

ON KITAEV'S DETERMINANT FORMULA

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ABSTRACT. We establish a sufficient condition under which $\det(ABA^{-1}B^{-1}) = 1$ for a pair of bounded, invertible operators A, B on a Hilbert space.

1. KITAEV'S FORMULA AND TRACES OF CERTAIN COMMUTATORS

In a finite dimensional Hilbert space $\mathcal{H}$, the determinantal formula

$$\det(ABA^{-1}B^{-1}) = 1 \quad (1.1)$$

holds for any invertible operators $A, B \in \mathcal{L}(\mathcal{H})$. Its naive generalization to the infinite dimensional case (via the Fredholm extension, see e.g., [S, Sections 3] for a background and basic properties) fails. A simple counterexample can be constructed using the Helton-Howe-Pincus formula, [E]: Let C and D be bounded operators on a Hilbert space $\mathcal{H}$ such that $[C, D] \in \mathcal{S}_1$ (the Schatten trace class), where $[C, D] = CD - DC$ stands for the commutator of C and D . Then $e^C e^D e^{-C-D} = I + S$, where I denotes the identity map and $S \in \mathcal{S}_1$. In particular, the Fredholm operator below is well defined and satisfies

$$\det(e^C e^D e^{-C-D}) = e^{\frac{1}{2}\text{tr}[C, D]}. \quad (1.2)$$

Thus $e^C e^D e^{-C-D} - I \in \mathcal{S}_1$ and using a basic property of the Fredholm determinant

$$\det(e^C e^D e^{-C-D}) = \det(e^C e^D e^{-C-D}) \det(e^{C+D} e^{-C} e^{-D}) = e^{\text{tr}[C, D]} \quad (1.3)$$

for such operators C and D .

Let R, L denote the forward and backward shift operators on $\ell^2(\mathbb{N})$ (with respect to the standard basis $\{e_n\}$), and let $z \in \mathbb{C}$. Then, the operators $A = e^{zR}$, $B = e^L$ are bounded and invertible, and moreover $[R, L] = P_1$, the orthogonal projection onto $\text{Span}(e_1)$. Hence, denoting by I the identity map, (1.3) implies that $ABA^{-1}B^{-1} - I \in \mathcal{S}_1$ and

$$\det(ABA^{-1}B^{-1}) = e^z,$$

i.e., the expression on the left hand side can take any non-zero complex value.

It is therefore an interesting question to determine under which conditions (1.1) actually holds. Another motivation to study this object comes from physics, where it can be linked to the quantization of transport in quantum systems, [K]. Indeed, if both (1.1) and (1.3) hold, one can deduce the quantization of $\text{tr}[C, D]$, i.e., $\text{tr}[C, D] \in 2\pi i\mathbb{Z}$. Kitaev observed via a formal computation that, if a pair of unitaries $U_1 = e^{iC}$, $U_2 = e^{iD}$ with bounded self-adjoint operators C, D satisfy $(U_1 - I)(U_2 - I), (U_2 - I)(U_1 - I) \in \mathcal{S}_1$, then (1.1) holds, implying the quantization for the case $[C, D] \in \mathcal{S}_1$.

This suggest the following

Conjecture 1.1. Let $\mathcal{H}$ be a complex, separable Hilbert space. Suppose that

- (i) $A, B \in \mathcal{L}(\mathcal{H})$ are invertible;
- (ii) $(A - I)(B - I), (B - I)(A - I) \in \mathcal{S}_1$.

Then (1.1) holds.

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While we don't know how to prove Conjecture [1.1](#), the purpose of this Note is to present an elementary derivation of the following weaker result.

Theorem 1.2. *Assume in addition to (i)-(ii) that*

$$(iii) \quad (A^* - I)(B - I), (B - I)(A^* - I) \in \mathcal{S}_1.$$

Then, [\(1.1\)](#) holds.

Let us stress that the condition (iii) is not a necessary, but only a sufficient condition. This can be seen from the following assertion (whose proof can be found in Section [2](#)).

Proposition 1.3. *Let C be a quasinilpotent operator and D bounded, such that*

$$(e^C - I)(e^D - I), (e^D - I)(e^C - I) \in \mathcal{S}^1. \quad (1.4)$$

Then

$$\det(e^C e^D e^{-C} e^{-D}) = 1.$$

We now construct a simple example, that shows that condition (iii) is not a necessary, based on this observation.

Example 1. Let $C = D = ML$, where L is the backward shift operator on $\ell^2(\mathbb{N})$, and M is a multiplication operator on the same space defined by

$$Me_n = \begin{cases} \frac{1}{\sqrt{n}}e_n & n \in 2\mathbb{N} \\ 0 & n \in 2\mathbb{N} - 1 \end{cases}.$$

Then [\(1.1\)](#) holds trivially for $A = e^C$, $B = e^D$ as C, D commute. We also note that $C^2 = 0$ (so C is nilpotent) and $e^C - I = C$ (so [\(1.4\)](#) holds as well). However, $(e^C - I)(e^{C^*} - I) = CC^* = M^2 \notin \mathcal{S}_1$, so (iii) in Theorem [1.2](#) is not satisfied.

Remark 1.4. We next note that if A (or B) is normal, then (ii) is equivalent to (iii), so in this case Conjecture [1.1](#) becomes a theorem, confirming Kitaev's formal observation. In fact, the proofs of Theorem [1.2](#) and Proposition [1.3](#) can be combined to show that Conjecture [1.1](#) is satisfied for the so-called spectral operators, introduced by Dunford, [\[D\]](#).

As we have already mentioned, [\(1.3\)](#) immediately implies

Corollary 1.5. *If $C, D \in \mathcal{L}(\mathcal{H})$ satisfy [\(1.4\)](#) and $[C, D] \in \mathcal{S}^1$, then $\text{tr}[C, D] \in 2\pi i\mathbb{Z}$.*

One can of course suspect, based on the vanishment of the trace of the commutator in the finite dimensional case, that in fact the only allowed value for $\text{tr}[C, D]$ in the statement above is zero. To this end, we construct

Example 2. There exist self-adjoint operators C, D satisfying the assumptions of Corollary [1.5](#) such that $\text{tr}[C, D] = -8\pi i$. Specifically, let $C = f(x) := 2\pi i \frac{x}{\langle x \rangle}$ where $\langle x \rangle = (1 + x^2)^{1/2}$ and let $D = f(p)$, where x and $p = -i \frac{d}{dx}$ are the position and momentum operators on $L(\mathbb{R})$, see [\[S, Section 4\]](#) for details (we note that here $f(p)$ is understood as a convolution operator, see [\[RS, Theorem IX.29\]](#)). Then [\(1.4\)](#) is satisfied, $[C, D] \in \mathcal{S}^1$, and $\text{tr}[C, D] = -8\pi i$.

We will verify the validity of this construction at the end of Section [2](#).

Since $U^k - I = (U - I) \sum_{j=0}^{k-1} U^j$ for any unitary U and any $k \in \mathbb{Z}$, one deduces from this example that there are operators C, D satisfying Corollary [1.5](#) above such that $\text{tr}[C, D] = 8k\pi i$ for any $k \in \mathbb{Z}$.

2. PROOFS

Proof of Theorem [1.2](#).

Lemma 2.1. Assume that the assumptions of Theorem 1.2 hold. Let $A = U|A|$ be the polar decomposition for A . Then U is in fact a unitary operator, $|A|$ is invertible, there are C, D that are normal and bounded such that $|A| = e^C$, $U = e^D$, and we have

$$(|A| - I)(B - I), (B - I)(|A| - I), (U - I)(B - I), (B - I)(U - I) \in \mathcal{S}_1. \quad (2.1)$$

In addition, the formula

$$\det(ABA^{-1}B^{-1}) = \det(|A|B|A|^{-1}B^{-1}) \det(BU^*B^{-1}U) \quad (2.2)$$

holds.

Proof. The fact that U and $|A|$ are invertible (and consequently have exponential representation in terms of normal operators) follows directly from the invertibility of A , so we only need to establish (2.1)–(2.2). To this end, we note that

$$(A^*A - I)(B - I) = (A^* + I)(A - I)(B - I) - (A - I)(B - I) + (A^* - I)(B - I) \in \mathcal{S}_1$$

by (ii-iii). Hence

$$(|A| - I)(B - I) = (|A| + I)^{-1}(A^*A - I)(B - I) \in \mathcal{S}_1$$

as well. An identical argument yields the inclusion $(B - I)(|A| - I) \in \mathcal{S}_1$. We also have

$$\begin{aligned} (B - I)(U - I) &= (B - I)(A - |A|)|A|^{-1} \\ &= (B - I)(A - I)|A|^{-1} - (B - I)(|A| - I)|A|^{-1} \in \mathcal{S}_1. \end{aligned}$$

Finally, we have

$$\begin{aligned} (U - I)(B - I) &= (A - |A|)|A|^{-1}(B - I) = (A - |A|)(B - I) + (A - |A|)(|A|^{-1} - I)(B - I) \\ &= (A - I)(B - I) - (|A| - I)(B - I) - (A - |A|)|A|^{-1}(|A| - I)(B - I) \in \mathcal{S}_1, \end{aligned}$$

so we established (2.1).

The relation (2.2) follows from the fact that $|A|B|A|^{-1}B^{-1} = I + K$, $BU^*B^{-1}U = I + M$ with $K, M \in \mathcal{S}_1$ by

$$ABA^{-1}B^{-1} = I + [A, B]A^{-1}B^{-1} \quad (2.3)$$

and (2.1), the representation

$$ABA^{-1}B^{-1} = U(|A|B|A|^{-1}B^{-1})(BU^*B^{-1}U)U^*,$$

as well as the basic properties of the Fredholm determinant. $\square$

Applying Lemma 2.1 twice, we see that the statement of Theorem 1.2 follows from

Proposition 2.2. Let A, B be bounded normal operators in $\mathcal{H}$ that satisfy

$$(e^A - I)(e^B - I), (e^B - I)(e^A - I) \in \mathcal{S}_1.$$

Then

$$\det(e^A e^B e^{-A} e^{-B}) = 1.$$

$\square$

Proof of Proposition 2.2. We will use the following:

Lemma 2.3. Let B, D be a pair of bounded operators on $\mathcal{H}$ that satisfy

$$D(e^B - I), (e^B - I)D \in \mathcal{S}_1.$$

Then

$$\det(e^D e^B e^{-D} e^{-B}) = 1.$$

Proof. Using a basic property of the Fredholm determinant, we have

$$\begin{aligned}\det(e^D e^B e^{-D} e^{-B}) &= \det\left(e^B e^{-D} e^{-B} e^D e^{(e^B D e^{-B} - D)} e^{-(e^B D e^{-B} - D)}\right) \\ &= \det\left(e^{-e^B D e^{-B}} e^D e^{(e^B D e^{-B} - D)}\right) \det\left(e^{-(e^B D e^{-B} - D)}\right),\end{aligned}$$

where both determinants on the right hand side are well-defined. We now use (1.2) to evaluate the first determinant on the right hand side:

$$\det\left(e^{-e^B D e^{-B}} e^D e^{(e^B D e^{-B} - D)}\right) = \exp\left(-\frac{1}{2} \operatorname{tr}[e^B D e^{-B}, D]\right) = 1,$$

since

$$\operatorname{tr}[e^B D e^{-B}, D] = \operatorname{tr}[e^B D e^{-B} - D, D] = 0,$$

where in the last step we used $e^B D e^{-B} - D = [e^B - I, D] e^{-B} \in \mathcal{S}_1$. We recall a consequence of Lidskii's theorem: If $X, Y \in \mathcal{L}(\mathcal{H})$ are such that $XY, YX \in \mathcal{S}_1$, then $\operatorname{tr}(XY) = \operatorname{tr}(YX)$. Thus

$$\operatorname{tr}(e^B D e^{-B} - D) = \operatorname{tr}((e^B - I)(D e^{-B})) - \operatorname{tr}((D e^{-B})(e^B - I)) = 0, \quad (2.4)$$

Using $\det(e^E) = e^{\operatorname{tr} E}$ for $E \in \mathcal{S}_1$ and (2.4), we get

$$\det\left(e^{-(e^B D e^{-B} - D)}\right) = 1.$$

□

Let P be the spectral projection $\chi_{2\pi i\mathbb{Z}}(A)$, where χ_W stands for the indicator of a set W . Then, $e^{AP} = I$ for a normal operator A and $\det(e^{AP} e^B e^{-AP} e^{-B}) = 1$. Let $\Delta \in (0, 1/2]$, let $W = \{x \in \mathbb{C} : \operatorname{dist}(x, 2\pi i\mathbb{Z}) \geq \Delta\}$, and let $Q_\Delta = \chi_W(A)$, and let $P_\Delta = I - Q_\Delta$.

We first observe that since $e^A - I$ is invertible on $\operatorname{Range}(Q_\Delta)$, we have

$$Q_\Delta(e^B - I) = \left((e^A - I)^{-1} Q_\Delta\right)(e^A - I)(e^B - I) \in \mathcal{S}_1,$$

and similarly

$$(e^B - I) Q_\Delta \in \mathcal{S}_1.$$

Thus, by Lemma 2.3 we deduce

$$\det(e^{AQ_\Delta} e^B e^{-AQ_\Delta} e^{-B}) = 1. \quad (2.5)$$

Next, we note that

$$[e^{AP_\Delta}, e^B] = [P_\Delta(e^A - I), (e^B - I)] \xrightarrow{\Delta \rightarrow 0} [P(e^A - I), (e^B - I)] = [e^{AP}, e^B],$$

where the convergence is in the trace norm sense (this follows from $SOT - \lim P_\Delta = P$ and the assumptions of Proposition 2.2). This implies

$$\det(e^{AP_\Delta} e^B e^{-AP_\Delta} e^{-B}) = \det(I + [e^{AP_\Delta}, e^B] e^{-AP_\Delta} e^{-B}) \rightarrow \det(e^{AP} e^B e^{-AP} e^{-B}) = 1 \quad (2.6)$$

as $\Delta \rightarrow 0$. We now can combine (2.5) and (2.6) to get

$$\det(e^A e^B e^{-A} e^{-B}) = \det(e^{AP_\Delta} e^B e^{-AP_\Delta} e^{-B}) \det(e^B e^{-AQ_\Delta} e^{-B} e^{AQ_\Delta}) \xrightarrow{\Delta \rightarrow 0} 1. \quad (2.7)$$

□

Proof of Proposition 1.3. The statement follows from

$$C(e^B - I), (e^B - I)C \in \mathcal{S}_1 \quad (2.8)$$

and Lemma 2.3.

To show (2.8), we observe that, denoting

$$D := \sum_{k=0}^{\infty} \frac{C^k}{(k+1)!},$$

we have $e^C - I = DC$. Hence, (2.8) will follow provided that D is invertible, as

$$C(e^B - I) = D^{-1}(e^C - I)(e^B - I).$$

To prove that D is invertible, it suffices to show that $D - I$ is a (quasi)nilpotent operator. To this end, we can bound

$$\|(I - D)^n\| = \|(CE)^n\| \leq \|C^n\| \|E\|^n, \quad E = \sum_{k=0}^{\infty} \frac{C^k}{(k+2)!}.$$

We have

$$\|E\| \leq \sum_{k=0}^{\infty} \frac{\|C\|^k}{(k+2)!} \leq e^{\|C\|},$$

so

$$\|(I - D)^n\|^{1/n} \leq \|C^n\|^{1/n} e^{\|C\|} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

so $D - I$ is indeed (quasi)nilpotent, and we are done. $\square$

Verification of Example 2. We first note that the conditions (1.4) are satisfied by [RS1, Theorem XI.21] since there exists a $C > 0$ such that

$$\left| e^{f(x)} - 1 \right| \langle x \rangle^2 \leq C.$$

We will use the integral representation

$$\frac{\pi}{2} \frac{1}{\langle p \rangle} = \int_0^\infty \frac{dt}{p^2 + 1 + t^2},$$

which implies

$$\begin{aligned} [C, D] &= [C, p] \frac{2\pi i}{\langle p \rangle} + p \left[C, \frac{2\pi i}{\langle p \rangle} \right] \\ &= -f'(x) \frac{2\pi}{\langle p \rangle} + 4 \int_0^\infty \frac{p}{p^2 + 1 + t^2} (f'(x)p + pf'(x)) \frac{1}{p^2 + 1 + t^2} dt. \end{aligned}$$

The integral can be written as

$$\begin{aligned} \int_0^\infty f'(x) \frac{2p^2}{(p^2 + 1 + t^2)^2} dt - \int_0^\infty \left[f'(x), \frac{p}{p^2 + 1 + t^2} \right] \frac{p}{p^2 + 1 + t^2} dt \\ - \int_0^\infty \left[f'(x), \frac{p^2}{p^2 + 1 + t^2} \right] \frac{1}{p^2 + 1 + t^2} dt. \end{aligned}$$

We note that the integrands in the second and third terms have trace norms decaying faster than $\frac{1}{t^2+1}$ in t , in particular these terms are trace class, see, e.g., [S, Section 4] for the trace class properties of the products of functions $F(x)G(p)$. Hence, we get

$$[C, D] = -4f'(x) \int_0^\infty \left(\frac{1}{p^2 + 1 + t^2} - \frac{2p^2}{(p^2 + 1 + t^2)^2} \right) dt + \mathcal{T} = -if'(x)f'(p) + \mathcal{T},$$

where $\mathcal{T}$ is a trace class operator. Since $f'(x) = \frac{2\pi i}{\langle x \rangle^3}$, we see that $f'(x)f'(p) \in \mathcal{L}^1$, so $[C, D] \in \mathcal{S}_1$ as well. In fact, $\text{tr } \mathcal{T} = 0$ (this term originates from the commutator of $f'(x)$ with functions of p that decay in p sufficiently fast), so

$$\text{tr } [C, D] = i \text{tr } f'(x)f'(p) = \frac{i}{2\pi} \left(\int_{\mathbb{R}} f' \right)^2 = -8\pi i,$$

where in the second step we have used the fact that $f'(p)$ is a convolution operator,

$$(f'(p)\phi)(x) = (2\pi)^{-1/2} \int (\check{f}')(x-y)\phi(y)dy, \quad (\check{f}')(x) := (2\pi)^{-1/2} \int e^{ixp}(f')(p)dp,$$

see [RS, Theorem IX.29]. $\square$

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