Maximal Distortion of Geodesic Diameters in Polygonal Domains

Adrian Dumitrescu[1] and Csaba D. Tóth[2,3]

[1] Algoresearch L.L.C., Milwaukee, WI, USA
ad.dumitrescu@algoresearch.org
[2] California State University Northridge, Los Angeles, CA, USA
[3] Tufts University, Medford, MA, USA
csaba.toth@csun.edu

Abstract. For a polygon P with holes in the plane, we denote by $\varrho(P)$ the ratio between the geodesic and the Euclidean diameters of P. It is shown that over all convex polygons with h convex holes, the supremum of $\varrho(P)$ is between $\Omega(h^{1/3})$ and $O(h^{1/2})$. The upper bound improves to $O(1+\min\{h^{3/4}\Delta, h^{1/2}\Delta^{1/2}\})$ if every hole has diameter at most $\Delta \cdot \mathrm{diam}_2(P)$; and to $O(1)$ if every hole is a *fat* convex polygon. Furthermore, we show that the function $g(h) = \sup_P \varrho(P)$ over convex polygons with h convex holes has the same growth rate as an analogous quantity over geometric triangulations with h vertices when $h \to \infty$.

1 Introduction

Determining the maximum distortion between two metrics on the same ground set is a fundamental problem in metric geometry. Here we study the maximum ratio between the geodesic (i.e., shortest path) diameter and the Euclidean diameter over polygons with holes. A *polygon P with h holes* (also known as a *polygonal domain*) is defined as follows. Let P_0 be a simple polygon, and let $P_1, \ldots, P_h$ be pairwise disjoint simple polygons in the interior of P_0. Then $P = P_0 \setminus \left(\bigcup_{i=1}^h P_i\right)$.

The Euclidean distance between two points $s, t \in P$ is $|st| = \|s-t\|_2$, and the shortest path distance $\mathrm{geod}(s,t)$ is the minimum arclength of a polygonal path between s and t contained in P. The triangle inequality implies that $|st| \le \mathrm{geod}(s,t)$ for all $s, t \in P$. The *geometric dilation* (also known as the *stretch factor*) between the two distances is $\sup_{s,t\in P} \mathrm{geod}(s,t)/|st|$. The geometric dilation of P can be arbitrarily large, even if P is a (nonconvex) quadrilateral.

The *Euclidean diameter* of P is $\mathrm{diam}_2(P) = \sup_{s,t\in P} |st|$ and its *geodesic diameter* is $\mathrm{diam}_g(P) = \sup_{s,t\in P} \mathrm{geod}(s,t)$. It is clear that $\mathrm{diam}_2(P) \le \mathrm{diam}_g(P)$. We are interested in the *distortion*

$$\varrho(P) = \frac{\mathrm{diam}_g(P)}{\mathrm{diam}_2(P)}. \tag{1}$$

Note that $\varrho(P)$ is unbounded, even for simple polygons. Indeed, if P is a zig-zag polygon with n vertices, contained in a disk of unit diameter, then $\mathrm{diam}_2(P) \le 1$

and $\mathrm{diam}_g(P) = \Omega(n)$, hence $\varrho(P) \geq \Omega(n)$. It is not difficult to see that this bound is the best possible, that is, $\varrho(P) \leq O(n)$.

In this paper, we consider convex polygons with convex holes. Specifically, let $\mathcal{C}(h)$ denote the family of polygonal domains $P = P_0 \setminus \left(\bigcup_{i=1}^h P_i\right)$, where $P_0, P_1, \ldots, P_h$ are convex polygons; and let

$$g(h) = \sup_{P \in \mathcal{C}(h)} \varrho(P). \tag{2}$$

It is clear that if $h = 0$, then $\mathrm{geod}(s,t) = |st|$ for all $s, t \in P$, which implies $g(0) = 1$. Or main result is the following.

Theorem 1. *For every $h \in \mathbb{N}$, we have $\Omega(h^{1/3}) \leq g(h) \leq O(h^{1/2})$.*

The lower bound construction is a polygonal domain in which all h holes have about the same diameter $\Theta(h^{-1/3}) \cdot \mathrm{diam}_2(P)$. We prove a matching upper bound for all polygons P with holes of diameter $\Theta(h^{-1/3}) \cdot \mathrm{diam}_2(P)$. In general, if the diameter of every hole is $o(1) \cdot \mathrm{diam}_2(P)$, we can improve upon the bound $g(h) \leq O(h^{1/2})$ in Theorem 1.

Theorem 2. *If $P \in \mathcal{C}(h)$ and the diameter of every hole is at most $\Delta \cdot \mathrm{diam}_2(P)$, then $\varrho(P) \leq O(1 + \min\{h^{3/4}\Delta, h^{1/2}\Delta^{1/2}\})$. In particular for $\Delta = O(h^{-1/3})$, we have $\varrho(P) \leq O(h^{1/3})$.*

However, if we further restrict the holes to be *fat* convex polygons, we can show that $\varrho(P) = O(1)$ for all $h \in \mathbb{N}$. In fact for every $s, t \in P$, the distortion $\mathrm{geod}(s,t)/|st|$ is also bounded by a constant.

Informally, a convex body is *fat* if its width is comparable with its diameter. The *width* of a convex body C is the minimum width of a parallel slab enclosing C. For $0 \leq \lambda \leq 1$, a convex body C is λ-*fat* if the ratio of its width to its diameter is at least λ, that is, $\mathrm{width}(C)/\mathrm{diam}_2(C) \geq \lambda$; and C is *fat* if the inequality holds for a constant λ. For instance, a disk is 1-fat, a 3×4 rectangle is $\frac{3}{5}$-fat and a line segment is 0-fat. Let $\mathcal{F}_\lambda(h)$ be the family of polygonal domain $P = P_0 \setminus \left(\bigcup_{i=1}^h P_i\right)$, where $P_0, P_1, \ldots, P_h$ are λ-fat convex polygons.

Proposition 1. *For every $h \in \mathbb{N}$ and $P \in \mathcal{F}_\lambda(h)$, we have $\varrho(P) \leq O(\lambda^{-1})$.*

The special case when all holes are axis-aligned rectangles is also easy.

Proposition 2. *Let $P \in \mathcal{C}(h)$, $h \in \mathbb{N}$, such that all holes are axis-aligned rectangles. Then $\varrho(P) \leq O(1)$.*

Triangulations. In this paper, we focus on the diameter distortion $\varrho(P) = \mathrm{diam}_g(P)/\mathrm{diam}_2(P)$ for polygons $P \in \mathcal{C}(h)$ with h holes. Alternatively, we can also compare the geodesic and Euclidean diameters in n-vertex triangulations. In a *geometric graph* $G = (V, E)$, the vertices are distinct points in the plane, and the edges are straight-line segment between pairs of vertices. The *Euclidean diameter* of G, $\mathrm{diam}_2(G) = \max_{u,v \in V} |uv|$ is the maximum distance between two vertices, and the *geodesic diameter* $\mathrm{diam}_g(G) = \max_{u,v\in V} \mathrm{dist}(u, v)$, where $\mathrm{dist}(u, v)$ is the shortest path distance in G, i.e., the minimum Euclidean length of a uv-path in G. With this notation, we define $\varrho(G) = \mathrm{diam}_g(G)/\mathrm{diam}_2(G)$,

A Euclidean triangulation $T = (V, E)$ of a point set V is a planar straight-line graph where all bounded faces are triangles, and their union is the convex hull $\mathrm{conv}(V)$. Let

$$f(n) = \sup_{G\in\mathcal{T}(n)} \varrho(G), \tag{3}$$

where the supremum is taken over the set $\mathcal{T}(n)$ all n-vertex triangulations. Recall that $g(n)$ is the supremum of diameter distortions over polygons with n convex holes; see (2). We prove that $f(n)$ and $g(n)$ have the same growth rate.

Theorem 3. *We have* $g(n) = \Theta(f(n))$.

Alternative problem formulation. The following version of the question studied here may be more attractive to the escape community [9, 16]. Given n pairwise disjoint convex obstacles in a convex polygon of unit diameter (e.g., a square), what is the maximum length of a (shortest) escape route from any given point in the polygon to its boundary? According to Theorem 1, it is always $O(n^{1/2})$ and sometimes $\Omega(n^{1/3})$.

Related work. The geodesic distance in polygons with or without holes has been studied extensively from the algorithmic perspective; see [19] for a comprehensive survey. In a simple polygon P with n vertices, the geodesic distance between two given points can be computed in $O(n)$ time [17]; trade-offs are also available between time and workspace [12]. A shortest-path data structure can report the geodesic distance between any two query points in $O(\log n)$ time after $O(n)$ preprocessing time [11]. In $O(n)$ time, one can also compute the geodesic diameter [13] and radius [1].

For polygons with holes, more involved techniques are needed. Let P be a polygon with h holes, and a total of n vertices. For any $s, t \in P$, one can compute $\mathrm{geod}(s, t)$ in $O(n + h \log h)$ time and $O(n)$ space [23], improving earlier bounds in [14, 15, 18, 24]. A shortest-path data structure can report the geodesic distance between two query points in $O(\log n)$ query time using $O(n^{11})$ space; or in $O(h \log n)$ query time with $O(n + h^5)$ space [6]. The geodesic radius can be computed in $O(n^{11} \log n)$ time [3, 22], and the geodesic diameter in $O(n^{7.73})$ or $O(n^7(\log n + h))$ time [2]. One can find an $(1 + \varepsilon)$-approximation in $O((n/\varepsilon^2 + n^2/\varepsilon) \log n)$ time [2, 3]. The geodesic diameter may be attained by a point pair $s, t \in P$, where both s and t lie in the interior or P; in which case it is known [2] that there are at least five different geodesic paths between s and t.

The diameter of an n-vertex triangulation with Euclidean weights can be computed in $\tilde{O}(n^{5/3})$ time [5, 10]. For unweighted graphs in general, the diameter problem has been intensely studied in the fine-grained complexity community. For a graph with n vertices and m edges, breadth-first search (BFS) yields a 2-approximation in $O(m)$ time. Under the Strong Exponential Time Hypothesis (SETH), for any integer $k \geq 2$ and $\varepsilon > 0$, a $(2 - \frac{1}{k} - \varepsilon)$-approximation requires $mn^{1+1/(k-1)-o(1)}$ time [7]; see also [20].

2 Convex Polygons with Convex Holes

In this section, we prove Theorem 1. A lower bound construction is presented in Lemma 1, and the upper bound is established in Lemma 2 below.

Lower Bound. The lower bound is based on the following construction.

Lemma 1. *For every $h \in \mathbb{N}$, there exists a polygonal domain $P \in \mathcal{C}(h)$ such that $g(P) \geq \Omega(h^{1/3})$.*

Proof. We may assume w.l.o.g. that $h = k^3$ for some integer $k \geq 3$. We construct a polygon P with h holes, where the outer polygon P_0 is a regular k-gon of unit diameter, hence $\mathrm{diam}_2(P) = \mathrm{diam}_2(P_0) = 1$. Let $Q_0, Q_1, \ldots, Q_{k^2}$ be a sequence of $k^2 + 1$ regular k-gons with a common center such that $Q_0 = P_0$, and for every $i \in \{1, \ldots, k^2\}$, Q_i is inscribed in Q_{i-1} such that the vertices of Q_i are the midpoints of the edges of Q_{i-1}; see Fig. 1. Enumerate the k^3 edges of $Q_1, \ldots, Q_{k^2}$ as $e_1, \ldots, e_{k^3}$. For every $j = 1, \ldots, k^3$, we construct a hole as follows: Let P_j be an $(|e| - 2\varepsilon) \times \frac{\varepsilon}{2}$ rectangle with symmetry axis e that contains e with the exception of the ε-neighborhoods of its endpoints. Then $P_1, \ldots, P_{k^3}$ are pairwise disjoint. Finally, let $P = P_0 \setminus \bigcup_{j=1}^{k^3} P_j$.

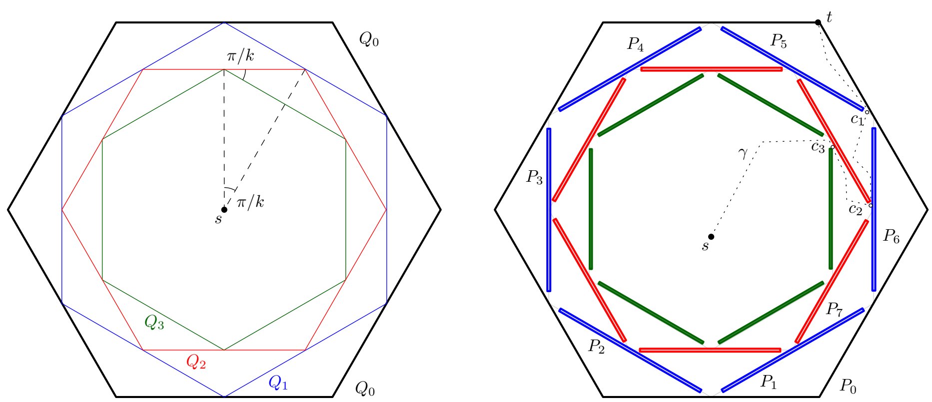

Fig. 1. Left: hexagons $Q_0, \ldots, Q_3$ for $k = 6$. Right: The 18 holes corresponding to the edges of $Q_1, \ldots, Q_3$.

Assume, w.l.o.g., that e_i is an edge of Q_i for $i \in \{0, 1, \ldots, k^2\}$. As $P_0 = Q_0$ is a regular k-gon of unit diameter, then $|e_0| \geq \Omega(1/k)$. Let us compare the edge lengths in two consecutive k-gons. Since Q_{i+1} is inscribed in Q_i, we have

$$|e_{i+1}| = |e_i| \cos \frac{\pi}{k} \geq |e_i| \left(1 - \frac{\pi^2}{2k^2}\right)$$

using the Taylor estimate $\cos x \geq 1-x^2/2$. Consequently, for every $i \in \{0,1,\ldots,k^2\}$,

$$|e_i| \geq |e_0| \cdot \left(1 - \frac{\pi^2}{2k^2}\right)^{k^2} \geq |e_0| \cdot \Omega(1) \geq \Omega\left(\frac{1}{k}\right).$$

It remains to show that $\mathrm{diam}_g(P) \geq \Omega(k)$. Let s be the center of P_0 and t and arbitrary vertex of P_0. Consider an st-path γ in P, and for any two points a, b along γ, let $\gamma(a,b)$ denote the subpath of γ between a and b. Let c_i be the first point where γ crosses the boundary of Q_i for $i \in \{1,\ldots,k^2\}$. By construction, c_i must be in an ε-neighborhood of a vertex of Q_i. Since the vertices of Q_{i+1} are at the midpoints of the edges of Q_i, then $|\gamma(c_i, c_{i+1})| \geq \frac{1}{2}|e_i| - 2\varepsilon \geq \Omega(|e_i|) \geq \Omega(1/k)$. Summation over $i = 0,\ldots,k^2-1$ yields $|\gamma| \geq \sum_{i=0}^{k^2-1} |\gamma(c_i, c_{i+1})| \geq k^2 \cdot \Omega(1/k) \geq \Omega(k) = \Omega(h^{1/3})$, as required. □

Upper Bound. Let $P \in \mathcal{C}(h)$ for some $h \in \mathbb{N}$ and let $s \in P$. For every hole P_i, let ℓ_i and r_i be points on the boundary of P_i such that $\overrightarrow{s\ell_i}$ and $\overrightarrow{sr_i}$ are tangent to P_i, and P_i lies on the left (resp., right) side of the ray $\overrightarrow{s\ell_i}$ (resp., $\overrightarrow{sr_i}$).

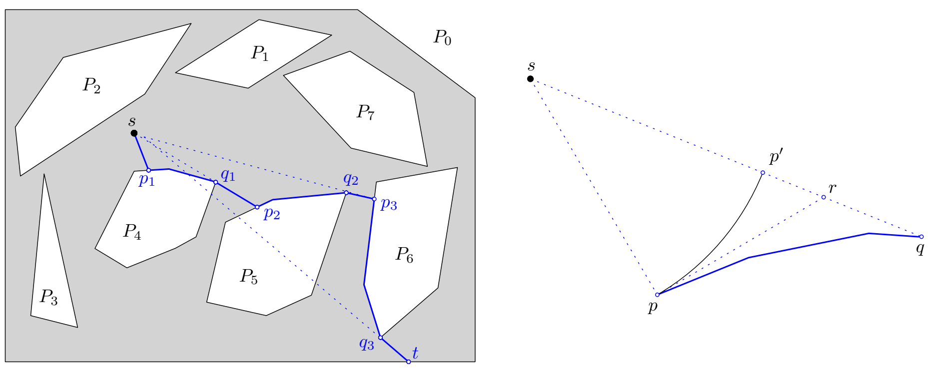

Fig. 2. Left: A polygon $P \in \mathcal{C}(7)$ with 7 convex holes, a point $s \in P$, and a path $\mathrm{greedy}_P(s, \boldsymbol{u})$ from s to a point t on the outer boundary of P. Right: A boundary arc $\widehat{pq}$, where $|\widehat{pq}| \leq |pr| + |rq|$.

We construct a path from s to some point in the outer boundary of P by the following recursive algorithm; refer to Fig. 2 (left). For a unit vector $\boldsymbol{u} \in \mathbb{S}^1$, we construct path $\mathrm{greedy}_P(s, \boldsymbol{u})$ as follows. Start from s along a ray emanating from s in direction $\boldsymbol{u}$ until reaching the boundary of P at some point p. While $p \notin \partial P_0$ do: Assume that $p \in \partial P_i$ for some $1 \leq i \leq h$. Extend the path along ∂P_i to the point ℓ_i or r_i such that the distance from s monotonically increases; and then continue along the ray $\overrightarrow{s\ell_i}$ or $\overrightarrow{sr_i}$ until reaching the boundary of P again. When $p \in \partial P_0$, the path $\mathrm{greedy}_P(s, \boldsymbol{u})$ terminates at p.

Lemma 2. *For every $P \in \mathcal{C}(h)$, every $s \in P$ and every $\boldsymbol{u} \in \mathbb{S}^1$, we have $|\mathrm{greedy}_P(s, \boldsymbol{u})| \leq O(h^{1/2}) \cdot \mathrm{diam}_2(P)$, and this bound is the best possible.*

Proof. Let P be a polygonal domain with a convex outer polygon P_0 and h convex holes. We may assume w.l.o.g. that $\mathrm{diam}_2(P) = 1$. For a point $s \in P$ and a unit vector $\boldsymbol{u}$, consider the path $\mathrm{greedy}_P(s, \boldsymbol{u})$. By construction, the distance from s monotonically increases along $\mathrm{greedy}_P(s, \boldsymbol{u})$, and so the path has no self-intersections. It it composed of *radial segments* that lie along rays emanating from s, and *boundary arcs* that lie on the boundaries of holes. By monotonicity, the total length of all radial segments is at most $\mathrm{diam}_2(P)$. Since every boundary arc ends at a point of tangency ℓ_i or r_i, for some $i \in \{1, \ldots, h\}$, the path $\mathrm{greedy}_P(s, \boldsymbol{u})$ contains at most two boundary arcs along each hole, thus the number of boundary arcs is at most $2h$. Let $\mathcal{A}$ denote the set of all boundary arcs along $\mathrm{greedy}_P(s, \boldsymbol{u})$; then $|\mathcal{A}| \leq 2h$.

Along each boundary arc $\widehat{pq} \in \mathcal{A}$, from p to q, the distance from s increases by $\Delta_{pq} = |sq| - |sp|$. By monotonicity, we have $\sum_{\widehat{pq} \in \mathcal{A}} \Delta_{pq} \leq \mathrm{diam}_2(P)$. We now give an upper bound for the length of $\widehat{pq}$. Let p' be a point in sq such that $|sp| = |sp'|$, and let r be the intersection of sq with a line orthogonal to sp passing through p; see Fig. 2 (right). Note that $|sp| < |sr|$. Since the distance from s monotonically increases along the arc $\widehat{pq}$, then q is in the closed halfplane bounded by pr that does not contain s. Combined with $|sp| < |sr|$, this implies that r lies between p' and q on the line sq, consequently $|p'r| < |p'q| = \Delta_{pq}$ and $|rq| < |p'q| = \Delta_{pq}$. By the triangle inequality and the Pythagorean theorem, these estimates give an upper bound

$$
\begin{aligned}
|\widehat{pq}| &\leq |pr| + |rq| = \sqrt{|sr|^2 - |sp|^2} + |rq| \leq \sqrt{(|sp'| + |p'r|)^2 - |sp|^2} + |rq| \\
&\leq \sqrt{(|sp| + \Delta_{pq})^2 - |sp|^2} + \Delta_{pq} \leq O\left(\sqrt{|sp|\Delta_{pq}} + \Delta_{pq}\right) \\
&\leq O\left(\sqrt{\mathrm{diam}_2(P) \cdot \Delta_{pq}} + \Delta_{pq}\right).
\end{aligned}
$$

Summation over all boundary arcs, using Jensen's inequality, yields

$$
\begin{aligned}
\sum_{\widehat{pq} \in \mathcal{A}} |\widehat{pq}| &\leq \sum_{\widehat{pq} \in \mathcal{A}} O\left(\sqrt{\mathrm{diam}_2(P) \cdot \Delta_{pq}} + \Delta_{pq}\right) \\
&\leq \sqrt{\mathrm{diam}_2(P)} \cdot O\left(\sum_{\widehat{pq} \in \mathcal{A}} \sqrt{\Delta_{pq}}\right) + O\left(\sum_{\widehat{pq} \in \mathcal{A}} \Delta_{pq}\right) \\
&\leq \sqrt{\mathrm{diam}_2(P)} \cdot O\left(|\mathcal{A}| \cdot \sqrt{\frac{1}{|\mathcal{A}|} \sum_{\widehat{pq} \in \mathcal{A}} \Delta_{pq}}\right) + O(\mathrm{diam}_2(P)) \\
&\leq \sqrt{\mathrm{diam}_2(P)} \cdot O\left(\sqrt{|\mathcal{A}| \cdot \mathrm{diam}_2(P)}\right) + O(\mathrm{diam}_2(P)) \\
&\leq O\left(\sqrt{|\mathcal{A}|}\right) \cdot \mathrm{diam}_2(P) \leq O\left(\sqrt{h}\right) \cdot \mathrm{diam}_2(P),
\end{aligned}
$$

as claimed.

We now show that the bound $|\mathrm{greedy}_P(s, \boldsymbol{u})| \leq O(h^{1/2}) \cdot \mathrm{diam}_2(P)$ is the best possible. For every $h \in \mathbb{N}$, we construct a polygon $P \in \mathcal{C}(h)$ and a point s

such that for every $\boldsymbol{u} \in \mathbb{S}^1$, we have $|\text{greedy}_P(s, \boldsymbol{u})| \geq \Omega(h^{1/2})$. Without loss of generality, we may assume $\text{diam}_2(P) = 1$ and $h = 3(k^2+1)$ for some $k \in \mathbb{N}$.

We start with the construction in Lemma 1 with k^3 rectangular holes in a regular k-gon P_0, where s is the center of P_0. We modify the construction in three steps: (1) Let T be a small equilateral triangle centered at s, and construct three rectangular holes around the edges of T; to obtain a total of k^3+3 holes. (2) Rotate each hole P_j counterclockwise by a small angle, such that when the greedy path reaches P_j in an ε-neighborhood of its center, it would always turn left. (3) For any $\boldsymbol{u} \in \mathbb{S}^1$, the path $\text{greedy}_P(s, \boldsymbol{u})$ exit the triangle T at a small neighborhood of a corner of T. From each corner of T, $\text{greedy}_P(s, \boldsymbol{u})$ continues to the outer boundary along the same k^2 holes. We delete all holes $\text{greedy}_P(s, \boldsymbol{u})$ does not touch for any $\boldsymbol{u} \in \mathbb{S}^1$, thus we retain $h = 3k^3 + 3$ holes. For every $\boldsymbol{u} \in \mathbb{S}^1$, we have $|\text{greedy}_P(s, \boldsymbol{u})| \geq \Omega(k)$ according to the analysis in Lemma 1, hence $|\text{greedy}_P(s, \boldsymbol{u})| \geq \Omega(h^{1/2})$, as required. □

Corollary 1. *For every $h \in \mathbb{N}$ and every polygon $P \in \mathcal{C}(h)$, we have $\text{diam}_g(P) \leq O(h^{1/2}) \cdot \text{diam}_2(P)$.*

Proof. Let $P \in \mathcal{C}(h)$ and $s_1, s_2 \in P$. By Lemma 2, there exist points $t_1, t_2 \in \partial P_0$ such that $\text{geod}(s_1, t_1) \leq O(h^{1/2}) \cdot \text{diam}_2(P)$ and $\text{geod}(s_2, t_2) \leq O(h^{1/2}) \cdot \text{diam}_2(P)$. There is a path between t_1 and t_2 along the perimeter of P_0. It is well known [21, 25] that $|\partial P_0| \leq \pi \cdot \text{diam}_2(P_0)$ for every convex body P_0, hence $\text{geod}(t_1, t_2) \leq O(\text{diam}_2(P))$. The concatenation of these three paths yields a path in P connecting s_1 and s_2, of length $\text{geod}(s_1, s_2) \leq O(h^{1/2}) \cdot \text{diam}_2(P)$. □

3 Improved Upper Bound for Holes of Bounded Diameter

In this section we prove Theorem 2. Similar to the proof of Theorem 1, it is enough to bound the geodesic distance from an arbitrary point in P to the outer boundary. We give three such bounds in Lemmas 3, 4 and 7.

Lemma 3. *Let $P \in \mathcal{C}(h)$ such that $\text{diam}_2(P_i) \leq \Delta \cdot \text{diam}_2(P)$ for every hole P_i. If $\Delta \leq O(h^{-1})$, then there exists a path of length $O(\text{diam}_2(P))$ in P from any point $s \in P$ to the outer boundary ∂P_0.*

Proof. Let $s \in P$ and $t \in \partial P_0$. Construct an st-path γ as follows: Start with the straight line segment st, and whenever st intersects the interior of a hole P_i, then the segment $st \cap P_i$ is replaced by an arc along ∂P_i. Since $|\partial P_i| \leq \pi \cdot \text{diam}_2(P_i)$ for every convex hole P_i [21, 25], then $|\gamma| \leq |st| + \sum_{i=1}^{h} |\partial P_i| \leq \text{diam}_2(P) + \sum_{i=1}^{h} O(\text{diam}_2(P_i)) \leq \mathcal{O}(1 + h\Delta) \cdot \text{diam}_2(P) \leq O(\text{diam}_2(P))$, as claimed. □

Lemma 4. *Let $P \in \mathcal{C}(h)$ such that $\text{diam}_2(P_i) \leq \Delta \cdot \text{diam}_2(P)$ for every hole P_i. Then there exists a path of length $O(1 + h^{3/4}\Delta) \cdot \text{diam}_2(P)$ in P from any point $s \in P$ to the outer boundary ∂P_0.*

Proof. Assume without loss of generality that $\text{diam}_2(P) = 1$, and s is the origin. Let $\ell \in \mathbb{N}$ be a parameter to be specified later. For $i \in \{-\ell, -\ell+1, \ldots, \ell\}$, let

$H_i : y = i \cdot \Delta$ be a horizontal line, and $V_i : x = i \cdot \Delta$ a vertical line. Since any two consecutive horizontal (resp., vertical) lines are distance Δ apart, and the diameter of each hole is at most Δ, then the interior of each hole intersects at most one horizontal and at most one vertical line. By the pigeonhole principle, there are integers $a, b, c, d \in \{1, \ldots, \ell\}$ such that H_{-a}, H_b, V_{-c}, and V_d each intersects the interior of at most h/ℓ holes; see Fig. 3.

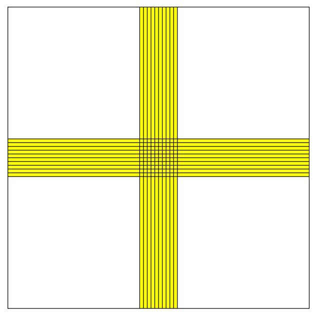

Fig. 3. Illustration for $\ell = 5$ (assuming that P is a unit square centered at s).

Let B be the axis-aligned rectangle bounded by the lines H_{-a}, H_b, V_{-c}, and V_d. Due to the spacing of the lines, we have $\text{diam}_2(B) \leq 2 \cdot \sqrt{2} \cdot \ell\Delta = O(\ell\Delta)$.

We construct a path from s to ∂P_0 as a concatenation of two paths $\gamma = \gamma_1 \oplus \gamma_2$. Let γ_1 be the initial part of $\text{greedy}_P(s, \boldsymbol{u})$ from s until reaching the boundary of $B \cap P_0$ at some point p. If $p \in \partial P_0$, then $\gamma_2 = (p)$ is a trivial one-point path. Otherwise p lies on a line $L \in \{H_{-a}, H_b, V_{-c}, V_d\}$ that intersects the interior of at most h/ℓ holes. Let γ_2 follow L from p to the boundary of P_0 such that when it encounters a hole P_i, it makes a detour along ∂P_i.

It remains to analyze the length of γ. By Lemma 2, we have $|\gamma_1| \leq O(\sqrt{h}) \cdot \text{diam}_2(B) \leq O(h^{1/2}\ell\Delta)$. The path γ_2 has edges along the line L and along the boundaries of holes whose interior intersect L. The total length of all edges along L is at most $\text{diam}_2(P) = 1$. It is well known that $\text{per}(C) \leq \pi \cdot \text{diam}_2(C)$ for every convex body [21, 25], and so the length of each detour is $O(\text{diam}_2(P_i)) \leq O(\Delta)$, and the total length of $O(h/\ell)$ detours is $O(h\Delta/\ell)$. Consequently,

$$|\gamma| \leq O(h^{1/2}\ell\Delta + h\Delta/\ell + 1). \tag{4}$$

Finally, we set $\ell = \lceil h^{1/4} \rceil$ to balance the first two terms in (4), and obtain $|\gamma| \leq O(h^{3/4}\Delta + 1)$, as claimed. □

When all holes are line segments, we construct a monotone path from s to the outer boundary. A polygonal path $\gamma = (p_0, p_1, \ldots, p_m)$ is $\boldsymbol{u}$-*monotone* for a unit vector $\boldsymbol{u} \in \mathbb{S}^1$ if $\boldsymbol{u} \cdot \overrightarrow{v_{i-1}v_i} \geq 0$ for all $i \in \{1, \ldots, m\}$; and γ is *monotone* if it is $\boldsymbol{u}$-monotone for some $\boldsymbol{u} \in \mathbb{S}^1$.

Lemma 5. *Let $P \in \mathcal{C}(h)$ such that every hole is a line segment of length at most $\Delta \cdot \mathrm{diam}_2(P)$. If $\Delta \geq h^{-1}$, then there exists a monotone path of length $O(h^{1/2}\Delta^{1/2}) \cdot \mathrm{diam}_2(P)$ in P from any point $s \in P$ to the outer boundary ∂P_0.*

Proof. We may assume w.l.o.g. that $\mathrm{diam}_2(P) = 1$. Denote the line segments by a_ib_i, for $i = 1, \ldots, h$, such that $x(a_i) \leq b(x_i)$. Let $\ell = \lceil h^{1/2}\Delta^{1/2} \rceil$, and note that $\ell = \Theta(h^{1/2}\Delta^{1/2})$ when $\Delta \geq h^{-1}$. Partition the right halfplane (i.e., right of the y-axis) into ℓ wedges with aperture π/ℓ and apex at the origin, denoted $W_1, \ldots, W_\ell$. For each wedge W_i, let $\boldsymbol{w}_i \in \mathbb{S}$ be the direction vector of its axis of symmetry.

Partition the h segments as follows: For $j = 1, \ldots, \ell$, let $\mathcal{H}_j$ be the set of segments a_ib_i such that $\overrightarrow{a_ib_i}$ is in W_j. Finally, let $\mathcal{H}_{j^*}$ be a set with minimal cardinality, that is, $|\mathcal{H}_{j^*}| \leq h/\ell = O(h^{1/2}/\Delta^{1/2})$. Let $\boldsymbol{v} = \boldsymbol{w}_{j^*}^{\perp}$. We construct a $\boldsymbol{v}$-monotone path γ from s to the outer boundary ∂P_0 as follows. Start in direction $\boldsymbol{v}$ until reaching a hole a_ib_i at some point p. While $p \notin \partial P_0$, continue along a_ib_i to one of the endpoints: to a_i if $\boldsymbol{v} \cdot \overrightarrow{a_ib_i} \geq 0$, and to b_i otherwise; then continue in direction $\boldsymbol{v}$. By monotonicity, γ visits every edge at most once.

It remains to analyze the length of γ. We distinguish between two types of edges: let E_1 be the set of edges of γ contained in $\mathcal{H}_{j^*}$, and E_2 be the set of all other edges of γ. The total length of edges in E_1 is at most the total length of all segments in $\mathcal{H}_{j^*}$, that is,

$$\sum_{e \in E_1} |e| \leq |\mathcal{H}_{j^*}| \cdot \Delta \leq O(h^{1/2}/\Delta^{1/2}) \cdot \Delta = O(h^{1/2}\Delta^{1/2}).$$

Every edge $e \in E_2$ makes an angle at least $\pi/(2\ell)$ with vector $\boldsymbol{v}$. Let $\mathrm{proj}(e)$ denote the orthogonal projection of e to a line of direction $\boldsymbol{v}$. Then $|\mathrm{proj}(e)| \geq |e|\sin(\pi/(2\ell))$. By monotonicity, the projections of distinct edges have disjoint interiors. Consequently, $\sum_{e \in E_2} |\mathrm{proj}(e)| \leq \mathrm{diam}_2(P) = 1$. This yields

$$\begin{aligned}\sum_{e \in E_2} |e| &\leq \sum_{e \in E_2} \frac{|\mathrm{proj}(e)|}{\sin(\pi/(2\ell))} = \frac{1}{\sin(\pi/(2\ell))} \sum_{e \in E_2} |\mathrm{proj}(e)| \\ &= O(\ell) = O(h^{1/2}\Delta^{1/2}).\end{aligned}$$

Overall, $|\gamma| = \sum_{e \in E_1} |e| + \sum_{e \in E_2} |e| = O(h^{1/2}\Delta^{1/2})$, as claimed. □

For extending Lemma 5 to arbitrary convex holes, we need the following technical lemma. (All omitted proofs are available in the full paper [8].)

Lemma 6. *Let P be a convex polygon with a diametral pair $a, b \in \partial P$, where $|ab| = \mathrm{diam}_2(P)$. Suppose that a line L intersects the interior of P, but does not cross the line segment ab. Let $p, q \in \partial P$ such that $pq = L \cap P$, and points a, p, q, and b appear in this counterclockwise order in ∂P; and let $\widehat{pq}$ be the counterclockwise pq-arc of ∂P. Then $|\widehat{pq}| \leq \frac{4\pi\sqrt{3}}{9}|pq| < 2.42|pq|$.*

The final result is as follows.

Lemma 7. *Let $P \in \mathcal{C}(h)$ such that $\mathrm{diam}_2(P_i) \leq \Delta \cdot \mathrm{diam}_2(P)$ for every hole P_i. If $\Delta \geq h^{-1}$, then there exists a path of length $O(h^{1/2}\Delta^{1/2}) \cdot \mathrm{diam}_2(P)$ in P from any point $s \in P$ to the outer boundary ∂P_0.*

4 Polygons with Fat or Axis-Aligned Convex Holes

In this section, we show that in a polygonal domain P with fat convex holes, the distortion $\mathrm{geod}(s,t)/|st|$ is bounded by a constant for all $s,t \in P$. Let C be a convex body in the plane. The *geometric dilation* of C is $\delta(C) = \sup_{s,t\in\partial C} \frac{\mathrm{geod}(s,t)}{|st|}$, where $\mathrm{geod}(s,t)$ is the shortest st-path along the boundary of C.

Lemma 8. *Let C be a λ-fat convex body. Then $\delta(C) \leq \min\{\pi\lambda^{-1}, 2(\lambda^{-1}+1)\} = O(\lambda^{-1})$.*

Corollary 2. *Let $P = P_0 \setminus \left(\bigcup_{i=1}^{h} P_i\right)$ be a polygonal domain, where $P_0, P_1, \ldots, P_h$ are λ-fat convex polygons. Then for any $s,t \in P$, we have $\mathrm{geod}(s,t) \leq O(\lambda^{-1}|st|)$.*

Proof. If the line segment st is contained in P, then $\mathrm{geod}(s,t) = |st|$, and the proof is complete. Otherwise, segment st is the concatenation of line segments contained in P and line segments $p_iq_i \subset P_i$ with $p_i, q_i \in \partial P_i$, for some indices $i \in \{1,\ldots,h\}$. By replacing each segment p_iq_i with the shortest path on the boundary of the hole P_i, we obtain an st-path γ in P. Since each hole is λ-fat, we replaced each line segment p_iq_i with a path of length $O(|p_iq_i|/\lambda)$ by Lemma 8. Overall, we have $|\gamma| \leq O(|st|/\lambda)$, as required. □

Corollary 3. *If $P = P_0 \setminus \left(\bigcup_{i=1}^{h} P_i\right)$ be a polygonal domain, where $P_0, P_1, \ldots, P_h$ are λ-fat convex polygons for some $0 < \lambda \leq 1$, then $\mathrm{diam}_g(P) \leq O(\lambda^{-1}\mathrm{diam}_2(P))$, hence $\varrho(P) \leq O(\lambda^{-1})$.*

Proposition 3. *Let $P \in \mathcal{C}(h)$, $h \in \mathbb{N}$, such that every hole is an axis-aligned rectangle. Then from any point $s \in P$, there exists a path of length at most $\mathrm{diam}_2(P)$ in P to the outer boundary ∂P_0.*

Proof. Let $B = [0,a] \times [0,b]$ be a minimal axis-parallel bounding box containing P. We may assume w.l.o.g. that $x(s) \geq a/2$, $y(s) \geq b/2$, and $b \leq a$. We construct a staircase path γ as follows. Start from s in horizontal direction $\boldsymbol{d}_1 = (1,0)$ until reaching the boundary ∂P at some point p. While $p \notin \partial P_0$, make a 90° turn from $\boldsymbol{d}_1 = (1,0)$ to $\boldsymbol{d}_2 = (0,1)$ or vice versa, and continue. We have $|\gamma| \leq \frac{a+b}{2} \leq a \leq \mathrm{diam}_2(P)$, as claimed. □

5 Polygons with Holes versus Triangulations

The proof of Theorem 3 is the combination of Lemmas 9 and 10 below (the proof of Lemma 9 is deferred to the full version of this paper [8]).

Lemma 9. *For every triangulation $T \in \mathcal{T}(n)$, there exists a polygonal domain $P \in \mathcal{C}(h)$ with $h = \Theta(n)$ holes such that $\varrho(P) = \Theta(\varrho(T))$.*

Every planar straight-line graph $G = (V, E)$ can be augmented to a triangulation $T = (V, E')$, with $E \subseteq E'$. A notable triangulation is the *Constrained Delaunay Triangulation*, for short, CDT(G). Bose and Keil [4] proved that CDT(G) has bounded stretch for so-called *visibility* edges: if $u, v \in V$ and uv does not cross any edge of G, then CDT(G) contains a uv-path of length $O(|uv|)$.

Lemma 10. *For every polygonal domain $P \in \mathcal{C}(h)$, there exists a triangulation $T \in \mathcal{T}(n)$ with $n = \Theta(h)$ vertices such that $\varrho(T) = \Theta(\varrho(P))$.*

Proof. Assume that $P = P_0 \setminus \bigcup_{i=1}^{h} P_i$. For all $j = 1, \ldots, h$, let $a_i, b_i \in \partial P_i$ be a diametral pair, that is, $|a_i b_i| = \mathrm{diam}_2(P_i)$. The line segments $\{a_i b_i : i = 1, \ldots, h\}$, together with the four vertices of a minimum axis-aligned bounding box of P, form a planar straight-line graph G with $2h + 4$ vertices. Let $T = \mathrm{CDT}(G)$ be the constrained Delaunay triangulation of G.

We claim that $\varrho(T) = \Theta(\varrho(P))$. We prove this claim in two steps. For an intermediate step, we define a polygon with h line segment holes: $P' = P_0 \setminus \bigcup_{i=1}^{h} \{a_i b_i\}$. For any point pair $s, t \in P$, denote by $\mathrm{dist}(s,t)$ and $\mathrm{dist}'(s,t)$, resp., the shortest distance in P and P'. Since $P \subseteq P'$, we have $\mathrm{dist}'(s,t) \leq \mathrm{dist}(s,t)$. By Lemma 6, $\mathrm{dist}(s,t) < 2.42 \cdot \mathrm{dist}'(s,t)$ so $\mathrm{dist}'(s,t) = \Theta(\mathrm{dist}(s,t))$, $\forall s, t \in P$.

Every point $s \in P$ lies in one or more triangles in T; let s' denote a closest vertex of a triangle in T that contains s. For $s, t \in P$, let $\mathrm{dist}''(s,t)$ be the length of the st-path γ composed of the segment ss', a shortest $s't'$-path in the triangulation T, and the segment $t't$.

Since γ does not cross any of the line segments $a_j b_j$, we have $\mathrm{dist}'(s,t) \leq \mathrm{dist}''(s,t)$ for any pair of points $s, t \in P$. Conversely, every vertex in the shortest $s't'$-path in P' is an endpoint of an obstacle $a_j b_j$. Consequently, every edge is either an obstacle segment $a_j b_j$, or a visibility edge between the endpoints of two distinct obstacles. By the result of Bose and Keil [4], for every such edge pq, T contains a pq-path τ_{pq} of length $|\tau_{pq}| \leq O(|pq|)$. The concatenation of these paths is an $s't'$-path τ of length $|\tau| \leq O(\mathrm{dist}'(s', t'))$. Finally, note that the diameter of each triangle in T is at most $\mathrm{diam}_2(P')$. Consequently, if $s, t \in P$ maximizes $\mathrm{dist}(s,t)$, then $\mathrm{dist}''(s,t) = |ss'| + |\gamma| + |t't| \leq 2 \cdot \mathrm{diam}_2(P) + |\tau| \leq O(\mathrm{dist}'(s't'))$. Consequently, $\mathrm{diam}_g(T) = \Theta(\mathrm{diam}_g(P))$, which yields $\varrho(T) = \Theta(\varrho(P))$. ☐

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