymptotic to $2(n+4)$ maximal flats in* $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$.

Proof. We start by proving that in each standard half-plane (U, w) given by Proposition 3.2 the surface S is asymptotic to four maximal flats, one for each interval of stable directions.

We give the detailed proof for the sector J_+, the other cases being analogous. We recall that S is parameterized by the map

$$f(w) = \overline{P^t(\psi(w)^{-1})^t}\mathrm{diag}(h_1(w), h_2^{-1}(w), h_1^{-1}(w), h_2(w))\psi(w)^{-1}P$$

for some $P \in \mathrm{Sp}(4,\mathbb{C})$. We compare $f(w)$ with the flat parameterized by

$$f_0(w) = \overline{(\psi_0(w)^{-1})^t}\psi_0(w)^{-1} \ :$$

by Corollary 2.6 and Lemma 3.6, the limit

$$\lim_{\substack{|w|\to+\infty \\ w\in J_+}} P^{-1}\psi(w)H^{-\frac{1}{2}}(w)\psi_0(w)^{-1} := M_+$$

exists for some M_+ in the standard copy of $\mathrm{Sp}(4,\mathbb{R})$ fixed in Section 1. We now claim that S is asymptotic to the flat parameterized by

$$f_0(w) = \overline{(M_+^{-1})^t(\psi_0(w)^{-1})^t}\psi_0(w)^{-1}M_+^{-1} \ .$$

Namely, the map $g(w) = P^{-1}\psi(w)H(w)^{-\frac{1}{2}}\psi_0(w)^{-1}M_+^{-1}$ is an isometry sending $f_0(w)$ to $f(w)$ such that

$$\lim_{\substack{|w|\to+\infty \\ w\in J_+}} g(w) = \mathrm{Id} \ .$$

This would give a total number of $4(n+4)$ maximal flats to which S is asymptotic, but we can actually see that in two overlapping standard half-planes U_k and U_{k+1} two of the four flats coincide. In fact, by the above discussion, we can notice that the asymptotic flat depends only on the limit of $\psi(z)\psi_0^{-1}(z)$, which itself only depends on the half-plane U_k in which z eventually lies and on the specific sector $J^k_{++}, J^k_+, J^k_-, J^k_{--}$ in which z is approaching infinity. Since in the intersection of the two charts U_k and U_{k+1} the natural coordinates differ by a multiplication by i and by an additive constant, a quasi-ray of angle θ in the w_k-coordinate has direction $\theta + \frac{\pi}{2}$ in the w_{k+1}-coordinate. Therefore, the sector J^{k+1}_{--} gets identified with J^k_+ and the sector J^{k+1}_- gets identified with J^k_{++} by the change of coordinates. Hence, the limits in those directions coincide and it follows that we only have a total number of $2(n+4)$ maximal flats. □

The theorem, and especially the discussion in the final paragraph of the proof, provides a natural meaning to a pair of "consecutive flats asymptotic to the minimal surface S".

We can also describe precisely the combinatorics of the collection of flats at infinity.

Proposition 3.10. *Two consecutive flats asymptotic to the minimal surface S share four adjacent Weyl chambers at infinity.*

Proof. Recall that a Weyl chamber at infinity is the stabilizer of a minimal parabolic subgroup $P \subset \mathrm{Sp}(4,\mathbb{R})$ acting on the boundary at infinity of the symmetric space. In our case, a Weyl chamber at infinity can thus be described by a complete Lagrangian flag, that is by a collection of vector subspaces of $\mathbb{R}^4$

$$\mathcal{F} = \{\{0\} \subset \ell \subset L \subset \ell^{\perp_\omega} \subset \mathbb{R}^4\}$$

where ℓ is a line, L is a Lagrangian plane and $\ell^{\perp_\omega}$ denotes the hyperplane orthogonal to ℓ with respect to the symplectic form ω on $\mathbb{R}^4$ that $\mathrm{Sp}(4,\mathbb{R})$ preserves. Notice

that the data of ℓ and L already determine the flag uniquely.
We are going to show that the intersection of the two consecutive flats

$$F_1(w) = \overline{(M_-^{-1})^t(\psi_0(w)^{-1})^t}\psi_0(w)^{-1}M_-^{-1}$$

and

$$F_2(w) = \overline{(M_+^{-1})^t(\psi_0(w)^{-1})^t}\psi_0(w)^{-1}M_+^{-1}$$

constructed in Theorem 3.9 share four Weyl chambers at infinity; the proof for the other cases in analogous. Recall that in Section 1, we identified $\mathrm{Sp}(4,\mathbb{R})$ as the subgroup of $\mathrm{Sp}(4,\mathbb{C})$ fixed by the anti-linear involution λ, and we pointed out that the map

$$\begin{aligned}\mathrm{Sp}(4,\mathbb{C})/\mathrm{SU}(4) &\to \mathrm{Sp}(4,\mathbb{C})/\mathrm{SU}(4)\\ [g] &\mapsto \overline{(g^{-1})^t}g^{-1}\end{aligned}$$

induces an isometry between two models of the $\mathrm{Sp}(4,\mathbb{R})$-symmetric space: as cosets and also as the space of Q-symmetric and Ω-symplectic hermitian metrics on $\mathbb{C}^4$. Using such correspondence, the flats F_1 and F_2 can equivalently be described by the matrices

$$F_1(w) = M_-\psi_0(w) \quad \text{and} \quad F_2(w) = M_+\psi_0(w)\ .$$

Moreover, from Lemma 3.6 and Lemma 3.7 we know that $M_- = P^{-1}L_+SU_0S^{-1}$ and $M_+ = P^{-1}L_+$, so, together with the fact that

$$\psi_0(w) = S\mathrm{diag}(e^{2\Re(z)}, e^{-2\Im(z)}, e^{-2\Re(z)}, e^{2\Im(z)})S^{-1}\ ,$$

we deduce that the intersection at infinity of the flats F_1 and F_2 only depends on how the matrix U_0 acts on the Weyl chambers at infinity of the maximal flat of diagonal matrices. Since $U_0 = \mathrm{Id} + \mu E_{42}$, for some $\mu \neq 0$, it is straightforward to check that U_0 does not preserve the Weyl chambers at infinity corresponding to the Lagrangian flags (that involve e_2)

$$\begin{aligned}&\{0\} \subset \mathrm{Span}(e_3) \subset \mathrm{Span}(e_3, e_2) \subset \mathrm{Span}(e_3, e_2, e_4) \subset \mathbb{R}^4\\ &\{0\} \subset \mathrm{Span}(e_2) \subset \mathrm{Span}(e_3, e_2) \subset \mathrm{Span}(e_3, e_2, e_1) \subset \mathbb{R}^4\\ &\{0\} \subset \mathrm{Span}(e_2) \subset \mathrm{Span}(e_1, e_2) \subset \mathrm{Span}(e_3, e_2, e_1) \subset \mathbb{R}^4\\ &\{0\} \subset \mathrm{Span}(e_1) \subset \mathrm{Span}(e_1, e_2) \subset \mathrm{Span}(e_4, e_2, e_1) \subset \mathbb{R}^4\ .\end{aligned}$$

Therefore, F_1 and F_2 share four adjacent Weyl chambers at infinity because $\mathrm{Sp}(4,\mathbb{R})$ is a Lie group with root system of type B_2. □

3.3.4. *The induced metric on the minimal surface.* Using the bounds (2.4) and (2.5) we prove that the harmonic map $f : \mathbb{C} \to \mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ is a quasi-isometric embedding if $\mathbb{C}$ is endowed with the flat metric with cone singularities $|q|^{\frac{1}{2}}$.

Proposition 3.11. *The induced metric on the minimal surface $S = f(\mathbb{C})$ is quasi-isometric to $4|q|^{\frac{1}{2}}$, with quasi-isometric constant $1 + O(|q|^{-2})$ on the end of S. In particular, it is complete.*

Proof. Recall that the induced metric g_f on S can be expressed in terms of the Higgs field φ

$$g_f = \mathrm{tr}(\varphi\varphi^{*H}) = h_1^2|q|^2 + 2h_2h_1^{-1} + h_2^{-2} \ .$$

Moreover, the sub-solution and super-solution found in Section 2 provide the following upper- and lower-bounds for the metric H satisfying Hitchin's equations

$$(|q|^2 + C)^{-\frac{3}{8}} \leq h_1 \leq |q|^{-\frac{3}{4}}$$
$$(|q|^2 + 3C)^{-\frac{1}{8}} \leq h_2 \leq |q|^{-\frac{1}{4}}$$

for some positive constant C. We deduce that, as $|z| \to \infty$, we have

$$\begin{aligned} g_f = h_1^2|q|^2 + 2h_2h_1^{-1} + h_2^{-2} &\geq (|q|^2 + C)^{-\frac{3}{4}}|q|^2 + 2(|q|^2 + 3C)^{-\frac{1}{8}}|q|^{\frac{3}{4}} + |q|^{\frac{1}{2}} \\ &\geq |q|^{\frac{1}{2}}(1-\epsilon) + 2|q|^{\frac{1}{2}}(1-3\epsilon) + |q|^{\frac{1}{2}} \\ &\geq 4|q|^{\frac{1}{2}}(1+\epsilon) \ . \end{aligned}$$

where the terms ϵ stand for terms that satisfy $\epsilon \asymp |q|^{-2}$. As for the upper-bound, a similar argument shows that

$$\begin{aligned} g_f = h_1^2|q|^2 + 2h_2h_1^{-1} + h_2^{-2} &\leq |q|^{\frac{1}{2}} + 2|q|^{-\frac{1}{4}}(|q|^2 + C)^{\frac{3}{8}} + (|q|^2 + 3C)^{\frac{1}{4}} \\ &\leq |q|^{\frac{1}{2}} + 2|q|^{\frac{1}{2}}(1+\epsilon) + |q|^{\frac{1}{2}}(1+3\epsilon) \\ &\leq 4|q|^{\frac{1}{2}}(1+\epsilon) \ . \end{aligned}$$

where again the terms ϵ represent terms of the comparability class $\epsilon \asymp |q|^{-2}$. It thus follows that outside a compact set K the induced metric is controlled by a multiple of $4|q|^{\frac{1}{2}}$, and decays to that quantity at a rate comparable to $|q|^{-2}$. The quasi-isometry between $4|q|^{\frac{1}{2}}$ and g_f is then obtained by noticing that, since K is compact, the metrics $4|q|^{\frac{1}{2}}$ and g_f are trivially quasi-isometric on K; the claimed asymptotics of the quasi-isometric ratio is immediate. In particular, the induced metric g_f on the minimal surface S is complete because the metric $|q|^{\frac{1}{2}}$ is complete. □

3.3.5. *Estimates of the error term.* This section is dedicated to the proof of Proposition 3.4. Let us define the following auxiliary functions

$$w_1 = -\frac{1}{2}\sum_{j\in\mathbb{Z}_4} i^j(\tilde{u}_j - \tilde{u}_{j+1}) = \tilde{u}_1 + \tilde{u}_2 = -w_3$$
$$w_2 = -\frac{1}{2}\sum_{j\in\mathbb{Z}_4} (-1)^j(\tilde{u}_j - \tilde{u}_{j+1}) = 2(\tilde{u}_1 - \tilde{u}_2) \ ,$$

where we are using the notation $\tilde{u}_3 = -\tilde{u}_2$ and $\tilde{u}_4 = -\tilde{u}_1$. We recall that in a natural coordinate w on a half-plane, the functions $\tilde{u}_1$ and $\tilde{u}_2$ satisfy the following system of PDE

$$\begin{cases} \Delta\tilde{u}_1 = e^{\tilde{u}_1 - \tilde{u}_2} - e^{-2\tilde{u}_1} \\ \Delta\tilde{u}_2 = e^{2\tilde{u}_2} - e^{\tilde{u}_1 - \tilde{u}_2} \ . \end{cases} \tag{3.5}$$

Therefore, a simple computation shows that the error term R'_{kl} (cfr. Equation (3.4)) can be written in terms of the derivatives of w_{k-l}, when $k > l$. In fact,

$$R'_{kl} = -\frac{i^{l-k}}{2(1-i^{l-k})}\partial w_{k-l} + \frac{i^{1-k}}{2(1-i^{l-k})(1-i^{k-l})}\Delta w_{k-l} \ . \tag{3.6}$$

Notice that the symmetries of the matrix R' imply that the asymptotic behaviour of R'_{kl} depends only on $k-l$ and it is sufficient to estimate the cases where $k-l = 1, 2$.

Proposition 3.4 will then be a consequence of the following estimate:

Lemma 3.12. *Let r be the distance from the zeros of q. Then for $r \to +\infty$ we have*

$$w_1 = O\left(\frac{e^{-2\sqrt{2}r}}{\sqrt{r}}\right) \ , \ w_2 = O\left(\frac{e^{-4r}}{\sqrt{r}}\right) \qquad \text{and} \qquad \Delta w_j = O(w_j) \text{ for } j = 1, 2 \ .$$

Proof of Proposition 3.4. Let us start with the terms R'_{kl} for $k \neq l$. In view of Equation (3.6), it is sufficient to estimate Δw_{k-l} and ∂w_{k-l}. Lemma 3.12 asserts that $\Delta w_{k-l} = O(w_{k-l})$ for $r \to +\infty$, hence

$$\Delta w_{k-l} = O\left(\frac{e^{-2|1-i^{k-l}|r}}{\sqrt{r}}\right)$$

by Lemma 3.12. The bound on ∂w_{k-l} follows then from the Schauder estimates applied to Δw_{k-l} in a ball of radius $r_0 \asymp 1$ about a point at distance comparable to r from the zeroes of q.

As for the terms on the diagonal,

$$R'_{kk} = \frac{e^{-i\theta}(-i)^{k-1}}{4}(e^{-2\tilde{u}_1} + 2e^{\tilde{u}_1-\tilde{u}_2} + e^{2\tilde{u}_2} - 4) \ ,$$

since $\tilde{u}_1$ and $\tilde{u}_2$ are infinitesimal as $r \to +\infty$, we deduce that $R'_{kk} = o(\tilde{u}_j)$ for $j = 1, 2$. In particular, $R'_{kk} = o(w_1)$ and the estimate follows. □

The proof of Lemma 3.12 relies on some results already proved in [DW15].

Lemma 3.13. *Let $g \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$ be a positive function. Then for every positive constant k there exists a function $h \in C^\infty(\mathbb{H}) \cap C^0(\overline{\mathbb{H}})$ such that*

$$\begin{cases} \Delta h = kh \\ h_{|\mathbb{R}} = g \end{cases} \ .$$

Moreover, h satisfies

$$0 \leq h \leq \sup(g)$$
$$h = O\left(\|g\|_1 \frac{e^{-2\sqrt{k}y}}{\sqrt{y}}\right) \quad \text{for} \quad y = \Im(z) \to +\infty \ .$$

Lemma 3.14. *Let $g \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$ be a positive function satisfying $g \leq \frac{1}{4k'}$ for some $k' > 0$. Then, there exists a function $v \in C^\infty(\mathbb{H}) \cap C^0(\overline{\mathbb{H}})$ such that*

$$\begin{aligned} &v_{|\mathbb{R}} \geq g \\ &\Delta v \leq kv - kk'v^2 \\ &v = O\left(\|g\|_1 \frac{e^{-2\sqrt{k}y}}{\sqrt{y}}\right) \quad \textit{for} \quad y = \Im(z) \to +\infty \ . \end{aligned}$$

Proof. Consider an arbitrary solution h of the equation $\Delta h = kh$. The function $v = h - k'h^2$ satisfies

$$\begin{aligned} \Delta v - kv + k'kv^2 &= \Delta h - k'\Delta h^2 - kh + kk'h^2 + kk'(h - k'h^2)^2 \\ &= -k'(2h\Delta h + 2|\nabla h|^2) + 2kk'h^2 + kk'^3h^4 - 2kk'^2h^3 \\ &= -2k'|\nabla h|^2 + kk'^3h^4 - 2kk'^2h^3 \\ &\leq kk'^3h^4 - 2kk'^2h^3 \leq 0 \end{aligned}$$

provided $h \leq \frac{2}{k'}$. This condition is satisfied if we take as h the solution provided by Lemma 3.13 with boundary value $2g$, as

$$0 \leq h \leq 2\sup(g) \leq \frac{1}{2k'} \leq \frac{2}{k'} \ .$$

Therefore, $v = h - k'h^2$ satisfies

$$\begin{aligned} &0 < v < h \\ &\Delta v \leq kv - kk'v^2 \ . \end{aligned}$$

In particular, by Lemma 3.13, we deduce that $v = O\left(\|g\|_1 \frac{e^{-2\sqrt{k}}y}{\sqrt{y}}\right)$ for $y \to +\infty$. Moreover, the condition $g \leq \frac{1}{4k'}$ implies that $v_{|\mathbb{R}} \geq g$. □

Proof of Lemma 3.12. Let us start with w_1.

By Corollary 2.6, there exists a compact subset K, outside of which

$$\tilde{u}_1 = u_1 - \frac{3}{8}\log(|q|^2) \leq \frac{1}{16} \quad \text{and} \quad \tilde{u}_2 = u_2 - \frac{1}{8}\log(|q|^2) \leq \frac{1}{16} \ .$$

By Proposition 3.3, every point p sufficiently far from K lies in a half-plane (U, w) with $U \cap K = \emptyset$ and $\Im(w(p)) \geq r(p) - C$ for some $C > 0$ independent from p. We identify (U, w) with $\mathbb{H}^2$ and we work in the w-coordinates. In particular, in these coordinates the functions $\tilde{u}_j$ satisfy the system of PDE (3.5). Moreover, again by Proposition 3.3, the function $w_1 = \tilde{u}_1 + \tilde{u}_2$ is positive and the restriction of w_1 to the real axis is integrable. Moreover, its L^1-norm can be bounded by some constant that depends only on the coefficients of q. We can thus apply Lemma 3.14 with boundary condition $g = w_1$ and $k = k' = 2$, thus getting a function v which satisfies $v = O\left(\frac{e^{-2\sqrt{2}r}}{\sqrt{r}}\right)$. It is now sufficient to prove that $w_1 \leq v$, or, equivalently, that

$\eta_1 = w_1 - v$ is always non-positive. Notice that $\eta_1 \in C^\infty(\mathbb{H}^2) \cap C^0(\overline{\mathbb{H}}^2)$, and, since $\tilde{u}_1$ and $\tilde{u}_2$ are positive, the following inequality holds

$$\Delta w_1 = e^{2\tilde{u}_2} - e^{-2\tilde{u}_1} \geq 2w_1 + 2(\tilde{u}_2^2 - \tilde{u}_1^2) \geq 2w_1 - 2w_1^2 \ .$$

Suppose by contradiction that η_1 is positive in some point, so that the set $Q = \eta_1^{-1}([\epsilon, +\infty)) \neq \emptyset$. Since $\eta_1 \leq 0$ on $\partial\mathbb{H}^2$ and η_1 decays to zero for $|z| \to +\infty$, the set Q is compact and η_1 has a maximum at some $p \in Q$. In this point, we have

$$\begin{aligned} 0 \geq \Delta\eta_1(p) &= \Delta w_1(p) - \Delta v(p) \\ &\geq 2w_1(p) - 2w_1(p)^2 - 2v(p) + 4v(p)^2 \\ &\geq 2\eta_1(p) - 2\eta_1(p)(w_1(p) + v(p)) \\ &\geq 2\eta_1(p) - \eta_1(p) = \eta_1(p) \end{aligned}$$

and this contradicts the fact that $p \in Q$.

We now use the estimate for w_1 to deduce the asymptotic behaviour of w_2. Since we do not know if w_2 is positive, we work with the function $w_2' = |w_2| \in C^0(\mathbb{C}) \cap H^1_{loc}(\mathbb{C})$. Let K be a compact set containing the roots of the quartic differential q such that $w_2' \leq 1$ on $\mathbb{C} \setminus K$: this is possible because $w_2' \leq w_1$, and we proved that w_1 decays to zero. By Proposition 3.2, we can cover the complement of a neighbourhood of K with a finite number of standard half-planes (U_i, ζ_i). In each of these, the function w_2 satisfies the following PDE

$$\Delta w_2 = 2(2e^{\tilde{u}_1 - \tilde{u}_2} - e^{-2\tilde{u}_1} - e^{2\tilde{u}_2}) \ .$$

Now, where $w_2 \geq 0$, we have that

$$\begin{aligned} e^{\tilde{u}_1 - \tilde{u}_2} &\geq 1 + \tilde{u}_1 - \tilde{u}_2 \\ e^{-2\tilde{u}_1} &\leq 1 - 2\tilde{u}_1 + 2\tilde{u}_1^2 \\ e^{2\tilde{u}_2} &\leq 1 + 2\tilde{u}_2 + 3\tilde{u}_2^2 \end{aligned}$$

where the last inequality is true for $|\zeta_i|$ large, since the functions $\tilde{u}_1$ and $\tilde{u}_2$ decay to zero. Therefore, if $w_2 \geq 0$, we have

$$\Delta w_2 \geq w_2 - 3(\tilde{u}_1^2 + \tilde{u}_2^2) \geq 4w_2 - 4w_1^2$$

for $|\zeta_i|$ large enough. Similarly, when $w_2 < 0$ we have

$$\begin{aligned} e^{-2\tilde{u}_1} &\geq 1 - 2\tilde{u}_1 \quad \text{for } |\zeta_i| \text{ large enough} \\ e^{2\tilde{u}_2} &\geq 1 + 2\tilde{u}_2 \\ e^{\tilde{u}_1 - \tilde{u}_2} &\leq 1 + \tilde{u}_1 - \tilde{u}_2 + \frac{(\tilde{u}_1 - \tilde{u}_2)^2}{2} \quad \text{for } |\zeta_i| \text{ large enough} \ , \end{aligned}$$

thus

$$\Delta(-w_2) \geq 4(-w_2) - (-w_2)^2 \geq 4(-w_2) - 4w_1^2 \ .$$

We deduce that, in each standard half-plane (U_i, ζ_i), outside a compact set, the function $w_2' = |w_2|$ satisfies

$$\Delta w_2' \geq 4w_2' - 4w_1^2 \ .$$

Moreover, from the estimates for w_1, we know that, for ζ_i sufficiently large, we have

$$w_1^2 \leq C_i \frac{e^{-4\sqrt{2}|\zeta_i|}}{|\zeta_i|} .$$

Let v be the solution of the boundary value problem

$$\begin{cases} \Delta v = 4v - 4w_1^2 \\ v_{|\mathbb{R}} = w_2' \end{cases} .$$

By a similar argument as that used for w_1, we have that $w_2' \leq v$ and the estimate for w_2' is then a consequence of the following lemma. □

Lemma 3.15. *Let $g \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$ be a positive function such that $g \leq 1$ and let $g' \in C^\infty(\mathbb{H})$ be such that $g' = O(e^{-4\sqrt{2}r}/r)$ when r goes to infinity. Then, the solution $v \in C^\infty(\mathbb{H}) \cap C^0(\overline{\mathbb{H}})$ to the boundary value problem*

$$\begin{cases} \Delta v - 4v = -g' \\ v_{|\mathbb{R}} = g \end{cases}$$

satisfies $v = O\left(\|g\|_1 \frac{e^{-4r}}{\sqrt{r}}\right)$ for $r \to +\infty$.

Proof. It is sufficient to prove that there exists a constant $C > 0$ and $1 < \alpha < \sqrt{2}$ so that, $\eta = (C+1)h - Ch^\alpha$ is a super-solution, where h is the function provided by Lemma 3.13 with $k = 4$. First notice that, by our assumption on g, we have

$$\eta_{|\mathbb{R}} = (C+1)g - Cg^\alpha \geq (C+1)g - Cg = g$$

Moreover,

$$\begin{aligned} \Delta\eta - 4\eta + g' &= 4(C+1)h - C\Delta(h^\alpha) - 4(C+1)h + 4Ch^\alpha + g' \\ &= -C\alpha(\alpha-1)|\nabla h|^2 h^{\alpha-2} - C\alpha h^{\alpha-1}\Delta h + 4Ch^\alpha + g' \\ &\leq -4\alpha C h^\alpha + 4Ch^\alpha + g' = 4Ch^\alpha(1-\alpha) + g' \end{aligned}$$

is negative for r sufficiently large by our assumption on the asymptotic decay on g' and by Lemma 3.13. We can thus choose C sufficiently large so that it is negative everywhere, and η is a super-solution as claimed. □

4. Immersions into the Grassmannian of symplectic planes

In this section, we begin the proof of Theorem C. We need to relate the solutions of the Hitchin equations (2.1) to boundary values of maximal surfaces in $\mathbb{H}^{2,2}$. We accomplish this association via an intermediary identification (Proposition 4.2 and subsequent remarks) of solutions of the Hitchin equations to convex embeddings of the plane $\mathbb{C}$ into a Grassmannian $\mathrm{Gr}_2(\mathcal{E}_\mathbb{R})$. We then relate (Proposition 4.5) such a convex embedding to a maximal surface in $\mathbb{H}^{2,2}$.

Notation. From this point on, we will denote by $\mathrm{Sp}(4,\mathbb{R})$ the group of real matrices preserving the symplectic form Ω. Recall that this differs by conjugation by $A \in \mathrm{SU}(4)$ from the group that we have used so far (see Remark 1.1).

4.1. **Surfaces in the Grassmannian of symplectic planes.** Let us start with the data of an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle $(\mathcal{E},\varphi)$ over $\mathbb{C}$ with polynomial quartic differential $q = q(z)dz^4$ (see Section 1 for the definition). We denote by W the (trivial) circle bundle over $\mathbb{C}$ and with $\pi : W \to \mathbb{C}$ the canonical projection. We define a global section of the pullback bundle $\pi^*\mathcal{E} \to W$ by

$$s(z,\theta) = \pi^* \begin{pmatrix} 0 \\ h_2^{\frac{1}{2}} e^{i\theta} \\ 0 \\ 0 \end{pmatrix} + \pi^*\tau \begin{pmatrix} 0 \\ h_2^{\frac{1}{2}} e^{i\theta} \\ 0 \\ 0 \end{pmatrix} , \tag{4.1}$$

where $\tau : \mathcal{E} \to \overline{\mathcal{E}}$ is the real involution preserved by the flat connection ∇ and the coordinates are expressed with respect to the frame $\{F(z)\}_{z\in\mathbb{C}}$ constructed in section 3.1. Recalling that $\tau(v) = H^{-1}Q\overline{v}$ ([Bar10]), with Q defined in Section 1.2, we obtain

$$s(z,\theta) = \pi^* \begin{pmatrix} 0 \\ h_2^{\frac{1}{2}} e^{i\theta} \\ 0 \\ h_2^{-\frac{1}{2}} e^{-i\theta} \end{pmatrix} .$$

Define $\mathcal{E}_{\mathbb{R}}$ to be the fixed point set of τ in $\mathcal{E}$.

Notice that by equation (4.1), we have that the image of s lies in the real subbundle $\pi^*\mathcal{E}_{\mathbb{R}} = \mathrm{Fix}(\pi^*\tau)$, which is preserved by $\hat{\nabla} = \pi^*\nabla$. Then we compute

$$\hat{\nabla}_{\frac{\partial}{\partial\theta}} \hat{\nabla}_{\frac{\partial}{\partial\theta}} s(z,\theta) = -s(z,\theta) .$$

to conclude that the fibres of W are developed onto real lines.

Therefore, if we denote by $\mathrm{Gr}_2(\mathcal{E}_{\mathbb{R}})$ the bundle over $\mathbb{C}$, whose fibre over each point $z \in \mathbb{C}$ is the Grassmannian of 2-planes in $(\mathcal{E}_{\mathbb{R}})_z$, the independence on θ of the development of s implies that the map

$$f(z) = s(z,\theta) \wedge \hat{\nabla}_{\frac{\partial}{\partial\theta}} s(z,\theta)$$

is a well-defined section of $\mathrm{Gr}_2(\mathcal{E}_{\mathbb{R}})$; here we are identifying fiber-wise the Grassmannian of 2-planes with the space of decomposable tensors in $\Lambda^2\mathcal{E}_{\mathbb{R}}$ via the Plücker embedding. If we introduce the following H-unitary, real, global frame of $\mathcal{E}_{\mathbb{R}}$

$$u_1(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h_2^{\frac{1}{2}} \\ 0 \\ h_2^{-\frac{1}{2}} \end{pmatrix} \qquad u_2(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} h_1^{-\frac{1}{2}} \\ 0 \\ h_1^{\frac{1}{2}} \\ 0 \end{pmatrix}$$

$$u_3(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ ih_2^{\frac{1}{2}} \\ 0 \\ -ih_2^{-\frac{1}{2}} \end{pmatrix} \qquad u_4(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} ih_1^{-\frac{1}{2}} \\ 0 \\ -ih_1^{\frac{1}{2}} \\ 0 \end{pmatrix} ,$$

it is straightforward to verify that $f(z) = u_1(z) \wedge u_3(z)$, hence f selects in each fibre of $\mathcal{E}_{\mathbb{R}}$ the plane generated by $u_1(z)$ and $u_3(z)$. Now, recall the definition of Ω in Equation (1.1): the ∇-parallel symplectic form Ω induces a ∇-parallel symplectic form on $\mathcal{E}_{\mathbb{R}}$, which, in the above frame, is expressed by the matrix

$$\omega_{\mathbb{R}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ;$$

thus the image of f entirely lies in the space of symplectic 2-planes of $\mathcal{E}_{\mathbb{R}}$.

Let us now underline some properties of this map that will allow to recover the minimal surface in the $\mathrm{Sp}(4, \mathbb{R})$-symmetric space. We learned the following from François Labourie.

Definition 4.1. Let Σ be a Riemann surface. An immersion $f : \Sigma \to \mathrm{Gr}_2(\mathbb{R}^4)$ is convex if for any point $p \in \Sigma$ and any tangent vector $X \in T_p\Sigma$ the map

$$B(X) = d_p f(X) \in T_{f(p)}\mathrm{Gr}_2(\mathbb{R}^4)$$

is invertible.

We develop next an interpretation of $T\mathrm{Gr}_2(\mathbb{R}^4)$. We recall that given a plane $L \in \mathrm{Gr}_2(\mathbb{R}^4)$, the tangent space $T_L\mathrm{Gr}_2(\mathbb{R}^4)$ is identified with the vector space of linear homomorphisms $\mathrm{Hom}(L, \mathbb{R}^4/L)$ in the following way. Let $L = \mathrm{Span}(v, w)$ and let $\gamma : [0, 1] \to \mathrm{Gr}_2(\mathbb{R}^4)$ be a smooth path such that $\gamma(0) = L$. For every $t \in [0, 1]$ we choose smoothly two vectors $v(t)$ and $w(t)$ so that $\gamma(t) = \mathrm{Span}(v(t), w(t))$. Using the Plücker embedding, we can thus write $\gamma(t) = v(t) \wedge w(t)$. Now, the tangent vector at $t = 0$ is given by

$$\gamma'(0) = \frac{d}{dt}_{|t=0} v(t) \wedge w(t) = v'(0) \wedge w(0) + v(0) \wedge w'(0) \ .$$

The variation of the plane L is expressed only by the components of $v'(0)$ and $w'(0)$ that do not lie in the plane L. Thus the tangent vector $\gamma'(0)$ is completely determined by the linear map

$$B(X) : L \to \mathbb{R}^4/L$$

where we construe $X \in T_p\Sigma$ as tangent to a curve γ (as described above) with $f \circ \gamma \subset \mathrm{Gr}_2(\mathbb{R}^4)$, and suppressing some of the notation, we take $B(\dot\gamma)v = v'(0) \pmod L$ and $B(\dot\gamma)w = w'(0) \pmod L$.

Proposition 4.2. *The immersion $f : \mathbb{C} \to \mathrm{Gr}_2(\mathbb{R}^4)$ defined above is convex.*

Proof. Recall that the flat connection ∇^{Gr} on $\mathrm{Gr}_2(\mathcal{E}_{\mathbb{R}})$ may be defined in terms of the connection $\nabla = H^{-1}\partial H + \varphi + \varphi^{*H}$ on $\mathcal{E}$: in particular $\nabla^{Gr}(v \wedge w)$ is defined in terms of ∇v and ∇w. To that end, suppose we have a basis $\{u_1(z), u_3(z)\}$ of

$f(z)$ and $\{u_2(z), u_4(z)\}$ of $\mathbb{R}^4/f(z)$: to compute $\nabla^{Gr} u_i \wedge u_j$ for $i \neq j$, it suffices to compute ∇u_k for $k = 1$ and $k = 3$. For these particular vectors, we thus compute

$$
\begin{aligned}
\nabla_{\frac{\partial}{\partial x}} u_1(z) &= h_2^{-1} u_1(z) + \frac{1}{2} h_2^{-1} \partial_y h_2 u_3(z) + h_1^{-\frac{1}{2}} h_2^{\frac{1}{2}} u_2(z) \\
\nabla_{\frac{\partial}{\partial x}} u_3(z) &= -\frac{1}{2} h_2^{-1} \partial_y h_2 u_1(z) - h_2^{-1} u_3(z) - h_1^{-\frac{1}{2}} h_2^{\frac{1}{2}} u_4(z) \\
\nabla_{\frac{\partial}{\partial y}} u_1(z) &= (h_2^{-1} - \frac{1}{2} h_2^{-1} \partial_x h_2) u_3(z) - h_1^{-\frac{1}{2}} h_2^{\frac{1}{2}} u_4(z) \\
\nabla_{\frac{\partial}{\partial y}} u_3(z) &= (h_2^{-1} + \frac{1}{2} h_2^{-1} \partial_x h_2) u_1(z) - h_1^{-\frac{1}{2}} h_2^{\frac{1}{2}} u_2(z)
\end{aligned}
\tag{4.2}
$$

and deduce that the homomorphisms $B(\frac{\partial}{\partial x})$ and $B(\frac{\partial}{\partial y})$ are represented by the matrices

$$
B\left(\frac{\partial}{\partial x}\right) = h_1^{-\frac{1}{2}} h_2^{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B\left(\frac{\partial}{\partial y}\right) = h_1^{-\frac{1}{2}} h_2^{\frac{1}{2}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
$$

with respect to the basis $\{u_1(z), u_3(z)\}$ of $f(z)$ and $\{u_2(z), u_4(z)\}$ of $\mathbb{R}^4/f(z)$. Since

$$
\det B\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) = -h_1^{-1} h_2 (a^2 + b^2)
$$

vanishes only for $a = b = 0$, the result follows. □

Lemma 4.3. *Let $f : \mathbb{C} \to \mathrm{Gr}_2(\mathbb{R}^4)$ be the convex immersion constructed above. Then there exist complex structures J_1 on $f(z)$ and J_2 on $\mathbb{R}^4/f(z)$ such that, for every $z \in \mathbb{C}$, the map $B : T_z\mathbb{C} \to \mathrm{Hom}_{\mathbb{C}}(f(z), \mathbb{R}^4/f(z))$ is an isomorphism that intertwines with J_1 and J_2.*

Proof. Let us choose the basis $\{u_1(z), u_3(z)\}$ for $f(z)$ and let us identify $\mathbb{R}^4/f(z)$ with the plane generated by $\{u_2(z), u_4(z)\}$. We define the complex structures on these planes as follows

$$
J_1(z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad J_2(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .
$$

Using the explicit formulas for B found in the previous proposition, it is easy to check that for every $X \in T_z\mathbb{C}$ the map $B(X)$ is $\mathbb{C}$-linear, i.e. $B(X)J_1(z) = J_2(z)B(X)$. Moreover, if J denotes the standard complex structure on $\mathbb{C}$, we notice that

$$
B\left(J\frac{\partial}{\partial x}\right) = J_2(z) B\left(\frac{\partial}{\partial x}\right) \quad \text{and} \quad B\left(J\frac{\partial}{\partial y}\right) = J_2(z) B\left(\frac{\partial}{\partial y}\right) ,
$$

which implies that the linear map

$$
\begin{aligned}
B : T_z\mathbb{C} &\to \mathrm{Hom}_{\mathbb{C}}(f(z), \mathbb{R}^4/f(z)) \\
X &\to B(X)
\end{aligned}
$$

is well-defined and $\mathbb{C}$-linear for every $z \in \mathbb{C}$. Since it is not trivial, it is an isomorphism. □

Remark 4.4. We note that the convex embedding of $\mathbb{C}$ into $\mathrm{Gr}_2(\mathbb{R}^4)$ contructed above induces a minimal immersion of $\mathbb{C}$ into the symmetric space $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$. Indeed, since $\mathbb{R}^4 = f(z) \oplus \mathbb{R}^4/f(z)$ for every $z \in \mathbb{C}$, the complex structures provided by the previous lemma enables us to define a complex structure J on $\mathbb{R}^4$, depending on the point $z \in \mathbb{C}$. Precisely, $J(z) = J_1(z) \oplus -J_2(z)$. We can also define a family of metrics on $\mathbb{R}^4$ depending on the point $z \in \mathbb{C}$ by

$$H_{\omega_{\mathbb{R}},J}(z)(v,w) = \omega_{\mathbb{R}}(v, J(z)w) \ .$$

It follows that the H-unitary frame $\{u_1(z), u_2(z), u_3(z), u_4(z)\}$ is $H_{\omega_{\mathbb{R}},J}(z)$-unitary at every point, thus $H_{\omega_{\mathbb{R}},J}(z)$ coincides with the harmonic metric H, and thus the ∇-parallel transport of $H_{\omega_{\mathbb{R}},J}$, or equivalently of the complex structure J, produces the minimal surface in $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ associated to the given $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle data.

4.2. **Explicit parameterization for $q = dz^4$.** In the special case when the quartic differential is constant we can write explicitly a parameterization of the surface constructed previously. In analogy with affine spheres ([DW15]), we will refer to it as the standard flat maximal surface T_0.

To this aim, it is convenient to work in the global frame, say $\{w_i(z)\}_{i=1}^4$, for $\mathcal{E}$ in which the matrix connection of the flat connection ∇_0 is diagonal (see Section 3.2). The change of frame is expressed by the constant unitary matrix S introduced in Section 3.2. We obtain, for the section s in (4.1),

$$\begin{aligned} s(z,\theta) &= \frac{1}{2}e^{i\theta}(w_1(z) - w_2(z) + w_3(z) - w_4(z)) \\ &+ \frac{1}{2}\tau(e^{i\theta}(w_1(z) - w_2(z) + w_3(z) - w_4(z)) = \frac{1}{2}\begin{pmatrix} e^{i\theta} \\ -e^{i\theta} \\ e^{i\theta} \\ -e^{i\theta} \end{pmatrix} + \frac{1}{2}\tau \begin{pmatrix} e^{i\theta} \\ -e^{i\theta} \\ e^{i\theta} \\ -e^{i\theta} \end{pmatrix} \end{aligned}$$

where the coordinates are now expressed with respect to the frame $\{w_i(z)\}_{i=1,\dots,4}$. In this frame the real involution $\tau = H^{-1}Q$ is given by

$$\tau \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \overline{z}_1 \\ -i\overline{z}_2 \\ -\overline{z}_3 \\ i\overline{z}_4 \end{pmatrix} \ ,$$

thus the frame $\{e_1(z) = w_1(z), e_2(z) = \frac{(1-i)}{\sqrt{2}}w_2(z), e_3(z) = iw_3(z), e_4(z) = \frac{(1+i)}{\sqrt{2}}w_4(z)\}$ is real and still diagonalizes the flat connection ∇_0. Since we know that $s(z,\theta)$ will take value in $\mathcal{E}_{\mathbb{R}}$, we will use coordinates with respect to this frame from now on. Moreover, the restriction of the ∇_0-parallel symplectic form Ω induces a ∇_0-parallel

symplectic form on $\mathcal{E}_{\mathbb{R}}$, that is given by the matrix

$$\omega_{\mathbb{R}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} .$$

We can thus identify $\mathbb{R}^4$ endowed with the above symplectic form, with the fibre of $\mathcal{E}_{\mathbb{R}}$ over a base point $0 \in \mathbb{C}$. By ∇_0-parallel transporting s at (z, θ) to this fibre over $0 \in \mathbb{C}$, we can first parameterize the image of s (as images in the fixed $\mathbb{R}^4$) as

$$\begin{aligned} s(z,\theta) &= \frac{1}{2} e^{i\theta} D(z)(w_1(z) - w_2(z) + w_3(z) - w_4(z)) \\ &\quad + \frac{1}{2}\tau(e^{i\theta} D(z)(w_1(z) - w_2(z) + w_3(z) - w_4(z))) \\ &= \tfrac{1}{2}(2\cos(\theta)e^{2\Re(z)}, \sqrt{2}(\sin(\theta) - \cos(\theta))e^{-2\Im(z)}, 2\sin(\theta)e^{-2\Re(z)}, \sqrt{2}(-\sin(\theta) - \cos(\theta))e^{2\Im(z)}) \end{aligned}$$

where $D(z) = \operatorname{diag}(e^{2\Re(z)}, e^{-\Im(z)}, e^{-2\Re(z)}, e^{2\Im(z)})$ for $z \in \mathbb{C}$. Consequently, the standard flat maximal surface $T_0 = f(\mathbb{C})$ in the Grassmannian of symplectic planes (identified with a submanifold of $\mathbb{P}(\Lambda^2\mathbb{R}^4)$ via the Plücker embedding)

$$\begin{aligned} f(z) &= s(z,\theta) \wedge \nabla_{\frac{\partial}{\partial\theta}} s(z,\theta) \\ &= e_1 \wedge e_3 + e_2 \wedge e_4 + \frac{1}{\sqrt{2}} e^{2\Re(z) - 2\Im(z)} e_1 \wedge e_2 - \frac{1}{\sqrt{2}} e^{2\Re(z) + 2\Im(z)} e_1 \wedge e_4 \\ &\quad + \frac{1}{\sqrt{2}} e^{-2\Re(z) + 2\Im(z)} e_3 \wedge e_4 - \frac{1}{\sqrt{2}} e^{-2\Re(z) - 2\Im(z)} e_2 \wedge e_3 \ , \end{aligned}$$

where $e_i = e_i(0)$ for $i = 1, \dots, 4$. We observe from this explicit formula that T_0 coincides with the orbit of the point $f(0) \in \mathrm{Gr}_2(\mathbb{R}^4)$ under the action of the diagonal matrices $D(z) = \operatorname{diag}(e^{2\Re(z)}, e^{-\Im(z)}, e^{-2\Re(z)}, e^{2\Im(z)})$ for $z \in \mathbb{C}$.

Moreover, by looking at limits along (quasi)-rays, we can describe the boundary at infinity of T_0 as a quadrilateral in the space of Lagrangians of $\mathbb{R}^4$, as Table 1 shows.

4.3. **Relation with the maximal surface in $\mathbb{H}^{2,2}$.** Exploiting the low-dimensional isomorphism $\mathbb{P}\mathrm{Sp}(4, \mathbb{R}) \cong \mathrm{SO}_0(2, 3)$, we can relate the convex surface Σ in the Grassmannian of symplectic planes in $\mathbb{R}^4$ with a unique maximal surface in $\mathbb{H}^{2,2}$. We will see, in particular, that under the identification between the boundary at infinity of $\mathbb{H}^{2,2}$ (i.e. the Einstein Universe $\mathrm{Ein}^{1,2}$) and the space of Lagrangians of $\mathbb{R}^4$, the two surfaces share the same boundary at infinity.

Let us first recall how the low-dimensional isomorphism $\mathbb{P}\mathrm{Sp}(4, \mathbb{R}) \cong \mathrm{SO}_0(2, 3)$ is accomplished. We denote by $\{e_i\}_{i=1,\dots,4}$ the canonical basis of $\mathbb{R}^4$ and we consider the symplectic form $\omega = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$. Let $V = \Lambda^2\mathbb{R}^4$ be the vector space of

Type of path γ	**Direction θ**	**Projective limit p_γ of $f_0(\gamma)$**
Quasi-ray	$\theta \in (0, \frac{\pi}{2})$	$p_\gamma = [e_1 \wedge e_4]$
Ray (of height y)	$\theta = \frac{\pi}{2}$	$p_\gamma = [-e_1 \wedge e_4 + e^{-4y} e_3 \wedge e_4]$
		$(p_\gamma \to [e_1 \wedge e_4]$ as $y \to \infty)$
Quasi-ray	$\theta \in (\frac{\pi}{2}, \pi)$	$p_\gamma = [e_3 \wedge e_4]$
Ray (of height iy)	$\theta = \pi$	$p_\gamma = [e_3 \wedge e_4 - e^{-4y} e_2 \wedge e_3]$
		$(p_\gamma \to [e_3 \wedge e_4]$ as $y \to \infty)$
Quasi-ray	$\theta \in (\pi, \frac{3\pi}{2})$	$p_\gamma = [e_2 \wedge e_3]$
Ray (of height y)	$\theta = \frac{3\pi}{2}$	$p_\gamma = [e_1 \wedge e_2 - e^{-4y} e_2 \wedge e_3]$
		$(p_\gamma \to [e_1 \wedge e_2]$ as $y \to \infty)$
Quasi-ray	$\theta \in (\frac{3\pi}{2}, 2\pi)$	$p_\gamma = [e_1 \wedge e_2]$
Ray (of height iy)	$\theta = 0$	$p_\gamma = [e^{-4y} e_1 \wedge e_2 - e_1 \wedge e_4]$
		$(p_\gamma \to [e_1 \wedge e_4]$ as $y \to \infty)$

TABLE 1. Limits of the standard flat maximal surface along rays

skew-symmetric 2-tensors on $\mathbb{R}^4$. A standard basis for V is given by $\{e_i \wedge e_j\}_{1 \le i < j \le 4}$. The symplectic form ω induces an inner product on V via the relation

$$-2\langle \phi, \psi \rangle e_1 \wedge e_2 \wedge e_3 \wedge e_4 = \phi \wedge \psi \ . \tag{4.3}$$

It turns out that $\langle \cdot, \cdot \rangle$ is non-degenerate and has signature $(3,3)$. The non-degeneracy allows us to define a canonical 2-tensor ω^* dual to the symplectic form ω by requiring that

$$-2\langle \omega^*, v \wedge w \rangle = \omega(v, w)$$

for every $v, w \in \mathbb{R}^4$. In our case, we have

$$\omega^* = e_1 \wedge e_3 + e_2 \wedge e_4 \ ,$$

and we notice that $\langle \omega^*, \omega^* \rangle = 1$. The group $\mathrm{Sp}(4,\mathbb{R})$ acts naturally by isometries on $(V, \langle \cdot, \cdot \rangle)$, and preserves ω^*. Therefore, it acts isometrically on $(\omega^*)^\perp$, which is a five-dimensional real vector space endowed with an inner product of signature $(2,3)$. Tracing this action, we can define a continuous group homomorphism $\mathrm{Sp}(4,\mathbb{R}) \to \mathrm{SO}_0(2,3)$, whose kernel only contains $\{\pm \mathrm{Id}\}$, thus giving the aforementioned isomorphism.

We emphasize that inside the projective space $\mathbb{P}(V)$ we can embed:

- the Grassmannian of 2-planes in $\mathbb{R}^4$, which correspond to the submanifold of decomposable 2-tensors;
- the Grassmannian of symplectic planes in $\mathbb{R}^4$, which can be characterized as those decomposable tensors ϕ such that $\langle \phi, \omega^* \rangle \neq 0$;
- the Lagrangians of $\mathbb{R}^4$, which are in bijection with decomposable 2-tensors orthogonal to ω^*;

- the Einstein Universe $\mathrm{Ein}^{1,2}$ can be identified with the points $p \in \mathbb{P}((\omega^*)^\perp)$ such that $p \wedge p = 0$;
- the pseudo-hyperbolic space $\mathbb{H}^{2,2}$, as the projectivization of the vectors $\phi \in V$ such that $\langle \phi, \phi \rangle < 0$.

We notice, in particular, that this identifies Lagrangians planes in $\mathbb{R}^4$ with points of the Einstein Universe.

We wish to interpret our embedding $f : \mathbb{C} \to \mathrm{Gr}_2(\mathbb{R}^4)$ in terms of pseudo-Riemannian geometry.

Let $f : \mathbb{C} \to \mathrm{Gr}_2(\mathbb{R}^4)$ be the (convex) embedding, depending on a choice of quartic differential q, constructed in Section 4.1. Since it takes values in the Grassmannian of symplectic planes, there exists a unique lift $\tilde{f} : \mathbb{C} \to V$ such that $\langle \tilde{f}(z), \omega_{\mathbb{R}} \rangle = \frac{1}{2}$. We define $\tilde{\sigma} : \mathbb{C} \to (\omega_{\mathbb{R}}^*)^\perp$ by $\tilde{\sigma}(z) = 2\tilde{f}(z) - \omega_{\mathbb{R}}^*$, and we denote by σ its projection into $\mathbb{P}(V)$. By construction $\langle \tilde{\sigma}(z), \tilde{\sigma}(z) \rangle = -1$ for every $z \in \mathbb{C}$, hence σ defines an embedding of the complex plane into $\mathbb{H}^{2,2}$. Of course, the map σ still depends on the choice of quartic differential q.

Proposition 4.5. *The map $\sigma : \mathbb{C} \to \mathbb{H}^{2,2}$ is harmonic and conformal, hence $\sigma(\mathbb{C})$ is a maximal surface in $\mathbb{H}^{2,2}$.*

Proof. Since $\omega_{\mathbb{R}}$ is ∇-parallel, we can write, using that $f(z) = u_1(z) \wedge u_3(z)$ and $\omega_{\mathbb{R}}^* = u_1(z) \wedge u_3(z) + u_2(z) \wedge u_4(z)$, [1]

$$\sigma(z) = 2f(z) - \omega_{\mathbb{R}}^* = u_1(z) \wedge u_3(z) - u_2(z) \wedge u_4(z) \ .$$

Using Equation (4.2) and the following covariant derivatives of $u_2(z)$ and $u_4(z)$

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x}} u_2(z) &= h_2^{\frac{1}{2}} h_1^{-\frac{1}{2}} u_1(z) + \Re(q) h_1 u_2(z) + (\Im(q) h_1 - \frac{1}{2} h_1^{-1} \partial_y h_1) u_4(z) \\
\nabla_{\frac{\partial}{\partial x}} u_4(z) &= -h_2^{\frac{1}{2}} h_1^{-\frac{1}{2}} u_3(z) - \Re(q) h_1 u_4(z) + (\Im(q) h_1 + \frac{1}{2} h_1^{-1} \partial_y h_1) u_2(z) \\
\nabla_{\frac{\partial}{\partial y}} u_2(z) &= -h_2^{\frac{1}{2}} h_1^{-\frac{1}{2}} u_3(z) - \Im(q) h_1 u_2(z) + (\Re(q) h_1 + \frac{1}{2} h_1^{-1} \partial_x h_1) u_4(z) \\
\nabla_{\frac{\partial}{\partial y}} u_4(z) &= -h_2^{\frac{1}{2}} h_1^{-\frac{1}{2}} u_1(z) + \Im(q) h_1 u_4(z) + (\Re(q) h_1 - \frac{1}{2} h_1^{-1} \partial_x h_1) u_2(z)
\end{aligned} \tag{4.4}$$

we deduce that

$$\begin{aligned}
\langle \nabla_{\frac{\partial}{\partial x}} \sigma, \nabla_{\frac{\partial}{\partial y}} \sigma \rangle &= 0 \\
\langle \nabla_{\frac{\partial}{\partial x}} \sigma, \nabla_{\frac{\partial}{\partial x}} \sigma \rangle &= \langle \nabla_{\frac{\partial}{\partial y}} \sigma, \nabla_{\frac{\partial}{\partial y}} \sigma \rangle = 4 h_1^{-1} h_2
\end{aligned}$$

which means that the embedding is conformal.
As for the harmonic condition, since $\mathbb{H}^{2,2}$ is umbilical in $\mathbb{R}^{2,3}$, it is sufficient to check that

$$\nabla_{\frac{\partial}{\partial \bar{z}}} \nabla_{\frac{\partial}{\partial z}} \sigma = 0 \pmod{\sigma} \ .$$

[1] the definition of $\omega_{\mathbb{R}}^*$ is correct based on formula (4.3) and the one below

Again, using Equations (4.2) and (4.4), a direct computation shows that

$$\nabla_{\frac{\partial}{\partial \bar{z}}} \nabla_{\frac{\partial}{\partial z}} \sigma = 4h_1^{-1} h_2 \sigma$$

and the proof is complete. □

Remark 4.6. The proof of the proposition above also shows that the induced metric on the maximal surface is $4h^{-1}h_2|dz|^2$. In particular, following the argument of Proposition 3.11, it is quasi-isometric to $(\mathbb{C}, 4|q|^{\frac{1}{2}})$, hence complete.

We remark that, viewing $\sigma(\mathbb{C})$ and $f(\mathbb{C})$ as embedded inside $\mathbb{P}(V)$, since we can choose lifts to V that differ only by a translation by $\omega^*_{\mathbb{R}}$, they share the same boundary at infinity, which is a curve in the Einstein Universe, or, equivalently, in the space of Lagrangians. We will study the properties of the boundary curve in the next section.

Remark 4.7. The maximal surface in $\mathbb{H}^{2,2}$ with constant quartic differential coincides with the horospherical surface, described in [Tam19], embedded in a copy of anti-de Sitter space inside $\mathbb{H}^{2,2}$. In particular, its boundary at infinity is a future-directed, negative, light-like polygon in the Einstein Universe (see next section).

Moreover, the proof of Proposition 4.5 shows that the embedding data of the maximal surface $\sigma(\mathbb{C})$, which determines it up to post-compostion by global isometries, only depend on the quartic differential q on the complex plane and on the solution to Equation (2.1). We record this remark for future use as a proposition.

Proposition 4.8. *Two planar maximal surfaces $\sigma_1 : \mathbb{C} \to \mathbb{H}^{2,2}$ and $\sigma_2 : \mathbb{C} \to \mathbb{H}^{2,2}$, defined as in Proposition 4.5, which share quartic differential and solutions h to* (2.1) *agree up to post-composition by a global isometry.*

5. Moduli space of future-directed negative light-like polygons in $\mathrm{Ein}^{1,2}$

As we will see in the next section, a future-directed, negative, light-like polygon will appear as a boundary at infinity of a maximal embedding of the complex plane in $\mathbb{H}^{2,2}$ with an associated polynomial quartic differential. In this section we define these geometric objects and parameterize their moduli space under the conformal action of $\mathrm{SO}_0(2,3)$.

Definition 5.1. A light-like polygon in the Einstein Universe $\mathrm{Ein}^{1,2}$ is an oriented embedded one-dimensional simplicial complex Δ with a finite number of vertices such that every edge is a photon (i.e. contained in the projection of an isotropic plane of $\mathbb{R}^{2,3}$). We will also assume that Δ is a generator of $\pi_1(\mathrm{Ein}^{1,2}) \cong \mathbb{Z}$. We say that a light-like polygon is negative if it can be lifted to a cone $\widetilde{\Delta}$ in $\mathbb{R}^{2,3} \setminus \{0\}$, such that the inner product of any two non-collinear points is non-positive and vanishes if and only if their projections belong to the same edge of Δ.

In the above definition, a cone in $\mathbb{R}^{2,3} \setminus \{0\}$ is intended as subset of $\mathbb{R}^{2,3}$ that is invariant under multiplication by positive scalars. There are two cones that occur as possible lifts, but the condition of being negative is unaffected by the choice of cone. Moreover, an orientation of Δ will be given by an enumeration in the set of vertices.

Remark 5.2. It is sufficient to check the negativity condition between pairs of non-consecutive vertices. Namely, if $\{v_i\}_{i=1,\dots,n}$ are vectors that generate the half-lines of the cone $\widetilde{\Delta}$ that project to vertices, then every other point in $\widetilde{\Delta}$ can be written as $p = \lambda(tv_i + (1-t)v_{i+1})$ for some $i \in \{1, \dots, n\}$ (where the indices are to be intended modulo n), $\lambda > 0$ and $t \in [0, 1]$. Therefore,

$$\langle p, q\rangle = \langle \lambda(tv_i + (1-t)v_{i+1}), \mu(sv_j + (1-s)v_{j+1})\rangle < 0$$

as soon as p and q do not project onto the same edge, under the assumption that the inner product between any pair of non-consecutive vertices is negative.

Definition 5.3. Fix an oriented time-like 3-plane $W = \mathrm{Span}(w_1, w_2, w_3)$ in $\mathbb{R}^{2,3}$. A negative light-like polygon is future-directed if for every pair of vectors $v_i, v_{i+1} \in \widetilde{\Delta}$ that project to consecutive vertices of Δ, we have

$$\mathrm{dVol}(v_i, v_{i+1}, w_1, w_2, w_3) > 0 \ ,$$

where dVol denotes the standard volume form in $\mathbb{R}^{2,3}$.

We observe that this definition is independent of the initial choice of plane W.

We will denote by $\mathcal{LP}_k^-$, the space of future-directed negative light-like polygons in $\mathrm{Ein}^{1,2}$ with k vertices. The group $\mathrm{SO}_0(2,3)$ acts on $\mathcal{LP}_k^-$, since its conformal action on the Einstein Universe sends photons into photons, preserves the sign of the inner products and preserves the orientation and the time-orientation of $\mathbb{R}^{2,3}$. We indicate with $\mathcal{MLP}_k^-$ the quotient by this action. We note that we can see $\mathcal{MLP}_k^-$ as the quotient $\mathcal{TLP}_k^-/\mathbb{Z}_k$, where $\mathcal{TLP}_-^k$ denotes the moduli space of future-directed negative light-like polygons with a marked vertex and $\mathbb{Z}_k$ acts by change of marking.

Proposition 5.4. *There exists a unique future-directed negative light-like quadrilateral in* $\mathrm{Ein}^{1,2}$*, up to the action of* $\mathrm{SO}_0(2,3)$*, i.e. the space* $\mathcal{MLP}_4^-$ *consists of only one point.*

Proof. Let Δ be a future-directed negative light-like polygon in $\mathrm{Ein}^{2,1}$ with vertices $\{p_1, p_2, p_3, p_4\}$. Let $\widetilde{\Delta}$ be the cone in $\mathbb{R}^{2,3} \setminus \{0\}$ given by Definition 5.1. We denote by $\{v_1, v_2, v_3, v_4\}$ some light-like vectors in $\mathbb{R}^{2,3}$ that generate the lifts of the vertices, i.e. the half-line that projects to the vertex p_i is given by $\{tv_i \mid t > 0\}$. We can arrange v_i so that

$$\langle v_1, v_3\rangle = \langle v_2, v_4\rangle = -1 \ .$$

In particular, the vectors v_i are linearly independent.
Let us denote by $\{e_1, e_2, e_3, e_4, e_5\}$ a basis of $\mathbb{R}^{2,3}$ such that the inner product in these coordinates is given by

$$\langle x, y\rangle = x_1y_3 + x_2y_4 + x_3y_1 + x_4y_2 - x_5y_5 \ .$$

We fix the orientation on $\mathbb{R}^{2,3}$ given by the volume form

$$\mathrm{dVol} = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5$$

and the time-like orientation induced by the time-like 3-plane

$$W = \mathrm{Span}(e_1 - e_3, e_2 - e_4, e_5) \ .$$

Now, the linear map

$$\begin{aligned}\mathrm{Span}(\{v_i\}_{i=1,\dots 4}) &\to \mathrm{Span}(\{e_i\}_{i=1,\dots,4})\\ v_i &\mapsto \epsilon(i)e_i \quad \text{with } \epsilon(i)=1 \text{ if } i=1,2 \text{ and } -1 \text{ otherwise}\end{aligned}$$

preserves the inner product and the time-orientation, hence it can be extended to an element of $\mathrm{SO}_0(2,3)$. We thus deduce that every future-directed negative light-like polygon with 4 vertices is equivalent to the standard quadrilateral Δ_4 with vertices $([e_1],[e_2],[e_3],[e_4])$ and associated cone

$$\widetilde{\Delta_4} = \mathbb{R}^+([e_1,e_2]\cup[e_2,-e_3]\cup[-e_3,-e_4]\cup[-e_4,e_1])$$

where $[e_i,\pm e_j]$ denotes the Euclidean segment joining e_i and $\pm e_j$. □

Proposition 5.5. *There exists a unique future-directed negative light-like polygon in* $\mathrm{Ein}^{1,2}$ *with* 5 *vertices, up to the action of* $\mathrm{SO}_0(2,3)$.

Proof. We use the same notation as in the previous proposition. It follows from the proof of Proposition 5.4 that we can assume that $v_1 = e_1, v_2 = e_2$ and $v_3 = -e_3$. Now the lift of the fourth vertex satisfies:

$$\langle v_4, v_4\rangle = 0 \quad \langle v_4, v_5\rangle = 0 \quad \langle v_4, e_3\rangle = 0$$

and

$$\langle e_1, v_4\rangle < 0 \quad \langle e_2, v_4\rangle < 0\ ,$$

hence it must belong to the set

$$U = \{x \in \mathbb{R}^{2,3} \mid 2x_1x_3 + 2x_2x_4 - x_5^2 = 0, x_1 = 0, x_4 < 0, x_3 < 0\}.$$

We claim that the subgroup $H = \mathrm{Stab}(p_1)\cap\mathrm{Stab}(p_2)\cap\mathrm{Stab}(p_3) < \mathrm{SO}_0(2,3)$ acts transitively on this set. Namely,

i) if $x_2 = 0$, then necessarily $x_5 = 0$, leaving x_3 and x_4 to be specified arbitrarily – yet these are in the orbit of the action of the diagonal matrix subgroup of H;
ii) if $x_2 \neq 0$, then we must have $x_5 \neq 0$ and we can choose a representative of v_4 such that $x_5 = \pm\sqrt{2}$. It is easy to check that diagonal matrices (which we recall to be of the form $\mathrm{diag}(\lambda,\mu,\lambda^{-1},\mu^{-1},1)$) act transitively on points with $x_5 = \sqrt{2}$ and on those with $x_5 = -\sqrt{2}$. We then notice that if $v \in U$ with $x_5 = \sqrt{2}$ and $v' \in U$, which differs from v only in the sign of the last component, then the polygon Δ with vertices p_1, p_2, p_3 and $[v]$ is the image of the polygon Δ' with vertices p_1, p_2, p_3 and $[-v']$ under the diagonal matrix $D = \mathrm{diag}(-1,-1,-1,-1,1) \in H$.

Therefore, it is enough to show that there exists an element $A \in H$ that sends a point satisfying *i*) to a point satisfying *ii*). Now, the linear transformation A determined by

$$A(e_i) = e_i \quad \text{for } i = 1,2,3 \quad A(e_4) = e_2 + e_4 + \sqrt{2}e_5 \quad \text{and} \quad A(e_5) = \sqrt{2}e_2 + e_5$$

preserves the inner product and sends e_3+e_4 to $e_2+e_3+e_4+\sqrt{2}e_5$. Moreover, it lies in $\mathrm{SO}_0(2,3)$, an explicit path connecting A to the identity being given by the linear transformations $\{A_t\}_{t\in[0,1]}$ such that

$$A_t(e_i)=e_i \quad \text{for } i=1,2,3 \quad A_t(e_4)=te_2+e_4+\sqrt{2t}e_5 \quad \text{and} \quad A_t(e_5)=\sqrt{2t}e_2+e_5\ .$$

Hence it fulfills all our requirements. As a consequence, we can suppose that the fourth vertex is $p_4=[e_3+e_4]$.
By a similar reasoning, the lift of the last vertex belongs to

$$W=\{x\in\mathbb{R}^5 \mid 2x_1x_3+2x_2x_4-x_5^2=0, x_3=0, x_1+x_2=0, x_1>0, x_4<0\}.$$

As before, we can suppose that $x_5=\sqrt{2}$ and it is clear that the diagonal matrices of the form $\mathrm{diag}(a,a,a^{-1},a^{-1},1)$ with $a\in\mathbb{R}^+$ act transitively on W.
We thus deduce that every future-directed, negative light-like polygon with 5 vertices is equivalent to the standard light-like penthagon Δ_5 with vertices $([e_1],[e_2],[e_3],[e_3+e_4],[e_1-e_2-e_4+\sqrt{2}e_5])$ and associated cone

$$\widetilde{\Delta_5}=\mathbb{R}^+([e_1,e_2]\cup[e_2,-e_3]\cup[-e_3,-e_3-e_4]\cup [-e_3-e_4,e_1-e_2-e_4+\sqrt{2}e_5]\cup[e_1-e_2-e_4+\sqrt{2}e_5,e_1])\ .$$

□

Let us now consider the first non-trivial case of hexagons. We find an explicit parametrization of $\mathcal{TLP}_6^-$ that surprisingly shows that this moduli space has two connected components.

Proposition 5.6. *The moduli space of marked future-directed negative light-like hexagons in* $\mathrm{Ein}^{1,2}$ *is a topological manifold homeomorphic to*

$$\{(s,t)\in\mathbb{R}^2 \mid s\geq 0,\ st\neq 2\}/(0,t)\sim(0,-t)$$

Proof. It follows from the proof of Proposition 5.5 that, up to the action of $\mathrm{SO}_0(2,3)$, we can assume that $v_1=e_1$, $v_2=e_2$, $v_3=-e_3$ and $v_4=-e_3-e_4$. The lift of the fifth vertex v_5 must now satisfy

$$\langle v_1,v_5\rangle<0 \quad \langle v_2,v_5\rangle<0 \quad \langle v_3,v_5\rangle<0$$

and

$$\langle v_4,v_5\rangle=0 \quad \langle v_5,v_5\rangle=0\ ,$$

hence v_5 belongs to the set

$$U=\{x\in\mathbb{R}^{2,3} \mid x_1>0,\ x_1=-x_2,\ x_3<0,\ x_4<0,\ 2x_1x_3+2x_2x_4-x_5^2=0\}\ .$$

We can still renormalize partly the position of v_5 acting by the group $H=\mathrm{Stab}(p_1)\cap\mathrm{Stab}(p_2)\cap\mathrm{Stab}(p_3)\cap\mathrm{Stab}(p_4)$ which, by our conventions on the p_i, consists of diagonal matrices of the form $\mathrm{diag}(a,a,a^{-1},a^{-1},1)$ with $a\neq 0$. Because $x_1>0$ and $x_3<0$, we can find $a>0$ such that $a^2=\frac{-x_3}{x_1}$. This implies that $ax_1=-a^{-1}x_3$. Therefore, after renormalizing by the action of H we can assume that v_5 has coordinates

$$v_5=(x_1,-x_1,-x_1,x_4,x_5)$$

for some $x_1 > 0$. Since we are interested only in the projective class of v_5 we can assume that $x_1 = 1$. Moreover, because v_5 must be isotropic, we deduce that

$$x_4 = -1 - \frac{x_5^2}{2} \ .$$

The lift of a generic fifth vertex has thus coordinates

$$v_5 = \left(1, -1, -1, -1 - \frac{x_5^2}{2}, x_5\right)$$

for some $x_5 \in \mathbb{R}$. However, we notice that the matrix $\mathrm{diag}(-1,-1,-1,-1,1) \in \mathrm{SO}_0(2,3)$ fixes the vertices $p_i = [v_i]$ for $i = 1, \dots, 4$ and sends $[v_5]$ to $[v_5']$ where v_5' only differs from v_5 by the sign of the last component. Hence, we can assume $x_5 \geq 0$.

The lift of the sixth vertex v_6 must be chosen so that

$$\langle v_2, v_6\rangle < 0 \quad \langle v_3, v_6\rangle < 0 \quad \langle v_4, v_6\rangle < 0$$

and

$$\langle v_1, v_5\rangle = 0 \quad \langle v_5, v_6\rangle = 0 \quad \langle v_6, v_6\rangle = 0 \ ,$$

hence v_6 belongs to the set

$$U = \{y \in \mathbb{R}^{2,3} \mid y_1 > 0,\ y_1 > -y_2,\ y_3 = 0,\ y_4 < 0,\ 2y_1y_3 + 2y_2y_4 - y_5^2 = 0,\ \langle v_5, v_6\rangle = 0\} \ .$$

Because $y_4 < 0$ and we are only interested in the projective class of v_6 we can assume that $y_4 = -1$. In particular, we deduce that v_6 is isotropic only if $y_2 = -\frac{y_5^2}{2}$. Moreover,

$$0 = \langle v_5, v_6\rangle = 1 - y_1 - \frac{y_5^2}{2}\left(-1 - \frac{x_5^2}{2}\right) - x_5y_5$$

implies that

$$y_1 = 1 + \frac{y_5^2}{2} + \frac{y_5^2x_5^2}{4} - x_5y_5 \ .$$

On the other hand, we must have $y_1 > -y_2$, and, imposing this condition, one finds

$$\frac{x_5^2y_5^2}{4} - x_5y_5 + 1 > 0$$

which gives the constraint $x_5y_5 \neq 2$. Therefore, a generic sixth vertex has coordinates

$$v_6 = \left(1 + \frac{y_5^2}{2} + \frac{y_5^2x_5^2}{4} - x_5y_5, -\frac{y_5^2}{2}, 0, -1, y_5\right)$$

with $x_5 \geq 0$ and $x_5y_5 \neq 2$. It can be easily verified that for every such choice of x_5 and y_5 the associated polygons are future-directed.

Moreover, we observe that, if $x_5 = 0$, the matrix $\mathrm{diag}(-1,-1,-1,-1,1) \in \mathrm{SO}_0(2,3)$ stabilizes the first five vertices and sends $p_6 = [v_6]$ to $[v_6']$ where v_6' only differs from v_6 by the sign of the last component. Therefore, we conclude that

$$\mathcal{TLP}_6^- = \{(s,t) \in \mathbb{R}^2 \mid s \geq 0,\ st \neq 2\}/(0,t) \sim (0,-t)$$

which has two connected components. □

Remark 5.7. The special point in $\mathcal{TLP}_6^-$ with $s = t = 0$ coorresponds to the unique light-like hexagon in $\mathrm{Ein}^{1,1} \subset \mathrm{Ein}^{1,2}$ (cfr. [Tam19]).

In general, we can prove the following

Theorem 5.8. *The moduli space of future-directed negative light-like polygons with $k \geq 6$ vertices is a (possibly disconnected) real orbifold of dimension $2(k-5)$.*

Proof. Recall that we can see the moduli space as the quotient

$$\mathcal{ML}\mathcal{P}_k^- = \mathcal{TLP}_k^-/\mathbb{Z}_k$$

where $\mathcal{TLP}_k^-$ denotes the moduli space of future-directed negative light-like polygons with a marked vertex. Thus it is enough to show that $\mathcal{TLP}_k^-$ is a (topological) manifold of dimension $2(k-5)$.
The marking on the polygon induces a natural enumeration of the vertices that is compatible with the orientation. From the proof of Proposition 5.6, we learn that, up to the action of $\mathrm{SO}_0(2,3)$, we can assume that $(p_1, p_2, p_3, p_4, p_5) = ([e_1], [e_2], [e_3], [e_3 + e_4], [e_1 - e_2 - e_3 - (1 + s^2/2)e_4 + se_5])$ with $s \geq 0$. Now, all other vertices p_j with $6 \leq j \neq k-1$ must be chosen in an open subset of the light-cone of p_{j-1} (the open subset is determined by the negativity conditions of the inner product with the previous non-consecutive vertices, which is sufficient to obtain a negative light-like polygon by Remark 5.2, and by the future-directed condition). The only exception is given by p_k which must lie in the intersection between the light-cone of p_{k-1} and p_1, which is a real manifold of dimension 1. Therefore, the moduli space of marked future-directed negative light-like polygon in $\mathrm{Ein}^{1,2}$ with k vertices is a manifold of dimension

$$\dim_{\mathbb{R}} \mathcal{TLP}_k^- = 1 + 2(k-6) + 1 = 2(k-5) \ .$$

□

Remark 5.9. As shown in the case of hexagons (Proposition 5.6), the open sets where the vertices p_k with $k \geq 6$ can be chosen may not be connected, hence $\mathcal{TLP}_k^-$ may be disconnected.

6. From polynomial quartic differentials to light-like polygons

We define a map

$$\tilde{\alpha} : \mathcal{TQ}_n \to \mathcal{TLP}_{n+4}^-$$

between the space $\mathcal{TQ}_n$ of monic, centered polynomial quartic differentials of degree n and the space $\mathcal{TLP}_{n+4}^-$ of future-directed negative light-like polygons with $(n+4)$ vertices (and a marked vertex) by associating, to a polynomial quartic differential, the boundary at infinity of the maximal surface in $\mathbb{H}^{2,2}$ constructed in Section 4.3. Equivalently, we could imagine the target as the boundary at infinity of the convex immersion in the Grassmannian of symplectic planes of $\mathbb{R}^4$ described in Section 4.1. This map $\tilde{\alpha}$ is equivariant with respect to the $\mathbb{Z}_{n+4}$ action and so induces a map

$$\alpha : \mathcal{MQ}_n \to \mathcal{ML}\mathcal{P}_{n+4}^-$$

between the moduli space of polynomial quartic differentials of degree n and future-directed negative light-like polygons in the Einstein Universe with $(n+4)$-vertices.

The aim of this section is to show that this latter map α is a homeomorphism onto a connected component of $\mathcal{MLP}^-_{n+4}$. This is the content of Theorem C.

The section begins by showing that the maximal surface defined by a polynomial quartic differential has the asymptotic geometry we claim above: this requires demonstrating a 'Stokes' phenomenon for solutions to (2.1). The proof of Theorem C then proceeds as a succession of results of properties of the map α: we prove α is continuous (Proposition 6.6) and proper (Corollary 6.9), while $\tilde{\alpha}$ is injective (Proposition 6.13). These properties together then imply Theorem C, which appears here as the summary Theorem 6.14.

6.1. **The boundary at infinity of a maximal surface in $\mathbb{H}^{2,2}$ with polynomial growth.** Given a polyonimal quartic differential q, in Section 4.1, we constructed a convex embedding of $\mathbb{C}$ in the Grassmannian of symplectic planes, and showed that it shares the same boundary at infinity of a complete maximal surface Σ in $\mathbb{H}^{2,2}$. We will refer to Σ as the maximal surface in $\mathbb{H}^{2,2}$ with polynomial growth q.

We recall some general facts about maximal surfaces in $\mathbb{H}^{2,2}$. We denote by $\widehat{\mathbb{H}}^{2,2}$ the space of unitary time-like vectors in $\mathbb{R}^{2,3}$, which is a double cover of $\mathbb{H}^{2,2}$ under the natural projection $\pi : \mathbb{R}^{2,3} \setminus \{0\} \to \mathbb{RP}^5$. Similarly we denote by $\widehat{\mathrm{Ein}}^{1,2}$ the space of isotropic vectors of $\mathbb{R}^{2,3}$ up to positive scalar multiples. Again the projection $\pi : \widehat{\mathrm{Ein}}^{1,2} \to \mathrm{Ein}^{1,2}$ is a double cover, and we can identify $\widehat{\mathrm{Ein}}^{1,2}$ with the boundary at infinity of $\widehat{\mathbb{H}}^{2,2}$. The map

$$\begin{aligned} F : D^2 \times S^2 &\to \widehat{\mathbb{H}}^{2,2} \\ (z,w) &\mapsto \left(\frac{2}{1-\|z\|^2} z, \frac{1+\|z\|^2}{1-\|z\|^2} w \right) \end{aligned}$$

is a diffeomorphism, hence $D^2 \times S^2$ is a model for $\widehat{\mathbb{H}}^{2,2}$, if endowed with the pull-back metric

$$F^* g_{\widehat{\mathbb{H}}^{2,2}} = \frac{4}{(1-\|z\|^2)^2} |dz|^2 - \left(\frac{1+\|z\|^2}{1-\|z\|^2} \right)^2 g_{S^2}.$$

Here g_{S^2} is the standard round metric on the unit sphere. The map F extends to homeomorphism of the boundary

$$\begin{aligned} \partial_\infty F : S^1 \times S^2 &\to \widehat{\mathrm{Ein}}^{1,2} \\ (z,w) &\mapsto (z,w) \ . \end{aligned}$$

Lemma 6.1. *Under the homeomorphism $\partial_\infty F$, graphs of parameterized geodesic arcs in S^2 correspond to light-like segments in $\widehat{\mathrm{Ein}}^{1,2}$.*

Proof. Let $\gamma : [0, \theta_0] \subset S^1 \to S^2$ be a unit-length parameterization of a geodesic. We can suppose that $\gamma(0) = (0,0,1)$ and $\gamma(\theta) = (\sin(\theta)\cos(\alpha), \sin(\theta)\sin(\alpha), \cos(\theta))$ for

some $\alpha \in [0, 2\pi]$ for every $\theta \in [0, \theta_0]$. Now, the graph of γ consists of points of the form

$$(\cos(\theta), \sin(\theta), \sin(\theta)\cos(\alpha), \sin(\theta)\sin(\alpha), \cos(\theta)) \in S^1 \times S^2 \ ,$$

for $\theta \in [0, \theta_0]$. On the other hand, the light-like segment joining $(1, 0, 0, 0, 1)$ and $(\cos(\theta_0), \sin(\theta_0), \sin(\theta_0)\cos(\alpha), \sin(\theta_0)\sin(\alpha), \cos(\theta_0)) \in \widehat{\mathrm{Ein}}^{1,2}$ is given by

$$\tfrac{1}{\sqrt{t^2+(1-t)^2}}((1-t)+t\cos(\theta_0), t\sin(\theta_0), t\sin(\theta_0)\cos(\alpha), t\sin(\theta_0)\sin(\alpha), (1-t)+t\cos(\theta_0)) \quad t \in [0,1]$$

and this coincides exactly with the graph of γ if we define θ so that

$$\cos(\theta) = \frac{(1-t)+t\cos(\theta_0)}{\sqrt{t^2+(1-t)^2}} \quad \text{and} \quad \sin(\theta) = \frac{t\sin(\theta_0)}{\sqrt{t^2+(1-t)^2}} \ .$$

Notice that this is the correspondence between points in $\widehat{\mathrm{Ein}}^{1,2}$ and $S \times S^2$ given by the homeomorphism $\partial_\infty F$. □

Lemma 6.2. *Let $v, v' \in \widehat{\mathrm{Ein}}^{1,2}$ such that $\langle v, v' \rangle < 0$. Then $\partial_\infty F^{-1}(v) = (\theta, p)$ and $\partial_\infty F^{-1}(v') = (\theta', p')$ with $d_{S^2}(p, p') < d_{S^1}(\theta, \theta')$.*

Proof. Without loss of generality we can suppose $v = (1, 0, 0, 0, 1)$, so that $\theta = 0$ and $p = (0, 0, 1)$. A general point $v' \in \widehat{\mathrm{Ein}}^{1,2}$ can be written as

$$v' = (\cos(\theta'), \pm\sin(\theta'), \sin(\alpha)\cos(\beta), \sin(\alpha)\sin(\beta), \cos(\alpha))$$

for some $\beta \in [0, 2\pi]$ and $\theta', \alpha \in [0, \pi]$. By assumption

$$\langle v, v' \rangle = \cos(\theta') - \cos(\alpha) < 0$$

hence $\alpha < \theta' \leq \pi$. By construction

$$\partial_\infty F^{-1}(v') = (\theta', p'),$$

with $p' = (\sin(\alpha)\cos(\beta), \sin(\alpha)\sin(\beta), \cos(\alpha))$. Therefore,

$$d_{S^2}(p, p') = \alpha < \theta' = d_{S^1}(\theta, \theta') \leq \pi$$

as claimed. □

This model is also useful to understand complete space-like surfaces in $\widehat{\mathbb{H}}^{2,2}$.

Proposition 6.3. *Let $\widehat{\Sigma}$ be a complete, connected, space-like surface in $\widehat{\mathbb{H}}^{2,2}$. Then $\widehat{\Sigma}$ is the graph of a 2-Lipschitz map $f : D^2 \to S^2$.*

Proof. Let $\mathrm{pr}_1 : \widehat{\Sigma} \to D^2$ denote the restriction of the projection onto the first factor. Let g_Σ be the induced metric on $\widehat{\Sigma}$. Then

$$g_\Sigma \leq \mathrm{pr}_1^* \left(\frac{4}{(1-\|z\|^2)^2} |dz|^2 \right) \ .$$

Since g_Σ is complete, we deduce that $\mathrm{pr}_1 : \widehat{\Sigma} \to D^2$ is a proper immersion, hence a covering. Since D^2 is simply connected and $\widehat{\Sigma}$ is connected, pr_1 is a diffeomorphism.

Therefore, $\widehat{\Sigma}$ is the graph of a function $f : D^2 \to S^2$. Since $\widehat{\Sigma}$ is space-like, for every $z \in D^2$ and $v \in T_z D^2$ we must have

$$\frac{4}{(1-\|z\|^2)^2}\|v\|^2 - \left(\frac{1+\|z\|^2}{1-\|z\|^2}\right)^2 \|df_z(v)\|^2 > 0$$

which implies that

$$\|df_z(v)\| \leq \frac{2}{1+\|z\|^2}\|v\| \leq 2\|v\| \ ,$$

so f is 2-Lipschitz. □

In particular, it follows from the proof of the above proposition, that the boundary at infinity of $\widehat{\Sigma}$ is the graph of a 1-Lipschitz map $\partial f : S^1 \to S^2$.

Proposition 6.4. *The boundary at infinity of a maximal surface Σ in $\mathbb{H}^{2,2}$ is always non-positive. Moreover, the boundary is negative if and only if it does not contain any light-like segments.*

Proof. Consider a lift $\widehat{\Sigma}$ of Σ in $\widehat{\mathbb{H}}^{2,2}$. By the previous discussion, the lift $\widehat{\Sigma}$ is the graph of a 2-Lipschitz map, and its boundary at infinity $\widehat{\Delta}$ is the graph of a 1-Lipschitz map $\partial f : S^1 \to S^2$. Let $v_0 = (\theta_0, p_0 = \partial f(\theta_0)) \in \widehat{\Delta}$. Let $v \in \widehat{\Delta}$ which corresponds to a point of the form $(\theta, \partial f(\theta) = p)$. The positivity condition on the inner-product is equivalent to (Lemma 6.2)

$$d_{S^2}(p_0, p) \geq d_{S^1}(\theta, \theta_0).$$

But on the other hand, since ∂f is 1-Lipschitz we know that

$$d_{S^2}(p_0, p) = d_{S^2}(\partial f(\theta_0), \partial f(\theta)) \leq d_{S^1}(\theta, \theta_0).$$

with the equality if and only if the restriction of ∂f to the arc between θ_0 and θ is the parameterisation of a geodesic between p_0 and p. Therefore, the boundary at infinity of Σ is always non-positive and fails to be negative if and only if it contains light-like segments.

□

Proposition 6.5. *If q is a polynomial quartic differential of degree n, the boundary at infinity of Σ is a future-directed, negative, light-like polygon with $n+4$ vertices.*

Proof. By Proposition 6.4, we only need to show that the boundary at infinity is a future-directed light-like polygon of $n+4$ vertices. For simplicity we describe the boundary at infinity of the embedding in the Grassmannian of symplectic planes. Recall that this is obtained by ∇-parallel transport of the section

$$f(z) = u_1(z) \wedge u_3(z)$$

of $\mathbb{P}(\Lambda^2 E_{\mathbb{R}})$ to the fibre over a base point $0 \in \mathbb{C}$. Let $A \in \mathrm{SU}(4)$ be the following unitary matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \end{pmatrix}$$

expressed in the frame $F(z) = \{v_i(z)\}_{i=1,\dots 4}$ introduced in Section 3.1. By definition (see Section 4.2),

$$u_1(z) = H(z)^{-\frac{1}{2}} A v_2(z) \quad \text{and} \quad u_3(z) = H(z)^{-\frac{1}{2}} A v_4(z) \ .$$

The parallel transport $T(z) : (\mathcal{E}_{\mathbb{R}})_z \to (\mathcal{E}_{\mathbb{R}})_0$ expressed in the real frame $\{u_i(z)\}_{i=1,\dots,4}$ is given by

$$T(z) = A^{-1} H(0)^{\frac{1}{2}} \psi(z) H(z)^{-\frac{1}{2}} A \ .$$

Now, recall from Theorem 3.9 that in each standard half-plane (U, w) and in each stable direction J_α, the limit M_α of $H(0)^{\frac{1}{2}} \psi(w) H(w)^{-\frac{1}{2}} \psi_0(w)^{-1}$ exists as $|w| \to +\infty$. Note that we may write

$$\begin{aligned} &\lim_{|w|\to\infty} A^{-1} H(0)^{\frac{1}{2}} \psi(w) H(w)^{-\frac{1}{2}} A \\ &\lim_{|w|\to\infty} = A^{-1} H(0)^{\frac{1}{2}} \psi(w) H(w)^{-\frac{1}{2}} \psi_0(w)^{-1} A [A^{-1} \psi_0(w) A] \\ &= A^{-1} M_\alpha A \lim_{|w|\to\infty} A^{-1} \psi_0(w) A \ . \end{aligned} \tag{6.1}$$

Therefore, working in this coordinate chart, we deduce that as $|w| \to +\infty$ along a stable direction, the embedding $f(w)$ has a limit point p_α, given by the image of the limit point of the standard flat maximal surface, under the composition of $M'_\alpha = A^{-1} M_\alpha A$ and $A^{-1} S' \in \mathrm{Sp}(4, \mathbb{R})$, where M_α is the limit found in the proof of Theorem 3.9 and $S' = S\mathrm{diag}(1, \frac{1-i}{\sqrt{2}}, i, \frac{1+i}{\sqrt{2}})$. We thus obtain the limits described in Table 2.

Type of path γ	**Direction θ**	**Projective limit p_γ of $\sigma(\gamma)$**
Quasi-ray	$\theta \in (0, \frac{\pi}{4})$	$p_{++} = M'_{++} A^{-1} S' [e_1 \wedge e_4]$
Quasi-ray	$\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$	$p_{+} = M'_{+} A^{-1} S' [e_1 \wedge e_4]$
Quasi-ray	$\theta \in (\frac{\pi}{2}, \frac{3\pi}{4})$	$p_{-} = M'_{-} A^{-1} S' [e_3 \wedge e_4]$
Quasi-ray	$\theta \in (\frac{3\pi}{4}, \pi)$	$p_{--} = M'_{--} A^{-1} S' [e_3 \wedge e_4]$

TABLE 2. Limits of the maximal surface Σ in a standard half-plane

Using that $M_\alpha = H(0)^{\frac{1}{2}} L_\alpha$, with L_α being the limit of $\psi(w) H(w)^{-\frac{1}{2}} \psi_0(w)^{-1}$ for $|w| \to +\infty$ along a stable direction in the sector J_α, as well as the relations between the matrices M_{++}, M_+, M_-, M_{--} displayed in Lemma 3.7, we obtain that

$$\begin{aligned} &M'_{--} A^{-1} S' [e_3 \wedge e_4] = M'_{-} A^{-1} S' [e_3 \wedge e_4] = M'_{+} A^{-1} S' [e_3 \wedge e_4] \\ &M'_{++} A^{-1} S' [e_1 \wedge e_4] = M'_{+} A^{-1} S' [e_1 \wedge e_4] : \end{aligned}$$

the key point in this computation is that the unipotents U_α that appear when one expands the expressions fix the planes on which they are acting. (For example, one sees from the proof of Lemma 3.7 that U_0 fixes the plane $e_3 \wedge e_4$, concluding the second inequality on the first line displayed above.)

A summary of the computation is that only two different points appear as limits along stable directions in each standard half-plane and those are obtained as images of the Stokes matrix M'_+.

Since M'_+, being an element of $\mathrm{Sp}(4,\mathbb{R})$, preserves the symplectic form

$$\omega^* = e_1 \wedge e_3 + e_2 \wedge e_4$$

the two points $p_{--} = p_-$ and $p_+ = p_{++}$ lie on a common photon: from the relationships above and equation (4.3), we find that

$$\begin{aligned}\langle p_-, p_+\rangle &= \langle M'_+A^{-1}S'[e_3 \wedge e_4], M'_+A^{-1}S'[e_1 \wedge e_4]\rangle \\ &= \langle [e_3 \wedge e_4], [e_1 \wedge e_4]\rangle \\ &= 0.\end{aligned}$$

Here the middle equality follows from the invariance of the bracket by an action of an element $M'_+A^{-1}S' \in \mathrm{Sp}(4,\mathbb{R})$.

The displayed computation of course expresses that p_- and p_+ are orthogonal and so we deduce by the proof of Proposition 6.4 that the light-like segment joining them is contained entirely in the boundary at infinity of Σ. Therefore, the asymptotic boundary of Σ in each standard half-plane consists of a light-like segment with extreme points $M'_+A^{-1}S'[e_3 \wedge e_4]$ and $M'_+A^{-1}S'[e_1 \wedge e_4]$.

Now, two segments belonging to two consecutive standard half-planes share a common vertex, because two standard half-planes intersect in only an open sector of angle $\pi/2$. (It is critical here that the intersection is restricted to but a single open sector angle $\pi/2$; a more relaxed definition of standard half-plane would likely not permit us to conclude that the intersection is but a single vertex.) Since there are $n+4$ standard half-planes, we conclude that the boundary at infinity of Σ is a negative light-like polygon Δ with $n+4$ vertices.

Moreover, Δ is future-directed because, by definition, the time-orientation of the polygon depends only on the positions of two consecutive vertices and these are inherited from the standard flat maximal surface. □

6.2. **Definition of the map and continuity.** The previous propositions allow us to define a map

$$\tilde{\alpha} : \mathcal{TQ}_n \to \mathcal{TLP}^-_{n+4}$$

that associates, to a polynomial quartic differential of degree n, the boundary at infinity of the maximal surface in $\mathbb{H}^{2,2}$ with polynomial growth q. In order to show that $\tilde{\alpha}$ induces a map α between the moduli spaces, we need to check the equivariance of $\tilde{\alpha}$ with respect to the $\mathbb{Z}_{n+4}$-action. First of all, let us describe how to encode the $\mathbb{Z}_{n+4}$-action. Given a monic polynomial quartic differential q of degree n in the complex plane, there are $n+4$ canonical directions corresponding to the set

$$D = \left\{ z \in \mathbb{C} \mid \arg(z) = \frac{2\pi j}{n+4},\ j = 0, \dots, n+3 \right\} .$$

Those can be understood as follows: if $q = z^n dz^4$, these are exactly the directions in which the quartic differential takes positive real values; in the general case, these

directions are characterized by the fact that they are contained eventually in a unique standard q-half-plane, where they correspond to quasi-rays with angle 0. If we fix one direction $\theta_0 = \arg(z_0) + \epsilon$ with $z_0 \in D$ and $\epsilon > 0$ small, we can see the action of the cyclic group $\mathbb{Z}_{n+4}$ as a rotation in this set.

Let $\sigma : \mathbb{C} \to \mathbb{H}^{2,2}$ be a maximal embedding associated to q. Let Δ denote the future-directed negative light-like polygon in the boundary at infinity of $\Sigma = \sigma(\mathbb{C})$. The direction θ_0 gives a marking on Δ as follows: the path $\sigma(e^{i\theta_0}t)$ converges to a point in Δ as $t \to +\infty$. By Proposition 6.4, the limit point is a vertex $v_0 \in \Delta$. We can then define

$$\begin{aligned} \tilde{\alpha} : \mathcal{TQ}_n &\to \mathcal{TLP}^-_{n+4} \\ q &\mapsto (\Delta, v_0) \ . \end{aligned}$$

If we change σ to σ' by post-composition with an isometry g of $\mathbb{H}^{2,2}$, the boundary at infinity becomes $\Delta' = g(\Delta)$, hence the two marked light-like polygons (Δ, v_0) and (Δ', v_0') are equivalent under the action of $\mathrm{SO}_0(2,3)$, hence the map $\tilde{\alpha}$ is well-defined. Moreover, if we change σ by a pre-composing with the generator of the $\mathbb{Z}_{n+4}$-action $T(z) = \zeta_{n+4}^{-1} z$, then $\sigma' = \sigma \circ T$ is a maximal embedding with polynomial growth $T_* q$. Its boundary at infinity remains Δ, but the limit point of the ray $\sigma'(e^{i\theta_0}t)$ changes to v_0', which coincides with the limit point of the ray $\sigma(e^{i(\theta_0 + 2\pi/(n+4))}t)$. By the description of the limit points along rays, given in Proposition 6.4, we find $v_0' = v_1$, hence the map $\tilde{\alpha}$ is $\mathbb{Z}_{n+4}$-equivariant. Notice, moreover, that if $T_*^j(q) = q$ for some $j = 1, \dots, n+4$, then σ and $\sigma' = \sigma \circ T^j$ are maximal embedding of $\mathbb{C}$ into $\mathbb{H}^{2,2}$ with the same embedding data, hence they differ by post-composition with an isometry of $\mathbb{H}^{2,2}$. Therefore, $\tilde{\alpha}(q) = \tilde{\alpha}(T_*^j(q))$.

Proposition 6.6. *The map $\tilde{\alpha}$ is continuous.*

Proof. Let q_n be a sequence of monic and centered polynomial quartic differentials converging to q. Let (Δ_n, v_n) and (Δ, v) be marked future-directed, negative, light-like polygons representing $\alpha(q_n)$ and $\alpha(q)$, respectively.

We need to show that (Δ_n, v_n) converges to (Δ, v) up to the conformal action of $\mathrm{SO}_0(2,3)$. We claim first that the maximal surfaces $\Sigma_n = \sigma_n(\mathbb{C})$ in $\mathbb{H}^{2,2}$ associated to q_n converge to the maximal surface $\Sigma = \sigma(\mathbb{C})$ associated to q, up to isometries. In fact, since q_n is convergent, the supersolution estimates of Lemma 2.4 and standard Schauder estimates give a uniform bound on the $C^{1,\alpha}$ norm of the functions $(u_1)_n$ and $(u_2)_n$, solutions to Equation (2.1) on compact sets. Hence they weakly converge to a weak solution (u_1, u_2) of

$$\begin{cases} \Delta u_1 = e^{u_1 - u_2} - e^{-2u_1}|q|^2 \\ \Delta u_2 = e^{2u_2} - e^{u_1 - u_2} \end{cases} .$$

By elliptic regularity, the limits are strong solutions, and by uniqueness they must coincide with those found in Section 2. Therefore, $(u_1)_n$ and $(u_2)_n$ actually converge smoothly on compact sets to u_1 and u_2. Recalling that the induced metric on the maximal surface Σ is given by $4e^{u_1 - u_2}|dz|^2$ and the second fundamental form only

depends on q, we deduce (cf. Proposition 4.8) that the embedding data of Σ_n converge to the embedding data of Σ, thus Σ_n converges to Σ up to global isometries of $\mathbb{H}^{2,2}$.

Since Σ_n and Σ are space-like surfaces, they can be seen, together with their boundary at infinity, as graphs of 2-Lipschitz functions $f_n, f : D^2 \to S^2$. In particular, uniform convergence on compact sets of Σ_n to Σ implies that Δ_n converges to Δ. Moreover, we remarked, just prior to the statement of Proposition 6.4, that these graphs extend to uniformly Lipschitz maps of the closed disk. Thus the limit points of the rays $\sigma_n(e^{i\theta_0}t)$ converge to the limit point of the ray $\sigma(e^{i\theta_0}t)$, hence v_n converges to v. □

6.3. **Properness.** Let $[q_i]$ be a sequence of polynomial quartic differentials of degree n that leaves every compact set in the moduli space $\mathcal{MQ}_n$. We consider a representative $q_i \in [q_i]$ that has a root at the origin. Therefore, we can write

$$q_i = q_i(z)dz^4 = (z^n + a_{n-1,i}z^{n-1} + \cdots + a_{1,i}z)dz^4$$

for some $a_{j,i} \in \mathbb{C}$, and we must necessarily have that $|a_{j,i}| \to +\infty$ as $i \to +\infty$, for some $j = 1, \dots, n-1$, up to subsequences. The idea now is to re-scale the variable z appropriately, so that q_i converges to a polynomial quartic differential of lower degree.

Lemma 6.7. *There exists a unique sequence of complex numbers λ_i such that*

$$q_i(\lambda_i^{-1}w)\lambda_i^{-4}dw^4$$

converges, up to subseqeunces, to a monic non-constant polynomial quartic differential q_∞ of lower degree.

Proof. We describe an algorithm to find such a sequence λ_i. Let j be the largest index such that $|a_{j,i}|$ diverges as $i \to +\infty$ and define λ_i so that $\lambda_i^{-j-4}a_{j,i} = 1$. Let us then consider the index $j-1$. Two things can happen:

(1) if $\lambda_i^{-j-3}a_{j-1,i}$ is uniformly bounded, we keep the same sequence λ_i and we move to the index $j-2$;
(2) if $\lambda_i^{-j-3}a_{j-1,i}$ is unbounded, we replace λ_i with λ_i' such that $(\lambda_i')^{-j-3}a_{j-1,i} = 1$. Notice that we must necessarily have

$$\lim_{i\to\infty} \frac{\lambda_i}{\lambda_i'} = 0 \ ,$$

therefore $\lambda_i^{-j-4}a_{j,i}$ (where we use the new λ_i) tends to 0. We then move to the index $j-2$.

When we arrive at the index 1, the sequence λ_i that we end up with has the property that the product $\lambda_i^{-j-4}a_{j,i}$ is uniformly bounded for every $j = 1, \dots, n-1$, and every subsequential limit of $q_i(\lambda_i^{-1}w)\lambda_i^{-4}dw^4$ is monic, non-constant and of degree strictly less than n. □

We will say that q_∞ is a re-scaled limit of q_i. We will show the following:

Proposition 6.8. *Let $[q_i] \in \mathcal{MQ}_n$ be a diverging sequence. Let $1 \leq m \leq n-1$ be the degree of its rescaled limit q_∞. Let Δ_i be the boundary at infinity of the maximal surface Σ_i corresponding to $[q_i]$. Then there is a sequence $A_i \in \mathrm{SO}_0(2,3)$ such that $A_i\Delta_i$ converges to a future-directed, negative, light-like polygon in $\mathrm{Ein}^{1,2}$ with $m+4$ vertices.*

In particular, we deduce

Corollary 6.9. *The map α is proper.*

Proof. We have to show that, if $[q_i] \in \mathcal{MQ}_n$ is a diverging sequence of polynomial quartic differentials of degree n, then $\alpha([q_i])$ is a diverging sequence in $\mathcal{MLP}^-_{n+4}$. Let Δ_i be polygons representing $\alpha([q_i])$. By Proposition 6.8, there is a sequence $A_i \in \mathrm{SO}_0(2,3)$ such that $A_i\Delta_i$ converges to a future-directed negative light-like polygon in $\mathrm{Ein}^{1,2}$ with fewer vertices that is not a quadrilateral. The following lemma shows that $[\Delta_i]$ cannot be contained in a compact set of $\mathcal{MLP}^-_{n+4}$. □

Lemma 6.10. *Let $[\Delta_n]$ be a sequence of equivalence classes of future-directed negative light-like polygons in $\mathrm{Ein}^{1,2}$ with k-vertices converging to $[\Delta] \in \mathcal{MLP}^-_k$. Let B_n be any sequence in $\mathrm{SO}_0(2,3)$. Then, the only light-like polygons that can appear as limits of subsequences of $B_n\Delta_n$ are*

- *either equivalent to Δ in the moduli space $\mathcal{MLP}^-_k$;*
- *or quadrilaterals.*

Proof. For this proof, it is convenient to use the quadratic form in $\mathbb{R}^5$ with signature $(2,3)$ given by

$$\langle x, x\rangle = 2x_1x_5 + 2x_2x_4 - x_3^2$$

with respect to the canonical basis $\{e_1, \dots, e_5\}$ of $\mathbb{R}^5$. We denote by A the group of diagonal matrices in $\mathrm{SO}_0(2,3)$ and by $A^+ \subset A$ the semigroup of diagonal matrices with eigenvalues $\lambda_1 \geq \lambda_2 \geq 1 \geq \frac{1}{\lambda_2} \geq \frac{1}{\lambda_1}$ in this order. This choice induces a KAK-decomposition, where K is a maximal compact subgroup of $\mathrm{SO}_0(2,3)$. We can thus write $B_n = K'_n D_n K_n$ with $D_n \in A^+$ and $K_n, K'_n \in K$. Up to subsequences, we can assume that K_n and K'_n converge to K and K' respectively. Moreover, we can assume that the vertices of Δ_n converge to the vertices of Δ. We will see that the subsequential limits of $B_n\Delta_n$ depend on the behaviour of the eigenvalues $\lambda_{i,n}$ of D_n.

Let us first assume that $\lambda_{1,n}$ is uniformly bounded. Then, up to subsequences, we can assume that D_n converges to a diagonal matrix D so that the isometries B_n converge to $B = K'DK$. It is clear then that $B_n\Delta_n$ converges to $B\Delta$. This, in particular, means that acting on Δ_n by a converging sequence of isometries does not change the equivalence class in $\mathcal{MLP}^-_k$ of the limit.

Let us now consider the case of $\lambda_{2,n}$ unbounded and choose a subsequence so that $\lambda_{2,n} \to +\infty$ as $n \to +\infty$. By the previous remark, it is sufficient to understand the behaviour of D_n on the polygons $\Delta'_n = K_n\Delta_n$ which converge to $\Delta' = K\Delta$. Let us choose vectors $v_{i,n}$ and v_i for $i = 1, \dots, k$ in $\mathbb{R}^5$ such that each $v_{i,n}$ projects to a vertex $p_{i,n} \in \Delta'_n$, the vector v_i projects to the vertex $p_i \in \Delta'$ and $v_{i,n}$ converges to v_i as $n \to +\infty$. This means that if we denote by $a_{j,i,n}$ (resp. $a_{j,i}$) the component

of $v_{i,n}$ (resp. v_i) along e_j, we have $\lim_{n\to+\infty} a_{j,i,n} = a_{j,i}$ for every $j = 1, \dots, 5$. We then note the following:

(i) if $\lambda_{1,n}a_{1,i,n}$ and $\lambda_{2,n}a_{2,i,n}$ are not both uniformly bounded, then $D_n v_{i,n}$ limits to a point on the photon $\mathbb{P}(\mathrm{Span}(e_1, e_2))$;
(ii) if both $\lambda_{1,n}a_{1,i,n}$ and $\lambda_{2,n}a_{2,i,n}$ are bounded, we denote by q_i the projective limit of $D_n v_{i,n}$.

We notice that (ii) can only happen for those indices $1 \leq i \leq k$ such that $a_{1,i} = a_{2,i} = 0$. Since the vectors v_i project to vertices of a negative light-like polygon, they are isotropic, thus necessarily $a_{3,i} = 0$ as well. This implies that such vectors v_i are orthogonal to each other, so there can be at most two of them in Δ'. This shows that, the only light-like polygon that can be the limit of Δ'_n is a quadrilateral.

The argument is similar if $\lambda_{2,n}$ is bounded but $\lambda_{1,n}$ tends to $+\infty$. In this case, up to subsequences, all vertices $[v_{i,n}]$ that do not converge to $[e_5]$ (so all vertices, except at most one) necessarily limit to a point in the light-cone of e_1. The only light-like polygon with this configuration of vertices is a quadrilateral. □

Proof of Proposition 6.8. Consider the change of coordinates on $\mathbb{C}$ given by $w = \lambda_i z$, where λ_i is the sequence found in Lemma 6.7. In the w-plane, the quartic differential can be written as

$$q_i = \hat{q}_i(w)dw^4 \quad \text{where} \quad \hat{q}_i(w) = \lambda_i^{-4} q_i(\lambda_i^{-1}w),$$

hence it subconverges uniformly on compact sets to some q_∞ of degree $1 \leq m \leq n-1$. Recall that the induced metric on the maximal surface Σ_i is given by $4e^{u_{1,i}-u_{2,i}}|dz|^2$ where $u_{1,i}$ and $u_{2,i}$ are the solution of the system of PDE

$$\begin{cases} \Delta u_{1,i}(z) = e^{u_{1,i}(z)-u_{2,i}(z)} - e^{-2u_{1,i}(z)}|q_i(z)|^2 \\ \Delta u_{2,i}(z) = e^{2u_{2,i}(z)} - e^{u_{1,i}(z)-u_{2,i}(z)} \end{cases}.$$

Changing to the w-coordinates, the functions

$$v_{1,i}(w) = u_{1,i}(\lambda_i^{-1}w) - 3\log(|\lambda_i|) \quad \text{and} \quad v_{2,i}(w) = u_{2,i}(\lambda_i^{-1}w) - \log(|\lambda_i|)$$

satisfy the differential equations

$$\begin{cases} \Delta v_{1,i}(w) = e^{v_{1,i}(w)-v_{2,i}(w)} - e^{-2v_{1,i}(w)}|\hat{q}_i(w)|^2 \\ \Delta v_{2,i}(w) = e^{2v_{2,i}(w)} - e^{v_{1,i}(w)-v_{2,i}(w)} \end{cases}. \tag{6.2}$$

Because the coefficients of polynomial $\hat{q}_i(w)$ converge, the sub-solution and super-solutions found in Section 2 show that

$$\begin{aligned} \frac{3}{8}\log(|\hat{q}_i(w)|) \leq v_{1,i}(w) \leq \frac{3}{8}\log(|\hat{q}_i(w)| + C) \\ \frac{1}{8}\log(|\hat{q}_i(w)|) \leq v_{2,i}(w) \leq \frac{1}{8}\log(|\hat{q}_i(w)| + 3C) \end{aligned},$$

hence $v_{1,i}$ and $v_{2,i}$ are uniformly bounded on compact sets, and from Equation (6.2) we also have a uniform bound on their laplacian. Therefore, standard elliptic theory

tells us that $v_{1,i}$ and $v_{2,i}$ converge smoothly to solutions $v_{1,\infty}$ and $v_{2,\infty}$ of the limiting system of PDE

$$\begin{cases} \Delta v_{1,\infty}(w) = e^{v_{1,\infty}(w)-v_{2,\infty}(w)} - e^{-2v_{1,\infty}(w)}|q_\infty(w)|^2 \\ \Delta v_{2,\infty}(w) = e^{2v_{2,\infty}(w)} - e^{v_{1,\infty}(w)-v_{2,\infty}(w)} \end{cases}.$$

On the other hand, by uniqueness (Remark 2.7) of the solution to (2.1), the pair $(v_{1,\infty}, v_{2,\infty})$ defines the embedding data of a maximal surface Σ_∞ in $\mathbb{H}^{2,2}$ with polynomial quartic differential q_∞, and hence polygonal boundary with $\deg(q_\infty) + 4$ vertices. It follows that the sequence of maximal surfaces Σ_i converges smoothly on compact sets to Σ_∞, up to global isometries of $\mathbb{H}^{2,2}$. In particular, as in the proof of the continuity of α, the boundary at infinity of Σ_i converges to the boundary at infinity of Σ_∞, which is a light-like polygon with $m+4$ vertices, up to the conformal action of $\mathrm{SO}_0(2,3)$. □

6.4. **Injectivity.** The following lemma is crucial in order to prove the injectivity of the map $\tilde{\alpha}$:

Lemma 6.11. *Let $\Gamma \subset \mathrm{Ein}^{1,2}$ be the graph of a 1-Lipschitz map. If there exists a complete maximal surface $\Sigma \subset \mathbb{H}^{2,2}$ spanning Γ, then it is unique.*

The proof will be an adaptation "at infinity" of the argument used in [CTT19] in the case of boundary curves invariant by a cocompact group, which consisted of an appplication of the maximum principle to a carefully chosen function defined on $\widehat{\Sigma} \times \widehat{\Sigma}'$. In our non-compact context, we will need the following version of the maximum principle:

Theorem 6.12 ([Omo67])**.** *Let M be a complete Riemannian manifold with sectional curvature bounded below. If g is a smooth function on M with $\sup(g) < +\infty$, then there exists a sequence of points $x_k \in M$ such that*

$$\lim_{k\to\infty} g(x_k) = \sup(g) \qquad |\mathrm{grad}(g)_{x_k}| \leq \frac{1}{k} \qquad \mathrm{Hess}(g)_{x_k}(w,w) \leq \frac{\|w\|^2}{k}\ \forall w \in T_{x_k}M$$

Proof of Lemma 6.11. Suppose, by contradiction, that there exists another complete maximal surface Σ' with boundary at infinity Γ. We choose their lifts $\widehat{\Sigma}$ and $\widehat{\Sigma}'$ to $\widehat{\mathbb{H}}^{2,2}$ in such a way that they share the same boundary at infinity. As a consequence, the function

$$\begin{aligned} B : \widehat{\Sigma} \times \widehat{\Sigma}' &\to \mathbb{R} \\ (u, v) &\mapsto \langle u, v \rangle \end{aligned}$$

is always non-positive ([CTT19, Lemma 3.24]). Moreover, if $\widehat{\Sigma}$ and $\widehat{\Sigma}'$ are distinct, then we can find a pair of points $(u_0, v_0) \in \widehat{\Sigma} \times \widehat{\Sigma}'$ such that $B(u_0, v_0) > -1$ ([CTT19, Lemma 3.28]). In particular, $-1 < \sup(B) \leq 0$ ([CTT19, Lemma 3.25]). Notice that by a general result of Ishihara ([Ish88]), maximal surfaces in $\mathbb{H}^{2,2}$ have uniformly bounded second fundamental form, thus the Riemannian manifold $M = \widehat{\Sigma} \times \widehat{\Sigma}'$

has bounded sectional curvature. By the Omori maximum principle, we can find a sequence of points (u_n, v_n) such that

$$\lim_{k\to\infty} B(u_n, v_n) = \sup(B) \qquad |\mathrm{grad}(B)_{(u_n,v_n)}| \leq \frac{1}{n} \qquad \mathrm{Hess}(B)_{(u_n,v_n)}(\dot\gamma_n, \dot\gamma_n) \leq \frac{\|\dot\gamma\|^2}{n}$$

for every geodesic path $\gamma_n : [-\epsilon, \epsilon] \to M$ with $\gamma_n(0) = (u_n, v_n)$. We will follow the construction of ([CTT19]) in order to find a sequence of paths γ_n which will give a contradiction.
The second derivative of B along a geodesic path $\gamma(t) = (u(t), v(t))$ is given by

$$\begin{aligned}\mathrm{Hess}(B)(\dot u, \dot v)_{(u(0),v(0))} &= \frac{d}{dt^2}_{|t=0} B(u(t), v(t)) \\ &= 2\langle \dot u, \dot v\rangle + \|\dot u\|^2\langle u(0), v(0)\rangle + \|\dot v\|^2\langle u(0), v(0)\rangle \\ &+ \langle II(\dot u, \dot u), v(0)\rangle + \langle II'(\dot v, \dot v), u(0)\rangle \ ,\end{aligned}$$

where II and II' denote the second fundamental form of $\widehat\Sigma$ and $\widehat\Sigma'$ respectively. Since $\widehat\Sigma$ and $\widehat\Sigma'$ are maximal surfaces, the quadratic forms $\beta(\dot u) = \langle II(\dot u, \dot u), v(0)\rangle$ and $\beta'(\dot v) = \langle II'(\dot v, \dot v), u(0)\rangle$ are traceless. Let λ and λ' be their positive eigenvalues. Let (u_n, v_n) be the sequence of points given by the Omori maximum principle. We explain how to choose γ_n assuming that the positive eigenvalue λ_n of β is larger than the positive eigenvalue λ'_n of β' at the point (u_n, v_n); for the other case it will be sufficient to interchange the role of $\dot{u_n}$ and $\dot{v_n}$ described below.
We choose tangent vectors $(\dot{u_n}, \dot{v_n}) \in T_{(u_n,v_n)}M$ such that

$$\|\dot u_n\| = 1 \qquad \dot v_n = \frac{p(\dot u_n)}{\|p(\dot u_n)\|}$$

where $\beta(\dot u_n) = \lambda_n$ and p is the orthogonal projection onto $T_{v_n}\widehat\Sigma'$. This choice of $\dot u_n$ and consequent choice of $\dot v_n$ will force the terms $\langle II(\dot u, \dot u), v(0)\rangle + \langle II'(\dot v, \dot v), u(0)\rangle \ \geq 0$, as in [CTT19]. Next, since we have an orthogonal decomposition

$$\mathbb{R}^{2,3} = \mathrm{Span}(v_n) \perp T_{v_n}\widehat\Sigma' \perp N_{v_n}\widehat\Sigma'$$

we may write $\dot u_n = k_n v_n + p(\dot u_n) + w_n$, with $w_n \in N_{v_n}\widehat\Sigma'$. Since the normal bundle of a space-like surface in $\mathbb{H}^{2,2}$ is negative definite, we have that $\|w_n\| \leq 0$. Hence,

$$1 = \|\dot u_n\|^2 = -k_n^2 + \|p(\dot u_n)\|^2 + \|w_n\| \leq -k_n^2 + \|p(\dot u_n)\|^2$$

which implies that $\|p(\dot u_n)\|^2 \geq 1 + k_n^2$. Therefore,

$$\langle \dot u_n, \dot v_n\rangle = \langle k_n v_n + p(\dot u_n) + w_n, \frac{p(\dot u_n)}{\|p(\dot u_n)\|}\rangle = \|p(\dot u_n)\| \geq \sqrt{1 + k_n^2}.$$

We notice that k_n decays to zero as n goes to infinity because

$$|k_n| = |\langle \dot u_n, v_n\rangle| = |dB_{(u_n,v_n)}(\dot u_n, 0)| = |g(\mathrm{grad}(B)_{(u_n,v_n)}, (\dot u_n, 0))| \leq \frac{\|\dot u_n\|}{n}$$

where we denoted with g the Riemannian metric on M.
The Omori maximum principle then gives that

$$\frac{2}{n} \geq \mathrm{Hess}(B)_{(u_n,v_n)}(\dot{u}_n, \dot{v}_n) \geq 2\sqrt{1+k_n^2} + 2B(u_n, v_n)$$

and letting n go to infinity we obtain that

$$0 \geq \limsup_{n\to+\infty} 2\sqrt{1+k_n^2} + 2B(u_n, v_n) = 2 + 2\sup(B) \ .$$

Thus, $\sup(B) \leq -1$, but this contradicts the fact that $-1 < \sup(B) \leq 0$. □

Proposition 6.13. *The map $\tilde{\alpha}$ is injective.*

Proof. Let $q, q' \in \mathcal{TQ}_n$ be different monic and centered polynomial quartic differentials. If there exists $j \in \{1, \dots, n\}$ such that $q' = T_*^j q$, where $T(z) = \zeta_{n+4} z$ is a generator of the $\mathbb{Z}_{n+4}$-action, then the equivariance of the map already implies that $\tilde{\alpha}(q) \neq \tilde{\alpha}(q')$, because the marking of the polygon at infinity is changed. Otherwise, suppose by contradiction that $\tilde{\alpha}(q) = \tilde{\alpha}(q')$. Then, we can choose maximal surfaces Σ and Σ' with polynomial growth q and q' with the same boundary at infinity Δ. By Lemma 6.11, the surfaces Σ and Σ' must coincide, and, in particular, have the same embedding data. Therefore, there exists a biholomorphism T' of $\mathbb{C}$ such that $T'_* q' = q$, but this is impossible because q and q' do not lie in the same $\mathbb{Z}_{n+4}$-orbit and they are both monic and centered. □

Theorem 6.14. *The map $\tilde{\alpha}$ induces a homeomorphism*

$$\alpha : \mathcal{MQ}_n \to \mathcal{ML}\mathcal{P}^-_{n+4}$$

between the moduli space of polynomial quartic differential on the complex plane of degree n and a connected component of the moduli space of future-directed, negative light-like polygons in the Einstein Universe with $n+4$ vertices.

Proof. The map $\tilde{\alpha} : \mathcal{TQ}_n \to \mathcal{TL}\mathcal{P}^-_{n+4}$ is continuous and injective by Proposition 6.13. It is also proper by Corollary 6.9, hence it is a homeomorphism onto a connected component of $\mathcal{TL}\mathcal{P}^-_{n+4}$ by the Invariance of the Domain. Since it is $\mathbb{Z}_{n+4}$-equivariant, it decends to a homeomorphism $\alpha : \mathcal{MQ}_n \to \mathcal{ML}\mathcal{P}^-_{n+4}$ between connected components of the moduli spaces. □

7. Estimates along rays

We wish to apply our results in the (new) setting of closed surfaces. In this direction, let $X = (S, J)$ be a closed Riemann surface and let q be a holomorphic quartic differential on X. Recall from Section 1 that, out of these data, one can construct an $\mathrm{Sp}(4, \mathbb{R})$-Higgs bundle $(\mathcal{E}, \varphi)$ over X in the $\mathrm{Sp}(4, \mathbb{R})$-Hitchin component where $\mathcal{E} = K^{\frac{3}{2}} \oplus K^{-\frac{1}{2}} \oplus K^{-\frac{3}{2}} \oplus K^{\frac{1}{2}}$ and

$$\varphi = \begin{pmatrix} 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ .$$

In this setting, it is well-known that the solution of Hitchin's self-duality equations is unique and diagonal of the form $g = \mathrm{diag}(g_1, g_2^{-1}, g_1^{-1}, g_2)$. Works of Collier-Li ([CL17]) and Mochizuki ([Moc14]) describe the asymptotic behaviour of the metric g_s along rays of quartic differentials $q_s = sq_0$ away from the zeros of q_0. Here, we use the harmonic metric H found in Theorem 2.1 in order to construct sub- and supersolutions that will describe the asymptotics of the harmonic metrics g_s at and near a zero. We will prove the following result:

Theorem 7.1. *Assume $p \in X$ is a zero of order $k \geq 1$ of the quartic differential q_0. Let σ denote the conformal hyperbolic metric on X. Then, there is a sequence of radii $r_s \to 0$ such that*

$$g_{1,s}^{-1}|_{B(p,r_s)} = O(s^{\frac{3}{k+4}}\sigma^{\frac{3}{2}}) \quad \textit{and} \quad g_{2,s}^{-1}|_{B(p,r_s)} = O(s^{\frac{1}{k+4}}\sigma^{\frac{1}{2}})$$

along the ray $q_s = sq_0$ as $s \to +\infty$.

As a geometric corollary to this analytic result, we will find that that the harmonic metrics g_s "localize" in the sense that the maximal surfaces associated to the quartic differentials sq_0, equipped with a basepoint p, converge in the Gromov-Hausdorff sense to the polygonal maximal surface associated to the divisor of q_0 at p. We describe this more carefully in Corollary 7.8 at the end of the section.

To begin the proof of Theorem 7.1, let us fix local coordinates (V, z) around p such that $q_0(z) = z^k dz^4$. We introduce new coordinates (V, ζ_s) defined by

$$\zeta_s = s^{\frac{1}{k+4}} z$$

so that

$$\hat{q}_s(\zeta_s) := (\zeta_s)_* q_s = \zeta_s^k d\zeta_s^4 \ .$$

We notice that if $U = z(V) = \{z \in \mathbb{C} \mid |z| < \epsilon\}$, then $U_s = \zeta_s(V) = \{z \in \mathbb{C} \mid |z| < \epsilon s^{\frac{1}{k+4}}\}$, thus the sequence $\{(V, sq_0, p)\}_{s \geq 0}$ converges geometrically to $(\mathbb{C}, \zeta^k d\zeta^4, 0)$.

Let σ be the hyperbolic metric on X compatible with the complex structure. We denote

$$\begin{aligned} \sigma &= \sigma(z)|dz|^2 = \hat{\sigma}(\zeta_s)|d\zeta_s|^2 \quad \text{where} \quad \hat{\sigma}(\zeta_s) = s^{\frac{-2}{k+4}}\sigma(z) \\ \Delta_\sigma &= \sigma(z)^{-1}\partial_z\partial_{\bar{z}} \\ \Delta_{\hat{\sigma}} &= \hat{\sigma}(\zeta_s)^{-1}\partial_{\zeta_s}\partial_{\bar{\zeta_s}}. \end{aligned}$$

Let $g = (g_1, g_2^{-1}, g_1^{-1}, g_2)$ be the solutions of Hitchin's equations on $(\mathcal{E}, \varphi)$. We define two functions $\psi_1, \psi_2 : X \to \mathbb{R}$ by the property that

$$\frac{1}{g_1} = e^{\psi_1}\sigma^{\frac{3}{2}} \quad \text{and} \quad \frac{1}{g_2} = e^{\psi_2}\sigma^{\frac{1}{2}} \ .$$

The pair (ψ_1, ψ_2) is the solution of the system of PDEs, defined on the whole surface,

$$\begin{cases} \Delta_\sigma \psi_1 = e^{\psi_1 - \psi_2} - e^{-2\psi_1}\frac{|q|^2}{\sigma^4} + \frac{3}{4}\kappa(\sigma) \\ \Delta_\sigma \psi_2 = e^{2\psi_2} - e^{\psi_1 - \psi_2} + \frac{1}{4}\kappa(\sigma) \end{cases} \tag{7.1}$$

where $\kappa(\sigma)$ denotes the Gaussian curvature of σ.

We denote by $\{(\psi_1^s, \psi_2^s)\}_{s\geq 0}$ the solution to the above system along the ray $\{q_s\}_{s\geq 0}$. When studying the equation on V, or in general in a neighbourhood of a zero of order k for q_0, it will be convenient to rescale the background metric σ to a metric σ_s so that

$$\kappa(\sigma_s) = -s^{\frac{-2}{k+4}} \ .$$

It is straightforward to verify that the metric $\sigma_s = s^{\frac{2}{k+4}}\sigma$ satisfies the above condition and we can write in local coordinates

$$\sigma_s = \hat{\sigma}_s(\zeta_s)|d\zeta_s|^2 \qquad \text{where} \quad \hat{\sigma}_s(\zeta_s) = s^{\frac{2}{k+4}}\sigma(\zeta_s) \ .$$

Rewriting Equation (7.1) using the background metric σ_s in the coordinate ζ_s, we obtain

$$\begin{cases} \Delta_{\hat{\sigma}_s}\left(\psi_1^s - \frac{3}{k+4}\log(s)\right) = e^{\psi_1^s - \psi_2^s - \frac{2}{k+4}\log(s)} - e^{-2(\psi_1^s - \frac{3}{k+4}\log(s))}\frac{|\hat{q}_s|^2}{\hat{\sigma}^4 s^{\frac{8}{k+4}}} + \frac{3}{4}\frac{\kappa(\sigma)}{s^{\frac{2}{k+4}}} \\ \Delta_{\hat{\sigma}_s}\left(\psi_2^s - \frac{1}{k+4}\log(s)\right) = e^{2(\psi_2^s - \frac{1}{k+4}\log(s))} - e^{(\psi_1^s - \frac{3}{k+4}\log(s))} - e^{(\psi_2^s - \frac{1}{k+4}\log(s))} + \frac{1}{4}\frac{\kappa(\sigma)}{s^{\frac{2}{k+4}}} \ . \end{cases}$$

Therefore, we deduce that the functions

$$v_1^s(\zeta_s) = \psi_1^s(\zeta_s) - \frac{3}{k+4}\log(s) \qquad v_2^s(\zeta_s) = \psi_2^s(\zeta_s) - \frac{1}{k+4}\log(s)$$

are solutions of

$$\begin{cases} \Delta_{\hat{\sigma}_s} v_1^s = e^{v_1^s - v_2^s} - e^{-2v_1^s}\frac{|\hat{q}_s|^2}{\hat{\sigma}_s^4} + \frac{3}{4}\kappa(\hat{\sigma}_s) \\ \Delta_{\hat{\sigma}_s} v_2 = e^{2v_2^s} - e^{v_1^s - v_2^s} + \frac{1}{4}\kappa(\hat{\sigma}_s) \ . \end{cases} \tag{7.2}$$

We notice that the coefficients of the above equations converge to the planar Hitchin's equations with polynomial quartic differential $q_\infty = \zeta^k d\zeta^4$ because, as s tends to $+\infty$, we have

$$\begin{aligned} |\hat{q}_s|^2 = |\zeta_s^k|^2|d\zeta_s^4|^2 &\to |q_\infty|^2 \\ \kappa(\hat{\sigma}_s) &\to 0 \\ \hat{\sigma}_s(\zeta_s) &\to 1 \ . \end{aligned}$$

This suggests that v_j^s restricted on V should converge to the solutions of the planar equations, here making crucial use of the uniqueness result of Remark 2.7. We prove this using the sub- and supersolution method, together with the uniqueness result Proposition 6.13.

Let us start with the subsolutions. Let $\{U_i\}_{i=1}^N$ be pairwise disjoint natural coordinate charts such that each of them is centered at a zero of q_0 of order k_i. We can assume that each natural coordinate identifies U_i with $\{|z| < \epsilon\} \subset \mathbb{C}$. We define the following functions on X:

$$w_1^s(z) = \max\left(w_1^{s,U_i}(z), \frac{3}{8}\log\left(\frac{|q_s|^2}{\sigma^4}\right)\right) \qquad w_2^s(z) = \max\left(w_2^{s,U_i}(z), \frac{1}{8}\log\left(\frac{|q_s|^2}{\sigma^4}\right)\right).$$

Here, for $z \in U_i$

$$w_1^{s,U_i}(z) = \frac{3}{k_i+4}\log(s) + u_{1,i}(s^{\frac{1}{k_i+4}}z) - B_i$$

$$w_2^{s,U_i}(z) = \frac{1}{k_i+4}\log(s) + u_{2,i}(s^{\frac{1}{k_i+4}}z) - B_i$$

where $B_i > 0$ needs to be chosen, and $(u_{1,i}, u_{2,i})$ is the solution to the planar Hitchin equations with polynomial quartic differential $\zeta^{k_i}d\zeta^4$ found in Theorem 2.1.

Lemma 7.2. *There exists a constant $B_0 > 0$ such that for all $B_i > B_0$ the functions w_j^s for $j = 1, 2$ are continuous for every s sufficiently large.*

Proof. Let us first notice that at a zero of q_0, the functions w_j^s are well-defined because each w_j^{s,U_i} takes a finite value at $z = 0$. We need to show that we can choose $B_i > 0$ so that if $z \in \partial U_i$ we have

$$w_1^{s,U_i}(z) \leq \frac{3}{8}\log\left(\frac{|q_s|^2}{\sigma^4}\right) \quad \text{and} \quad w_2^{s,U_i}(z) \leq \frac{1}{8}\log\left(\frac{|q_s|^2}{\sigma^4}\right)$$

for every s large enough. We give the details for w_1^{s,U_i}, the other case being analogous. By definition of w_1^{s,U_i} the above inequality can be re-written as

$$\frac{3}{k_i+4}\log(s) + u_{1,i}(s^{\frac{1}{k_i+4}}z) - B_i \leq \frac{3}{8}\log\left(\frac{|q_s|^2}{\sigma^4}\right) . \tag{7.3}$$

The estimates on $u_{1,i}$ (see Equation (2.4)) tell us that there exists $c_i > 0$ and $s_0 > 0$ such that for every $s > s_0$ and for every $z \in \partial U_i$ we have

$$\begin{aligned} w_1^{s,U_i}(z) &= \frac{3}{k_i+4}\log(s) + u_{1,i}(s^{\frac{1}{k_i+4}}z) - B_i \\ &\leq \frac{3}{k_i+4}\log(s) + \frac{3}{4}\log(s^{\frac{k_i}{k_i+4}}|z|^{k_i}) + c_i - B_i \\ &= \frac{3}{4}\log(s) + \frac{3k_i}{4}\log|\epsilon| + c_i - B_i . \end{aligned} \tag{7.4}$$

On the other hand, if $z \in \partial U_i$, the right-hand side of Equation (7.3) becomes

$$\begin{aligned} \frac{3}{8}\log\left(\frac{|q_s|^2}{\sigma^4}\right) &= \frac{3}{8}\log\left(\frac{s^2|z|^{2k_i}}{\sigma^4(z)}\right) \\ &\geq \frac{3}{4}\log(s) + \frac{3k_i}{4}\log(|\epsilon|) + \frac{3}{8}\min_{|z|=\epsilon}\log\left(\frac{1}{\sigma^4(z)}\right) . \end{aligned} \tag{7.5}$$

Comparing Equation (7.4) and Equation (7.5), we observe that the inequality in (7.3) is satisfied for $s > s_0$ if we choose a positive $B_i > B_0$ with

$$B_0 \geq c_i - \frac{3}{8}\min_{|z|=\epsilon}\log\left(\frac{1}{\sigma^4(z)}\right) .$$

A similar inequality on B_0 is obtained when studying the function w_2^{s,U_i}, and it is then sufficient to take B_0 large enough to satisfy all the inequalities found in this way for each U_i. □

Lemma 7.3. *There exist constants $B_i > 0$ such that the functions w_j^s are subsolutions to Equation (7.1) for s large enough.*

Proof. Let us first show that the functions w_j^{s,U_i} for $j = 1, 2$ are subsolutions on U_i. By the discussion at the beginning of this subsection, it is actually easier to consider the functions

$$w_1^{s,U_i} - \frac{3}{k_i+4}\log(s) \quad \text{and} \quad w_2^{s,U_i} - \frac{1}{k_i+4}\log(s)\ ,$$

and show that they are subsolutions for Equation (7.2). Let $F_1 = F_1(v_1^s, v_2^s)$ and $F_2 = F_2(v_1^s, v_2^s)$ denote the functions on the right-hand side of Equation (7.2). We have to show that we can choose $B_i > B_0$ so that

$$\begin{cases} \Delta_{\hat\sigma_s}\left(w_1^{s,U_i} - \frac{3}{k_i+4}\log(s)\right) \geq F_1\left(w_1^{s,U_i} - \frac{3}{k_i+4}\log(s), w_2^{s,U_i} - \frac{1}{k_i+4}\log(s)\right) \\ \Delta_{\hat\sigma_s}\left(w_2^{s,U_i} - \frac{1}{k_i+4}\log(s)\right) \geq F_2\left(w_1^{s,U_i} - \frac{3}{k_i+4}\log(s), w_2^{s,U_i} - \frac{1}{k_i+4}\log(s)\right) \end{cases}.$$

Recalling that the pair $u_{1,i}$ and $u_{2,i}$ are the solution to the planar Hitchin equations with polynomial quartic differential $\zeta^{k_i}d\zeta^4$, the above system is equivalent to

$$\begin{cases} e^{u_{1,i}-u_{2,i}} - e^{-2u_{1,i}}e^{2B_i}\frac{|\zeta^{k_i}|^2}{\hat\sigma_s^4} + \frac{3}{4}\kappa(\sigma_s) \leq e^{u_{1,i}-u_{2,i}} - e^{-2u_{1,i}}|\zeta^{k_i}|^2 \\ e^{2u_{2,i}}e^{-2B_i} - e^{u_{1,i}-u_{2,i}} + \frac{1}{4}\kappa(\sigma_s) \leq e^{2u_{2,i}} - e^{u_{1,i}-u_{2,i}} \end{cases}$$

which gives

$$\begin{cases} -e^{-2u_{1,i}}|\zeta^{k_i}|^2(\frac{e^{2B_i}}{\hat\sigma_s^4} - 1) + \frac{3}{4}\kappa(\sigma_s) \leq 0 \\ e^{2u_{2,i}}(e^{-2B_i} - 1) + \frac{1}{4}\kappa(\sigma_s) \leq 0\ . \end{cases}$$

The above conditions are satisfied provided $B_i > 0$ and $s > 0$ are large enough because $\hat\sigma_s \to 1$ and $\kappa(\sigma_s) \to 0$ as $s \to +\infty$, and of course, the function $u_{j,i}$ are bounded on U_i. Clearly, if necessary, we can increase B_i so that $B_i > B_0$.
It is straighforward to check that the pair $\left(\frac{3}{8}\log\left(\frac{|q_s|^2}{\sigma^4}\right), \frac{1}{8}\log\left(\frac{|q_s|^2}{\sigma^4}\right)\right)$ is a solution outside the zeros of q_0, hence, in particular, it is a subsolution. The functions (w_1^s, w_2^s) are then subsolutions as well because of the monotonicity property of the functions G_i in the right hand side of Equation (7.1). Assume, for instance, that, at a point $z \in U_i$, we have

$$w_1^s(z) = w_1^{s,U_i}(z) \quad \text{and} \quad w_2^s(z) = \frac{1}{8}\log\left(\frac{|q_s(z)|^2}{\sigma^4}\right)$$

Then we have

$$\nabla_\sigma w_1^s(z) = \nabla_\sigma w_1^{s,U_i}(z) \geq G_1(w_1^{s,U_i}(z), w_2^{s,U_i}(z)) \geq G_1\left(w_1^{s,U_i}(z), \frac{1}{8}\log\left(\frac{|q_s(z)|^2}{\sigma^4}\right)\right)$$

where the first inequality comes from the fact that the pair $(w_1^{s,U_i}, w_2^{s,U})$ is a subsolution and the second inequality follows from G_1 being decreasing in the second variable. Similarly,

$$\begin{aligned}\nabla_\sigma w_2^s(z) &= \frac{1}{8}\nabla_\sigma \log\left(\frac{|q_s(z)|^2}{\sigma^4}\right) \geq G_2\left(\frac{3}{8}\log\left(\frac{|q_s(z)|^2}{\sigma^4}\right), \frac{1}{8}\log\left(\frac{|q_s(z)|^2}{\sigma^4}\right)\right)\\ &\geq G_2\left(w_1^{s,U_i}, \frac{1}{8}\log\left(\frac{|q_s(z)|^2}{\sigma^4}\right)\right) \ .\end{aligned}$$

The other cases can be proved analogously. □

Let us now move on to the supersolutions.

Lemma 7.4. *There exist positive constants $C_1(s)$ and $C_2(s)$ such that $(C_1(s), C_2(s))$ is a supersolution of Equation (7.1) along the ray $q_s = sq_0$ with respect to the background metric $\sigma_s = s^{\frac{2}{k+4}}\sigma$. Moreover, we can choose them so that, as $s \to +\infty$, we have*

$$C_1(s) - 3C_2(s) = O(s^{-\frac{2}{k+4}}) \quad \text{and} \quad C_2(s) - \frac{k}{4(k+4)}\log(s) = o(1) \ .$$

Proof. The pair $(C_1(s), C_2(s))$ is a supersolution of Equation (7.1) with respect to the background metric σ_s if it satisfies

$$\begin{cases} e^{C_1(s)-C_2(s)} - e^{-2C_1(s)}\frac{|q_s|^2}{\sigma_s^4} + \frac{3}{4}\kappa(\sigma_s) \geq 0\\ e^{2C(s)} - e^{C_1(s)-C_2(s)} + \frac{1}{4}\kappa(\sigma_s) \geq 0 \end{cases} \tag{7.6}$$

Dividing the second equation by $e^{2C(s)}$, we get

$$0 \leq 1 - e^{C_1(s)-3C_2(s)} - \frac{1}{4}e^{-2C_2(s)}s^{\frac{-2}{k+4}} \ .$$

This is satisfied for s sufficiently large if we choose for example

$$C_1(s) = 3C_2(s) - \frac{1}{2}s^{\frac{-2}{k+4}} \ ,$$

provided $C_2(s) > 0$. Let us now verify that this choice makes also the first inequality in (7.6) true for s large enough. We can estimate

$$\begin{aligned} e^{C_1(s)-C_2(s)} &- e^{-2C_1(s)}\frac{|q_s|^2}{\sigma_s^4} - \frac{3}{4}s^{\frac{-2}{k+4}}\\ &\geq e^{C_1(s)-C_2(s)} - e^{-2C_1(s)}\max_X\left(\frac{|q_s|^2}{\sigma_s^4}\right) - \frac{3}{4}s^{\frac{-2}{k+4}}\\ &= e^{2C_2(s)}e^{-\frac{s^{\frac{-2}{k+4}}}{2}} - e^{-6C_2(s)}e^{-\frac{s^{\frac{-2}{k+4}}}{2}}O(s^{\frac{2k}{k+4}}) - \frac{3}{4}s^{\frac{-2}{k+4}}\\ &= e^{2C_2(s)}\left(e^{-\frac{s^{\frac{-2}{k+4}}}{2}} - e^{-8C_2(s)}e^{-\frac{s^{\frac{-2}{k+4}}}{2}}O(s^{\frac{2k}{k+4}}) - \frac{3}{4}e^{-2C_2(s)}s^{\frac{-2}{k+4}}\right)\end{aligned}$$

and we can simply define $C_2(s)$ by the property that

$$e^{-s^{\frac{-2}{k+4}}} - e^{-8C_2(s)} e^{-s^{\frac{-2}{k+4}}} O(s^{\frac{2k}{k+4}}) - \frac{3}{4} e^{-2C_2(s)} s^{\frac{-2}{k+4}} = 0$$

Notice that this implies that necessarily $C_2(s)$ diverges as $s \to +\infty$, precisely,

$$C_2(s) - \frac{k}{4(k+4)} \log(s) = O(1) \ .$$

□

Corollary 7.5. *The constants*

$$\widehat{C_1}(s) = C_1(s) + \frac{3}{k+4} \log(s) \quad \textit{and} \quad \widehat{C_2}(s) = C_2(s) + \frac{1}{k+4} \log(s)$$

are supersolutions for Equation (7.1) with respect to the background metric σ.

Recall that (V, z) is a coordinate chart centered at the zero p of q_0 of order k. We then improve these supersolutions $(\widehat{C_1}(s), \widehat{C_2}(s))$ on V using the solutions to the planar Hitchin's equations. We define the functions

$$W_1^s = \min(W_1^{s,V}, \widehat{C_1}(s)) \qquad W_2^s = \min(W_2^{s,V}, \widehat{C_2}(s))$$

with

$$W_1^{s,V}(z) = \frac{3}{k+4} \log(s) + u_1(s^{\frac{1}{k+4}} z) + 2A$$
$$W_2^{s,V}(z) = \frac{1}{k+4} \log(s) + u_2(s^{\frac{1}{k+4}} z) + A \ ,$$

where A is a positive constant to be chosen later and (u_1, u_2) is the solution to the planar Hitchin's equations with quartic differential $\zeta^k d\zeta^4$.

Lemma 7.6. *There exists a constant $A_0 > 0$ such that for every $A > A_0$ the functions W_j^s are continuous for every s large enough.*

Proof. The argument is similar to that in the proof of Lemma 7.2. We first notice that at the point p, we will always have $W_j^s(p) = W_j^{s,V}(p)$ for s large enough because u_j is uniformly bounded in s at $z = 0$, whereas $C_j(s)$ diverge as $s \to +\infty$. It is then sufficient to check that we can find $A > 0$ such that for s sufficiently large and for all $z \in \partial V$ the following inequalities hold

$$W_1^{s,V}(z) \geq \widehat{C_1}(s) \quad \text{and} \quad W_2^{s,V}(z) \geq \widehat{C_2}(s) \ . \tag{7.7}$$

Let us consider the first condition. Now, of course,

$$W_1^{s,V} = \frac{3}{k+4} \log(s) + u_1(s^{\frac{1}{k+4}} z) + 2A$$

and so the estimates on u_1 (see Equation (2.4)) tell us that there exists $c_1 > 0$ and $s_0 > 0$ such that for every $s > s_0$ and for every $z \in \partial V$ we have

$$W_1^{s,V} \geq \frac{3}{k+4} \log(s) + \frac{3k}{4(k+4)} \log(s) + \frac{3k}{4} \log(|\epsilon|) - c_1 + 2A \ . \tag{7.8}$$

On the other hand, by Lemma 7.4, there exists a constant d_1 such that for $s \geq s_0$ we have

$$\widehat{C_1}(s) = C_1(s) + \frac{3}{k+4}\log(s) \leq \frac{3k}{4(k+4)}\log(s) + d_1 + \frac{3}{k+4}\log(s) \ . \tag{7.9}$$

Comparing Equations (7.8) and (7.9), the inequality in (7.7) is satisfied if we choose

$$A > \frac{1}{2}\left(d_1 - c_1 - \frac{3k}{4}\log(|\epsilon|)\right) \ .$$

The same argument applied to $W_2^{s,V}$ gives

$$A > d_2 - c_2 - \frac{k}{4}\log(|\epsilon|) \ .$$

The proof follows by choosing

$$A_0 = \max\left(\frac{1}{2}\left(d_1 - c_1 - \frac{3k}{4}\log(|\epsilon|)\right), d_2 - c_2 - \frac{k}{4}\log(|\epsilon|), 0\right) \ .$$

□

Lemma 7.7. *There exists a constant $A > A_0$ such that the functions W_j^s are supersolutions to Equation (7.1) for s large enough.*

Proof. It is sufficient to prove that $W_j^{s,V}$ for $j = 1, 2$ are supersolutions on V, because then the same argument as Lemma 7.3 applies. By the discussion at the beginning of this subsection, it is actually easier to consider the functions

$$W_1^{s,V} - \frac{3}{k_i+4}\log(s) \quad \text{and} \quad W_2^{s,V} - \frac{1}{k_i+4}\log(s) \ ,$$

and show that they are supersolutions to Equation (7.2). Let F_1 and F_2 denote the functions on the right-hand side of Equation (7.2). We have to show that we can choose $A > A_0$ so that

$$\begin{cases} \Delta_{\hat{\sigma}_s}\left(W_1^{s,V} - \frac{3}{k+4}\log(s)\right) \leq F_1\left(W_1^{s,V} - \frac{3}{k+4}\log(s), W_2^{s,V} - \frac{1}{k+4}\log(s)\right) \\ \Delta_{\hat{\sigma}_s}\left(W_2^{s,V} - \frac{1}{k+4}\log(s)\right) \leq F_2\left(W_1^{s,V} - \frac{3}{4}\log(s), W_2^{s,V} - \frac{1}{k+4}\log(s)\right) \end{cases} .$$

Recalling that (u_1, u_2) is the solution to the planar Hitchin equations with polynomial quartic differential $\zeta^{k_i} d\zeta^4$, the above system is equivalent to

$$\begin{cases} e^{u_1-u_2}(e^A - 1) - e^{-2u_1}|\zeta^k|^2\left(\frac{e^{-4A}}{\hat{\sigma}_s} - 1\right) + \frac{3}{4}\kappa(\sigma_s) \geq 0 \\ e^{2u_2}(e^{2A} - 1) + e^{u_1-u_2}(1 - e^A) + \frac{1}{4}\kappa(\sigma_s) \geq 0 \end{cases} .$$

In the first inequality we notice that, since $\kappa(\sigma_s)$ tends to 0, for every A large enough we can make the sum of the first and last term positive. Note that u_2 is bounded away from $-\infty$ by Lemma 2.3, so we can take the sum of the first and third terms to grow like e^{2A} in A. Moreover, the second term can be made non-negative for A

large enough and s large enough because $\hat{\sigma}_s$ tends to 1. Dividing by e^{2u_2}, the second inequality is equivalent to

$$(e^{2A}-1)+e^{u_1-3u_2}(1-e^A)+\frac{1}{4}\kappa(\sigma_s)e^{-2u_2}\geq 0\ .$$

From (2.4) and (2.5), we know that $u_1(\zeta)-3u_2(\zeta)=o(1)$ as $|\zeta|\to+\infty$. Therefore the condition holds for A large enough, because the coefficient $e^{2A}-1$ is dominant. □

Proof of Theorem 7.1. In the local chart (V,ζ_s) around the zero p of order k of the quartic differential q_0, the sub- and super-solutions found in Lemma 7.3 and Lemma 7.7 imply that the densities (ψ_1^s,ψ_2^s) satisfy

$$w_1^s\leq\psi_1^s\leq W_1^s\quad\text{and}\quad w_2^s\leq\psi_2^s\leq W_2^s\ . \tag{7.10}$$

We already remarked that the sequence of coordinate charts (V,ζ_s) converges geometrically to $(\mathbb{C},\zeta)$. Consider a point $q\in\mathbb{C}$ that is the limit of the sequence $\zeta_s=s^{\frac{1}{k+4}}z_s$, with $z_s=s^{-\frac{1}{k+4}}\zeta(q)\in U$. Evaluating Equation 7.10 at ζ_s, by definition of the functions w_j^s, W_j^s and ψ_j, we obtain

$$\begin{aligned}u_1^s(\zeta_s)-B\leq&v_1^s(z_s)\leq u_1^s(\zeta_s)+2A\\ u_2^s(\zeta_s)-B\leq&v_2^s(z_s)\leq u_2^s(\zeta_s)+A\ .\end{aligned} \tag{7.11}$$

Since $\zeta_s\to\zeta(q)$ as $s\to+\infty$, the above inequalities give uniform bound (independent of s) on every compact set in V for the functions v_1^s and v_2^s and their Laplacian. Therefore, they converge $C^{1,\alpha}$ on compact sets to functions v_1^∞ and v_2^∞ defined on the limiting plane $(\mathbb{C},\zeta)$. Since (v_1^s,v_2^s) is a sequence of solutions of Equation 7.2, the limit (v_1^∞,v_2^∞) is a weak solution of the system of PDEs obtained by taking the limit as $s\to+\infty$ of the coefficients. As observed before, this is the planar Hitchin's equation with polynomial quartic differential $q_\infty=\zeta^k d\zeta^4$. Hence, by the injectivity portion of Theorem C (see especially Proposition 6.13), the functions v_i^∞ are the solutions found in Theorem 2.1. In particular they are smooth and the convergence of v_i^s to v_i^∞ is smooth as well. This shows that for the sequence of radii $r_s=s^{-\frac{1}{k+4}}$, the harmonic metric g_s on the ball centered at p and radius r_s satisfies

$$g_{1,s}^{-1}=e^{\psi_1^s}\sigma^{\frac{3}{2}}=e^{v_1^s}s^{\frac{3}{k+4}}\sigma^{\frac{3}{2}}=O(s^{\frac{3}{k+4}}\sigma^{\frac{3}{2}})$$

and similarly

$$g_{2,s}^{-1}=e^{\psi_2^s}\sigma^{\frac{1}{2}}=e^{v_2^s}s^{\frac{1}{k+4}}\sigma^{\frac{1}{2}}=O(s^{\frac{1}{k+4}}\sigma^{\frac{1}{2}})$$

as $s\to+\infty$. □

As a consequence of this proof – especially the use of the uniqueness of the solutions v_i^∞ in the last paragraph – we can describe the pointed Gromov-Hausdorff limit of the family of maximal surfaces in $\mathbb{H}^{2,2}$ associated to the Higgs bundles $(\mathcal{E},\varphi_s)$, as defined in [CTT19].

Corollary 7.8. *Let $p\in X$ be a zero of order k of the quartic differential q_0. Let Σ_s denote the maximal surfaces in $\mathbb{H}^{2,2}$ associated to the family of Higgs bundles $(\mathcal{E},\varphi_s)$ and denote by I_s its induced metric. Let (Σ_∞,I_∞) be the conformally planar*

maximal surface (with polygonal boundary) with polynomial quartic differential $\zeta^k d\zeta^4$ endowed with its induced metric. Then (Σ_s, I_s, p) converges, up to subsequences and composition by global isometries, to $(\Sigma_\infty, I_\infty, p)$ in the pointed Gromov-Hausdorff topology.

Proof. Let (V, z) be a natural coordinate chart for q_0 so that $q_0(z) = z^k dz^4$. We already showed that the new coordinate charts (V, ζ_s) defined by $\zeta_s = s^{\frac{1}{k+4}} z$ converge geometrically to the complex plane $(\mathbb{C}, \zeta)$ and in these coordinates $q_s = \zeta_s^k d\zeta^4$ tends to $q_\infty = \zeta^k d\zeta^4$ uniformly on compact sets. It is thus sufficient to prove that the induced metrics I_s restricted to V converge to I_∞ smoothly on compact sets, up to subsequences. This follows immediately from the proof of Theorem 7.1. In fact, up to subsequences,

$$I_{s|_V} = 4e^{\psi_1^s - \psi_2^s}\sigma(z)|dz|^2 = 4e^{v_1^s - v_2^s}\hat{\sigma}_s(\zeta_s)|d\zeta_s|^2 \to 4e^{v_1^\infty - v_2^\infty}|d\zeta|^2 = I_\infty \ .$$

□

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AT: Department of Mathematics, University of Pisa
E-mail address: andrea_tamburelli@libero.it

MW: School of Mathematics, Georgia Institute of Technology
E-mail address: mwolf40@gatech.edu