From Curves to Words

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Abstract

Blank, in his Ph.D. thesis on determining whether a planar closed curve γ is self-overlapping, constructed a combinatorial word *geometrically* over the faces of γ by drawing cuts from each face to a point at infinity and tracing their intersection points with γ. Independently, Nie, in an unpublished manuscript, gave an algorithm to determine the minimum area swept out by any homotopy from a closed curve γ to a point. Nie constructed a combinatorial word *algebraically* over the faces of γ inspired by ideas from geometric group theory, followed by dynamic programming over the subwords. In this paper, we examine the definitions of the two words and prove the equivalence between Blank's word and Nie's word under the right assumptions.

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1 Introduction

A *closed curve* in the plane is a continuous map γ from the circle $\mathbb{S}^1$ to the plane $\mathbb{R}^2$. In this paper, we are given a *generic* planar curve meaning there are finitely many self-crossing points in the curve, and each of them is a *transverse* intersection.

In order to work with planar curves, one must choose a *representation*. Blank [1] determines if a curve is *self-overlapping*—boundary of an immersed disk—by checking whether a property holds on a combinatorial word, constructed by drawing cables from each face to infinity then traversing the curve and recording the signed intersection sequence of the curve and the cables. Nie [3] computes the minimum null homotopy area by performing dynamic programming on a combinatorial word. Nie represents a curve algebraically as a word in $\pi_1(\gamma)$. Motivated by simplifying Nie's proof of correctness, we show that Blank and Nie's word constructions are equivalent under modest assumptions.

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2 Curves and Words

We now describe Blank's word construction [1, page 5]. Let γ be a generic closed curve in the plane. Pick a point in the unbounded face of γ and call it the *basepoint* p_0. From each bounded face f_i, pick a *representative point* p_i. Now connect each p_i to p_0 by a simple path in such a way that no two paths intersect each other. We call the collection of such simple paths a *cable system*, denoted as Π, and each individual path π_i from p_i to p_0 as a *cable*.

Orient each π_i from p_i to p_0. Now traverse γ from an arbitrary *starting point* of γ and construct a cyclic word by writing down the indices of γ crossing the cables π_i in the order they appear on γ; each index i has a *positive* sign if we cross π_i from right to left and a *negative* sign if from left to right. We denote negative crossing with an overline $\bar{\texttt{i}}$. We call the resulting combinatorial word over the faces a *Blank word* of γ with respect to Π, denoted as $[\gamma]_B(\Pi)$. Figure 1 provides an example. Note that changing the starting point corresponds to a cyclic permutation of the word. Words with this property are called *cyclic words*. We make additional assumptions to organize the cable system onto a tree, an organized cable system is said to be *managed*.

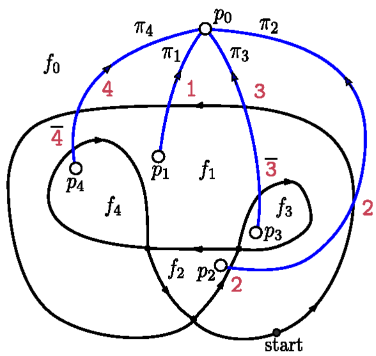

Figure 1 A curve γ with labeled faces and edges, Π_a is drawn in blue. The Blank word of γ corresponding to Π_a is $[\gamma]_B(\Pi_a) = [\texttt{2314}\texttt{2}\overline{\texttt{3}}\overline{\texttt{4}}]$.

We next describe Nie's word construction [3]. Choose a point p_i for each bounded face f_i of γ; denote the collection of points as P. Consider the punctured plane $X := \mathbb{R}^2 \setminus P$ and its fundamental group $\pi_1(X)$. Choose a basepoint x_0 in X and a set of generators Σ for $\pi_1(X)$, where each x_i in Σ represents the generator of $\pi_1(\mathbb{R}^2 \setminus \{p_i\}, x_0) \cong \mathbb{Z}$. These choices, along with a choice of a path from x_0 to each p_i, determine a map from each generator of $\pi_1(\mathbb{R}^2 \setminus \{p_i\})$ into $\pi_1(X)$ [4]. The fundamental group $\pi_1(X)$ is a free group over such generators, and the curve γ can be represented as a word over generators of $\pi_1(X)$. Elements of $\pi_1(X)$ are *free words*.

Consider the curve γ as a four-regular plane graph. Decompose γ into a spanning tree T and the complementary cotree (spanning tree of the dual graph) T^*, the trees (T, T^*) are a *tree-cotree* pair [2]. There are two natural sets of generators for $\pi_1(X)$: (1) the set of all cotree edges, and (2) the set of all face boundaries. We describe the change-of-basis between the two sets of generators in graph-theoretic terms. Traverse γ from some arbitrary starting point and orient each edge of γ accordingly. For each face f_i, define the *boundary operator* ∂ by mapping face f_i to the signed cyclic sequence of edges around face f_i, where each edge is signed positively if it is oriented counter-clockwise and negatively otherwise.

Now, write the curve γ as a cyclic word over the cotree edges T^* by traversing γ, ignoring all tree edges in T. We perform the following procedure inductively on the cotree T^* to construct another cyclic word, this time as an element in the free group over the faces of γ. Starting from the leaves f of T^*, rewrite each edge e bounding the face f as a singleton word, with positive sign if edge e is oriented counter-clockwise, or with negative sign otherwise. Next, for any internal node f of T^*, the boundary ∂f consists of a sequence of (1) tree edges, (2) cotree edges to children of f in T^* denoted as $e_1, e_2, \ldots, e_r$, and (3) (a unique) cotree edge to parent of f denoted as $e_f : \partial f = [e_f e_1 e_2 \ldots e_r]$.

We inductively rewrite each child cotree edge e_i as a free word w_i over the faces. Such words are *face words*. We emphasize that each word for the child cotree edge constructed inductively is a free word, not a cyclic word. Choose a particular but arbitrary way to break the cyclic sequence of faces and rewrite the equation: $e_f = \bar{w}_r \cdot \cdots \cdot \bar{w}_{j+1} \cdot (\bar{w}_j)' \cdot \partial f \cdot (\bar{w}_j)'' \cdot \bar{w}_{j-1} \cdot \cdots \cdot \bar{w}_1$, where $\bar{w}_j = (\bar{w}_j)'(\bar{w}_j)''$ is a particular way of breaking the face word $\bar{w}_j$ in two.

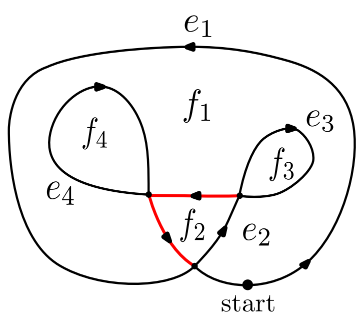

Figure 2 A curve γ with labeled faces and edges and tree in red. One cycle flattening of the boundaries gives $\partial(f_1) = e_3\bar{e}_2e_1e_4, \partial(f_2) = e_2, \partial(f_3) = \bar{e}_3$ and $\partial(f_4) = \bar{e}_4$. Write $\gamma = e_1e_2e_3e_4$ then use the boundaries to change the basis. The Nie word of γ is $[\gamma]_N(\Sigma) = [2314\overline{2}\overline{3}\overline{4}]$.

This gives us a free word over the faces for edge e_f, and, by induction, we have rewritten γ as a free word over the faces. Finally, we can turn the free word back into a cyclic word, by observing that the cyclic permutation of the constructed free word over the faces does not affect the element we are getting in $\pi_1(X)$ (but as a side effect of choosing the basepoint p_0 of γ). We call the resulting signed sequence of faces the *Nie word* and denoted as $[\gamma]_N(\Sigma)$, where Σ is the choices we made when breaking up the cyclic word at each cotree edge, referred to as a *cycle flattening*. Notice that the definition of $[\gamma]_N$ depends on how we choose to break the cyclic edge sequences, and thus is not well-defined without specifying the choices. The proof follows by induction.

▶ **Theorem 1** (Word Equivalence)**.** *Let γ be any plane curve. For a Nie word $[\gamma]_N(\Sigma)$ with a fixed cycle flattening Σ, there is a managed cable system Π such that the Blank word $[\gamma]_B(\Pi)$ is equal to $[\gamma]_N(\Sigma)$. Conversely, any managed cable system Π induces a cycle flattening Σ such that $[\gamma]_B(\Pi)$ and $[\gamma]_N(\Sigma)$ are equal.*

References

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