CONTRASTING CASES IN GEOMETRY: OPPORTUNITIES TO EXPLORE DIFFERENT STUDENT SOLUTION STRATEGIES

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Utilizing an innovative and theoretically-grounded approach, we extend the work of cognitive scientists and mathematics educators who have previously documented the impact of comparison on students' learning in algebra with the goal of transforming the learning that occurs in eighth-grade geometry classrooms. The purpose of this paper is to examine the types of comparisons participants made during think aloud interviews when engaging with curricular materials that have them examine multiple solution strategies. This research seeks to extend the work of using comparisons in algebra to determine if using comparisons in geometry will help improve students' mathematical understanding.

Keywords: Curriculum, Geometry and Spatial Reasoning, Middle School Education

Introduction

Dissonance is defined, in part, as "an instance of such inconsistency or disagreement" (Merriam-Webster, 2022). There is much to be learned through these inconsistencies, especially in regards to learning mathematics, for it is through this dissonance that students work to make sense of the inconsistencies and develop stronger mathematical arguments. By simply comparing different solution strategies to a mathematics problem, students can draw out inconsistencies to deepen their mathematical thinking.

Utilizing an innovative and theoretically grounded approach, we extend the work of cognitive scientists and mathematics educators who have previously documented the impact of comparison on students' learning in algebra (Star, Pollack, et al., 2015), with the goal of transforming the learning that occurs in middle grade geometry classrooms. Comparing and contrasting objects is a powerful learning tool with deep roots in cognitive science literature. Goldstone, Day, and Son (2010) stated, "research has demonstrated that the simple act of comparing two things can produce important changes in our knowledge" (p. 103). There is empirical support from cognitive scientist literature for the use of comparing contrasting examples for learning about business negotiations (Gentner et al., 2003), heat flow in science (Kurtz et al., 2001), children's learning (Loewenstein & Gentner, 2001; Namy & Gentner, 2002), and in studies of infants 4 to 6-months old (Oakes & Ribar, 2005). In mathematics education, research on comparing has proven effective in learning: estimation (Star & Rittle-Johnson, 2008, March, 2009), the concept of an altitude for a triangle (Guo & Pang, 2011), and equation solving (Rittle-Johnson & Star, 2007, 2009; Rittle-Johnson, Star, & Durkin, 2012).

Theoretical Framework

Comparisons are a powerful way to improve learning across disciplines; having students compare and contrast different strategies has produced gains in students' knowledge of algebra (Lynch & Star, 2014; Star, Newton, et al., 2015; Star, Pollack, et al., 2015). Furthermore, comparison of multiple strategies also plays a prominent role in policy documents. In particular, one of the five recommendations for improving mathematical problem solving for middle grade students noted in the released Practice Guide from the U.S. Department of Education

(Woodward et al., 2012) was to have students examine multiple problem solving strategies. Similarly, one of the Common Core State Standards for Mathematics (NGA, 2010) mathematical practice standards is for students to be able to construct viable arguments and critique the reasoning of others, specifically that, "Mathematically proficient students are also able to compare the effectiveness of two plausible arguments" (p. 10). When students share their ideas and evaluate the thinking of others in class discussions, they develop the ability to construct mathematical arguments; the practice of critiquing peers is thought to enhance mathematical understanding (Lampert, 1990; Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005).

Researchers have developed contrasting case materials to address typical Algebra I content that students struggle with or that elicit student misconceptions (see Star, Pollack, et al., 2015; Star, Rittle-Johnson, & Durkin, 2016). Star and colleagues' Algebra I curriculum is focused around worked example pairs (WEPs), where each WEP shows two fictitious students solving one or more algebra problems. The intent of each WEP is for students to directly compare and contrast, line-by-line, the methods that the two fictitious students use in solving each problem. Star and colleagues (2015) have demonstrated that teachers' implementation of the contrasting cases materials, including whole class and small group discussions around the WEPs, can lead to increased procedural knowledge, procedural flexibility, and conceptual knowledge of Algebra I topics.

The research in this proposal seeks to extend this work to geometry content and to further enhance the effectiveness of this approach for geometry by creating digital materials with animations. Because a major goal of the project is to establish a scientific foundation for animated contrasting cases as a basis for the learning of geometry, this paper seeks to answer an initial question: *What types of comparisons between mathematical strategies do students make?*

Design of Materials

Our digital curricular materials center two fictitious students' voices at the center of mathematics learning, and each lesson includes five unique features: a page for the first fictitious student's solution strategy on a given geometry task, a page for the second fictitious student's solution to a geometry task (which could be the same or different task shown on first student's page), a page with both students' strategies side-by-side, a discussion sheet with four questions for the students to answer, and a thought bubble page summarizing the key mathematical concepts in the problem. The discussion sheet and thought bubble page are designed to make the instructional goal of each WEP more explicit and for students to summarize their work from the WEPs (Star, Pollack, et al., 2015).

There are four units, each containing either five or six WEPs, that cover the geometry content in the 8th grade CCSSM (NGA, 2010). Each unit introduces two new characters; for this paper we will refer to the characters as Jaxon and Maxine. As we created the materials, we considered several design features: animations and colors to draw student's attention to the geometric content in meaningful ways, characters' methods purposefully selected to spark comparisons, geometric thinking of the WEP characters, and diversity of characters throughout the units.

Methods

After fully developing the 8th grade geometry materials, and due to shifts in classroom-based research due to COVID, we conducted 56 hour-long open-ended semi-structured clinical interviews (Piaget, 1976; Opper, 1977), in the form of think alouds, with individual participants ($n = 42$). Our goal was to elicit student thinking as participants engaged with the materials and discussion questions, not to get them to a "correct" response (Opper, 1977). Consistent with

Opper's description of the clinical method, we used the students' language, paced the interview to each student's speed, and encouraged students to elaborate on their thinking. In order to engage participants in each phase of the WEP during the interviews, we followed a detailed protocol: examine Jaxon's method, examine Maxine's method, horizontally compare Jaxon and Maxine's methods, solve the problems on the discussion page, and read the thought bubble at the end. If needed, we had questions for each phase of the protocol to probe student thinking. At the end, we asked if they had any questions for us or any final thoughts on the materials.

We transcribed each interview and began *a priori* (Saldaña, 2013) coding based on our key design features (Animations, Colors, Comparison Between Characters, Diversity of Characters, Geometric-Thinking of WEP Characters). We then added emergent (Saldaña, 2013) Level 1 codes for the students' geometric thinking and curricular form and Level 2, 3, and 4 codes as appropriate. We met to develop the codebook and then each coded several interviews individually. We then met to discuss any discrepancies and came to agreement on all codes. Once we had finalized our codebook, two coders independently coded each transcript and the third researcher resolved all disagreements. The average initial agreement between the pairs of coders was: 88.92% for Level 1 codes, 82.76 for Level 2, 81.04% for Level 3, and 88.09% for Level 4. This paper specifically analyzes the Level 1 Comparison Between Characters codes to determine what types of comparisons the participants made regarding Jaxon and Maxine's solutions strategies. Below, we report on the types of comparisons they made and give examples of each type.

Findings

Comparisons Between Characters

We observed 756 (23.27% of all Level 1 codes) instances where students were making comparisons between the WEP characters (Table 1). Most often they were discussing differences between the characters ($n = 484$), but they also noted similarities ($n = 267$) and used both WEP characters' strategies to verify a mathematical idea ($n = 5$). Looking at the Level 3 codes, regardless if students were pointing out a similarity or difference in Jaxon and Maxine's strategies, participants often referred directly to the method they were using to solve a problem.

Specifically, when pointing out differences, students most often described differences in the methods the characters used to solve a problem ($n = 380$). For example, when analyzing strategies related to translating a figure, one student stated, "Jackson is more plotting it out, while Maxine is subtracting the values to go left or down. They both had it in the same spot, which is good; I think that's the idea." This student realized Jackson is using a visual geometric method, while Maxine is using an algebraic approach, yet they arrive at the same answer. This student was attending to the visual/algebraic aspects of Jaxon and Maxine's approaches. Students noted differences in the students' methods regarding WEP specific content. For example, in a WEP designed to have students understand why the interior angle sum in a triangle is 180 degrees, one student said, "Alex like ripped his triangle apart and… what did Morgan do? And Morgan, just drew the line and just used like the parallel cut by transversal stuff to figure everything out. To figure out that it was 180 degrees." Here the student is attending to specific mathematics content in the WEP.

Students noted similarities based on the method students were utilizing in the WEP. In a WEP focused on verifying similarity using transformations, one student noted, "they both show that the triangle is similar, I guess. They both did show that the side lengths are proportional. As you can see." Another student, working in a WEP focused on reflections, stated, "They're both, they're both flipped to the other side, depending on if it's the x or y axis, they're both being flipped.” Here the student noticed that Jaxon reflected his triangle over the x-axis, whereas Maxine reflected hers over the y-axis. This student was later able to make a generalization about what happens to the coordinates of a figure when it is reflected over the x-axis and y-axis. These comparisons helped students analyze benefits and drawbacks of the WEP characters’ methods and make informed decisions about their preferred strategy for solving similar problems.

Table 1: Number Codes at Each Level

Level 1	*n*	Level 2	*n*	Level 3	*n*	Level 4	*n*
Comparison Between Characters	756	Similarities	267	Answer	55		
				Method	162		
				Problem	50		
		Differences	484	Answer	69		
				Method	380	Liked Better	37
						Same Result	20
						Way I Would Do It	33
						Easier	55
						Unsure	3
						Visual/Algebraic	26
						WEP Content Specific	206
				Problem	35		
		Led to Understanding	5				

Conclusions and Implications for Future Work

We have begun to document the types of comparisons students made during think aloud interviews regarding fictitious student methods to mathematics problems. This research shows a viable scientific basis for using comparisons to explore multiple solution strategies of students, as students were able to note similarities and differences in the strategies. Given critiquing reasoning is important to deepening mathematical understanding (Lampert, 1990; Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005), these findings are a step towards documenting the ways in which contrasting cases can be used in geometry. Future research will analyze if these comparisons of the fictitious students’ solution strategies advance students’ knowledge of geometry content, provide them with more flexible solution strategies, and equip them to better critique the reasoning of their peers.

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