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Spectral gaps of the Laplacian on differential forms

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Abstract. In this short article, we explore some basic results associated to the Generalized Weyl criterion for the essential spectrum of the Laplacian on Riemannian manifolds. We use the language of Gromov-Hausdorff convergence to recall a spectral gap theorem. Finally, we make the necessary adjustments to extend our main results, and construct a class of complete noncompact manifolds with an arbitrarily large number of gaps in the spectrum of the Hodge Laplacian acting on differential forms.

1. Introduction

Let X be a complete noncompact Riemannian manifold of dimension n and denote by Δ the Laplace-Beltrami operator acting on smooth functions with compact support $\mathcal{C}_0^\infty(X)$. It is well-known that the unique self-adjoint extension of Δ on $L^2(X)$ is a nonnegative definite and densely defined linear operator.

The spectrum of Δ, denoted by $\sigma(\Delta)$, consists of all $\lambda \in \mathbb{C}$ for which $\Delta - \lambda I$ fails to be invertible. The essential spectrum of Δ, $\sigma_{\text{ess}}(\Delta)$, consists of the cluster points in the spectrum and of isolated eigenvalues of infinite multiplicity. The pure point spectrum is defined by

$$\sigma_{\text{pp}}(\Delta) = \sigma(\Delta) \backslash \sigma_{\text{ess}}(\Delta).$$

A consequence of the Hodge Decomposition Theorem is that, on a compact manifold,

$$\sigma(\Delta) = \sigma_{\text{pp}}(\Delta).$$

In the case of a noncompact manifold, however, the spectral structure is generally more complex than in the compact case.

Nonetheless, while it is impossibe to precisely compute the pure point spectrum for most compact manifolds, it is possible to locate the essential spectrum of the Laplacian for a large class of complete, noncompact Riemannian manifolds.

Historically, there are many results that exhibit a large class of noncompact manifolds whose essential spectrum is a connected subset of the real line. See, for example [**7**, **9**–**11**, **15**, **23**]. Likewise, one can find large sets of manifolds for which the essential spectrum has an arbitrarily large number of "gaps" (that is, the number of connected components of $\mathbb{R} \backslash \sigma_{\text{ess}}(X)$ can be arbitrarily large).

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In [2], we explore this set of manifolds, in collaboration with N. Charalambous, by first considering spectral continuity and then observing the evolution of the spectrum of a sequence of manifolds under Gromov-Hausdorff convergence. Using these ideas, we prove the existence of gaps in the essential spectrum of a periodic manifold. Our result is, in essence, a new proof to that by Schoen and Tran [22], as well as Post [19], Lledó and Post [16] who used Floquet theory; Khrabustovskyi [14] who presented more refined results; and Exner and Post [12] who exhibit limiting results.

It is a more complicated task to obtain similar results for the spectrum of the Hodge Laplacian acting on k-forms on a manifold, due to the stronger connection of the operator to the curvature.

In this note, we generalize the ideas present in [2] to the k-form spectrum, and construct a class of complete noncompact manifolds with an arbitrary number of gaps in the spectrum of the Hodge-Laplacian on k-forms.

2. The spectrum of the Laplacian on k-forms

Given k-forms $\omega = a_{i_1 \cdots i_k}\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$ and $\eta = b_{i_1 \cdots i_k}\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$, where $(\omega_1 \cdots, \omega_n)$ is a orthonormal co-frame, the L^2 *inner product* in $\Lambda^k M$ is defined as

$$(\omega, \eta) = k! \int_M a_I b_I \; dV,$$

where dV is the volume form, $\{e_1, \cdots, e_n\}$ is a global orthonormal frame and $\{\omega_1, \cdots, \omega_n\}$ its dual frame, and I is the corresponding multi-index.

Let $d : \Lambda^k M \to \Lambda^{k+1} M$ be the exterior differential. The adjoint δ of d, which is called the *codifferential operator*, is $\delta : \Lambda^{k+1} M \to \Lambda^k M$ satisfying

$$(d\omega, \eta) = (\omega, \delta\eta)$$

for all $\omega \in \Lambda^k M, \eta \in \Lambda^{k+1} M$ with compact support. It is worth noting that, contrary to the differential operator d, its adjoint operator δ depends on the metric g on the manifold. We have

$$(\delta\eta)_{i_1 \cdots i_k} = -(k+1)\nabla_s \eta_{s i_1 \cdots i_k}.$$

The *Hodge Laplacian*, also known as the *Laplace–de Rham* operator, is defined as

$$\Delta_k = \Delta \coloneqq d\delta + \delta d = (d + \delta)^2. \tag{1}$$

Similar to the case of the Laplace-Beltrami operator, it is well-known that the Hodge Laplacian extends to a self-adjoint, nonnegative operator densely defined over $L^2(\Lambda^k M)$. In particular, when $k = 0$, the Hodge Laplacian coincides with the Laplace-Beltrami operator acting on functions.

The Weitzenböck formula (see [18], Theorem 9.4.1) gives that

$$\Delta\omega = \nabla^*\nabla\omega + E(\mathcal{R}),$$

where $\mathcal{R}$ is the curvature operator and $E(\mathcal{R})$ is an algebraic operator of $\mathcal{R}$. Here

$$\nabla^*\nabla = -\sum_i \nabla^2_{E_i},$$

where $\nabla^2_{E_i} = \nabla_{E_i}\nabla_{E_i} - \nabla_{\nabla_{E_i} E_i}$ for an orthonormal frame $\{E_i\}$, is called the connection Laplacian. The results of this paper also hold for connection Laplacian.

We denote $\sigma(k, \Delta)$ the spectrum of the Hodge Laplacian Δ_k. In order to use the language of Gromov-Hausdorff convergence (see [4]), we abuse notation and use $\sigma(k, \Delta)$ to refer to the pointed metric space

$$(\sigma(k, \Delta) \cup \{-1\}, -1)\,.$$

Similarly, we use $\sigma_{\mathrm{ess}}(k, \Delta)$ for the pointed metric space

$$(\sigma_{\mathrm{ess}}(k, \Delta) \cup \{-1\}, -1)\,.$$

Note that these definitions imply

$$\sigma(-1, \Delta) = \sigma(n+1, \Delta) = \sigma_{\mathrm{ess}}(-1, \Delta) = \sigma_{\mathrm{ess}}(n+1, \Delta) = \emptyset,$$

meaning that the set consists of a single point metric space $\{-1\}$.

We will omit the degree k of the differential form when there is no risk of confusion.

As mentioned above, directly computing the essential spectrum of the Laplacian on forms has been a complicated task, even for the case of 1-forms, due to their stronger connection to the curvature, and because of the lack of good test forms on the manifold.

In view of the above obstacle, it would be more efficient to study the evolution of the spectrum under various deformations of the manifolds. The first natural case to consider is the evolution of eigenvalues under the continuous deformation its Riemannian metric. J. Dodziuk proved the following result

THEOREM 2.1 ([8]). *Let X be a compact manifold and let g_t be a family of Riemannian metrics on X. Assume that*

$$g_t \to g$$

in the $\mathcal{C}^0$ topology. Then the spectrum (eigenvalues) of g_t converges to the spectrum of g (as pointed metric spaces in the Gromov-Hausdorff sense).

A remarkable feature of the above theorem is that it doesn't depend on the curvature of the family of deformed manifolds. For the application in this paper, we need to use a generalized version Dodziuk's result. In the paper [4] (also see [20] for related results), N. Charalambous and the second author generalized spectral continuity to the case when the quadratic forms of two self-adjoint operators are ε-close.

Let $\mathcal{H}$ be a Hilbert space with two inner products $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$. Consider two densely defined nonnegative operators H_0 and H_1 on $\mathcal{H}$ that are self-adjoint with respect to the inner products $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$ respectively. Let Q_0, Q_1 be their respective quadratic forms and denote the two norms on $\mathcal{H}$ by $\|\cdot\|_0$ and $\|\cdot\|_1$. Note that both Q_0 and Q_1 are nonnegative.

Denote the domain of the Friedrichs extension of H_0 and H_1 by $\mathfrak{Dom}(H_0)$ and $\mathfrak{Dom}(H_1)$ respectively. We assume that there exists a dense subspace $\mathcal{C} \subset \mathcal{H}$ such that $\mathcal{C}$ is contained in $\mathfrak{Dom}(H_0) \cap \mathfrak{Dom}(H_1)$. In the case of the Laplacian operators $H_0 = \Delta_{g_0}$ and $H_1 = \Delta_{g_1}$ associated to two different metrics g_0 and g_1, $\mathcal{C}$ will be the space of smooth functions/forms with compact support.

DEFINITION 1 ([4]). We say that the operators H_0, H_1 are ε-close, if there exists a positive constant $0 < \varepsilon < 1$ such that for all $u \in \mathcal{C}$ the following two inequalities hold

$$(1-\varepsilon)\,\|u\|_0^2 \le \|u\|_1^2 \le (1+\varepsilon)\,\|u\|_0^2; \tag{2}$$

$$(1-\varepsilon)\,Q_0(u,u) \le Q_1(u,u) \le (1+\varepsilon)\,Q_0(u,u). \tag{3}$$

We note that if H_0, H_1 are ε-close, then for any $u, v \in \mathcal{C}$

$$|(u,v)_1 - (u,v)_0| \le \varepsilon(\|u\|_0\,\|v\|_0); \tag{4}$$

$$|Q_1(u,v) - Q_0(u,v)| \le \varepsilon\,[Q_0(u,u)\,Q_0(v,v)]^{1/2}. \tag{5}$$

It has been shown in [**4**] that two ε-close operators have nearby spectra. This result has an important application in the context of the Hodge Laplacian on k-forms over a Riemannian manifold with two ε-close metrics over it. In particular, it allows for the proof of the following theorem which holds even in the noncompact case.

THEOREM 2.2 ([**4**]). *Let X be an orientable manifold, and let g_0, g_1 be two smooth complete Riemannian metrics on X that are ε-close for some $0 < \varepsilon < 1/2$. Fix $A > 0$. Then for any $\lambda \in \sigma(k, \Delta_1) \cap [0, A]$,*

$$\mathrm{dist}(\lambda, \sigma(k, \Delta_0)) < c(A, n)\,\varepsilon^{\frac{1}{3}}$$

for some constant $c(A)$ depending only on A. A similar result holds for the essential spectra of the operators. In particular,

$$d_{\mathfrak{h}}(\sigma(k, \Delta_1), \sigma(k, \Delta_0)) = o(1),$$

where $o(1) \to 0$, as $\varepsilon \to 0$.

To clarify the notation in the above theorem, $d_{\mathfrak{h}}$ denotes the pointed Gromov-Hausdorff distance between the spectra as subsets of the real line with a common fixed point -1. $\sigma(k, \Delta_i)$ denotes the spectrum of nonnegative definite Hodge Laplacian Δ_i acting on k-forms which corresponds to the metric g_i for $i = 0, 1$.

In contrast, in the setting of a family of compact Riemannian manifolds which is convergent in the Gromov-Hausdorff sense, we have the following important results. The first result is due to K. Fukaya

THEOREM 2.3 ([**13**]). *Let X_t be a family of compact Riemannian manifolds which is Gromov-Hausdorff convergent to a compact metric space X. We assume that X is not a point. Assume that the curvatures of the manifolds X_t are uniformly bounded. Then the eigenvalues of X_t converge to those of X.*

The above result was later generalized by J. Cheeger and T. H. Colding

THEOREM 2.4 ([**6**]). *Let X_t be a family of compact Riemannian manifolds which is Gromov-Hausdorff convergent to a compact metric space X. We assume that X is not a point. Assume that the Ricci curvatures of the manifolds X_t are uniformly bounded below. Then the eigenvalues of X_t converge to those of X.*

There is no known common generalization of Theorems 2.1, 2.2, 2.3 and 2.4. In [**2**] we studied a special case, which allowed us to find manifolds with gaps in their L^2 essential spectrum.

Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$. The norm and inner product in $\mathcal{H}$ are noted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. Let $\sigma(H)$, $\sigma_{\mathrm{ess}}(H)$ be the spectrum and the essential spectrum of H, respectively. Let $\mathfrak{Dom}(H)$ be the domain of H. The Generalized Weyl criterion states the following.

THEOREM 2.5 (Charalambous-Lu [3]). *Let f be a bounded positive continuous function over $[0,\infty)$. A nonnegative real number λ belongs to the spectrum $\sigma(H)$ if, and only if, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ such that*

(1) $\|\psi\|$, $\forall n\in\mathbb{N}$;
(2) $(f(H)(H-\lambda)\psi_n,(H-\lambda)\psi_n)\to\infty$ *as* $n\to\infty$; *and*
(3) $(\psi_n,(H-\lambda)\psi_n)\to 0$ *as* $n\to\infty$.

Moreover, λ belongs to $\sigma_{\text{ess}}(H)$ of H if, and only if, in addition to the above properties

(4) $\psi_n\to 0$, *weakly as* $n\to\infty$ *in* $\mathcal{H}$.

One direct consequence is the following.

THEOREM 2.6 (Charalambous-Lu [4]). *Let M be a complete Riemannian manifold. Suppose that $\lambda>0$ belongs to the essential spectrum of the Laplacian on k-forms, $\sigma_{\text{ess}}(k,\Delta)$. Then one of the following holds:*

(1) $\lambda\in\sigma_{\text{ess}}(k-1,\Delta)$ *or*
(2) $\lambda\in\sigma_{\text{ess}}(k+1,\Delta)$.

3. Basic construction

We recreate the construction in [2]. Let $(X_1,g_1),(X_2,g_2)$ be two complete Riemannian manifolds. Let $x_1\in X_1$ and $x_2\in X_2$ be two fixed points on the manifolds respectively.

Let
$$N=S^{n-1}\times(-2,2)$$
be the product manifold equipped with the metric $g_N=\varepsilon^2 g_0$, where g_0 is the standard product metric.

For any $\varepsilon>0$, we construct the manifold X_ε by gluing the three manifolds X_1,X_2,N in the following way. Let $f_1:S^{n-1}\times(-2,-1)\to X_1$ be the function
$$f_1(\theta,t)=\exp_{x_1}(-t\varepsilon\theta),$$
where $\exp_{x_1}$ is the exponential map from $T_{x_1}X_1\to X_1$. In particular $\exp_{x_1}(0)=x_1$. Similarly, let $f_2:S^{n-1}\times(1,2)\to X_2$ be the function
$$f_2(\theta,t)=\exp_{x_2}(t\varepsilon\theta),$$
where $\exp_{x_2}$ is the exponential map from $T_{x_2}X_2\to X_2$. In particular $\exp_{x_2}(0)=x_2$. It is clear that f_i $(i=1,2)$ are diffeomorphisms between their domains and ranges. Let X_ε denote the composite manifold defined by (X_1,X_2,N,f_1,f_2), such that
$$X_\varepsilon=(X_1\backslash B_{x_1}(\varepsilon))\cup(X_2\backslash B_{x_2}(\varepsilon))\cup N/\sim, \tag{6}$$
where we identify f_i with their images respectively for $i=1,2$. Roughly speaking, X_ε is constructed from X_1,X_2 by removing two balls of radius ε and adding a neck connecting them.

Abusing notation, we will identify g_i with $f_i^*(g_i)$ for $i=1,2$ on the sets where they are defined.

We construct the metric g_ε on X_ε as follows. For $\delta\geq\varepsilon>0$, let $\rho_0^\delta,\rho_1^\delta,\rho_2^\delta$ be a partition of unity for X_ε in the following sense. Let
$$\text{supp}(\rho_0^\delta)\subset\{p\in X_\varepsilon\mid \text{dist}(p,S^{n-1}\times\{0\})<2\delta\},$$

and assume that $\rho_0^\delta \equiv 1$ on $\{p \in X_\varepsilon \mid \text{dist}(p, S^{n-1} \times \{0\}) < \delta\}$. Moreover, assume that $|\nabla \rho_0^\delta| \le C/\delta$. Then $X_\varepsilon \backslash \text{supp}(\rho_0^\delta)$ has two connected components X_1^δ and X_2^δ. Let

$$\rho_i^\delta = 1 - \rho_0^\delta$$

for $i = 1, 2$ be the functions on X_1^δ and X_2^δ, respectively. Then

$$\rho_0^\delta + \rho_1^\delta + \rho_2^\delta = 1$$

on X_ε.

We define

$$g_\varepsilon = \rho_0^\varepsilon g_N + \rho_1^\varepsilon g_1 + \rho_2^\varepsilon g_2.$$

Then we have

PROPOSITION 3.1 ([2]). *Let X_1, X_2 be two compact Riemannian manifolds. Using the above notations, we have*

(1) *$(X_\varepsilon, g_\varepsilon)$ is Gromov-Hausdorff convergent to the metric space X_0, which is the union $X_1 \cup X_2$ with x_1 identified with x_2 ;*
(2) *Let x_0 be a reference point in the middle of the neck N. The pointed Riemannian manifolds $(X_\varepsilon, \varepsilon^{-2} g_\varepsilon, x_0)$ are Gromov-Hausdorff convergent to $(S^{n-1} \times (-1, 1), x_0)$.*

We call $(X_\varepsilon, g_\varepsilon)$ a *smoothing* of X_0. Apparently, the curvature of X_ε are not bounded as $\varepsilon \to 0$. Nevertheless, The key property of the family of manifolds X_ε is that it has uniform local Sobolev constants. We shall use this fact to prove the spectrum continuity when the Ricci curvature of X_ε doesn't have a lower bound.

LEMMA 1 ([2]). *The (local) Sobolev constants for both $(X_\varepsilon, g_\varepsilon)$ and $(X_\varepsilon, \varepsilon^{-2} g_\varepsilon)$ are uniformly bounded.*

4. Proof of the main theorems

We begin this section with the following

LEMMA 2. *Let Δ_k be the Hodge Laplacian on k-forms of $(N, \varepsilon^2 g_0)$ with the Friedrichs extension, where g is the product metric. Then the first eigenvalue λ_1 of Δ_k diverges to $+\infty$ as $\varepsilon \to 0$.*

PROOF. We consider the first eigenvalue of (N, g_0) with the standard product metric. It is well known that if $k = 0$, then the first eigenvalue has a positive lower bound $\sigma > 0$. By scaling, for the metric, $\varepsilon^2 g_0$, the first eigenvalue is bounded below by $\sigma \varepsilon^{-2}$, which diverges to ∞ when $\varepsilon \to 0$. By duality, the same is true for $k = n$.

Similarly, we can work out the cases when $k \ne 0, n$. Since g_0 is the product metric, we have the following decomposition for $\Delta = \Delta_k$:

$$\Delta_N = \Delta_{S^{n-1}} \otimes 1 + 1 \otimes \Delta_{\mathbb{R}}.$$

Let $\lambda_1^k(N)$ denote the first eigenvalue of the manifold N on k-forms. Let ω be the first eigenform. Then we can write ω as

$$\omega = \omega_1 \wedge f(t)dt + g(t)\omega_2,$$

where $f(t), g(t)$ are functions such that $f(\pm 2) = g(\pm 2) = 0$ (by the Friedrichs extension) , and ω_1, ω_2 are $(k-1)$ and k eigenforms of S^{n-2} respectively.

If $g \not\equiv 0$, then

$$\lambda_1^k(S^{n-1} \times \mathbb{R}) \ge \lambda_1^0((-2, 2)) > 0$$

if $k \neq 0, n$, and the conclusion of the lemma proved. If $g \equiv 0$, then f must not be identically zero. As a result, we have

$$\lambda_1^k(S^{n-1} \times \mathbb{R}) \geq \lambda_1^1((-2,2)).$$

Since $f(\pm 2) = 0$, we have $\lambda_1^1((-2,2)) > 0$, thus completes the proof of the lemma. $\square$

Using the above, we shall prove the main technical result of this paper in the following.

THEOREM 4.1. *Let X_1, X_2 be two compact Riemannian manifolds and take $\lambda \notin \operatorname{Spec}(X_1) \cup \operatorname{Spec}(X_2)$. Consider the manifold $(X_\varepsilon, g_\varepsilon)$ defined above. Set $2\sigma = \operatorname{dist}(\lambda, \operatorname{Spec}(X_1) \cup \operatorname{Spec}(X_2))$ and take $\lambda' \in (\lambda - \sigma, \lambda + \sigma)$. Then, for any $\varepsilon > 0$ small enough, $\lambda' \notin \operatorname{Spec}(X_\varepsilon)$.*

PROOF. Let λ' be an eigenvalue of X_ε and let ω be an eigenform such that $\|\omega\|_{L^2} = 1$. Write

$$\omega = \rho_0^\delta \omega + \rho_1^\delta \omega + \rho_2^\delta \omega := \omega_0 + \omega_1 + \omega_2.$$

it turns out that

$$(\Delta\omega_0, \omega) + (\Delta\omega_1, \omega) + (\Delta\omega_2, \omega) = \lambda'. \tag{7}$$

Note that in fact, $\omega_i = \omega_i^{\delta,\varepsilon}$ depending on both ε, δ. For fixed $\delta > 0$, we let $\varepsilon \to 0$. Since $(\Delta\omega_i, \omega)/\|\omega\|_{L^2}^2$ is bounded for $i = 1, 2$, then using Lemma 1, by the elliptic regularity, we have $\mathcal{C}^{2,\alpha}$-estimate. Therefore, we have a sequence limit

$$\lim_{\varepsilon_j \to 0} \omega_i^{\delta,\varepsilon_j} = \omega_i^{\delta,0}$$

as $\varepsilon_j \to 0$ and $i = 1, 2$.

If any of $\omega_1^{\delta,0}$, or $\omega_2^{\delta,0}$ is not zero, say $\omega_1^{\delta,0} \neq 0$. Then we have

$$\Delta\omega_1^{\delta,0} = \lambda' \omega_1^{\delta,0}.$$

Letting $\delta \to 0$, the form $\omega_1^{\delta,0}$ will be convergent to an eigenform on $X_1 \backslash \{x_1\}$. By the removable singularity theorem, it would be extended to an eigenform of X_1, contradicting to the fact that λ' is away from $\operatorname{Spec}(X_1)$.

It remains to prove that it is not possible that both $\omega_1^{\delta,0}$ and $\omega_2^{\delta,0}$ are zero. Assume otherwise, then by (7), $(\Delta\omega_0, \omega)/\|\omega\|_{L^2}^2$ must be bounded when $\varepsilon \to 0$. But this would imply that

$$(\Delta\omega_0, \omega_0)/\|\omega_0\|_{L^2}^2$$

is bounded, which contradicts to Lemma 2. This completes the proof of the theorem. $\square$

Using the above technical lemma, we can prove the following

THEOREM 4.2. *There exists a complete non-compact Riemannian manifold M whose essential spectrum of the Hodge Laplacian on k-forms has an arbitrarily large number of connected components.*

PROOF. Let $X = S^n$ be the n-dimensional sphere and let N, S be the north pole and the south pole of X, respectively. Define a sequence of compact manifolds $X_i = X$, with the corresponding north and south poles N_i, S_i. Define the metric space Y by glueing the north pole of X_i to the south pole of X_{i+1} for any $i \geq 0$. Then the spectrum of Y is the same as the spectrum of $X = S^n$, which is discrete.

Using Theorem 4.1, after a smoothing of Y to Y_ε, then the essential spectrum of Y_ε would have arbitrarily large number of connected components. □

Using a similar argument, we can prove

COROLLARY 1. *There exists a complete non-compact Riemannian manifold M whose spectrum of the connection Laplacian on k-forms has an arbitrarily large number of connected components.*

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