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Relative Dynamics and Stability of Point Vortices on the Sphere

Tomoki Ohsawa

Department of Mathematical Sciences, The University of Texas at Dallas
800 W Campbell Rd, Richardson, TX 75080-3021, USA
Email: tomoki@utdallas.edu

Abstract—We exploit the SO(3)-symmetry of the Hamiltonian dynamics of N point vortices on the sphere to derive a Hamiltonian system for the relative dynamics of the vortices. The resulting system combined with the energy–Casimir method helps us prove the stability of the tetrahedron relative equilibria when all of their circulations have the same sign—a generalization of some existing results on tetrahedron relative equilibria of identical vortices.

1. Introduction

1.1. Dynamics of Point Vortices on Sphere

Consider N point vortices on the two-sphere $\mathbb{S}^2_R \subset \mathbb{R}^3$ with (fixed) radius $R > 0$ centered at the origin. Let $\{\mathbf{x}_i \in \mathbb{S}^2_R\}_{i=1}^N$ be the positions of the point vortices with circulations $\{\Gamma_i\}_{i=1}^N$. Then the equations of motion of the point vortices are

$$\dot{\mathbf{x}}_i = \frac{1}{2\pi R} \sum_{\substack{1\le j\le N\\ j\ne i}} \Gamma_j \frac{\mathbf{x}_j \times \mathbf{x}_i}{|\mathbf{x}_i - \mathbf{x}_j|^2} \qquad (1)$$

for $i \in \{1,\dots,N\}$; see, e.g., Bogomolov [1], Kimura and Okamoto [3], and Newton [7, Chapter 4].

This system of equations is Hamiltonian in the following sense: Let Ω_i be the area form of the i-th copy of $\mathbb{S}^2_R$ and define the following two-form on $(\mathbb{S}^2_R)^N$:

$$\Omega_{\mathbb{S}^2_R} := \sum_{i=1}^N \Gamma_i \pi_i^* \Omega_i \quad \text{with} \quad \Omega_i(\mathbf{x}_i)(\mathbf{v}_i, \mathbf{w}_i) := \frac{1}{R} \mathbf{x}_i \cdot (\mathbf{v}_i \times \mathbf{w}_i)$$

where $\pi_i\colon (\mathbb{S}^2_R)^N \to \mathbb{S}^2_R$ is the projection to the i-th copy. Define the Hamiltonian on $(\mathbb{S}^2_R)^N$ as

$$H_{\mathbb{S}^2_R}(\mathbf{x}_1,\dots,\mathbf{x}_N) := -\frac{1}{4\pi R^2} \sum_{1\le i<j\le N} \Gamma_i\Gamma_j \ln\!\left(2(R^2 - \mathbf{x}_i\cdot\mathbf{x}_j)\right).$$

Then we man write (1) as the following Hamiltonian system on $(\mathbb{S}^2_R)^N$:

$$\mathbf{i}_X \Omega_{\mathbb{S}^2_R} = \mathbf{d}H_{\mathbb{S}^2_R},$$

where X is a vector field on $(\mathbb{S}^2_R)^N$.

ORCID iDs Tomoki Ohsawa: 0000-0001-9406-132X

1.2. Relative Motion and Shape Dynamics

The focus of this paper is the relative motion or the *shape dynamics* of the point vortices, i.e., we are interested in the set of equations that governs the evolution of the "shape" or relative positions of the point vortices—regardless of where the vortices are located on the sphere. For example, for $N = 3$, it is the dynamics of the shape of the triangle formed by the three point vortices, regardless of its position and orientation on the sphere.

Defining the inter-vortex (Euclidean) distance $\ell_{ij} := |\mathbf{x}_i - \mathbf{x}_j|$ for $i, j \in \{1,\dots,N\}$ with $i \ne j$ and the (signed) volume $V_{ijk} := \mathbf{x}_i \cdot (\mathbf{x}_j \times \mathbf{x}_k)$ of the parallelepiped formed by vectors $\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k$ for $i, j, k \in \{1,\dots,N\}$ with $i \ne j \ne k$, we can derive the *equations of relative motion*

$$\frac{d}{dt}\ell_{ij}^2 = \frac{1}{\pi R} \sum_{\substack{1\le k\le N\\ k\ne i\ne j}} \Gamma_k V_{ijk} \left(\frac{1}{\ell_{jk}^2} - \frac{1}{\ell_{ki}^2} \right)$$

from (1); see, e.g., Newton [7, Section 4.2].

There are a couple of issues with this formulation: (i) The variables $\{\ell_{ij}\}_{1\le i<j\le N} \cup \{V_{ijk}\}_{1\le i<j<k\le N}$ are redundant as those to describe the shapes; (ii) it is not easy to find the invariants of the system. Our goal is to find a formulation of the shape dynamics that addresses these issues.

1.3. SO(3)-Symmetry

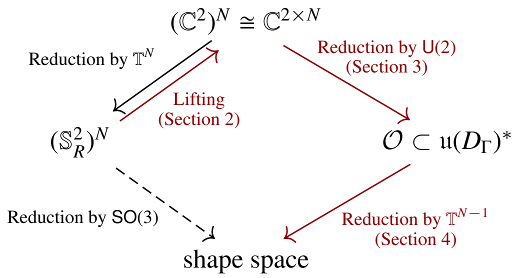
In principle, one may obtain the relative/shape dynamics of N point vortices by exploiting the invariance of (1) under the following SO(3)-action:

$$\begin{gathered}\mathsf{SO}(3) \times (\mathbb{S}^2_R)^N \to (\mathbb{S}^2_R)^N;\\ (R, (\mathbf{x}_1,\dots,\mathbf{x}_N)) \mapsto (R\mathbf{x}_1,\dots,R\mathbf{x}_N).\end{gathered}$$

However, the reduction by SO(3)-symmetry is quite intricate. Let $\mathbf{I}\colon (\mathbb{S}^2_R)^N \to \mathfrak{so}(3)^* \cong \mathbb{R}^3$ be the associated momentum map. Then the difficulty is that the reduced space or the Marsden–Weinstein quotient $\mathbf{I}^{-1}(\mathbf{c})/\mathsf{SO}(3)_{\mathbf{c}}$ with $\mathbf{c} \in \mathbb{R}^3$ is tricky to work with when describing the reduced dynamics; see Kirwan [4].

1.4. Lifting to $\mathbb{C}^2$ and Reduction by $U(2)$

We sidestep the difficulty of the SO(3)-reduction as described in the figure below.



Namely, instead of reducing the dynamics on $(\mathbb{S}^2_R)^N$ by SO(3) directly, first lift it to $(\mathbb{C}^2)^N$ (which picks up $\mathbb{T}^N$-symmetry) and then apply reduction by U(2) (which is facilitated by a dual pair); this results in a Lie–Poisson dynamics in a coadjoint orbit $\mathcal{O} \subset \mathfrak{u}(D_\Gamma)^*$. We may then further reduce the system by $\mathbb{T}^{N-1}$-symmetry to get rid of the extra symmetry picked up in the lifting process; see [8] for details.

This geometric treatment results in fewer variables for the shape dynamics compared to those "internal" variables of Borisov and Pavlov [2]. In fact, our shape dynamics is described using $(N-1)^2$ variables. On the other hand, the number of the "internal" variables of [2] is $N(N^2-1)/6$.

Another advantage of our formulation is that we can find a family of Casimirs exploiting the underlying algebraic structure of the Lie–Poisson bracket on $\mathcal{O}$. This is not easy with the Poisson bracket of [2] because its algebraic structure is not clear.

2. Lifted Vortex Dynamics in $\mathbb{C}^2$

We would like to first lift the vortex dynamics from $\mathbb{S}^2_R$ to $\mathbb{C}^2$. This idea is inspired by Vankerschaver and Leok [11], where they lift the dynamics from $\mathbb{S}^2_R$ to $\mathbb{S}^3_{\sqrt{R}}$ via the Hopf fibration $\mathbb{S}^3_{\sqrt{R}} \to \mathbb{S}^2_R$. As we have shown in [8], our approach naturally gives rise to the Hopf fibration by identifying the reduced space $\mathbb{S}^2_R = \mathbb{S}^3_{\sqrt{R}}/\mathbb{S}^1$ as a Marsden–Weinstein quotient.

2.1. Vortex Equations in $\mathbb{C}^2$

First define a Hamiltonian $H\colon (\mathbb{C}^2)^N \to \mathbb{R}$ as

$$H(\boldsymbol{\varphi}) := -\frac{1}{4\pi R^2} \sum_{1\le i<j\le N} \Gamma_i\Gamma_j \ln\left[\left(\|\varphi_i\|^2 + \|\varphi_j\|^2\right)^2 - 4|\varphi_i^*\varphi_j|^2\right], \quad (2)$$

where we used the shorthand $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_N) \in (\mathbb{C}^2)^N$, and defined the norm $\|\varphi\| := \sqrt{\varphi^*\varphi}$ induced by the natural inner product on $\mathbb{C}^2$. We also write

$$\varphi_i = \begin{bmatrix} z_i \\ u_i \end{bmatrix} \quad \text{with} \quad z_i, u_i \in \mathbb{C} \quad \forall i \in \{1, \dots, N\}.$$

We define a symplectic form Ω on $(\mathbb{C}^2)^N$ as follows:

$$\Omega := -\frac{2}{R}\sum_{i=1}^N \Gamma_i \operatorname{Im}(\mathbf{d}\varphi_i^* \wedge \mathbf{d}\varphi_i),$$

or $\Omega = -\mathbf{d}\Theta$ with

$$\Theta := -\frac{2}{R}\sum_{i=1}^N \Gamma_i \operatorname{Im}(\varphi_i^* \mathbf{d}\varphi_i). \quad (3)$$

Then the Hamiltonian vector field $X = \dot{\varphi}_i \frac{\partial}{\partial \varphi_i} + \text{c.c.}$ ("c.c." stands for the complex conjugate of the preceding term) defined by the Hamiltonian system $\mathbf{i}_X\Omega = \mathbf{d}H$ gives the following Schrödinger-like lifted vortex equation on $\mathbb{C}^2$ for $i = 1, \dots, N$:

$$\Gamma_i \dot{\varphi}_i = -\frac{\mathrm{i}}{2}\frac{\partial H}{\partial \varphi_i^*}. \quad (4)$$

3. U(2)-Reduction of N-vortex Dynamics in $\mathbb{C}^2$

The lifted dynamics (4) of N point vortices evolves in $(\mathbb{C}^2)^N$. We identify it with the space of $2 \times N$ complex matrices as follows:

$$(\mathbb{C}^2)^N \to \mathbb{C}^{2\times N}; \quad \boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_N) \mapsto \Phi = [\varphi_1 \dots \varphi_N].$$

Then we may rewrite the one-form (3) as

$$\Theta(\Phi) = -\frac{2}{R}\operatorname{Im}(\operatorname{tr}(D_\Gamma \Phi^* \mathbf{d}\Phi)), \quad (5)$$

where we defined

$$D_\Gamma := \operatorname{diag}(\Gamma_1, \dots, \Gamma_N). \quad (6)$$

3.1. U(2)-Reduction via a Dual Pair

Now consider the (left) U(2)-action on $\mathbb{C}^{2\times N}$ defined as

$$\mathsf{U}(2) \times \mathbb{C}^{2\times N} \to \mathbb{C}^{2\times N}; \quad (Y, \Phi) \mapsto Y\Phi.$$

One can easily check that this is a symplectic action and also that the Hamiltonian (2) is invariant under this action. Therefore, the associated momentum map

$$\mathbf{K}\colon \mathbb{C}^{2\times N} \to \mathfrak{u}(2)^*; \quad \mathbf{K}(\Phi) = -\frac{\mathrm{i}}{R}\sum_{i=1}^N \Gamma_i \varphi_i \varphi_i^*$$

is an invariant of the system (4), where we identified the dual $\mathfrak{u}(2)^*$ with $\mathfrak{u}(2)$ via this inner product $\langle \xi, \eta \rangle := 2\operatorname{tr}(\xi^*\eta)$.

Let us also define

$$\mathsf{U}(D_\Gamma) := \left\{ U \in \mathbb{C}^{N\times N} \mid U D_\Gamma U^* = D_\Gamma \right\},$$

using D_Γ from (6), and its (right) action on $\mathbb{C}^{2\times N}$:

$$\mathsf{U}(D_\Gamma) \times \mathbb{C}^{2\times N} \to \mathbb{C}^{2\times N}; \quad (Z, \Phi) \mapsto \Phi U. \quad (7)$$

The Lie algebra of $\mathsf{U}(D_\Gamma)$ is given by

$$\mathfrak{u}(D_\Gamma) := \left\{ \zeta \in \mathbb{C}^{N\times N} \mid \zeta D_\Gamma + D_\Gamma \zeta^* = 0 \right\}.$$

Now let us now equip the unitary algebra $\mathfrak{u}(N)$ with the modified Lie bracket

$$[\xi, \eta]_\Gamma := \xi D_\Gamma^{-1}\eta - \eta D_\Gamma^{-1}\xi.$$

to define a Lie algebra $\mathfrak{u}(N)_\Gamma$. Then we have the Lie algebra isomorphism $\mathfrak{u}(D_\Gamma) \to \mathfrak{u}(N)_\Gamma$; $\tilde{\zeta} \mapsto \tilde{\zeta} D_\Gamma =: \zeta$.

It is clear that the above action (7) leaves Θ invariant as well; see (5). The associated momentum map is then

$$\mathbf{L}\colon \mathbb{C}^{2\times N} \to \mathfrak{u}(N)^*_\Gamma; \quad \mathbf{L}(\Phi) = -\frac{\mathrm{i}}{R}\Phi^*\Phi.$$

However, the Hamiltonian (2) is *not* invariant under this action, and so $\mathbf{L}$ is *not* an invariant of the system (4).

It turns out that the above momentum maps $\mathbf{K}$ and $\mathbf{L}$ constitute a dual pair (see Skerritt [9] and Skerritt and Vizman [10]) defined on $\mathbb{C}^{2\times N}$:

$$\mathfrak{u}(2)^* \xleftarrow{\ \mathbf{K}\ } \mathbb{C}^{2\times N} \xrightarrow{\ \mathbf{L}\ } \mathfrak{u}(N)^*_\Gamma.$$

This implies that the $\mathsf{U}(2)$-reduced dynamics is given by a $(-)$-Lie–Poisson equation on $\mathfrak{u}(N)^*_\Gamma$:

$$\dot{\lambda} = \operatorname{ad}^*_{\delta h/\delta\lambda}\lambda \quad \text{or} \quad \dot{\lambda} = \{\lambda, h\}, \tag{8}$$

where $h\colon \mathfrak{u}(N)^*_\Gamma \to \mathbb{R}$ is defined so that $h \circ \mathbf{L} = H$:

$$h(\lambda) := -\frac{1}{4\pi R^2}\sum_{1\le i<j\le N}\Gamma_i\Gamma_j \ln\!\left(R^2\!\left(\left(\frac{1}{2}\left(\lambda_i+\lambda_j\right)\right)^2 - |\lambda_{ij}|^2\right)\right), \tag{9}$$

and the $(-)$-Lie–Poisson bracket on $\mathfrak{u}(N)^*_\Gamma$ is given by

$$\{f, h\}(\lambda) := -\left\langle \lambda, \left[\frac{\delta f}{\delta\lambda}, \frac{\delta h}{\delta\lambda}\right]_\Gamma \right\rangle$$

for any smooth $f, h\colon \mathfrak{u}(N)^*_\Gamma \to \mathbb{R}$.

Note that the Hamiltonian systems (4) and (8) are then related via

$$\lambda = -\frac{\mathrm{i}}{2}\begin{bmatrix} \sqrt{2}\lambda_1 & \lambda_{12} & \cdots & \lambda_{1N} \\ \bar{\lambda}_{12} & \sqrt{2}\lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda_{N-1,N} \\ \bar{\lambda}_{1N} & \cdots & \bar{\lambda}_{N-1,N} & \sqrt{2}\lambda_N \end{bmatrix} = \mathbf{L}(\Phi)$$

or more concretely, for $i, j \in \{1, \dots, N\}$,

$$\lambda_i = \frac{\sqrt{2}}{R}\,\|\varphi_i\|^2, \quad \lambda_{ij} = \frac{2}{R}\varphi_i^*\varphi_j.$$

Furthermore, one can show the above Lie–Poisson bracket has the following family of Casimirs [8]:

$$C_j(\lambda) := \operatorname{tr}\!\left((\mathrm{i}D_\Gamma\lambda)^j\right) \quad j \in \{1, \dots, N\}. \tag{10}$$

4. Further Reduction by $\mathbb{T}^{N-1}$-symmetry

4.1. $\mathbb{T}^{N-1}$-symmetry

Consider the action

$$\begin{gathered}\mathbb{T}^{N-1} \times \mathfrak{u}(N)^*_\Gamma \to \mathfrak{u}(N)^*_\Gamma \\ \left((e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_{N-1}}), \lambda\right) \mapsto \operatorname{Ad}^*_{e^{-\mathrm{i}\tilde{\theta}}}\lambda = e^{\mathrm{i}\tilde{\theta}}\lambda e^{-\mathrm{i}\tilde{\theta}}\end{gathered} \tag{11}$$

where

$$e^{\mathrm{i}\tilde{\theta}} := \operatorname{diag}\!\left(e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_{N-1}}, 1\right) \in \mathsf{U}(D_\Gamma).$$

It is easy to see that the Hamiltonian (9) is invariant under this action. However, note that the action (11) is not free. Thus we restrict the action to the open subset

$$\mathring{\mathfrak{u}}(N)^*_\Gamma := \{\lambda \in \mathfrak{u}(N)^*_\Gamma \mid \lambda_{ij} \neq 0 \text{ if } i \neq j\}$$

so that it becomes free.

Now, let us define

$$\mu_{ij} := \lambda_{ij}\bar{\lambda}_{iN}\lambda_{jN} = \lambda_{ij}\lambda_{Ni}\lambda_{jN} \in \mathring{\mathbb{C}} := \mathbb{C}\backslash\{0\}$$

for any $i, j \in \{1, \dots, N-1\}$ with $i < j$. Then we may parametrize $\lambda \in \mathring{\mathfrak{u}}(N)^*_\Gamma$ as follows:

$$\begin{aligned}\lambda = (\lambda_1, \dots, \lambda_N, \lambda_{1N}, \dots, \lambda_{N-1,N}, \mu_{12}, \dots, \mu_{N-2,N-1}) \\ \in \mathbb{R}^N \times \mathring{\mathbb{C}}^{N-1} \times \mathring{\mathbb{C}}^{(N-1)(N-2)/2}.\end{aligned}$$

Then the $\mathbb{T}^{N-1}$-action (11) becomes trivial on the variables $\{\mu_{ij}\}_{1\le i<j\le N-1}$, and hence we have

$$\begin{aligned}\mathring{\mathfrak{u}}(N)^*_\Gamma/\mathbb{T}^{N-1} &= \mathbb{R}^N \times \left(\mathring{\mathbb{C}}^{N-1}/\mathbb{T}^{N-1}\right) \times \mathring{\mathbb{C}}^{(N-1)(N-2)/2} \\ &= \mathbb{R}^N \times \mathbb{R}_+^{N-1} \times \mathring{\mathbb{C}}^{(N-1)(N-2)/2} \\ &= \left\{(\lambda_1, \dots, \lambda_N, |\lambda_{1N}|, \dots, |\lambda_{N-1,N}|, \mu_{12}, \dots, \mu_{N-2,N-1})\right\}.\end{aligned}$$

Then the Poisson bracket on $\mathring{\mathfrak{u}}(N)^*_\Gamma$ drops to the quotient by the standard Poisson reduction; see [8] for the resulting reduced Poisson bracket.

Furthermore, we may disregard $(\lambda_1, \dots, \lambda_N)$ from the variables because $\lambda_i = \frac{\sqrt{2}}{R}\|\varphi_i\|^2 = \sqrt{2}$ for $i = 1, \dots, N$. Also, since we have $|\lambda_{ij}|^2 = 4 - (\ell_{ij}/R)^2$, we impose that $0 < \ell_{ij} < 2R \iff 0 < |\lambda_{ij}| < 2$ to avoid collisions and having vortices at antipodal points. As a result, we have the following parametrization for the shape dynamics of N point vortices:

$$\begin{aligned}\mathcal{S}_N &:= (0,2)^{N-1} \times \mathring{\mathbb{C}}^{(N-1)(N-2)/2} \\ &= \{(|\lambda_{1N}|, \dots, |\lambda_{N-1,N}|, \mu_{12}, \dots, \mu_{N-2,N-1}) =: \zeta\}.\end{aligned}$$

Note that the dimension of this manifold is $(N-1)^2$, whereas the number of the "internal" variables $\{\ell_{ij}\}_{1\le i<j\le N} \cup \{V_{ijk}\}_{1\le i<j<k\le N}$ in Borisov and Pavlov [2] is $N(N^2-1)/6$.

Rewriting the collective Hamiltonian (9) in terms of our variables, we have the Hamiltonian $\mathcal{H}\colon \mathcal{S}_N \to \mathbb{R}$ defined as

$$\begin{aligned}\mathcal{H}(\zeta) := -\frac{1}{4\pi R^2}\Bigg(&\sum_{1\le i<j\le N-1}\Gamma_i\Gamma_j \ln\!\left(R^2\!\left(4 - \frac{|\mu_{ij}|^2}{|\lambda_{iN}|^2|\lambda_{jN}|^2}\right)\right) \\ &+ \Gamma_N\sum_{1\le i\le N-1}\Gamma_i \ln\!\left(R^2\!\left(4 - |\lambda_{iN}|^2\right)\right)\Bigg).\end{aligned} \tag{12}$$

To summarize our main result (see [8] for details), we have

Theorem 1.

(i) The relative/shape dynamics of N point vortices on the sphere is the Hamiltonian dynamics on

$$\begin{aligned}\mathcal{S}_N &:= (0,2)^{N-1} \times \mathring{\mathbb{C}}^{(N-1)(N-2)/2} \\ &= \{(|\lambda_{1N}|, \dots, |\lambda_{N-1,N}|, \mu_{12}, \dots, \mu_{N-2,N-1}) =: \zeta\}\end{aligned}$$

with respect to the reduced Poisson bracket (see [8]) and the Hamiltonian (12).

(ii) The Casimirs $\{C_j\}_{j\in\mathbb{N}}$ *from* (10) *are invariants of the shape dynamics.*

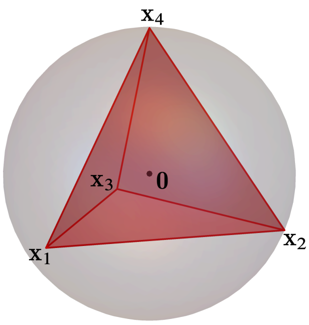
5. Application

5.1. Tetrahedron Relative Equilibria

Let us consider the special case with $N = 4$. The shape variables in this case are

$$\{\zeta = (|\lambda_{14}|, |\lambda_{24}|, |\lambda_{34}|, \mu_{12}, \mu_{13}, \mu_{23})\} \in \mathcal{S}_4 = (0,2)^3 \times \mathring{\mathbb{C}}^3.$$

We are particularly interested in the stability of the tetrahedron relative equilibrium as shown in the figure below.



Using our shape variables, let us set

$$|\lambda_{14}| = |\lambda_{24}| = |\lambda_{34}| = \frac{2}{\sqrt{3}}, \qquad \mu_{12} = -\mu_{13} = \mu_{23} = \frac{8}{3\sqrt{3}}\,\mathrm{i}.$$

Notice that $\operatorname{Im}\mu_{13}$ is the negative of $\operatorname{Im}\mu_{12}$ and $\operatorname{Im}\mu_{23}$ because the orientation of the triangle formed by $(\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4)$ is the opposite of those by $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4)$ and $(\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ as one can see (from the origin) in the figure. It is easy to check that we then have

$$\ell_{12} = \ell_{13} = \ell_{14} = \ell_{23} = \ell_{24} = \ell_{34} = 2\sqrt{\frac{2}{3}}R.$$

5.2. Stability of Tetrahedron Relative Equilibria

We would like to find a sufficient condition for stability of the tetrahedron relative equilibria. To our knowledge, existing stability results for tetrahedron equilibria are limited to the case with identical vortices, i.e., $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4$; see Kurakin [5] and Meleshko et al. [6]. We have generalized this result to the non-identical case with $N = 4$ as follows:

Proposition 2. *The tetrahedron configuration of four point vortices on the sphere is a stable equilibrium of the shape dynamics if all the circulations* $\{\Gamma_i\}_{i=1}^4$ *have the same sign.*

Proof. By the energy–Casimir method. See [8]. □

Acknowledgments

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