

Research Article

Suhyoung Choi*, Todd Drumm and William Goldman

Tameness of Margulis space-times with parabolics

<https://doi.org/10.1515/forum-2019-0331>

Received November 28, 2019; revised August 4, 2022

Abstract: Let E be a flat Lorentzian space of signature $(2, 1)$. A Margulis space-time is a noncompact complete Lorentz flat 3-manifold E/Γ with a free holonomy group Γ of rank g , $g \geq 2$. We consider the case when Γ contains a parabolic element. We obtain a characterization of proper Γ -actions in terms of Margulis and Charette–Drumm invariants. We show that E/Γ is homeomorphic to the interior of a compact handlebody of genus g generalizing our earlier result. Also, we obtain a bordification of the Margulis space-time with parabolics by adding a real projective surface at infinity giving us a compactification as a manifold relative to parabolic end neighborhoods. Our method is to estimate the translational parts of the affine transformation group and use some 3-manifold topology.

Keywords: Geometric structures, flat Lorentz space-time, 3-manifolds

MSC 2010: Primary 57M50; secondary 83A99

Communicated by: Jan Bruinier

Dedicated to the memory of Todd Drumm

1 Introduction

Let $\mathbf{Isom}^+(E)$ denote the group of orientation-preserving Lorentzian isometries on the oriented flat Lorentzian space E of the signature $(2, 1)$. Here, we have an exact sequence

$$1 \longrightarrow \mathbb{R}^{2,1} \longrightarrow \mathbf{Isom}^+(E) \xrightarrow{\mathcal{L}} \mathrm{SO}(2, 1) \longrightarrow 1,$$

where $\mathcal{L}$ is the homomorphism taking the linear parts of the isometries. A *parabolic* of $\mathbf{Isom}^+(E)$ is an element whose linear part is a parabolic element of $\mathrm{SO}(2, 1)$.

A discrete affine group Γ acting properly on E is either solvable or is free of rank ≥ 2 . (See Goldman and Labourie [27].) While we will assume that Γ is a free group of rank ≥ 2 , we say that Γ is a *proper affine free group of rank ≥ 2* .

We will often require $\mathcal{L}(\Gamma) \subset \mathrm{SO}(2, 1)^o$ for the subgroup $\mathrm{SO}(2, 1)^o$ of $\mathrm{SO}(2, 1)$ acting on the positive cone. Here, $\mathcal{L}(\Gamma)$ acts properly discontinuously and freely on a hyperbolic plane $\mathbb{H}^2$ formed by positive rays in the cone. We say that Γ is a *proper affine hyperbolic group of rank g with linear parts in $\mathrm{SO}(2, 1)^o$*

- if it acts properly discontinuously faithfully and freely on E , and
- $\mathcal{L}(\Gamma)$ is a free group of rank g , $g \geq 2$ in $\mathrm{SO}(2, 1)^o$, acting freely and discretely on $\mathbb{H}^2$.

It will be sufficient to prove tameness in this case.

***Corresponding author: Suhyoung Choi**, Department of Mathematical Sciences, KAIST, 305-701, Daejeon, Republic of Korea, e-mail: schoi@math.kaist.ac.kr

Todd Drumm, Department of Mathematics, Howard University, 20059, Washington DC., USA

William Goldman, Department of Mathematics, University of Maryland, 20742-4015, College Park MD, USA, e-mail: wmg@math.umd.edu

A *real projective structure* on a manifold is given by a maximal atlas of charts to $\mathbb{RP}^n$, $n \geq 1$, with transition maps in $\mathrm{PGL}(n+1, \mathbb{R})$. A *real projective manifold* is a manifold with a real projective structure.

Theorem 1.1. *Suppose that Γ is a proper affine free group of rank g , $g \geq 2$, with parabolics and linear parts in $\mathrm{SO}(2, 1)^\circ$. Then:*

- E/Γ is diffeomorphic to the interior of a compact handlebody of genus g .
- Moreover, it is the interior of a real projective 3-manifold M with boundary equal to a totally geodesic real projective surface, and M deformation retracts to a compact handlebody obtained by removing a union of finitely many end-neighborhoods homeomorphic to solid tori.

These real projective surfaces are from the paper of Goldman [26]. The second item is the so-called *relative compactification*.

For all cases of Margulis space-times, we have:

Corollary 1.2. *Let Γ be a proper affine free group of rank ≥ 2 with parabolics and linear parts in $\mathrm{SO}(2, 1)$. Then E/Γ is diffeomorphic to the interior of a compact handlebody of genus g . Moreover, it is the interior of a real projective 3-manifold M with boundary equal to a totally geodesic real projective surface, and M deformation-retracts to a compact handlebody obtained by removing a union of finitely many end neighborhoods homeomorphic to solid tori.*

We denote by $\mathbb{S}$ the sphere of directions in E , by $\mathbb{S}_+$ the space of directions of positive time-like directions and by $\mathbb{S}_-$ the space of directions of negative time-like directions. We will consider $\mathbb{S}_+$ as the projectivization of $\mathbb{S}_+ \cup \mathbb{S}_-$. Then the quotient space of $\mathbb{S}_+$ under Γ is a complete hyperbolic surface S . Let $\mathcal{P}_{\pi_1(S)}$ denote the set of parabolic elements and the identity element of $\pi_1(S)$. We denote by $l_{\mathbb{S}_+}(g)$ the length of the shortest closed geodesic in $\mathbb{S}_+/\Gamma$ corresponding to the element $g \in \Gamma$. By [6, Theorem 4.1] generalizing the Margulis opposite sign lemma [40], we will need the following criterion in this paper for our group Γ .

Criterion 1.1. Let Γ be an isometry group acting on E , and let $\alpha(g) \in \mathbb{R}$ for $g \in \Gamma$ denote the Margulis invariant of g . The isometry group Γ satisfies the following conditions:

- $\alpha(\gamma) > 0$ for every $\gamma \in \pi_1(S) \setminus \mathcal{P}_{\pi_1(S)}$,
- every $\gamma, \gamma \in \mathcal{P}_{\pi_1(S)} \setminus \{I\}$, has the positive Charette–Drumm invariant, and
- $\alpha(g) \geq c_{S \setminus E} l_{\mathbb{S}_+}(g)$ for every g realized as a closed geodesic in $S \setminus E$ for the union E of mutually disjoint cusp neighborhoods for a positive constant $c_{S \setminus E}$ depending on $S \setminus E$.

Of course, we can assume the negativity also since the change of the orientation of E changes the signs of Margulis invariants and Charette–Drumm invariants by [22] and [6].

Proposition 1.3. *Suppose that Γ acts properly on E . Then Criterion 1.1 holds up to changing the orientation of E .*

Proof. This is proved by [6, Theorem 4.1] and Lemma 1.4. □

Let US denote the inverse image of the projection $U\Sigma \rightarrow \Sigma$ for the subset $S \subset \Sigma$ and the unit tangent bundle $U\Sigma$ of a hyperbolic surface Σ . Let UE denote the bundle of unit space-like vectors over E .

Lemma 1.4. *Suppose that Γ acts properly on E . Let E' be the union of cusp neighborhoods in an ϵ -thin part of S . Then there exists a constant $c_{S \setminus E'}^{(1.4)}$ in $(0, 1)$ depending on E' such that for any closed curve g realized as a closed geodesic in $S \setminus E'$*

$$c_{S \setminus E'}^{(1.4)} l_{\mathbb{S}_+}(g) \leq \alpha(g) \leq \frac{1}{c_{S \setminus E'}^{(1.4)}} l_{\mathbb{S}_+}(g).$$

Proof. Consider the geodesic currents supported in a compact set $US \setminus UE'$. Then the argument of Goldman and Labourie [27] applies to this collection. We have a conjugacy homeomorphism from the set of geodesic currents on $US \setminus UE'$ with a compact set of neutral geodesic currents on UE/Γ . The length of each of these currents gives us the Margulis invariant. □

We prove the following characterization of a proper action of Γ in terms of Margulis and Charette–Drumm invariants.

Theorem 1.5. *An affine finitely generated free group Γ of rank ≥ 2 in $\mathbf{Isom}^+(\mathbb{E})$ acts properly discontinuously on $\mathbb{E}$ if and only if Criterion 1.1 holds up to a change of the orientation of $\mathbb{E}$.*

The forward part is Proposition 1.3. The converse follows from the main result Theorem 4.8 of Section 4. The proof is given at the end of Section 4.6

We mention that the tameness of geometrically finite hyperbolic manifolds was first shown by Marden [36] and later by Thurston [47]. (See Epstein and Marden [23].) Let $\mathbb{H}^3$ denote the hyperbolic 3-space. We take the convex hull $\mathcal{CH}(\Lambda)$ in $\mathbb{H}^3$ of the limit set Λ of the Kleinian group Γ , and there is a deformation retraction of $\mathbb{H}^3/\Gamma$ to the compact or finite volume $\mathcal{CH}(\Lambda)/\Gamma$ having a thick and thin decomposition. The paper here follows some of Marden's ideas. (See also Beardon and Maskit [3].)

Also, the approaches here are using thick and thin decomposition ideas of hyperbolic manifolds as suggested by Canary. However, we cannot find a canonical type of decomposition yet and artificially construct the parabolic regions. Only canonically defined regions in analogy to Margulis thin parts in the hyperbolic manifold theory are the regions bounded by parabolic cylinders. (See Section 3.1.2.)

Note that the tameness of Margulis space-times without parabolics was shown by Choi and Goldman [16] and Danciger, Guéritaud, and Kassel [18]. Danciger, Guéritaud, and Kassel have also announced a proof [20] for the tameness of Margulis space-times with parabolics, extending [19]. In addition, they give a proof [20] of the crooked plane conjecture in this setting, extending their proof in the setting without parabolics from [19]. Their methods, based on the deformation theory of hyperbolic surfaces, seem very different than those of the present paper.

Differently from them, we directly obtain 3-dimensional compactification relative to parabolic regions. We estimate by integrals the asymptotics of translation vectors of the affine holonomies. This is done by using the differential form version of the cocycles and estimating with geodesic flows on the vector bundles over the unit tangent bundle of the hyperbolic surface, the uniform Anosov nature of the flow (4.3), and the estimation of the cusp contributions in Appendix B. (See also Goldman and Labourie [27].) In the cusp neighborhoods, we replace the 1-form with the standard cusp 1-form and use this to estimate the growth of the cocycles. We use the exponential decreasing of a component of the differential form along the geodesic flows. Then we use estimates of the integration of the standard cusp 1-forms in Section 4.5.

Using this and the 3-manifold theory, we show that properly embedded disks and parabolic regions in $\mathbb{E}$ meet the inverse images of compact submanifolds in the Margulis space-time in compact subsets and find fundamental domains.

Since there are many proper affine actions of discrete groups not based on Lie algebraic situations as in [17–20], we hope that our method can generalize to these spaces with parabolics providing many points of view. (See Smilga [44–46] for example.)

Organization of the paper. The paper has three parts: the first two sections, Sections 2 and 3, are preliminary. Appendices A and B are only dependent on these two sections. Then the main argument parts follow: Section 4 discusses the geometry of the proper affine action, and Section 5 discusses the topology of the quotient space.

In Section 2, we review some projective geometry of Margulis space-times, the hyperbolic geometry of surfaces, Hausdorff convergences, and the Poincaré polyhedron theorem.

In Section 3, we first review the proper action of parabolic elements on the Lorentz space $\mathbb{R}^{2,1}$. We analyze the corresponding Lie algebra and vector fields. We introduce a canonical parabolic coordinate system of $\mathbb{R}^{2,1}$. In Section 3.2, we generalize the theory of Margulis invariants by Goldman, Labourie, and Margulis [28] and Ghosh and Treib [30] to groups with parabolics. That is, we introduce Charette–Drumm invariants which generalize the Margulis invariants for parabolic elements. In Section 3.3, we will study the parabolic regions and their ruled boundary components.

In Section 4, we will study the limit sets. We show that any sequence of the translation vectors of elements of Γ , i.e., cocycle elements, will accumulate in terms of directions only to $\mathbb{S}_0 := \mathbb{S} \setminus \mathbb{S}_+ \setminus \mathbb{S}_-$. In key result Corollary 4.9, we will prove that the limit points of a sequence of images of a compact set in $\mathbb{R}^{2,1}$ under elements of Γ are in $\mathbb{S}_0$. We will also prove the converse part of the equivalence of the properness of the action and Criterion 1.1, i.e., Theorem 1.5.

In Section 5, we will find the fundamental domain for M bounded by a finite union of properly embedded smooth surfaces showing that M is tame. We prove our main results Theorem 1.1 and Corollary 1.2 here. We make use of parabolic regions bounded by parabolic ruled surfaces. We avoid using almost crooked planes as in [16]. Instead, we are using disks that are partially ruled in parabolic regions to understand the intersections with parabolic regions. We will outline this major section in the beginning.

In Appendix A, we will prove facts about the parabolic regions.

In Appendix B, we will show how to modify 1-forms representing homology classes. We give estimates of some needed integrals here.

2 Preliminary

We will state some necessary facts here, mostly from the paper [16]. Let E denote the oriented flat Lorentzian space-time given as an affine space with a bilinear inner-product given by

$$B(\mathbf{x}, \mathbf{x}) := x_1^2 + x_2^2 - x_3^2, \quad \mathbf{x} = (x_1, x_2, x_3).$$

A Lorentzian norm $\|\mathbf{x}\|$ is given as $B(\mathbf{x}, \mathbf{x})^{\frac{1}{2}}$, where $(-1)^{\frac{1}{2}} = i$. We will fix a standard orientation on E and the associated vector space in this paper. Hence, E denote an oriented Lorentz space-time.

A Margulis space-time is a manifold of the form E/Γ , where Γ is a proper affine free subgroup of $\mathbf{Isom}(E)$ of rank $\mathbf{g}$, $\mathbf{g} \geq 2$. Elements of $\mathbf{PSO}(2, 1)$ are hyperbolic, parabolic, or elliptic. An element of $\mathbf{Isom}(E)$ is said to be *hyperbolic*, *parabolic*, or *elliptic* if its linear part is so.

The topological boundary $\text{bd}_X A$ of a subset A in another topological space X is given as $\text{Cl}(A)$ with the set of interior points of A removed. We denote by manifold boundary ∂A and the interior A° of a manifold A as usual. We define the manifold boundary $\partial A := \text{Cl}(A) \setminus A^\circ$ for any i -dimensional manifold A with i -dimensional manifold closure $\text{Cl}(A)$, $i = 1, 2, 3$, in a topological space X .

2.1 The projective geometry of the Margulis space-time

Let V be a vector space. Define $\mathbb{P}(V)$ as $V \setminus \{0\}/\sim$, where $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{x} = s\mathbf{y}$ for $s \in \mathbb{R} \setminus \{0\}$. Denote by $\text{PGL}(V)$ the group of automorphisms induced by $\text{GL}(V)$ on $\mathbb{P}(V)$.

Define the projective sphere $\mathbb{S}(V) := V \setminus \{0\}/\sim_+$, where $\mathbf{x} \sim_+ \mathbf{y}$ if and only if $\mathbf{x} = s\mathbf{y}$ for $s \in \mathbb{R}_+$. There is a double cover $\mathbb{S}(V) \rightarrow \mathbb{P}(V)$ with the deck transformation group generated by the antipodal map $\mathcal{A} : \mathbb{S}(V) \rightarrow \mathbb{S}(V)$. We will denote by $\langle \mathbf{v} \rangle$ the equivalence class of $\mathbf{v}$. Let $a_- = \mathcal{A}(a)$ denote the antipodal point of a . Also, given a set $A \subset \mathbb{S}(V)$, we define $A_- = \mathcal{A}(A)$. Let $\text{SL}_\pm(V)$ denote the group of linear maps of determinant ± 1 . The group $\text{SL}_\pm(V)$ acts on $\mathbb{S}(V)$ effectively and transitively.

We embed E as an open hemisphere in $\mathbb{S}(\mathbb{R}^4)$ by sending

$$(x_1, x_2, x_3) \mapsto \langle 1, x_1, x_2, x_3 \rangle \quad \text{for } x_1, x_2, x_3 \in \mathbb{R}.$$

The boundary of E is a great sphere $\mathbb{S}$ given by $x_0 = 0$. The rays of the positive cone end in an open disk $\mathbb{S}_+ \subset \mathbb{S}$, and the rays of the negative cone end in an open disk $\mathbb{S}_- \subset \mathbb{S}$, where $\mathcal{A}(\mathbb{S}_+) = \mathbb{S}_-$. The closure of E is a 3-hemisphere $\mathcal{H}$ bounded by $\mathbb{S}$.

The group $\mathbf{Isom}^+(E)$ of orientation-preserving isometries acts on E as a group of affine transformations and hence extends to a group $\text{SL}_\pm(\mathbb{R}^4)$ of projective automorphisms of $\mathbb{S}(\mathbb{R}^4)$. It restricts to the projective automorphism groups of $\mathcal{H}$ and of $\mathbb{S}$ and $\mathbb{S}_\pm$ respectively.

2.2 Thin parts of hyperbolic surfaces

As a subgroup of $\text{SL}_\pm(\mathbb{R}^3) \subset \text{SL}_\pm(\mathbb{R}^4)$, the Lorentz group $\text{SO}(2, 1)$ acts on $\mathbb{S}_+ \cup \mathbb{S}_-$, where $\text{SO}(2, 1)^o$ is the subgroup acting on $\mathbb{S}_+$ and is an index two subgroup. The space $\mathbb{S}_+ \cup \mathbb{S}_-$ carries a $\text{SO}(2, 1)$ -invariant hyperbolic

metric, and $\mathrm{SO}(2, 1)^0$ acting on $\mathbb{S}_+$ forms a Beltrami-Klein model of the hyperbolic plane. We denote the complete Beltrami-Klein metric by $d_{\mathbb{S}_+}$.

Given a nonelementary discrete subgroup Γ of $\mathrm{SO}(2, 1)^0$ acting freely on $\mathbb{S}_+$, we obtain a complete orientable hyperbolic surface $S := \mathbb{S}_+/\Gamma$ with the covering map $p_S : \mathbb{S}_+ \rightarrow S$. An *end neighborhood* of a manifold M is a component U of the complement of a compact subset of M that has a noncompact closure $\mathrm{Cl}(U)$.

Let $\epsilon > 0$ be the Margulis constant. Recall that the (ϵ) -thin part of S is the set of points through which essential loops with lengths $< \epsilon$ pass. The thin part is a union of open annuli. For a parabolic element, there is an embedded annulus that is a component of the thin part. It is a component of $S \setminus c$ for a simple closed curve c , and a horodisk H in the hyperbolic plane covers it. Here, $H/\langle g \rangle$ is isometric to the end-neighborhood for a parabolic isometry g acting on H . This end-neighborhood is called a *cuspid neighborhood*. For $\epsilon > 0$, a parabolic (ϵ) -end-neighborhood is a component of the ϵ -thin part of S that is an end-neighborhood.

We choose a union E of disjoint open cuspid-neighborhoods in S in an ϵ -thin part of S and its inverse image $\mathcal{H}$ in $\mathbb{S}_+$ which is a union of mutually disjoint horodisks.

2.2.1 Divergence functions

Definition 2.1. Let $\tilde{g} : I \rightarrow S$ be an arclength-parameterized geodesic and let $g : I \rightarrow S$ be a freely homotopic arc which is a closed arc whenever $\tilde{g}$ is closed. Suppose that there exists a continuous map $A : I \times \mathbb{R} \rightarrow S$ so that:

- $A(t, 0) = \tilde{g}(t)$ for each $t \in I$,
- if we define $A_t(s) := A(t, s)$ for each $t \in I, s \in \mathbb{R}$, then A_t is an arclength-parameterized geodesic perpendicular to $\tilde{g}$ at $\tilde{g}(t)$ for each $t \in I$, and
- $A(t, s_t) = g(t)$ for some s_t for each $t \in I$.

Then we say that we can *project* g to $\tilde{g}$ by the *perpendicular family of geodesics* A_t . If $|s_t| < \epsilon$ for all t , then we say that g is at a $d_{\mathbb{S}_+}$ -distance $< \epsilon$. The correspondence $g(t) \rightarrow \tilde{g}(t)$ for $t \in I$ to be called the *perpendicular projection*, and the geodesic segment between $g(t)$ to $\tilde{g}(t)$ for each t is called the *perpendicular projection path* and its length s_t the *perpendicular distance* at t .

Of course, the family of perpendicular geodesics may not be uniquely determined, but we make choices.

We call the f defined as below the *divergence function* from g_1 to g_2 .

Lemma 2.1. Let $g_1(t)$ and $g_2(t)$, $t \in [0, l]$, denote the parameterization of geodesics g_1 and g_2 , where g_1 is arclength parameterized. Suppose that we can project g_2 to g_1 by a perpendicular family of geodesics A_t . We orient these by the forward directions.

- We orient A_t so that the frame of its tangent vector and that of g_1 is positively oriented at $A_t(0) = g(t)$ for each $t \in I$. Define $f(t)$ to be the oriented path length on A_t from $g_1(t)$ to $g_2(t)$.
- Let $e_+ := f(l)$ and $e_- := f(0)$.
- Let α_+ and α_- denote $\pi/2$ minus the respective angles at the forward endpoint v_+ and the starting endpoint v_- of g_2 made by A_0 and A_l and g_2 respectively.

Assume $l \geq 1$. Then the following hold:

- If $|f(0)|, |f(l)| \leq C$, then $|f(t)| < C$ for $0 < t < l$. Furthermore, $|f|$ has at most one minimum.
- The integral of $|f(t)|$ over $[0, l]$ is less than $2|f(0)| + 2|f(l)|$.
- We have $\sum_{i=2}^{m-1} |f(t_i)| \leq 2|f(t_1)| + 2|f(t_m)|$ if $t_1, \dots, t_m, t_i \leq t_{i+1}$ for each $i = 1, \dots, m-1$, $m \geq 4$, satisfies $|t_{i+1} - t_i| \geq 1$.
- For the family of functions $l \geq 1, F_l : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ sending (α_+, α_-) to (e_+, e_-) for each $l \geq 1$ is 3.3 times a function decreasing the max norm provided $|\alpha_{\pm}| \leq 1/20$.

Proof. (i) We can show by [11] that

$$f(t) = g(y(t)), \quad g(y) := \frac{1}{2}(\log(1+y)) - \frac{1}{2}\log(1-y) \text{ and } y(t) := \pm \frac{\pm c_- s_+ \sinh(t) + c_+ s_- \sinh(l-t)}{c_- c_+ \sinh(l)},$$

where $c_i = \cosh(|e_i|)$, $s_i = \sinh(|e_i|)$, $i = -, +$. Notices that open geodesics become disjoint if only one of the

endpoints is changed. We may assume that e_- and e_+ are positive since old $|g(y(t))|$ is bounded above by the new $|g(y(t))|$ when we change all signs to be positive. We need to consider the case when the signs are + without loss of generality.

Now g has the expression as a Taylor series of y with only odd powers:

$$g(y) = y + \frac{y^3}{3} + \frac{y^5}{5} + \dots.$$

We see that y as a function of t can have exactly one interior minimum with only non-negative values or else it is strictly decreasing with some negative values. Since this property holds for the odd powers of y with the identical interior minimum point and zeros, our result follows for $e_-, e_+ \geq 0$. For other cases, we use hyperbolic trigonometry.

(ii) For (ii) and (iii), we can still look at $y(t)$ with positive coefficients only since we are seeking the upper bounds. We denote by $\tilde{y}$ the expression obtained from y by respectively replacing terms $\sinh(t)$ and $\sinh(l-t)$ by strictly larger $\frac{1}{2} \exp(t)$ and $\frac{1}{2} \exp(l-t)$ for $0 \leq t \leq l$. That is,

$$\tilde{y}(t) := \frac{c_- s_+ e^t + c_+ s_- e^{l-t}}{2c_- c_+ \sinh(l)}.$$

Now,

$$\tilde{y}(l) = \frac{e^l \tanh |e_+| + \tanh |e_-|}{2 \sinh l} \quad \text{and} \quad \tilde{y}(0) = \frac{\tanh |e_+| + e^l \tanh |e_-|}{2 \sinh l}.$$

Using $\tanh(x) < x$ for $x > 0$, and the fact that $1/(2 \sinh(l)) < 0.5$ and $e^l/(2 \sinh(l)) < 1.2$ for $l \geq 1$ while they are strictly decreasing functions of l , we can show

$$\tilde{y}(l) < (1.2)|e_+| + (0.5)|e_-| \quad \text{and} \quad \tilde{y}(0) < (0.5)|e_+| + (1.2)|e_-|. \quad (2.1)$$

By hyperbolic right triangle rules, we can show $|e_+|, |e_-| < 0.26$ provided $|\alpha_{\pm}| < 0.2$ for $l \geq 1$ by considering the contrapositive and the worst cases since it is again enough to consider the case $e_+, e_- \geq 0$. Hence $\tilde{y}(l), \tilde{y}(0) < 0.5$ and $\tilde{y}(t) < 0.5$ by the convexity of $\tilde{y}$.

Since g is strictly increasing, and $0 < y(t) < \tilde{y}(t)$ for $t > 0$, we obtain

$$\int_0^l |g(y(t))| dt \leq \int_0^l |g(\tilde{y}(t))| dt$$

provided $0 < \tilde{y}(t) < 1$. Since the Taylor series becomes a sum of terms that are positive number times $\exp(mt + nt)$ for $m, n \in \mathbb{Z}$, we obtain by a term-by-term argument

$$\int_0^l |g(y(t))| dt \leq \int_0^l |g(\tilde{y}(t))| dt \leq |g(\tilde{y}(l))| + |g(\tilde{y}(0))|.$$

Since $g(x) < 1.1x$ for $0 < x < 0.5$ by the convexity of g , (2.1) implies

$$|g(\tilde{y}(l))| + |g(\tilde{y}(0))| < 2(e_- + e_+) = 2|f(0)| + 2|f(l)|.$$

(iii) Note that $\sum_{i=2}^{m-1} |f(t_i)|$ is smaller than the integral of $|f|$ over t_1 to t_m since we can break up $|f|$ into parts as above and use the step functions dominated by $|f|$. (We may skip an interval containing the unique minimal point.) Hence, the sum is smaller than the twice of the sum of $|f(t_1)|$ and $|f(t_m)|$ by (ii).

(iv) Here again, we can look only at the cases when $e_+, e_- \geq 0$ and $\alpha_- \leq 0, \alpha_+ \geq 0$: Replacing the segments at v_+, v_- with ones with positive e_+, e_- , we can show by hyperbolic geometry that the max norm of old (α_+, α_-) is greater than or equal to that of new one while (e_+, e_-) does not change. In [15], we compute the map $[0, 1) \times (-1, 0] \rightarrow \mathbb{R}_+ \times \mathbb{R}_+$ which sends

$$(x_-, x_+) = \left(\cos\left(\frac{\pi}{2} + \alpha_-\right), \cos\left(\frac{\pi}{2} + \alpha_+\right) \right) = (-\sin(\alpha_-), -\sin(\alpha_+)) \mapsto (e_-, e_+).$$

We computed by analytic continuation

$$\begin{aligned} e_- &= \log \left(\frac{(x_- \coth(l) + x_+ \operatorname{csch}(l))}{\sqrt{1-x_-^2}} + \sqrt{1 + \frac{(x_- \coth(l) + x_+ \operatorname{csch}(l))^2}{1-x_-^2}} \right), \\ e_+ &= \log \left(\frac{(x_+ \coth(l) + x_- \operatorname{csch}(l))}{\sqrt{1-x_+^2}} + \sqrt{1 + \frac{(x_+ \coth(l) + x_- \operatorname{csch}(l))^2}{1-x_+^2}} \right), \end{aligned} \quad (2.2)$$

where there is a symmetry switching (e_-, x_-, x_+) with (e_+, x_+, x_-) , and we modified the computations in [15] to obtain an analytic continuation when x_+, x_- are very small. We use the series

$$\begin{aligned} \log(y + \sqrt{y^2 + 1}) &= \log(\sqrt{y^2 + 1}) + \log\left(1 + \frac{y}{\sqrt{1+y^2}}\right) \\ &= \frac{1}{2} \log(1+y^2) + \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{y}{\sqrt{y^2+1}} \right)^n \right), \end{aligned}$$

which is always absolutely convergent. We may plug into this

$$y = \frac{x_- \coth(l) + x_+ \operatorname{csch}(l)}{\sqrt{1-x_-^2}} \quad \text{and} \quad \frac{x_+ \coth(l) + x_- \operatorname{csch}(l)}{\sqrt{1-x_+^2}}$$

to obtain e_- and e_+ respectively in (2.2). Since $|x_+|, |x_-| < 1/\sqrt{2}$, the functions $|e_-|$ and $|e_+|$ respectively are bounded above by

$$\frac{1}{2} \log(1+2v^2) + \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\sqrt{2}v)^n \right) = \frac{1}{2} \log(1+2v^2) + \log(1+\sqrt{2}v)$$

for

$$v = (|x_-| \coth(l) + |x_+| \operatorname{csch}(l)) \quad \text{and} \quad (|x_+| \coth(l) + |x_-| \operatorname{csch}(l)).$$

By the Taylor analysis to order 1 and the Lagrange form of the error, the function is smaller than $\frac{3}{2}v$ for $v < 1/8$. (See [8].) Since $\coth(1) \leq 1.32$ and $\operatorname{csch}(1) \leq 0.86$, it follows that $(x_-, x_+) \mapsto (e_-, e_+)$ is $\frac{3}{2}(1.32 + 0.86)$ times a norm-nonincreasing function in terms of max norms provided $\max\{|x_-|, |x_+|\} < \frac{1}{8 \times 2.18}$. Since $x \rightarrow \sin(x)$ is strictly convex for $0 \leq x < 1/(8 \times 2.18)$, we take angles to satisfy $|\alpha_-|, |\alpha_+| \leq 0.05 < \arcsin(\frac{1}{8 \times 2.18})$. Then since $\arcsin(\alpha) < 1.00056\alpha$, $0 \leq \alpha < 0.05$, we are done. (See [15].) $\square$

A *broken geodesic* is a path consisting of parameterized geodesics except for isolated sets of points. For a broken geodesic, a *vertex* is a nonsmooth point of it. A *turning angle* at a vertex is the angle that the tangent vector the ending geodesic and one for the starting geodesic makes at the vertex. Since we are on an oriented surface S , we can say that the path can turn right or left at the vertex. The left-turning angle will be considered positive, and the right-turning angle will be considered negative.

Lemma 2.2. *Let g be a closed curve in S consisting of geodesic segments. Suppose that g is not parabolic. Suppose that the turning angles at vertices are within $(-\delta, \delta)$. Assume that $\delta < 1/40$. For the closed geodesic $\tilde{g}$ freely homotopic to g , suppose that each geodesic segment of g has a projected image with the length at least 1. Then $\tilde{g}$ has an arclength parameterization $\tilde{g}(t)$ with following properties:*

- *There is a corresponding perpendicular parametrization $g(t)$ of g so that $d_{\mathbb{S}^2}(g(t), \tilde{g}(t)) \leq \epsilon$ for $0 < \epsilon \leq 6.6\delta$.*
- *Let ζ be a bounded 1-form defined on a compact subset K . Let C_K denote the maximum value of the norm of ζ . Let α be a union of mutually disjoint geodesic subarcs in a geodesic subarc in g , going into K , corresponding to a union $\tilde{\alpha}$ of subarcs in $\tilde{g}$ where every perpendicular geodesic path between them is also going into K . Then the absolute value of the difference of respective integrals of ζ on α and $\tilde{\alpha}$ is less than $4C_K\epsilon$.*

Proof. Let $\tilde{g} : I \rightarrow S$ denote the closed geodesic. We draw the perpendicular lines at points of $\tilde{g}$ passing through broken points of g . A vertex $g(t_0)$ is *good* if the geodesic segments ending there has angles in $(\pi/2 - 2\delta, \pi/2 + 2\delta)$ with the perpendicular line to $\tilde{g}$ at $\tilde{g}(t_0)$. A geodesic segment e is *good at v* if it satisfies the condition for e for that side. We let $f : I \rightarrow \mathbb{R}$ be a function given by sending t to the perpendicular distance if $g(t)$ is in the right side of $\tilde{g}$ and to (-1) times that if $g(t)$ is in the left side.

We prove by induction on the number of vertices. If a vertex of g corresponds to a local maximum or the local minimum of the perpendicular distance function, then it is a good vertex since the turning angles are within $(-\delta, \delta)$. Since g is closed, there are at least two good vertices. For a broken geodesic, a local maximum of $|f|$ cannot occur in the interior point of a segment by hyperbolic geometry, but a local minimum of $|f|$ can occur.

We consider a maximal subarc m in g with no good interior vertex and f is either increasing or decreasing. Assume that the number of geodesic segments in m is ≥ 2 , or the signs on m are the same. Let v be the vertex with the maximal $|f|$ -value on m . Here, v is good since m is maximal. Suppose that the end vertex v' of the first geodesic segment e in m next to v has the same sign of the corresponding f -values. Then e is good at v and v' by elementary hyperbolic geometry using the hyperbolic right triangle with vertices v and v' and the right angle on the perpendicular line to $\tilde{g}$ passing v . Then the perpendicular distance function to e is given by above Lemma 2.1 and hence f -values of e are in $(-6.6\delta, 6.6\delta)$. Hence, so is m since we have the decreasing or increasing function where v has the maximum $|f|$ -value.

Suppose that $f(v)$ and $f(v')$ have different signs. Note that v' is not a local minimum or a local maximum. Now consider m' given by m with the edge e^o and v removed. Then the f -values have the same signs on m' and the maximal $|f|$ -value occurs at the other end which must be a good vertex also. Now the above applies and f -values on m' are in $(-6.6\delta, 6.6\delta)$. For e , we use the hyperbolic triangle with the vertex v and the two vertices that are perpendicular projections v_1 and v'_1 of v and v' on $\tilde{g}$ respectively. Let e' be the edge opposite v_1 . Now e' is good at v and v' since the angle sum of the triangle must be $< \pi$. Lemma 2.1 shows that $f(v) \in (-6.6\delta, 6.6\delta)$. Since f on e is strictly decreasing or increasing, we have the result for e .

We do these processes of estimation for such maximal subarcs. A segment e with a local minimum of $|f|$ in its interior can occur after the process ends. The vertices of e can be a vertex of such maximal subarcs or a good vertex. We need to work with quadrilateral obtained by projecting e to $\tilde{g}$ and the corresponding sides. We can use a reflection by the geodesic containing the shortest segment between e and its projection to $\tilde{g}$ and compare. We can show that either both angles at v and v' satisfy the premises of Lemma 2.1 or $|f|$ -values are both less than 6.6δ since the adjacent segments are as in the above paragraph or have a local minimum of $|f|$ in its interior.

Suppose that the number of segments in m is 1 with signs changing. Then both endpoints must be good. Otherwise, we can extend this segment at the other endpoint which is not good. If $|f|$ becomes zero, then we can use as above the right triangle with the hypotenuse obtained by extending the segment until $|f|$ becomes zero. If not, then there is a local minimum point where we can directly use Lemma 2.1.

The last item follows by using the divergence function. We obtain the bounds by (ii) of Lemma 2.1. $\square$

Lemma 2.3. *Let l be a maximal geodesic in a horodisk B in the upper half-space model given by $y > 1$. Suppose that the difference of the x -coordinates of the endpoints is t . Then the angle θ that l makes with the vertical line satisfies $\theta(t) = \pi/2 - \arctan(t/2)$. Also, $t \mapsto t\theta(t)$ is a strictly increasing function for $t \in (0, \infty)$, $t\theta(t) < 2$, and the limit is 2 as $t \rightarrow \infty$.*

Proof. The lemma follows from elementary geometry since the geodesics are circles perpendicular to $y = 0$ in the upper half-space model. (See [8].) $\square$

2.3 Hausdorff limits

The projective sphere $\mathbb{S}^3$ is a compact metric space, and has a natural standard metric $\mathbf{d}$. For a compact set $A \subset \mathbb{S}^3$, we define

$$\mathbf{d}(x, A) = \inf\{\mathbf{d}(x, y) : y \in A\}.$$

We define the ϵ - $\mathbf{d}$ -neighborhood $N_{\mathbf{d}, \epsilon}(A) := \{x : \mathbf{d}(x, A) < \epsilon\}$ for a point or a compact set A . We define the Hausdorff distance between two compact sets A and B as follows:

$$\mathbf{d}_H(A, B) = \inf\{\delta : \delta > 0, B \subset N_{\mathbf{d}, \delta}(A), A \subset N_{\mathbf{d}, \delta}(B)\}.$$

A sequence $\{A_i\}$ of compact sets *converges* to a compact subset A if $\{\mathbf{d}_H(A_i, A)\} \rightarrow 0$. The limit A is characterized as follows if it exists:

$$A := \{a \in \mathbb{S}^3 : a \text{ is a limit point of some sequence } \{a_i : a_i \in A_i\}\}.$$

See [4, Proposition E.12] for proof of this and Proposition 2.4 since the Chabauty topology for compact spaces is the Hausdorff topology. (See Munkres [41] also.)

Proposition 2.4 (Benedetti and Petronio [4]). *A sequence $\{A_i\}$ of compact sets converges to A in the Hausdorff topology if and only if both of the following hold*

- *If there is a sequence $\{x_{i_j}\}$, $x_{i_j} \in A_{i_j}$, where $x_{i_j} \rightarrow x$ for $i_j \rightarrow \infty$, then $x \in A$.*
- *If $x \in A$, then there exists a sequence $\{x_i\}$, $x_i \in A_i$, such that $\{x_i\} \rightarrow x$.*

Immediately we obtain:

Corollary 2.5. *Suppose that a sequence g_i of projective automorphisms of $\mathbb{S}^3$ converges to a projective automorphism g , and $\{K_i\} \rightarrow K$ for a sequence K_i of compact sets. Then $\{g_i(K_i)\} \rightarrow g(K)$.*

For example, a sequence of closed hemispheres will have a subsequence converging to a closed hemisphere.

2.4 The Poincaré polyhedron theorem

Definition 2.2. Let $\tilde{N}$ be an oriented manifold with empty or nonempty boundary on which a free group Γ acts properly and freely. Let S be a finite generating set $\{\gamma_1, \dots, \gamma_{2g}\}$ in Γ with $\gamma_{i+g} = \gamma_i^{-1}$ for indices in $\mathbb{Z}/2g\mathbb{Z}$. The collection of codimension-one submanifolds $A_1, \dots, A_{2g}$ satisfying the following properties is called a *matching collection of sets* under S :

- $\tilde{N}$ is a union of two submanifolds $\tilde{N}_1$ and $\tilde{N} \setminus \tilde{N}_1^\circ$ with $A_1 \cup \dots \cup A_g \subset \text{bd}_{\tilde{N}} \tilde{N}_1$ for $i \in \mathbb{Z}/2g\mathbb{Z}$,
- A_i is oriented by the boundary orientation from $\tilde{N}_1$,
- $\gamma_i(A_i) = A_{i+g}$ for $i \in \mathbb{Z}/2g\mathbb{Z}$,
- $\gamma_k(A_l) \cap A_m = \emptyset$ for $(k, l, m) \neq (i, i, i+g)$, and
- γ_i is orientation-preserving for each $i \in \mathbb{Z}/2g\mathbb{Z}$ and is orientation-reversing for A_i and A_{i+g} .

The following is a version of the Poincaré polyhedron theorem. We generalize Theorem 4.14 of Epstein and Petronio [24]. Here, we drop their distance lower-bound conditions, without which we can easily find counterexamples. However, we replace the condition with exhaustion by compact submanifolds where the lower-bounds hold. Thus, we give a proof. But we did not fully generalize the theorem by allowing sides of codimension ≥ 2 .

Proposition 2.6 (Poincaré). *Let N be a connected manifold with empty or nonempty boundary covered by a manifold $\tilde{N}$ with a free deck transformation group Γ_N .*

- *Let F be a connected codimension-zero submanifold with boundary in $\tilde{N}$ that is a union of mutually-disjoint, codimension-one, properly-embedded, two-sided submanifolds $A_1, \dots, A_{2g}$ with boundary orientation.*
- *Let $N_i \subset N$, $i = 1, 2, \dots$, be an exhausting sequence of compact submanifolds of N , where $N_i \subset N_{i+1}$ for $i = 1, 2, \dots$, and the inverse image $\tilde{N}_i$ of N_i in $\tilde{N}$ is connected.*
- *Let S be a finite generating subset of Γ_N and $\{A_1, \dots, A_{2g}\}$ is matched under S .*
- *$F \cap \tilde{N}_i$ is compact, and $F \cap \tilde{N}_i \cap A_j \neq \emptyset$ for each i and j .*

Then F is a fundamental domain of $\tilde{N}$ under Γ_N .

Proof. We define $X' := \bigsqcup_{\gamma \in \Gamma_N} \gamma(F)/\sim$, where we introduce an equivalence relation $\sim$ on $\bigsqcup_{\gamma \in \Gamma} \gamma(F)$ given by

$$x \in \gamma_1(F) \sim y \in \gamma_2(F) \iff \begin{cases} x = y \text{ and } \gamma_1 \gamma_2^{-1} \in S, \text{ or else} \\ x = y \text{ and } \gamma_1 = \gamma_2. \end{cases}$$

Thus,

$$X' := \bigsqcup_{\gamma \in \Gamma} \gamma(F)/\sim$$

is an open manifold immersing into N . We give a complete Riemannian metric on N where each ∂N_i is strictly convex. This induces a Γ -invariant Riemannian path-metric on X' and one on F .

Let $F_i = \tilde{N}_i \cap F$, a compact submanifold bounded by $A_j \cap \tilde{N}_i$ for $j = 1, \dots, 2g$ by a generic perturbation of N_i by small amounts. We define $X'_i := \bigsqcup_{\gamma \in \Gamma_N} \gamma(F_i) / \sim$, where we introduce an equivalence relation $\sim$ on $\bigsqcup_{\gamma \in \Gamma} \gamma(F_i)$ given by

$$x \in \gamma_1(F_i) \sim y \in \gamma_2(F_i) \iff \begin{cases} x = y \text{ and } \gamma_1 \gamma_2^{-1} \in \mathcal{S}, \text{ or else} \\ x = y \text{ and } \gamma_1 = \gamma_2. \end{cases}$$

We restrict the above Riemannian metric to X'_i as a submanifold of X' and obtain a Γ -invariant path metric d_i . We claim that d_i is metrically complete: Since $F \cap \tilde{N}_i$ is compact by the premise, it follows that $A_j \cap \tilde{N}_i$ is a compact subset. For every point in $x \in A_j \cap \tilde{N}_i$, the pathwise d_i -distance in $\tilde{N}_i$ to $A_k \cap \tilde{N}_i$, $k \neq j$ is bounded below by a positive number δ_i . Hence, each point of X'_i has a normal d_i -ball B'_i of fixed radius δ_i in the union of at most two images of F mapping isometric to a δ_i - d_i -ball B_i in N_i . Thus, given any Cauchy sequence x_l in X'_i , suppose that

$$d_i(x_k, x_l) < \frac{\delta_i}{3} \quad \text{for } l, k > L \text{ for some } L.$$

Then $d_i(x_j, x_{L+1}) < \delta_i/3$ for $j > L$. Since the ball of radius $\delta_i/3$ is in a union of two compact sets, it follows that x_l converges to a point of the δ_i - d_i -ball with center x_{L+1} . Hence, X'_i has a metrically complete path-metric d_i .

There is a natural local isometry $X'_i \rightarrow \tilde{N}_i$ given by sending $\gamma(F_i)$ to $\gamma(F_i)$ for each γ . Since $\{\gamma(F_i) | \gamma \in \Gamma\}$ is a locally finite collection of compact sets in $\tilde{N}_i$, the map is proper. The image in $\tilde{N}_i$ is open since each δ_i -ball is in the image of at most two sets of the form $\gamma(F_i)$. Since $\tilde{N}_i$ is connected, the openness and closedness show that $X'_i \rightarrow \tilde{N}_i$ is surjective. Therefore, $X'_i \rightarrow N_i$ is a covering map being a proper local homeomorphism. Now, $\tilde{N}_i$ and X'_i are covers of N_i with the identical deck transformation groups. We conclude $X'_i \rightarrow \tilde{N}_i$ is a homeomorphism.

There is a natural embedding $X'_i \rightarrow X'$. We identify X'_i with its image. We may identify X' with $\bigsqcup_{i=1}^{\infty} X'_i$. Since $\tilde{N} = \bigcup_{i=1}^{\infty} \tilde{N}_i$ holds, it follows that $X' \rightarrow \tilde{N}$ is a homeomorphism, and F is the fundamental domain. $\square$

3 Margulis invariants and Charette–Drumm invariants

We will first discuss parabolic group action in Section 3.1 and then discuss Charette–Drumm invariant ensuring their proper action in Section 3.2. In Section 3.3, we will introduce the parabolic ruled surfaces in E and the region bounded by them. We will also provide two transversal foliations on the regions.

3.1 Parabolic action

3.1.1 Understanding parabolic actions

Let V be a Lorentzian vector space of $\dim_{\mathbb{R}} V = 3$ with the inner product B . A linear endomorphism $N : V \rightarrow V$ is a *skew-adjoint endomorphism* of V if

$$B(N\mathbf{x}, \mathbf{y}) = -B(\mathbf{x}, N\mathbf{y}).$$

Lemma 3.1. *Suppose that N is a skew-adjoint endomorphism of V and $\mathbf{x} \in V$. Then $B(N\mathbf{x}, \mathbf{x}) = 0$.*

Proof. We have $B(N\mathbf{x}, \mathbf{x}) = -B(\mathbf{x}, N\mathbf{x}) = -B(N\mathbf{x}, \mathbf{x})$ by symmetry. Thus we obtain $B(N\mathbf{x}, \mathbf{x}) = 0$ as claimed. $\square$

Lemma 3.2. *Suppose that N is a nonzero nilpotent skew-adjoint endomorphism. Then $\text{rank}(N) = 2$.*

Proof. Since N is nilpotent, it is non-invertible and so $\text{rank}(N) < 3$. We have $\text{rank}(N) > 0$. Assume $\text{rank}(N) = 1$. Then $\dim \text{Ker}(N) = 3 - 1 = 2$. Since $\dim(V) = 3$, one of the following holds: $N(V) \cap \text{Ker}(N) = \{0\}$, or $N(V) \subset \text{Ker}(N)$. If $N(V) \cap \text{Ker}(N) = \{0\}$, then the restriction of N to $N(V)$ is nonzero, contradicting nilpotency. Thus,

$N(V) \subset \text{Ker}(N)$, that is, $N^2 = 0$. Then there exists $\mathbf{v} \in V$ with $N\mathbf{v} \neq 0$. Since $N^2\mathbf{v} = 0$, the set $\{\mathbf{v}, N\mathbf{v}\}$ is linearly independent. Complete $\{N\mathbf{v}\}$ to a basis $\{N\mathbf{v}, \mathbf{w}\}$ of $\text{Ker}(N)$. The set $\{\mathbf{v}, N\mathbf{v}, \mathbf{w}\}$ is a basis for V . We have:

- Lemma 3.1 implies $B(N\mathbf{v}, \mathbf{v}) = 0$.
- $N^2 = 0$ implies $B(N\mathbf{v}, N\mathbf{v}) = -B(N^2\mathbf{v}, \mathbf{v}) = 0$.
- $B(N\mathbf{v}, \mathbf{w}) = -B(\mathbf{v}, N\mathbf{w}) = 0$ since $N\mathbf{w} = 0$.

Thus, $N\mathbf{v}$ is a nonzero vector orthogonal to all of V , contradicting nondegeneracy. Hence, $\text{rank}(N) = 2$ as claimed. $\square$

Lemma 3.3. *We have $N^2 \neq 0$.*

Proof. Lemma 3.2 implies that $\dim \text{Ker}(N) = 1$ and $\dim N(V) = 2$. If $N^2 = 0$, then $N(V) \subset \text{Ker}(N)$, a contradiction. $\square$

Lemma 3.4. *We have $N(V) = \text{Ker}(N^2)$ and $N^2(V) = \text{Ker}(N)$.*

Proof. Since $\dim(V) = 3$, the nilpotency implies $N^3 = 0$. By Lemma 3.3, the invariant flag

$$V \supset N(V) \supset N^2(V) \supset \{0\} \quad (3.1)$$

is maximal; that is, $\dim V/N(V) = \dim N(V)/N^2(V) = 1$. Now, $N^3 = 0$ implies that $N(V) \subset \text{Ker}(N^2)$ and $N^2(V) \subset \text{Ker}(N)$. Hence, the invariant flag

$$V \supset \text{Ker}(N^2) \supset \text{Ker}(N) \supset \{0\} \quad (3.2)$$

is maximal. It follows that the flags (3.1) and (3.2) are equal, as claimed. $\square$

Lemma 3.5. *The group $\text{Ker}(N)$ is null.*

Proof. Lemma 3.4 implies $\text{Ker}(N) = N^2(V)$. Since N is skew-adjoint and $N^4 = 0$,

$$B(N^2(V), N^2(V)) \subset B(N^3(V), N(V)) = \{0\}$$

as desired. $\square$

Lemma 3.6. *We have $\text{Ker}(N) = N(V)^\perp$ and $N(V) = \text{Ker}(N)^\perp$.*

Proof. We have $B(N(V), \text{Ker}(N)) = B(V, N(\text{Ker}(N))) = \{0\}$ so that $\text{Ker}(N) \subset N(V)^\perp$ and $N(V) \subset \text{Ker}(N)^\perp$. Since $\text{Ker}(N)$ and $N(V)^\perp$ each have the dimension 1, and $N(V)$ and $\text{Ker}(N)^\perp$ each have the dimension 2, the lemma follows. $\square$

We find a canonical generator for the line $\text{Ker}(N)$ given N , together with a time-orientation.

Lemma 3.7. *There exists unique $\mathbf{c} \in \text{Ker}(N)$ such that:*

- $\mathbf{c} \neq 0$ is a causal null-vector,
- $\mathbf{c} = N(\mathbf{b})$ for a unit-space-like $\mathbf{b} \in V$ (that is, $B(\mathbf{b}, \mathbf{b}) = 1$).

Furthermore, the following hold:

- $\mathbf{b}$ is unique up to addition of $\lambda\mathbf{c}$, $\lambda \in \mathbb{R} - \{0\}$.
- We can choose the unique null vector $\mathbf{a}$ so that $N(\mathbf{a}) = \mathbf{b}$.
- $B(\mathbf{a}, \mathbf{b}) = 0 = B(\mathbf{b}, \mathbf{c})$, $B(\mathbf{a}, \mathbf{c}) = -1$.
- $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a basis.
- The Lorentz metric has an expression $g := dy^2 - 2dx dz$ with respect to the coordinate system given by $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Proof. Lemma 3.4 implies that N defines an isomorphism (of 1-dimensional vector spaces)

$$\bar{N} : N(V)/\text{Ker}(N) \rightarrow N^2(V) = \text{Ker}(N). \quad (3.3)$$

Now, $B|_{N(V) \times N(V)}$ is factored into the maps

$$N(V) \times N(V) \rightarrow N(V)/N(V)^\perp \times N(V)/N(V)^\perp \quad \text{and} \quad \hat{B} : N(V)/N(V)^\perp \times N(V)/N(V)^\perp \rightarrow \mathbb{R}.$$

Lemma 3.6 implies that the second map is

$$\hat{B} : N(V)/\text{Ker}N \times N(V)/\text{Ker}N \rightarrow \mathbb{R}.$$

Since $N(V)/\text{Ker}N$ is a 1-dimensional vector space, the quadratic map $\hat{B}$ is a square of an isomorphism $N(V)/\text{Ker}N \rightarrow \mathbb{R}$. Hence, the restriction to $N(V)$ of the quadratic form $\mathbf{u} \rightarrow B(\mathbf{u}, \mathbf{u})$ is the square of an isomorphism $N(V)/\text{Ker}(N) \rightarrow \mathbb{R}$ composed with the quotient map $N(V) \rightarrow N(V)/\text{Ker}(N)$.

Recall $\dim \text{Ker}(N) = 1$. Since $\tilde{N}$ is injective, the set of unit-space-like vectors in $N(V)$ is the union of two cosets of $\text{Ker}(N)$, mapped by N to two nonzero vectors in $\text{Ker}(N)$. By Lemma 3.5, the image is null. The image is a causal vector in $\text{Ker}(N)$ or a non-causal vector in $\text{Ker}(N)$. Take the causal one to be $\mathbf{c}$. Since the image has only two vectors, it follows that $\mathbf{c}$ is the unique one.

By (3.3), $\mathbf{b}$ can be chosen to be any in $N(V)$ in the coset of $\text{Ker}(N)$, and hence $\mathbf{b}$ can be changed to $\mathbf{b} + c_0\mathbf{c}$ since $\mathbf{c}$ generates $\text{Ker}(N)$.

By Lemma 3.1, $B(\mathbf{b}, \mathbf{c}) = B(N(\mathbf{c}), \mathbf{c}) = 0$.

The subspace $N^{-1}(\mathbf{b})$ is a line since $\dim \text{Ker}(N) = 1$ and is parallel to a null space and does not pass 0 since $\mathbf{b} \neq 0$. Hence, it meets a null cone at the unique point. Call this $\mathbf{a}$. By Lemma 3.1, $B(\mathbf{a}, \mathbf{b}) = 0$.

Finally,

$$B(\mathbf{a}, \mathbf{c}) = B(\mathbf{a}, N^2(\mathbf{a})) = -B(N(\mathbf{a}), N(\mathbf{a})) = -B(\mathbf{b}, \mathbf{b}) = -1.$$

The last statement follows by B-values which also implies the independence. $\square$

Definition 3.1. Let N be a nilpotent skew adjoint endomorphism. We will call the frame $\mathbf{a}, \mathbf{b}, \mathbf{c}$ satisfying the above properties:

- $\mathbf{b} = N(\mathbf{a}), \mathbf{c} = N(\mathbf{b})$,
- $\mathbf{a}, \mathbf{c}$ are null and $\mathbf{b}$ is of unit space-like,
- $B(\mathbf{a}, \mathbf{b}) = 0 = B(\mathbf{b}, \mathbf{c}), B(\mathbf{a}, \mathbf{c}) = -1$.

the *adopted frame* of N . We will say that N is *accordant* if the adopted frame has the standard orientation.

Corollary 3.8 shows that associated with N , there is a one-parameter family of frames. However, we remark that the orientation of $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is determined by N as we can see from exchanging N with $-N$ has the orientation-reversing effect.

Corollary 3.8. Let N be a nilpotent skew adjoint endomorphism. Then the Lorentzian vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ satisfying the property that

- $B(\mathbf{a}, \mathbf{b}) = 0 = B(\mathbf{b}, \mathbf{c}), B(\mathbf{a}, \mathbf{c}) = -1$,
- $\mathbf{c} = N(\mathbf{b}), \mathbf{b} = N(\mathbf{a})$, and
- $\mathbf{b}$ is a unit space-like vector, $\mathbf{c} \in \text{Ker}N$ is causally null, and $\mathbf{a}$ is null

are determined up to changes $\mathbf{b} \rightarrow \mathbf{b} + c_0\mathbf{c}, \mathbf{a} \rightarrow \mathbf{a} + c_0\mathbf{b} + \frac{c_0^2}{2}\mathbf{c}$ with respect to the a skew-symmetric nilpotent endomorphism N and $B : V \times V \rightarrow \mathbb{R}$. Furthermore, the adopted frame for N is determined only up to these changes and translations.

Proof. By Lemma 3.7, we can only change $\mathbf{b} \mapsto \mathbf{b} + c_0\mathbf{c}, \mathbf{a} \mapsto \mathbf{a} + c_0\mathbf{b} + d_0\mathbf{c}$. Since

$$B(\mathbf{a} + c_0\mathbf{b} + d_0\mathbf{c}, \mathbf{b} + c_0\mathbf{c}) = -c_0 + c_0 = 0,$$

and

$$B(\mathbf{a} + c_0\mathbf{b} + d_0\mathbf{c}, \mathbf{a} + c_0\mathbf{b} + d_0\mathbf{c}) = c_0^2 - 2d_0 = 0,$$

this is proved. $\square$

3.1.2 The action of the parabolic transformations

We represent an affine transformation with the formula $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{w}, \mathbf{x} \in \mathbb{R}^{2,1}$ by the matrix

$$\begin{pmatrix} A & \mathbf{w} \\ 0 & 1 \end{pmatrix}.$$

Let N be an accordant nilpotent element of the Lie algebra of $\text{SO}(2, 1)$: Let us use the frame $\mathbf{c}, \mathbf{b}, \mathbf{a}$ on E obtained by Corollary 3.8 as the vectors parallel to x -, y -, and z -axes respectively. Then the bilinear form B

takes the matrix form

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (3.4)$$

Let γ be a parabolic transformation $E \rightarrow E$. Then it must be of the form

$$\Phi(t) := \exp t \begin{pmatrix} N & \vec{v}_\gamma \\ 0 & 0 \end{pmatrix} \quad \text{for an accordant nilpotent skew adjoint element } N.$$

Using the frame given by Corollary 3.8 and shifting the origin by translation by (t, v_1, v_2) , $t \in \mathbb{R}$ when $\vec{v}_\gamma$ can be written as (v_1, v_2, μ) with respect to the frame, we obtain an affine coordinate system so that γ lies in a one-parameter group

$$\Phi(t) := \exp t \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t & t^2/2 & \mu t^3/6 \\ 0 & 1 & t & \mu t^2/2 \\ 0 & 0 & 1 & \mu t \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.5)$$

for $\mu \in \mathbb{R}$, where $\Phi(t) : E \rightarrow E$ is generated by a vector field

$$\phi := y\partial_x + z\partial_y + \mu\partial_z, \quad \text{where } B(\phi, \phi) = z^2 - 2\mu y.$$

For a parabolic element γ and $t \in \mathbb{R}$, we define $\gamma^t := \exp(t\eta)$, where $\gamma = \exp(\eta)$ for a unique Lie algebra element η of $\mathbf{Isom}^+(E)$.

Definition 3.2. For any parabolic element γ , the coordinate system where it can be written in the form (3.5) with the adopted frame for accordant nilpotent N , where $\gamma = \exp(tN)$, $t \in \mathbb{R}$ is called a *parabolic coordinate system adopted to γ* . Furthermore, γ is called *accordant* if $t > 0$.

Proposition 3.9. Any parabolic element γ has a parabolic coordinate system. All other parabolic coordinate system for γ is obtained by changing it by a 2-dimensional parameter family of isometries generated by the one-parameter family of translations along unique eigen-direction and the frame change given in Corollary 3.8.

Proof. The existence of the coordinate frame is already given. The fact that the 2-dimensional family of isometries preserves the form (3.5) is already shown in Corollary 3.8 and near (3.5). Also, from near (3.5) we obtain the translations must be the one-parameter ones along the unique eigen-direction. $\square$

This one-parameter subgroup $\{\Phi(t), t \in \mathbb{R}\}$ leaves invariant the two polynomials

$$\begin{aligned} F_2(x, y, z) &:= z^2 - 2\mu y, \\ F_3(x, y, z) &:= z^3 - 3\mu yz + 3\mu^2 x, \end{aligned}$$

and the diffeomorphism

$$F(x, y, z) := (F_3(x, y, z), F_2(x, y, z), z)$$

satisfies

$$F \circ \Phi(t) \circ F^{-1} : (x, y, z) \rightarrow (x, y, z + \mu t) \quad (3.6)$$

All the orbits are twisted cubic curves. In particular, every cyclic parabolic group leaves invariant no line and no plane for $\mu \neq 0$. (See Figure 1.)

Now, $Q := F_2$ is the unique quadratic ϕ -invariant function on E up to adding constants and scalar multiplications. If $Q(p) < 0$ for $p \in E$, then the trajectory $\Phi(t)(p)$ is time-like. If $Q(p) > 0$, then $\Phi(t)(p)$ is space-like. In addition, if $Q(p) = 0$, then $\Phi(t)(p)$ is a null-curve. The region $Q < k$ is defined canonically for γ for $k \in \mathbb{R}$. (k can be negative.) The region is a parabolic cylinder in the parabolic coordinate system of γ . We will call this a *parabolic cylinder* for γ .

Remark 3.1. The expression (3.5) can change by conjugation by a dilatation so that $\mu = \pm 1$. However, a dilatation is not a Lorentz isometry.

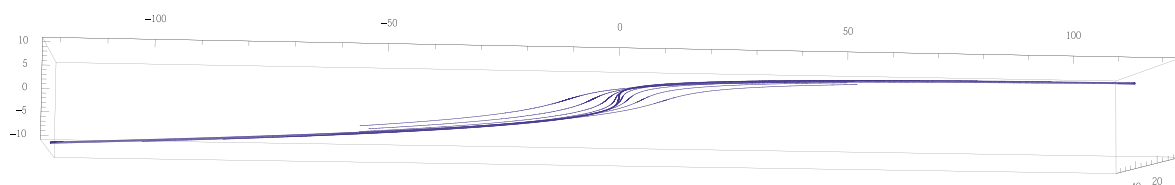


Figure 1: A number of orbits drawn horizontally.

Definition 3.3. A *semicircle tangent* to $\partial\mathbb{S}_+$ at $p \in \partial\mathbb{S}_-$ is the closure of a component of $S \setminus \{p, p_-\}$ of the great circle S tangent to $\partial\mathbb{S}_+$ at p which does not meet $\mathbb{S}_+$. An *accordant great segment* ζ_p to $\partial\mathbb{S}_+$ is an open semicircle tangent to $\partial\mathbb{S}_+$ starting from x in the direction of the orientation of $\partial\mathbb{S}_+$. (See [16, Section 3.4].)

We may refer to them as being *positively oriented* since we need to alter the construction when we change the orientation.

Remark 3.2. In the parabolic coordinate system of E for a parabolic γ , $\mathbb{S}_+$ is given by $(x, y, z, 0)$ in $\mathbb{S}$ with $y^2 - 2xz < 0$ with $x > 0$. Then it is easily shown that

$$\overline{\langle (1, 0, 0, 0) \rangle \langle (0, 1, 0, 0) \rangle} \cup \overline{\langle (0, 1, 0, 0) \rangle \langle (-1, 0, 0, 0) \rangle}$$

is the accordant great segment $\text{Cl}(\zeta_{(1,0,0,0)})$ to the boundary of $\mathbb{S}_+$ with the induced orientation.

For the following if γ is not accordant, we need to use γ^{-1} .

Proposition 3.10. Let γ be accordant parabolic transformation. We use the parabolic coordinate system of γ so that γ is of the form (3.5) with $\mu > 0$. Then the following hold:

- $\langle \gamma \rangle$ acts properly on E .
- The orbit $\{\gamma^n(p)\}$, $p \in E$, converges to the unique fixed point x_γ in $\partial\mathbb{S}_+$ as $n \rightarrow \infty$ and converges to its antipode $x_{\gamma^{-1}} \in \partial\mathbb{S}_-$ as $n \rightarrow -\infty$.
- The orbit lies on the parabolic cylinder

$$P_p := \{x \in E : Q(x) = Q(p)\},$$

where γ acts on.

- The set of lines in E parallel to the vector $\mathbf{x}_\gamma$ in the direction of $\mathbf{x}_\gamma$ foliates each parabolic cylinder and gives us equivalence classes. The space $P_p/\sim$ can be identified with a real line $\mathbb{R}$. The action of γ on $P_p/\sim$ corresponds to a translation action on $\mathbb{R}$.
- P_p can be compactified to a compact subspace in $\mathbb{S}^3$ homeomorphic to a 2-sphere by adding the great segment $\text{Cl}(\zeta_{\mathbf{x}_\gamma})$ accordant to $\partial\mathbb{S}_+$.

Proof. We have x_γ equal to $(1, 0, 0, 0)$ in this coordinate system. The properness follows since $\mu t^3/6$ dominates all other terms. The second item follows since F_2 is an invariant. Since F_2 is Φ_t -invariant, it follows that γ acts on the parabolic cylinder determined by F_2 . The third item follows by projecting to the z -value. The fourth item is straightforward from the third item.

Let H_0 be a great sphere given by $x = 0$ in $\mathbb{S}^3$. For each line l in the parabolic cylinder, $\{\gamma^t(\text{Cl}(l)) \cap H_0 : t \in \mathbb{R}\}$ is a parabola compactified by a single point $(0, 1, 0, 0)$ as we can see using (3.5). Let H_+ be the upper hemisphere bounded by H_0 and H_- the lower hemisphere. We have geometric convergence:

$$\begin{aligned} \{\gamma^t(\text{Cl}(l)) \cap H_+\} &\rightarrow \overline{\langle (1, 0, 0, 0) \rangle \langle (0, 1, 0, 0) \rangle} \quad \text{as } t \rightarrow \infty \text{ or } t \rightarrow -\infty, \\ \{\gamma^t(\text{Cl}(l)) \cap H_-\} &\rightarrow \overline{\langle (-1, 0, 0, 0) \rangle \langle (0, 1, 0, 0) \rangle} \quad \text{as } t \rightarrow \infty \text{ or } t \rightarrow -\infty. \end{aligned}$$

Hence, by Remark 3.2,

$$\{\gamma^t(\text{Cl}(l))\} \rightarrow \text{Cl}(\zeta_{(1,0,0,0)}) \quad \text{as } t \rightarrow \infty \text{ or } t \rightarrow -\infty.$$

For any sequence of points x_i on P_p , $x_i \in \gamma^{t_i}(\text{Cl}(l))$ for some $t_i \in \mathbb{R}$. If $|t_i|$ is bounded, then $\{x_i\}$ can accumulate only on P_p . If $|t_i|$ is unbounded, then $\{x_i\}$ can accumulate to $\text{Cl}(\zeta_{x_\gamma})$ by the above paragraph. The final part follows. $\square$

3.2 Proper affine deformations and Margulis and Charette–Drumm invariants

Let S be a complete orientable hyperbolic surface with $\chi(S) < 0$ and possibly some cusps. Let

$$h : \pi_1(S) \rightarrow \mathrm{SO}(2, 1)^o$$

be a discrete irreducible faithful representation. Now, the image is allowed to have parabolic elements. Each nonparabolic element γ of $\pi_1(S) \setminus \{I\}$ is represented by the unique closed geodesic in $S := \mathbb{S}_+/h(\pi_1(S))$ and hence is hyperbolic. Let Γ be a proper affine deformation of $h(\pi_1(S))$. For nonparabolic $\gamma \in \Gamma \setminus \{I\}$, we define

- $\mathbf{x}_+(\gamma)$ as an eigenvector of $\mathcal{L}(\gamma)$ in the causally null directions with the eigenvalue > 1 ,
- $\mathbf{x}_-(\gamma)$ as one of $\mathcal{L}(\gamma)$ with the eigenvalue < 1 , and
- $\mathbf{x}_0(\gamma)$ as a space-like positive eigenvector of $\mathcal{L}(\gamma)$ of the eigenvalue 1 which is given by

$$\mathbf{x}_0(\gamma) = \frac{\mathbf{x}_-(\gamma) \times \mathbf{x}_+(\gamma)}{\|\mathbf{x}_-(\gamma) \times \mathbf{x}_+(\gamma)\|}.$$

Here, $\times$ is the Lorentzian cross-product, and $\mathbf{x}_+(\gamma)$ and $\mathbf{x}_-(\gamma)$ are well-defined up to choices of sizes; however, $\mathbf{x}_0(\gamma)$ is well-defined since it has a unit Lorentz norm. They define the Margulis invariant

$$\alpha(\gamma) = B(\gamma(x) - x, \mathbf{x}_0(\gamma)), \quad x \in E, \quad (3.7)$$

where the value is independent of the choice of x .

In general, an *affine deformation* of a homomorphism $h : \pi_1(S) \rightarrow \mathrm{SO}(2, 1)$ is a homomorphism

$$h_{\mathbf{b}} : \pi_1(S) \rightarrow \mathbf{Isom}^+(E)$$

given by $h_{\mathbf{b}}(g)(x) = h(g)x + \mathbf{b}(g)$ for a cocycle $\mathbf{b} : \pi_1(S) \rightarrow \mathbb{R}^{2,1}$ in $Z^1(\pi_1(S), \mathbb{R}_h^{2,1})$. The vector space of co-boundary is denoted by $B^1(\pi_1(S), \mathbb{R}_h^{2,1})$. As usual, we define

$$H^1(\pi_1(S), \mathbb{R}_h^{2,1}) := \frac{Z^1(\pi_1(S), \mathbb{R}_h^{2,1})}{B^1(\pi_1(S), \mathbb{R}_h^{2,1})}.$$

Let $[\mathbf{u}]$ be the class of a cocycle in $H^1(\pi_1(S), \mathbb{R}_h^{2,1})$ with $\mathbf{u} \in Z^1(\pi_1(S), \mathbb{R}_h^{2,1})$. Let $h_{\mathbf{u}}$ denote the affine deformation of h according to a cocycle $\mathbf{u}$ in $[\mathbf{u}]$, and let $\Gamma_{\mathbf{u}}$ be the affine deformation $h_{\mathbf{u}}(\pi_1(S))$. There is a function $\alpha_{\mathbf{u}} : \pi_1(S) \setminus \mathcal{P}_{\pi_1(S)} \rightarrow \mathbb{R}$ with the following properties:

- $\alpha_{\mathbf{u}}(\gamma^n) = n\alpha_{\mathbf{u}}(\gamma)$, $n \in \mathbb{Z}$.
- $\alpha_{\mathbf{u}}(\gamma) = 0$ if and only if $h_{\mathbf{u}}(\gamma)$ fixes a point.
- The function $\alpha_{\mathbf{u}}$ depends linearly on $\mathbf{u}$.
- If $h_{\mathbf{u}}(\pi_1(S))$ acts properly and freely on E , then $|\alpha_{\mathbf{u}}(\gamma)|$ is the Lorentz length of the unique space-like closed geodesic in $E/h_{\mathbf{u}}(\pi_1(S))$ corresponding to γ . (See Goldman, Labourie, and Margulis [28].)

Charette and Drumm generalized the Margulis invariants for parabolic elements in [6], where the values are given only as “positive” or “negative”. Let $g \in \Gamma$ be a parabolic or hyperbolic element of an affine deformation of a linear group in $\mathrm{SO}(2, 1)^o$.

Definition 3.4. An eigenvector $\mathbf{v}$ of eigenvalue 1 of a linear hyperbolic or parabolic transformation g is said to be *positive* relative to g if $\{\mathbf{v}, \mathbf{x}, \mathcal{L}(g)\mathbf{x}\}$ is positively oriented when $\mathbf{x}$ is any null or time-like vector which is not an eigenvector of g .

It is easy to verify $\mathbf{v}$ is positive with respect to g if and only if $-\mathbf{v}$ is positive with respect to g^{-1} . Let $F(\mathcal{L}(g))$ be the oriented 1-dimensional space of eigenvectors of $\mathcal{L}(g)$ of eigenvalue 1. Define $\tilde{\alpha}(\gamma) : F(\mathcal{L}(\gamma)) \rightarrow \mathbb{R}$ by

$$\tilde{\alpha}(\gamma)(\cdot) = B(\gamma(x) - x, \cdot),$$

where $x \in E$ is any chosen point. Drumm [22] also shows

$$\alpha(\gamma) = \tilde{\alpha}(\gamma)(\mathbf{x}^0(\mathcal{L}(\gamma)))$$

provided γ is hyperbolic.

By Definition 3.4, components of $F(\mathcal{L}(\gamma)) \setminus \{0\}$ have well-defined signs. We say that the *Charette–Drumm invariant* $cd(\gamma) > 0$ if $\tilde{\alpha}(\gamma)$ is positive on positive eigenvectors in $F(\mathcal{L}(\gamma)) \setminus \{0\}$. Also, we note $cd(\gamma) > 0$ if and only if $cd(\gamma^{-1}) > 0$.

Lemma 3.11 (Charette–Drumm [6]). *Let $\gamma \in \Gamma$ be a parabolic or hyperbolic element. Then the following holds:*

- $\tilde{\alpha}(\gamma) = B(\gamma(x) - x, \cdot)$ is independent of the choice of x .
- $\tilde{\alpha}(\gamma) = 0$ if and only if γ has a fixed point in E .
- For any $\eta \in \mathbf{Aff}(E)$, $\tilde{\alpha}(\eta\gamma\eta^{-1})(\eta(\mathbf{v})) = \tilde{\alpha}(\gamma)(\mathbf{v})$ for $\mathbf{v} \in F(\mathcal{L}(\gamma))$.
- For any $n \in \mathbb{Z}$, $\mathbf{v} \in F(\mathcal{L}(\gamma))$, $\tilde{\alpha}(\gamma^n)(\mathbf{v}) = |n|\tilde{\alpha}(\gamma)(\mathbf{v})$.

In the parabolic coordinate system of γ , we obtain

$$\tilde{\alpha}(\gamma)(x, 0, 0) = -\mu tx \quad (3.8)$$

for μ, t given for γ as in (3.5) in Section 3.1.

Lemma 3.12. *Let γ be defined by (3.5) for $t > 0$ in the accordant parabolic coordinate system for γ . Then the following holds:*

- $\mu > 0$ if and only if γ has a positive Charette–Drumm invariant.
- $\mu < 0$ if and only if γ has a negative Charette–Drumm invariant.
- $\mu \neq 0$ if and only if $\langle \gamma \rangle$ acts properly on E .

Proof. We prove the first item: Choose $\mathbf{x} = (a, 0, c)$ with $ac > 0$, $a > 0$ so that $\mathbf{x}$ is a causal time-like vector. Then $\{\mathbf{i}, \mathbf{x}, \mathcal{L}(\gamma)\mathbf{x}\}$ is a negatively oriented frame, and $\mathbf{i}$ is the negative null eigenvector of $\mathcal{L}(\gamma)$ by Definition 3.4. By (3.8), the first item follows. The second item follows by the contrapositive of the first item. The final part follows by Proposition 3.10 and Lemma 3.11 and reversing the orientation of E . $\square$

3.3 Parabolic region and two transversal foliations on them

3.3.1 Parabolic regions

Let g be a parabolic element with the expression (3.5) for $t > 0$ under the parabolic coordinate system of Section 3.1.2. Assume that the Charette–Drumm invariant of g is positive. That is, $\mu > 0$ by Lemma 3.12. Recall from Section 3.1.2 that

$$F_2(x, y, z) = z^2 - 2\mu y \quad \text{and} \quad F_3(x, y, z) = z^3 - 3\mu yz + 3\mu^2 x$$

are invariants of g^t . Recall that $\Phi(t) : E \rightarrow E$ is generated by a vector field

$$\phi := y\partial_x + z\partial_y + \mu\partial_z$$

with the square of the Lorentzian norm $\|\phi\|^2 = z^2 - 2\mu y$.

The equation $F_2(x, y, z) = T$ gives us a parabolic cylinder P_T in the x -direction with the parabola in the yz -plane. The vector field ϕ satisfies

$$\phi(x, y_0, 0) = (y_0, 0, \mu) \quad \text{for all } x \quad \text{and} \quad T = -2\mu y_0.$$

Since we are looking for a g^t -invariant ruled surface, we take a line l tangent to P_T in the direction of $\mathbf{x} = (a, 0, c)$ starting at $(0, y_0, 0)$. Since $(\mathbf{x}) \in \mathbb{S}_+$ by the premise, we obtain $2ac > 0$ with $a > 0$, $c > 0$ under the parabolic coordinate system with the quadratic form (3.4). (See Figure 2.)

We define $\Psi(t, s) = g^t(l(s))$ so that

$$l(s) = (0, y_0, 0) + s(a, 0, c) = (sa, y_0, sc), \quad \phi(l(s)) = (y_0, sc, \mu).$$

Thus, ϕ is never parallel to $(a, 0, c)$ unless $s = 0$. We choose $(a, 0, c)$, $c \neq 0$, not parallel to $(y_0, 0, \mu)$, i.e.,

$$\frac{a}{c} \neq \frac{y_0}{\mu}.$$

Then $\phi|l$ is never parallel to the tangent vectors to l . Since $Dg^t(\phi) = \phi$, ϕ is never parallel to tangent vectors to $g^t(l)$, it follows that Ψ is an immersion in E .

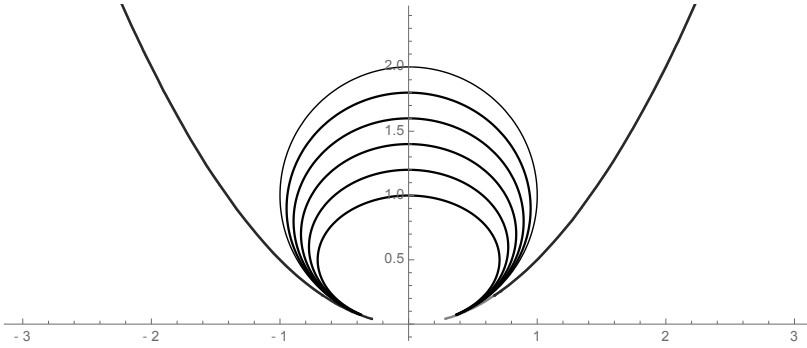


Figure 2: This shows the projective action of a 1-dimensional parabolic group on $\mathbb{S}_+$ with boundary represented as a parabola. We use the affine patch where $x > 0$ in the coordinate system. We normalize the homogeneous coordinates by setting $x = 1$. The parabola $2z = y^2$ describes the boundary of $\mathbb{S}_+$ given by $y^2 < 2xz$. See Remark 3.2.

Let $\mathcal{H}_{s_0, \kappa_1, \kappa_2}$ be the space of compact segments u passing E with the following properties:

- u has an antipodal pair of endpoints in $\mathbb{S}_+$ and in the antipodal set $\mathbb{S}_-$, and
- $u \cap E$ is equivalent under g^t for some t to a line $l(s)$ given by

$$l(s) = (sa, y_0, sc) \quad (3.9)$$

for $y_0 \geq s_0$, $a, c > 0$, $\frac{\kappa_1 a}{c} \leq \frac{y_0}{\mu} \leq \frac{\kappa_2 a}{c}$, and $a^2 + c^2 = 1$ for some pair $0 < \kappa_1 \leq \kappa_2 < 1$ and $s_0 > 0$.

This space has a metric coming from the Hausdorff metric $\mathbf{d}_H$.

We will prove the following in Appendix A.

Theorem 3.13. *Let $g, \mathcal{L}(g) \in \mathrm{SO}(2, 1)^0$, be an accordant parabolic element acting properly on E with the positive Charette–Drumm invariant. Let l be a line in $\mathcal{H}_{s_0, \kappa_1, \kappa_2}$ for the parabolic coordinate system for g . Then:*

- For each time-like line l in the ruling of S ,

$$\{g^t(\mathrm{Cl}(l))\} \rightarrow \mathrm{Cl}(\zeta_{\mathbf{x}_\infty}) \quad \text{as } t \rightarrow \infty \text{ and } t \rightarrow -\infty$$

geometrically.

- For any ϵ - $\mathbf{d}$ -neighborhood N of $\mathrm{Cl}(\zeta_{\mathbf{x}_\infty}) \subset \mathbb{S}$, we can find such a ruled surface S in $N \cap E$.
- There exists a $\{g^t : t \in \mathbb{R}\}$ -invariant surface S ruled by time-like lines containing l^0 properly embedded in E with boundary

$$\mathrm{Cl}(S) \setminus S = \{g^t(\mathbf{x}) : t \in \mathbb{R}\} \cup \{g^t(\mathbf{x}_-) : t \in \mathbb{R}\} \cup \mathrm{Cl}(\zeta_{\mathbf{x}_\infty})$$

for a point $\mathbf{x} \in \mathbb{S}_+$, and $\mathbf{x}_\infty$ is a parabolic fixed point of g in $\partial\mathbb{S}_+$ respectively. Furthermore, there exists a domain R homeomorphic to a 3-cell in E whose topological boundary in the hemisphere $\mathcal{H}$ equals $\mathrm{Cl}(S)$. Also, $R/\langle g \rangle$ is homeomorphic to a solid torus.

Definition 3.5. In Theorem 3.13, the surface denoted by S is called a *parabolic ruled surface*. (Compare with parabolic cylinders in Section 3.1.2.) The open region R in E bounded by a parabolic ruled surface is called the *parabolic region*. The generator of the parabolic group acting on a parabolic ruled surface fixes a point $p \in \partial\mathbb{S}_+$.

An immersed image $S/\langle g \rangle$ of the surfaces in a manifold E/Γ is also called a *parabolic ruled surface*. The embedded image $R/\langle g \rangle$ of R in a manifold E/Γ is called a *parabolic region*.

We can choose the parabolic surface and the parabolic regions so that they are in the ϵ - $\mathbf{d}$ -neighborhood N of $\bigcup_{x \in a} \mathrm{Cl}(\zeta_x) \subset \mathbb{S}$ by the last item of Theorem 3.13. Then we call the parabolic region $\frac{1}{\epsilon}$ -far away from the compact parts. The isometrically embedded images of such surfaces in E/Γ or E are described in the same manner.

3.3.2 Two transversal foliations

Assume

$$0 < \kappa_1 \leq \kappa_2 < 1.$$

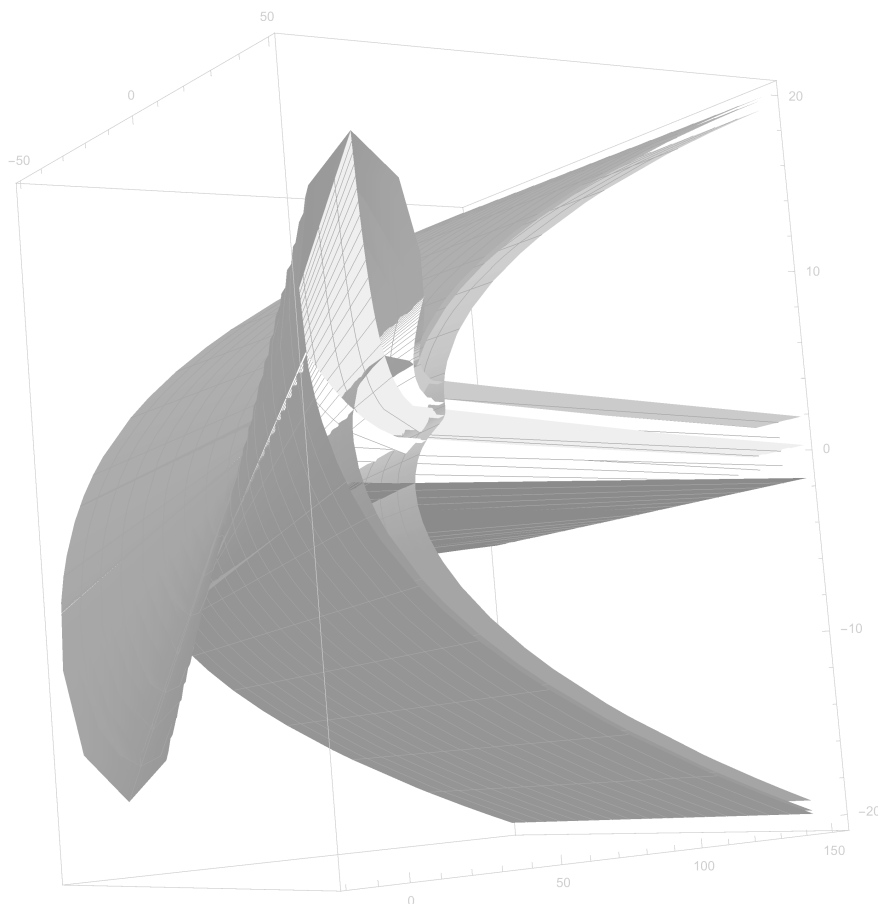


Figure 3: Three darker leaves of foliation $\mathcal{S}_{f,r_0}$ and five transversal light-gray leaves of $\mathcal{D}_{f,r_0}$, where $f(\rho) = \frac{3}{4} \frac{r}{\sqrt{1-r^2}}$ and $\mu = 1$. See [10].

Let $f : (0, 1) \rightarrow \mathbb{R}$ be a strictly increasing smooth function satisfying

$$\kappa_1 \mu \frac{r}{\sqrt{1-r^2}} \leq f(\rho) \leq \kappa_2 \mu \frac{r}{\sqrt{1-r^2}}.$$

Let $\mathcal{H}_f$ be the space of compact segments u passing E with the following properties:

- u has an antipodal pair of endpoints in $\mathbb{S}_+$ and in $\mathbb{S}_-$,
- $u \cap E$ is equivalent under g^t for some t to a line $l(s)$ given by $l_{f,r}(s) = (sa, y_f(\rho), sc)$, $s \in \mathbb{R}$, where

$$y_f(\rho) := f(\rho), a = r, c = \sqrt{1-r^2}, r \in (0, 1).$$

For fixed $r \in (0, 1)$, let $S_{f,r}$ denote the parabolic ruled surface given by

$$\bigcup_{t,s \in \mathbb{R}} g^t(l_{f,r}(s)).$$

Define $D_{f,r_0,t}$ for fixed $t \in \mathbb{R}$ to denote the surface

$$\bigcup_{s \in \mathbb{R}, r \in [r_0, 1)} g^t(l_{f,r}(s)).$$

We will prove the following in Appendix A.

Theorem 3.14. *Let $r_0 \in (0, 1)$. Then the following hold:*

- *The surfaces $S_{f,r}$ for $r \in [r_0, 1)$ are properly embedded leaves of a foliation $\tilde{\mathcal{S}}_{f,r_0}$ of the region R_{f,r_0} , closed in E , bounded by S_{f,r_0} where g^t acts on.*

- $\{D_{f,r_0,t} : t \in \mathbb{R}\}$ is the set of properly embedded leaves of a foliation $\tilde{\mathcal{D}}_{f,r_0}$ of R_{f,r_0} by disks meeting $S_{f,r}$ for each $r, r_0 < r < 1$, transversally.
 - $g^{t_0}(D_{f,r_0,t}) = D_{f,r_0,t+t_0}$.
 - $D_{f,r_0,t'} \cap D_{f,r_0,t} = \emptyset$ for $t, t', t \neq t'$.
 - $\text{Cl}(D_{f,r_0,t}) \cap \mathbb{S}_+$ is given as a geodesic ending at the parabolic fixed point of g .

Remark 3.3. The quotient $R_{f,r_0}/\langle g \rangle$ is foliated by the foliation $\mathcal{S}_{f,r_0}$ induced by $\tilde{\mathcal{S}}_{f,r_0}$ and $\mathcal{D}_{f,r_0}$ induced by $\tilde{\mathcal{D}}_{f,r_0}$. The leaves of $\mathcal{S}_{f,r_0}$ are annuli of the form $S_{f,r}/\langle g \rangle$ and the leaves of $\mathcal{D}_{f,r_0}$ are the embedded images of $D_{f,r_0,t}$ for $t \in \mathbb{R}$. The embedded image of $R_{f,r_0}/\langle g \rangle$ in E/Γ are foliated by induced foliations to be denoted by the same names. (See Figure 3.)

4 Orbits of proper affine deformations and translation vectors

We now come to the most important section of this paper. In this section, we assume $\mathcal{L}(\Gamma) \subset \text{SO}(2, 1)^0$ and work with Criterion 1.1 only without assuming the properness of the Γ -action. In Sections 4.1 and 4.2, we will present the objects of our discussion. In Section 4.3, we will discuss the Anosov properties of geodesic flows extended to a flat bundle $\mathbf{V}$. In Section 4.4, we will put the translation cocycle into an integral form. In Section 4.5, we will compute the translation parts of the holonomy representations. Theorem 4.8 is the main result where we will give an outline of the proof. We will prove the converse part of Theorem 1.5 at the end of Section 4.5. In Section 4.6, we obtain Corollary 4.9 which discusses all the accumulation points of Γ .

4.1 Convergence sequences

Let $g \in \Gamma$. Let $\lambda_1(g)$ denote the largest eigenvalue of $\mathcal{L}(g)$, which has eigenvalues $\lambda_1(g)$, 1, $1/\lambda_1(g)$. Note the relation

$$l_{\mathbb{S}_+}(g) = \log\left(\frac{\lambda_1(g)}{1/\lambda_1(g)}\right) = 2 \log \lambda_1(g). \quad (4.1)$$

Recall that Γ acts as a convergence group of a circle $\partial\mathbb{S}_+$. That is, if g_i is a sequence of mutually distinct elements of Γ , then there exists a subsequence g_{j_i} and points a, r in $\partial\mathbb{S}_+$ so that

- as $i \rightarrow \infty$, $\{g_{j_i}|\partial\mathbb{S}_+ \setminus \{r\}\}$ uniformly converges to a constant map with value a on every compact subset, and
 - as $i \rightarrow \infty$, $\{g_{j_i}^{-1}|\partial\mathbb{S}_+ \setminus \{a\}\}$ uniformly converges to a constant map with value r on every compact subset.
- Call a the *attractor* of $\{g_{j_i}\}$ and r the *repeller* of $\{g_{j_i}\}$. Here, a may or may not equal r . (See [1] for detail.) We call the sequence g_i satisfying the above properties the *convergence sequence*.

For a point $x \in E$, let $\Gamma(x)$ denote the orbit of x . We define the *Lorentzian limit set*

$$\Lambda_\Gamma := \bigcup_{x \in E} (\text{Cl}(\Gamma(x)) \setminus \Gamma(x)).$$

By the properness of the action, we obviously have:

Lemma 4.1. *Let Γ be a proper affine free group with rank ≥ 2 . Then Λ_Γ is a subset of $\mathbb{S}$.*

Recall $\mathbb{S}_0 = \mathbb{S} \setminus \mathbb{S}_+ \setminus \mathbb{S}_-$. For each point x of $\partial\mathbb{S}_+$, there exists an accordant great segment ζ_x (see Definition 3.3). We denote by $\Pi_+ : \mathbb{S}_0 \rightarrow \partial\mathbb{S}_+$ the map given by sending every point of $\text{Cl}(\zeta_x)$ to x . This is a fibration by [16, Section 3.4].

Let $\Lambda_{\Gamma, \mathbb{S}_+} \subset \text{Cl}(\mathbb{S}_+)$ be the limit set of the discrete faithful Fuchsian group action on $\mathbb{S}_+$ by $\mathcal{L}(\Gamma)$. (See [2].)

One of our main results of the section is Corollary 4.9 also giving us:

Theorem 4.2. *Let Γ be a proper affine free group of rank greater than or equal to 2 with or without parabolics. Assume $\mathcal{L}(\Gamma) \subset \text{SO}(2, 1)^0$. Then $\Lambda_\Gamma \subset \Pi_+^{-1}(\Lambda_{\Gamma, \mathbb{S}_+})$.*

4.2 The bundles $\mathbf{E}$ over $\mathbf{US}$

Let $\mathbf{US}_+$ denote the unit tangent bundle of $\mathbb{S}_+$, i.e., the space of direction vectors on $\mathbb{S}_+$. For any subset A of $\mathbb{S}_+$, we let $\mathbf{UA}$ denote the inverse image of A in $\mathbf{US}_+$ under the projection. The projection $\Pi_S : \mathbf{US} \rightarrow S$ lifts to the projection $\Pi_{\mathbb{S}_+} : \mathbf{US}_+ \rightarrow \mathbb{S}_+$.

Let $\Gamma := h_u(\pi_1(S))$ be a proper affine hyperbolic free group of rank ≥ 2 . We note that Γ acts on $\mathbf{US}_+$ as a deck transformation group over $\mathbf{US}$. An element $\gamma \in \Gamma$ goes to the differential map $D\gamma : \mathbf{US}_+ \rightarrow \mathbf{US}_+$ defined by

$$D\gamma(x, \mathbf{u}) = \left(\gamma(x), \frac{d\gamma(\beta(t))}{dt} \Big|_{t=0} \right), \quad x \in \mathbb{S}_+, \mathbf{u} \in \mathbf{U}_x \mathbb{S}_+$$

where $\beta(t)$ is a unit speed geodesic with $\beta(0) = x$ and $\dot{\beta}(0) = \mathbf{u}$. Goldman, Labourie and Margulis in [28] constructed a flat affine bundle $\mathbf{E}$ over the unit tangent bundle $\mathbf{US}$ of S . They took the quotient of $\mathbf{US}_+ \times \mathbf{E}$ by the diagonal action given by

$$\gamma(\mathbf{v}, x) = (D\gamma(x), \gamma(\mathbf{v})), \quad x \in \mathbf{US}_+, \mathbf{v} \in \mathbf{E}$$

for a deck transformation $\gamma \in \Gamma$. The cover $\mathbf{US}_+ \times \mathbf{E}$ of $\mathbf{E}$ is denoted by $\hat{\mathbf{E}}$ and is identical with $\mathbf{E} \times \mathbf{US}_+$. We denote by

$$\Pi_E : \hat{\mathbf{E}} = \mathbf{US}_+ \times \mathbf{E} \rightarrow \mathbf{E}$$

the projection.

4.3 The Anosov property of the geodesic flow

We denote the standard 3-vectors by

$$\mathbf{i} := (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1).$$

Definition 4.1. We say that two positive-valued functions $f(t)$ and $g(t)$, $t \in \mathbb{R}$, are compatible or satisfy $f \cong g$ if there exists $C > 1$ such that

$$\frac{1}{C} \leq \frac{f(t)}{g(t)} \leq C \quad \text{for } t \in \mathbb{R}.$$

Given $(\langle \mathbf{x} \rangle, \mathbf{u}) \in \mathbf{US}_+$,

- we denote by $l(\langle \mathbf{x} \rangle, \mathbf{u}) \subset \mathbb{S}_+$ the oriented complete geodesic passing through $\langle \mathbf{x} \rangle$ in the direction of $\mathbf{u}$,
- we denote by $\mathbf{v}_{+, (\langle \mathbf{k} \rangle, \mathbf{j})}$ and $\mathbf{v}_{-, (\langle \mathbf{k} \rangle, \mathbf{j})}$ the respective null vectors $\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$ and $\frac{-1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$ in the directions of the forward and backward endpoints of the oriented complete geodesic $l(\langle \mathbf{k} \rangle, \mathbf{j}) \subset \mathbb{S}_+$,
- we define $\mathbf{v}_{+, (\langle \mathbf{x} \rangle, \mathbf{u})}$ and $\mathbf{v}_{-, (\langle \mathbf{x} \rangle, \mathbf{u})}$ respectively as the images of $\mathbf{v}_{+, (\langle \mathbf{k} \rangle, \mathbf{j})}$ and $\mathbf{v}_{-, (\langle \mathbf{k} \rangle, \mathbf{j})}$ under an element $g \in \mathrm{SO}(2, 1)^o$ provided

$$g(\langle \mathbf{k} \rangle) = \langle \mathbf{x} \rangle \quad \text{and} \quad g(\mathbf{j}) = \mathbf{u}.$$

The well-definedness of these objects follows since there is a one-to-one correspondence of $\mathbf{US}_+$ with $\mathrm{SO}(2, 1)^o$.

Definition 4.2. We define $\mathbf{V}$ as the quotient space of $\tilde{\mathbf{V}} := \mathbf{US}_+ \times \mathbb{R}^{2,1}$ under the diagonal action defined by

$$\gamma(x, \mathbf{v}) = (D\gamma(x), \mathcal{L}(\gamma)(\mathbf{v})), \quad x \in \mathbf{US}_+, \mathbf{v} \in \mathbb{R}^{2,1}, \gamma \in \Gamma.$$

We will also need to define $\tilde{\mathcal{V}} := \mathbb{S}_+ \times \mathbb{R}^{2,1}$ and the quotient bundle $\mathcal{V} := \tilde{\mathcal{V}}/\Gamma$ where the action is given by

$$\gamma(x, \mathbf{v}) = (\gamma(x), \mathcal{L}(\gamma)(\mathbf{v})), \quad x \in \mathbb{S}_+, \mathbf{v} \in \mathbb{R}^{2,1}, \gamma \in \Gamma.$$

The vector bundle $\mathbf{V}$ has a fiberwise Riemannian metric $\|\cdot\|_{\text{fiber}}$ where Γ acts as an isometry group. At $(\langle \mathbf{x} \rangle, \mathbf{u}) \in \mathbf{US}_+$ with $\mathbf{x}$ satisfying $B(\mathbf{x}, \mathbf{x}) = -1$, we give as a basis

$$\left\{ \mathbf{v}_{+, (\langle \mathbf{x} \rangle, \mathbf{u})}, \mathbf{v}_{-, (\langle \mathbf{x} \rangle, \mathbf{u})}, \mathbf{v}_{0, (\langle \mathbf{x} \rangle, \mathbf{u})} := \frac{\mathbf{v}_{-, (\langle \mathbf{x} \rangle, \mathbf{u})} \times \mathbf{v}_{+, (\langle \mathbf{x} \rangle, \mathbf{u})}}{\|\mathbf{v}_{-, (\langle \mathbf{x} \rangle, \mathbf{u})} \times \mathbf{v}_{+, (\langle \mathbf{x} \rangle, \mathbf{u})}\|} \right\} \quad (4.2)$$

for the fiber over $((\mathbf{x}), \mathbf{u})$, where $\times$ is the Lorentzian cross product. We choose the positive definite metric $\|\cdot\|_{\text{fiber}}$ on $\tilde{\mathbf{V}}$ so that the above vector frame is orthonormal at the fiber of $\tilde{\mathbf{V}}$ over $((\mathbf{x}), \mathbf{u})$. The metric is $\text{SO}(2, 1)^0$ -invariant on US_+ . Thus, this induces a metric $\|\cdot\|_{\text{fiber}}$ on $\mathbf{V}$ as well.

Let $\tilde{\mathbf{V}}_\omega$ be the 1-dimensional subbundle of $\text{US}_+ \times \mathbb{R}^{2,1}$ containing $\mathbf{v}_{\omega, ((\mathbf{x}), \mathbf{u})}$ for each ω , $\omega = +, -, 0$. It is redundant to say that $\mathbf{v}_{\omega, ((\mathbf{x}), \mathbf{u})}$ is a fiber over the point $((\mathbf{x}), \mathbf{u})$ in US_+ for each ω .

We define a so-called *neutral map*

$$\tilde{\nu} : \text{US}_+ \rightarrow \text{US}_+ \times \mathbb{R}^{2,1}$$

given by $((\mathbf{x}), \mathbf{u}) \mapsto \mathbf{v}_{0, ((\mathbf{x}), \mathbf{u})}$. Here, $\tilde{\nu}$ is an $\text{SO}(2, 1)^0$ -equivariant map. By action of the isometry group Γ , we obtain a *neutral section*

$$\nu : \text{US} \rightarrow \mathbf{V}$$

by using the $\text{SO}(2, 1)^0$ -equivariance of the map. Hence, $\tilde{\mathbf{V}}_0$ coincides with the subspace generated by the image of the neutral section $\tilde{\nu}$.

For any smooth map $g : \text{US}_+ \rightarrow \text{US}_+$ or $\mathbb{S}_+ \rightarrow \mathbb{S}_+$, we denote by $\text{ID}g$ the induced automorphism $\text{US}_+ \times \mathbb{E}$ acting trivially on the $\mathbb{E}$ -factor.

Recall from [28, Section 4.4] the geodesic flow $\Psi_t : \text{US}_+ \rightarrow \text{US}_+$ denote the geodesic flow on US_+ defined by the hyperbolic metric. Let

$$\text{ID}\Psi_t : \text{US}_+ \times \mathbb{R}^{2,1} \rightarrow \text{US}_+ \times \mathbb{R}^{2,1}$$

denote the Goldman–Labourie–Margulis flow. This acts trivially on the second factor and as the geodesic flow on US_+ . The bundle $\mathbf{V}$ splits into three Ψ_t -invariant line bundles $\mathbf{V}_+$, $\mathbf{V}_-$ and $\mathbf{V}_0$, which are images of $\tilde{\mathbf{V}}_+$, $\tilde{\mathbf{V}}_-$ and $\tilde{\mathbf{V}}_0$. Our choice of $\|\cdot\|$ shows that $\text{ID}\Psi_t$ acts as uniform contractions in $\mathbf{V}_+$ as $t \rightarrow \infty$, $-\infty$, i.e.,

$$\begin{aligned} \|\text{ID}\Psi_t(\mathbf{v}_+)\|_{\text{fiber}} &\cong \exp(-t)\|\mathbf{v}_+\|_{\text{fiber}} & \text{for } \mathbf{v}_+ \in \tilde{\mathbf{V}}_+, \\ \|\text{ID}\Psi_t(\mathbf{v}_-)\|_{\text{fiber}} &\cong \exp(t)\|\mathbf{v}_-\|_{\text{fiber}} & \text{for } \mathbf{v}_- \in \tilde{\mathbf{V}}_-, \\ \|\text{ID}\Psi_t(\mathbf{v}_0)\|_{\text{fiber}} &\cong \|\mathbf{v}_0\|_{\text{fiber}} & \text{for } \mathbf{v}_0 \in \tilde{\mathbf{V}}_0. \end{aligned} \quad (4.3)$$

Here, k in [28] equals 1 since we can explicitly compute k from the framing above. The signs are different from [28] because we have slightly different objects. The fiberwise metric on US_+ is not dependent on the group Γ itself. See [28, last paragraph of Section 4.4].

Remark 4.1. The induced geodesic flow on $\mathbb{S}$ is denoted by Ψ_t and the induced action on $\mathbf{V}$ by $\text{ID}\Psi_t$. We may think of translating the picture of the flat bundle over US_+ to the bundle over US . As a bundle over US , $\text{ID}\Psi_t$ contracts and expands uniformly for $\mathbf{V}_\pm$ with respect to $\|\cdot\|_{\text{fiber}}$. However, in the picture over US_+ , $\text{ID}\Psi_t$ is the identity between fibers and objects lifted from $\mathbf{V}$ will uniformly increase or decrease exponentially with respect to any fixed Euclidean metric $\|\cdot\|_E$ on $\tilde{\mathbf{V}}$. (See Figure 4.)

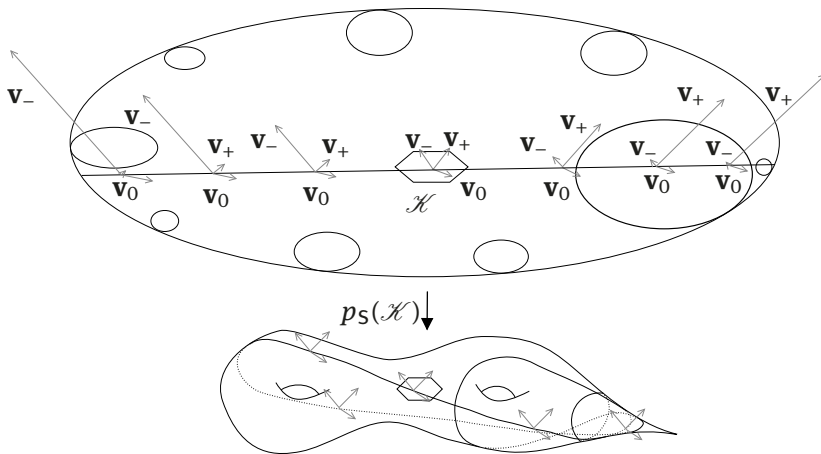


Figure 4: The frames on US_+ and on US . The circles bound horodisks covering the cusp neighborhoods below. The compact set $\mathcal{K}$ is some small compact set where the closed geodesics pass through. We drew only one closed geodesic.

Denote by

$$\tilde{\mathbf{V}}_+((\tilde{x}), \mathbf{u}), \quad \tilde{\mathbf{V}}_-((\tilde{x}), \mathbf{u}), \quad \tilde{\mathbf{V}}_0((\tilde{x}), \mathbf{u})$$

the fibers of $\tilde{\mathbf{V}}_+$, $\tilde{\mathbf{V}}_-$, $\tilde{\mathbf{V}}_0$ over $((\tilde{x}), \mathbf{u}) \in \text{US}$ respectively. We denote by

$$\Pi_{\tilde{\mathbf{V}}_+} : \tilde{\mathbf{V}} \rightarrow \tilde{\mathbf{V}}_+, \quad \Pi_{\tilde{\mathbf{V}}_-} : \tilde{\mathbf{V}} \rightarrow \tilde{\mathbf{V}}_-, \quad \Pi_{\tilde{\mathbf{V}}_0} : \tilde{\mathbf{V}} \rightarrow \tilde{\mathbf{V}}_0$$

the projections using the direct sum decomposition

$$\tilde{\mathbf{V}} = \tilde{\mathbf{V}}_+ \oplus \tilde{\mathbf{V}}_- \oplus \tilde{\mathbf{V}}_0.$$

4.4 Computing translation vectors

Here, we will write the cocycle in terms of an integral. Let g be a hyperbolic element. Let a_g denote the attracting fixed point of g in $\partial\mathbb{S}_+$ and r_g the repelling one. Let Σ_+ denote the surface

$$((\mathbb{S}_+ \cup \partial\mathbb{S}_+) \setminus \Lambda_{\Gamma, \mathbb{S}_+})/\Gamma.$$

The surface S is the dense subset of Σ_+ . The $\mathcal{V}$ -valued forms are differential forms with values in the fiber spaces of $\mathcal{V}$. (See Definition 4.2.) The $\tilde{\mathcal{V}}$ -valued forms on $\mathbb{S}_+$ are simply the $\mathbb{R}^{2,1}$ -valued forms on $\mathbb{S}_+$. However, the group Γ acts by

$$\gamma^*(\mathbf{v} \otimes dx) = \mathcal{L}(\gamma)^{-1}(\mathbf{v}) \otimes d(x \circ \gamma) = \mathcal{L}(\gamma)^{-1}(\mathbf{v}) \otimes \gamma^* dx, \quad \gamma \in \Gamma. \quad (4.4)$$

(See Labourie [34, Chapter 4].)

Let $\|\cdot\|_E$ denote a Euclidean metric on E by changing signs of the Lorentz metric which we fix from now on. Let g be a hyperbolic isometry. Let x_g be a point of the geodesic l_g in $\mathbb{S}_+$ on which g acts preserving an orientation direction $\mathbf{u}_g$. We define

$$\nu_g := \mathbf{v}_{0, (x_g, \mathbf{u}_g)} = \tilde{\nu}(x_g, \mathbf{u}_g),$$

which is independent of the choice of $(x_g, \mathbf{u}_g)$ on l_g by (4.2).

Recall from Section 3.2 the cocycle of $\Gamma = h_{\mathbf{b}}(\pi_1(S))$ for the holonomy homomorphism $h_{\mathbf{b}}$:

$$\mathbf{b} \in Z^1(\pi_1(S), \mathbb{R}_h^{2,1}).$$

We write every element g as $g(x) = A_g x + \mathbf{b}_g$, $x \in E$. Then the function $\mathbf{b} : \Gamma \rightarrow \mathbb{R}^{2,1}$ given by

$$g \mapsto \mathbf{b}_g \quad \text{for every } g$$

is a cocycle representing an element of

$$H^1(\pi_1(S), \mathbb{R}^{2,1}) = H^1(S, \mathcal{V})$$

using the de Rham isomorphism. (See Labourie [34, Theorem 4.2.3].) Let η denote the smooth $\mathcal{V}$ -valued 1-form on S representing the cocycle $\mathbf{b}$ in the de Rham sense.

Let $\tilde{\eta} : \mathbb{S}_+ \rightarrow \mathbb{R}^{2,1}$ denote the lift of η to $\mathbb{S}_+$. We can think of $\tilde{\eta}$, which is h -equivariant, as the differential of a section $s_{\tilde{\eta}} : \mathbb{S}_+ \rightarrow E$ which is $h_{\mathbf{b}}$ -equivariant:

$$\tilde{\eta} = ds_{\tilde{\eta}} \quad (4.5)$$

by [25, Theorem 1.14] and lifting to the cover $\mathbb{S}_+ \times E$.

Recall from Section 2.2, the end neighborhood E and its inverse image $\mathcal{H} \subset \mathbb{S}_+$. Let $\mathcal{CH}(\Lambda)$ denote the convex hull of a closed subset Λ of $\partial\mathbb{S}_+$ in $\mathbb{S}_+$. The surface $S_C := \mathcal{CH}(\Lambda_{\Gamma, \mathbb{S}_+})/\Gamma$ is a finite-volume connected hyperbolic surface with geodesic boundary and cusp ends. The boundary of S_C is a union of finitely many closed geodesic boundary components, and each end of S_C is a cusp. Assume that each component of E is a subset of S_C by choosing suitable cusp neighborhoods. We let F to denote a compact fundamental domain of $\mathcal{CH}(\Lambda_{\Gamma, \mathbb{S}_+}) \setminus \mathcal{H}$.

Let $\mathbf{US}_C$ denote the space of unit vectors on S with base points at S_C , and let $\mathbf{UC}\mathcal{H}(\Lambda_{\Gamma, \mathbb{S}_+})$ denote one for $\mathcal{CH}(\Lambda_{\Gamma, \mathbb{S}_+})$. We can compute the cocycle $\mathbf{b}$ by the following way:

Let $\mathcal{K}$ be a small fixed compact domain in $\mathcal{CH}(\Lambda_{\Gamma, \mathbb{S}_+}) \setminus \mathcal{H}$ in $\mathbb{S}_+$. Let $\tilde{\eta}$ denote the lift of η on $\mathbb{S}_+$. We may also assume that

$$\tilde{\eta}|_{\mathcal{K}} \equiv 0 \quad (4.6)$$

by locally changing η by (4.5). We simply need to change the section to a section that is a fixed parallel section on $p_S(\mathcal{K})$. This can obviously be achieved by using a partition of unity while this does not change the cohomology class of η . (See [28, Section 4].)

To simplify, we assume that $s_{\tilde{\eta}}$ at $\mathcal{K}$ takes the value of the origin O .

Definition 4.3. Let $\Gamma_{\hat{K}}$ denote the set of hyperbolic elements $g \in \Gamma$ that acts on a geodesic l_g in $\mathbb{S}_+$ passing a compact subset $\mathcal{K} \subset \mathbb{S}_+ \setminus \mathcal{H}$.

We lift the discussion to $\mathbf{US}_C$ and its cover $\mathbf{UC}\mathcal{H}(\Lambda_{\Gamma, \mathbb{S}_+}) \subset \mathbf{US}_+$. Let g be an element of $\Gamma_{\mathcal{K}}$ corresponding to a closed geodesic c_g . Let l_g be the unit speed geodesic in $\mathbb{S}_+$ in connecting $x_g \in \mathcal{K}$ to $g(x_g)$ covering c_g with the length t_g . Let $\Pi_{\mathbb{R}^{2,1}} : \mathbf{US}_+ \times \mathbb{R}^{2,1} \rightarrow \mathbb{R}^{2,1}$ denote the projection to the second factor. Then by the trivialization on $\mathcal{K}$

$$\mathbf{b}_g = \Pi_{\mathbb{R}^{2,1}} \left(\int_{[0, t_g]} \tilde{\eta} \left(\frac{dl_g(t)}{dt} \right) dt \right),$$

where t_g is the time needed to go from x_g to $g(x_g)$. (See Labourie [34, Section 4.2.2].) However, we will consider the case when x_g is anywhere in $\mathbb{S}_+$. Since

$$\Pi_{\mathbb{R}^{2,1}} \left(\int_{[0, t_g]} \tilde{\eta} \left(\frac{dl_g(t)}{dt} \right) dt \right) = g(\Pi_E \circ s_{\tilde{\eta}}(x_g)) - \Pi_E \circ s_{\tilde{\eta}}(x_g) = (\mathcal{L}(g) - I)(\Pi_E \circ s_{\tilde{\eta}}(x_g)) + \mathbf{b}_g,$$

we have

$$\mathbf{b}_g = \Pi_{\mathbb{R}^{2,1}} \left(\int_{[0, t_g]} \tilde{\eta} \left(\frac{dl_g(t)}{dt} \right) dt \right) + (I - \mathcal{L}(g))(\Pi_E \circ s_{\tilde{\eta}}(x_g)).$$

Thus, we obtain

$$\mathbf{b}_g = \Pi_{\mathbb{R}^{2,1}} \left(\int_{[0, t_g]} \mathbb{D}\Psi((x_g, \mathbf{u}_g), t)^{-1} \left(\tilde{\eta} \left(\frac{d\Psi((x_g, \mathbf{u}_g), t)}{dt} \right) \right) dt \right) + (I - \mathcal{L}(g))(\Pi_E \circ s_{\tilde{\eta}}(x_g)),$$

where the geodesic segment $\Psi((x_g, \mathbf{u}_g), [0, t_g])$ for a unit vector $\mathbf{u}_g$ at x_g , covers a closed curve representing g .

Using the origin O of E , we can consider it as $\mathbb{V}$ with a vector subspace $\mathbb{V}_{\omega}$, $\omega = +, -, 0$. Define

$$\Pi_{\omega, x_0} := \Pi_{\mathbb{R}^{2,1}} \circ \Pi_{\tilde{\mathbb{V}}_{\omega, x_0}} : \{x_0\} \times E \rightarrow \mathbb{V}_{\omega, x_0} \rightarrow \mathbb{R}^{2,1}$$

to denote the projection $\Pi_{\tilde{\mathbb{V}}_{\omega}}$ at the fiber E over $x_0 \in \mathbf{US}_+$. Define

$$\tilde{\eta}_{\omega}(x_0) = \Pi_{\tilde{\mathbb{V}}_{\omega, x_0}}(\tilde{\eta}(x_0)),$$

where $\omega = +, -, 0$. Since Ψ_t preserves the decomposition, $\mathbb{D}\Psi(x, t)$ commutes with these projections.

Definition 4.4. Let $\mathcal{K}$ be the compact subset of $\mathbb{S}_+ \setminus \mathcal{H}$. Let $g \in \Gamma_{\mathcal{K}}$. We choose $x_g \in \mathcal{K}$ so that the arc $\Psi((x_g, \mathbf{u}_g), [0, t_g])$ for a unit vector $\mathbf{u}_g$ at x_g covers a closed geodesic representing g , where

$$(g(x_g), Dg(\mathbf{u}_g)) = \Psi((x_g, \mathbf{u}_g), t_g).$$

The arc here is not necessarily in $\mathcal{K}$ of course. We define invariants:

$$\begin{aligned} \mathbf{b}_{g, \omega} &:= \Pi_{\tilde{\mathbb{V}}_{\omega, x_g}}(\mathbf{b}_g) \\ &= \Pi_{\mathbb{R}^{2,1}} \left(\int_{[0, t_g]} \mathbb{D}\Psi((x_g, \mathbf{u}_g), t)^{-1} \left(\tilde{\eta}_{\omega} \left(\frac{d\Psi((x_g, \mathbf{u}_g), t)}{dt} \right) \right) dt \right) + (I - \mathcal{L}(g))(\Pi_{\omega, x_g}(s_{\tilde{\eta}}(x_g))), \end{aligned} \quad (4.7)$$

where $\omega = +, -, 0$ respectively. The second equalities hold since $\mathbb{D}\Psi(x, t)$ and $\mathcal{L}(g)$ commute with projections $\Pi_{\tilde{\mathbb{V}}_+}$, $\Pi_{\tilde{\mathbb{V}}_-}$ and $\Pi_{\tilde{\mathbb{V}}_0}$.

Proposition 4.3. For nonparabolic $g \in \Gamma - \{I\}$, we have

$$\mathbf{b}_{g,0} = \alpha(g)v_g, \quad \|\mathbf{b}_{g,0}\| = \alpha(g). \quad (4.8)$$

Proof. First, $\mathbf{b}_{g,0}$ is parallel to v_g by (4.7). Since v_g is Lorentz orthogonal to the subspace spanned by $\mathbf{v}_{+, (x_g, \mathbf{u}_g)}$ and $\mathbf{v}_{-, (x_g, \mathbf{u}_g)}$, the component $\mathbf{b}_{g,0}$ is the image $\mathbf{b}_g$ under the Lorentzian projection to v_g . Since $\mathbf{b}_g = g(O) - O$ for the origin O by our choice of the $\mathbf{E}$ -section near (4.6), and $\|v_g\| = 1$, (3.7) and Criterion 1.1 imply the result. $\square$

The norm of a 1-form with values in $\mathbb{V}_0$ is given by the fiberwise norm of $\mathbb{V}_0$ and the norm of hyperbolic metric for the tangent bundle of $\mathbf{S}$. Finally, we will need:

Definition 4.5. Let K be a compact subset of $\mathbf{S}$, and let $\tilde{K}$ denote the inverse image of K in $\mathbb{S}_+$. The *neutral factor* of $\eta|_K$ is given as the maximum norm of $\tilde{\eta}_0$ on $U\tilde{K}$.

4.5 Translation vectors have direction limits in $\mathbb{S}_0$

We aim to prove Theorem 4.8 from Section 4.5.1 to Section 4.5.4. Section 4.5.1 discusses the standard cusp 1-forms and how to integrate along geodesics to obtain the Margulis invariants. Important Lemma 4.6 shows that long cusp geodesics can absorb many possibly negative perturbations during the argument that we will present. Section 4.5.2 outlines the proof of Theorem 4.8. In Section 4.5.3, we show $\alpha(g_i) \rightarrow \infty$ and $\alpha(g_i)/\|\mathbf{b}_{g_i}\| \rightarrow \infty$ if $l_{\mathbb{S}_+}(g_i) \rightarrow \infty$. We will use the fact that a sequence converges to $+\infty$ if we can show that a subsequence of any subsequence converges to $+\infty$. Hence, we will start with a subsequence and keep taking subsequences to obtain one that converges to $+\infty$. In Section 4.5.4, we finish the proof of the theorem on the limit of direction vectors.

4.5.1 Cusp forms

A *standard horodisk* D is an open disk bounded by a horocycle in $\mathbb{S}_+$ passing $(\mathbf{k})$ and ending at the unique point $(\mathbf{j} + \mathbf{k})$. We denote by $\partial_h D$ the horocycle $\text{Cl}(D) \setminus (D \cup \{(\mathbf{j} + \mathbf{k})\})$ for any horodisk D .

Let D' be a horodisk in $\mathbb{S}_+$. Let $\mathbf{p}$ denote a null-vector in the direction of $p \in \text{Cl}(D') \cap \partial\mathbb{S}_+$. Let us use an upper half-space model of the hyperbolic plane with the standard coordinates x, y and p corresponding to ∞ . Then we may assume without loss of generality that D' is given by $y > 1$.

Definition 4.6. Let g be an accordant parabolic transformation in Γ . Using the parabolic coordinates, let g be of the form (3.5) for some $t > 0$. Let E' be a cusp neighborhood covered by D' where $\langle g \rangle$ acts as the deck transformation group. On D' , we can find a $\mathcal{V}$ -valued 1-form

$$\mu(x^2/2, -x, 1)dx \quad (4.9)$$

that is closed but not exact and is g -invariant by (4.4) with respect to a coordinate system adopted to g . We call such a form on D' and the induced one on E' *standard cusp 1-forms*, $\mu > 0$ is the *cusp coefficient* of E' . (See [14] to check the form and the invariance.)

Here, $\mu > 0$ by Lemma 3.12 since $t > 0$ under the assumption.

Let $\mathcal{H}_j \subset \mathbb{S}_+$, $j = 1, 2, \dots$, denote the horodisks covering the components of E . Let p_j denote the parabolic fixed point corresponding to $\mathcal{H}_j$. Each $\mathcal{H}_j$ has standard coordinates x_j, y_j from the upper half-space model of $\mathbb{S}_+$ where p_j becomes ∞ , and $\mathcal{H}_j$ is given by $y_j > 1$.

Since $\mathbf{S}$ has finitely many cusps, we can choose horocyclic end neighborhoods with mutually disjoint closures. By taking even smaller ones, we may also assume that

$$d_{\mathbb{S}_+}(g(\mathcal{H}_i), k(\mathcal{H}_j)) > C_E^{(4.10)}, \quad C_E^{(4.10)} \geq \frac{5}{4}, \quad g, k \in \Gamma, \quad i, j = 1, \dots, m_0, \quad (4.10)$$

whenever $g(\mathcal{H}_i) \neq k(\mathcal{H}_j)$ for some fixed constant $C_E^{(4.10)}$ depending only on E .

There are only finitely many cusps in S_C . Thus, we can choose finitely many cusps in each orbit class of cusps whose closures meet the fundamental domain F . We may denote these by $\mathcal{H}_1, \dots, \mathcal{H}_{m_0}$ by reordering if necessary. We denote by $\mathbf{p}_1, \dots, \mathbf{p}_{m_0}$ the corresponding null vectors. We choose a parabolic coordinate system for each $\mathcal{H}_j$ in the Γ -equivariant manner.

Recall from Section 3.2 the cocycle of $\Gamma = h_{\mathbf{b}}(\pi_1(S))$ for the holonomy homomorphism $h_{\mathbf{b}}$:

$$\mathbf{b} \in Z^1(\pi_1(S), \mathbb{R}_h^{2,1}).$$

For each $\gamma \in \pi_1(S)$, $\mathbf{b}(\gamma) = h_{\mathbf{b}}(\gamma)(x_0) - x_0$ for a basepoint x_0 . For each peripheral element γ in the boundary orientation, let $\hat{\gamma}$ denote the corresponding deck transformation. We choose an adopted parabolic coordinate system where $h(\hat{\gamma})$ is accordant. Let E_{γ} be a component of E corresponding to γ . Let γ' be the homotopy class in E_{γ} of the simple closed curve $c_{\gamma'}$ bounding E_{γ} with a basepoint $x_{0,\gamma}$. If we choose a basepoint to be the origin of the coordinate system, we obtain a class $\mathbf{u}$ in $H^1(\langle \hat{\gamma} \rangle, \mathbb{R}_{\langle h(\hat{\gamma}) \rangle}^{2,1})$. Let $\tilde{c}_{\gamma'}$ denote the boundary horocycle corresponding to $\hat{\gamma}$. Using the partition of unity, we change the section $s_{\tilde{\eta}}$ associated with $\tilde{\eta}$ so that so that $s_{\tilde{\eta}}|_{\tilde{c}_{\gamma'}}$ is the orbit of the origin of the one-parameter group of parabolic affine transformations containing $h(\hat{\gamma})$. By (4.5), new η is obtained in E_{γ} . Since the de Rham class $[\eta_{\mu}^c] \in H^1(E', \mathcal{V})$ goes to $\mathbf{u} \in H^1(\langle \hat{\gamma} \rangle, \mathbb{R}_{\langle h(\hat{\gamma}) \rangle}^{2,1})$, we obtain by Propositions B.1 and B.2:

Corollary 4.4. *Let S , Γ , P , E , and γ be as above. Then we may replace a closed $\mathcal{V}$ -valued 1-form η on S with a cohomologous one η' so that $\eta'|_{E'}$ for each component E' of E is a standard cusp 1-form in a parabolic coordinate system adopted to the accordant holonomy element following the boundary orientation.*

We may choose the 1-form η representing the cohomology class so that $\tilde{\eta}$, its lift to $\mathbb{S}_+$, is a standard cusp 1-form on $\mathcal{H}_j$. Let μ_j denote the cusp coefficients for each j , $j = 1, 2, \dots$. Since there are only finitely many cusps in $\mathbb{S}_+/\Gamma$, there are only finitely many values of the cusp coefficients. Let $\mu_{\min}$ be the minimum of $\mu_1, \mu_2, \dots$, and let μ be the maximum of $\mu_1, \mu_2, \dots$.

Let $\|\cdot\|_E$ denote a Euclidean metric on E which we fix in this paper.

Lemma 4.5. *Let $\mathcal{K}$ be a compact subset of $\mathbb{S}_+ \setminus \mathcal{H}$. Suppose $x \in \mathcal{K}$. Then the matrix $\mathcal{C}_i$ with columns*

$$\mathbf{V}_{+, (x, \mathbf{u})}, \mathbf{V}_{0, (x, \mathbf{u})}, \mathbf{V}_{-, (x, \mathbf{u})} \quad \text{for every } \mathbf{u} \in \mathbb{U}_x \mathbb{S}_+$$

is in a compact subset of $\text{GL}(3, \mathbb{R})$ depending only on $\mathcal{K}$.

Proof. There is a uniformly bounded element of $\text{SO}(2, 1)^o$ sending a complete geodesic $\overline{\langle 0, -1, 1 \rangle} \overline{\langle 0, 1, 1 \rangle}$ to l_{g_i} and $\langle 1, 0, 0 \rangle$ to $\langle v_{g_i} \rangle$. From this and the way we define the frames in Section 4.3, the conclusion follows. $\square$

Let g be a hyperbolic element. We recall from (4.7) and (4.8),

$$\alpha(g) = \|\mathbf{b}_{g,0}\|, \mathbf{b}_{g,0} = \Pi_{\tilde{\mathbf{v}}_0, x_g}(\mathbf{b}_g) = \Pi_{\mathbb{R}^{2,1}} \left(\left(\int_{[0, t_g]} \mathbf{B} \left(v_{x_g, \mathbf{u}_g}, \tilde{\eta} \left(\frac{d\Psi((x_g, \mathbf{u}_g), t)}{dt} \right) \right) dt \right) v_{x_g, \mathbf{u}_g} \right)$$

since $(I - \mathcal{L}(g))(\Pi_{\tilde{\mathbf{v}}_0, x_g}(s_{\tilde{\eta}}(x_g))) = 0$.

For any subinterval ζ in a cusp with the cusp coefficient μ , we define $\alpha(\zeta)$ to be the corresponding part of the above integral from t_{ζ_0} and t_{ζ_1} for the corresponding arc-length parametrizing interval $[t_{\zeta_0}, t_{\zeta_1}]$. Define $R(\zeta)$ as the radius of ζ in the upper half-space model where the horocycle is given by $y = 1$. By Proposition B.4, and the compatibility (4.3), we can use

$$\alpha(\zeta) = \mu \left(\frac{\pm \sqrt{2} \sqrt{R(\zeta)^2 - 1}}{R(\zeta)} + 2R(\zeta) \sqrt{R(\zeta)^2 - 1} \right). \quad (4.11)$$

Definition 4.7. We define $r(\zeta) := \sqrt{R(\zeta)^2 - 1}$, which equals 1/2 times the absolute value of the difference of the x -coordinates of the endpoint of ζ in the upper half-space model where the horocycle is given by $y = 1$. The horospherical length h of a cusp neighborhood E is the $d_{\mathbb{S}_+}$ -length of ∂E . Note that if two maximal geodesics ζ and ζ' in a cusp E have the same endpoints, then $r(\zeta)$ and $r(\zeta')$ differ by a half an integer times h .

One useful result is [31, Theorem 4.6],

$$r(\hat{\zeta}) = \sinh \frac{1}{2}(l_{\mathbb{S}^+}(\hat{\zeta})). \quad (4.12)$$

From this, we can show that the difference of x -coordinates of the endpoints of an arc of length l is $\leq 2 \sinh(\frac{l}{2})$.

Heuristically, Lemma 4.6 states that the homotopy classes of maximal geodesics in a cusp neighborhood will give quadratic differences in α -values. In particular, item (ii) gives us the main estimations to absorb the negative contributions.

Lemma 4.6 (Large cusp radius). *Let ζ be a maximal geodesic in a cusp neighborhood E' with the standard cusp 1-form and a cusp coefficient μ' . Let h be the horospherical length of E' . There exists a positive constant R_c , independent of μ' but dependent on h and C , which is defined below so that for any $R_1 > R_c$ has the following properties:*

- (i) *For the set of maximal geodesics in E' , $r(\zeta') \mapsto \alpha(\zeta')$ for each ζ' in it forms a strictly increasing positive function of $r(\zeta')$ for $r(\zeta') > R_1$.*
- (ii) *Let ζ and ζ' be two maximal geodesics in E with the same endpoint as ζ but in the different homotopy classes with respect to endpoints. For any constant $0 \leq \eta_0 < C$ with*

$$R - h/2 < r(\zeta) < R < r(\zeta') \quad \text{for } R > R_1,$$

we have

$$\alpha(\zeta') - \alpha(\zeta) - \mu' \eta_0 \geq 2C_{R_1, C}^{(4.6)} \mu' r(\zeta')^2$$

for a constant $C_{R_1, C}^{(4.6)} > 0$ depending only on h , R_1 and C .

Proof. We choose a horoball $\tilde{E}'$ covering E' . Then we can compute $\alpha(\zeta)$ for a geodesic ζ by lifting ζ to $\tilde{E}'$.

Statement (i) is straightforward.

For (ii), the last term of (4.11) dominates the absolute values of other terms and $\mu\eta$ for sufficiently large R_1 : Using (4.11), the above term divided by μ' is bounded below by

$$r(\zeta')^2 - r(\zeta)^2 - \eta_0 - 2\sqrt{2}.$$

Since $(x - h/2)/x$ is an increasing function of x , the supremum on $x \in (R, R + h/2)$ is $R/(R + h/2)$. Hence, we have $r(\zeta) < C_R r(\zeta')$ for $C_R = R/(R + h/2)$ since the ratio $r(\zeta)/r(\zeta')$ is less than C_R for $r(\zeta') \geq R + h/2$. Then $\alpha(\zeta') - \alpha(\zeta) - \mu' \eta_0$ divided by μ' is bounded below by

$$r(\zeta')^2(1 - C_R^2) - C - 2\sqrt{2} \geq (1 - C_R^2) \left(r(\zeta')^2 - \frac{C + 2\sqrt{2}}{1 - C_R^2} \right).$$

Let $f_R(x)$ denote the polynomial given by the right side with x replacing $r(\zeta')$. The largest root of $f_R(x)$ is smaller than

$$\sqrt{\frac{(R + h)(C + 2\sqrt{2})}{h}}.$$

Since the function $R \mapsto R$ dominates any function given by the square root of the 1st order polynomial of R , there exists $R' > h$ so that for $R > R'$, we have

$$R > \sqrt{\frac{(R + h)(C + 2\sqrt{2})}{h}} \implies f_R(x) > 0 \text{ for } x > R.$$

Define

$$c := \frac{f_{R'+1}(R' + 1)}{(R' + 1)^2} > 0.$$

Then

$$f_{R'+1}(x) \geq cx^2 \quad \text{for } x \geq R' + 1$$

by an easy calculus argument. We take $R_1 = R' + 1$, and $C_{R_1, C}^{(4.6)} = c/2$. We can make R_1 as large as we wish to since we only need $c > 0$. $\square$

4.5.2 Summing up the contributions

Let $\{g_i\}$ be a sequence of elements in $\Gamma_{\mathcal{K}}$. We denote by $\hat{l}_{g_i}$ the lift of l_{g_i} to $\mathbb{U}\mathbb{S}_+$ directed towards the attracting fixed point of g_i in $\partial\mathbb{S}_+$.

Recalling (4.7), we estimate $\mathbf{b}_{g_i,-}(x)$. We give an outline of the rest of the long proof of Theorem 4.8 starting from Section 4.5.2:

- (I) First, we estimate the last term in the integral (4.7) for $\omega = -$.
- (II) We estimate the contribution of $\eta|_{S_C \setminus E}$ of the integral (4.7) for $\omega = -$.
- (III) We estimate the contribution of the arcs in $\mathcal{H}$
 - (a) We estimate the contribution of the arc when it is put into a standard position.
 - (b) We obtain the relationship of the contributions to the arc in the standard position and actual one by Lemma 4.7.
 - (c) We estimate the comparisons of sizes by length.
- (IV) Then we sum these results to estimate the integral (4.7) for $\omega = -$.
- (V) In Section 4.5.3, we show that $\alpha(g_i) \rightarrow \infty$ and $\alpha(g_i)/\|\mathbf{b}_{g_i,-}\| \rightarrow \infty$ as $l_{\mathbb{S}_+}(g_i) \rightarrow \infty$.
- (VI) Finally, we estimate the asymptotic direction as the last item in Section 4.5.4.

Let $(x, \mathbf{u}) \in \mathbb{U}\mathcal{K}$. The arc $\Psi((x, \mathbf{u}), [0, t])$ is a geodesic passing $\mathbb{U}\mathcal{K}$. We choose $x_i \in \mathcal{K} \cap l_{g_i}$ for each i and the unit vector $\mathbf{u}_i$ at x_i in the direction of $\hat{l}_{g_i}$. We let $t_{g_i} > 0$ be so that $\Psi((x_i, \mathbf{u}_i), [0, t_{g_i}]) \subset l_{g_i}$ corresponds to the closed geodesic corresponding to g_i .

Let $\mathbb{U}S_C$ denote the unit tangent bundle over S_C .

- We denote by $\mathcal{H}_{i,1}, \mathcal{H}_{i,2}, \dots$, the components of $\mathcal{H}$ meeting $\Pi_{\mathbb{S}_+}(\Psi(x_i, \mathbf{u}_i), t)$ as t increases.
- Let $\mathbf{p}_{i,j} dx_{i,j}$ denote $\tilde{\eta}|_{\mathcal{H}_{i,j}}$, where $(\mathbf{p}_{i,j})$ is the parabolic fixed point in the boundary of $\mathcal{H}_{i,j}$.
- Let $t_{i,j}$, $0 < t_{i,j} < t_{g_i}$, be the time the geodesic $\Psi((x_i, \mathbf{u}_i), t)$ enters $\mathbb{U}\mathcal{H}_{i,j}$, and $\hat{t}_{i,j}$ the time it leaves $\mathbb{U}\mathcal{H}_{i,j}$ for the first time after $t_{i,j}$.
- We denote $I_{i,j} = [t_{i,j}, \hat{t}_{i,j}]$.

- (I) We estimate $\|(I - \mathcal{L}(g_i))(\Pi_{-,x_g} \circ s_{\tilde{\eta}}(x_i))\|_E$ for $g \in \Gamma_{\mathcal{K}}$ from (4.7): The matrix of $\mathcal{L}(g_i)$ with the basis $\mathbf{v}_{+, (x_i, \mathbf{u}_i)}, \mathbf{v}_{0, (x_i, \mathbf{u}_i)}, \mathbf{v}_{-, (x_i, \mathbf{u}_i)}$ is a diagonal matrix with entries

$$\lambda_1(g_i), 1, 1/\lambda_1(g_i).$$

Hence, the above is given by

$$\left\| \left(1 - \frac{1}{\lambda_1(g_i)} \right) (\Pi_{-,x_g} \circ s_{\tilde{\eta}}(x_i)) \right\|_E < C_{\mathcal{K}}, \quad (4.13)$$

where we have a uniform constant $C_{\mathcal{K}}$ depending only on $\mathcal{K}$ by Lemma 4.5 and (4.1) since $\lambda_1(g_i) > 1$ and $\|s_{\tilde{\eta}}\|_E|_{\mathcal{K}}$ is bounded by a constant depending only on $\mathcal{K}$.

(II) Define

$$N(S_C \setminus E) := \max\{\|\eta(\mathbf{u})\|_{\text{fiber}} : \mathbf{u} \in \mathbb{U}S_C \setminus E\}.$$

We have

$$\left\| \int_{[0, t_{g_i}] \setminus \bigcup_j I_{i,j}} \mathbb{D}\Psi((x_i, \mathbf{u}_i), t)^{-1} \left(\tilde{\eta}_- \left(\frac{d\Psi((x_i, \mathbf{u}_i), t)}{dt} \right) \right) dt \right\|_{\text{fiber}} < C_1$$

for $C_1 < \infty$ by the second part of (4.3) applied to $\mathbb{D}\Psi((x_i, \mathbf{u}_i), t)^{-1}$ and the integrability of the exponential function. Here, $C_1 = C_1(N(S_C \setminus E))$ depends only on $N(S_C \setminus E)$.

Since these integrals have values in the fibers over $\mathcal{K}$, and $\|\cdot\|_{\text{fiber}}$ and $\|\cdot\|_E$ are uniformly compatible over $\mathcal{K}$, we have

$$\left\| \int_{[0, t_{g_i}] \setminus \bigcup_j I_{i,j}} \mathbb{D}\Psi((x_i, \mathbf{u}_i), t)^{-1} \left(\tilde{\eta}_- \left(\frac{d\Psi((x_i, \mathbf{u}_i), t)}{dt} \right) \right) dt \right\|_E < C_2 \quad (4.14)$$

for $C_2 < \infty$. (See Remark 4.1.) Hence, C_2 depends only on $\mathcal{K}$ and $N(S_C \setminus E)$. We write

$$C_2 = C_2(\mathcal{K}, N(S_C \setminus E)).$$

(III) For each $I_{i,j}$, we define for the maximal geodesic segment in $l_{g_i} \cap \mathcal{H}_{i,j}$

$$\eta_{i,j} := \Pi_{\mathbb{S}_+} \circ \Psi((x_i, \mathbf{u}_i), I_{i,j}) \subset \mathcal{H}_{i,j} \quad \text{and}$$

$$\mathbf{b}_{g_i, -}(\eta_{i,j}) := \int_{I_{i,j}} \mathbb{D}\Psi((x_i, \mathbf{u}_i), t)^{-1} \left(\tilde{\eta}_- \left(\frac{d\Psi((x_i, \mathbf{u}_i), t)}{dt} \right) \right) dt.$$

We now estimate $\mathbf{b}_{g_i, -}$ contributed by $I_{i,j}$ by looking at the situation of (4.17).

Recall the fundamental domain F of $\mathcal{CH}(\Lambda_{\Gamma, \mathbb{S}_+}) \setminus \mathcal{H}$ covering $S_C \setminus E$. Let p_i denote the beginning point in $\partial \mathbb{S}_+$ of l_{g_i} in $\mathbb{S}_+$, and p'_i denote the forward endpoint of l_{g_i} in $\partial \mathbb{S}_+$. Let $q_{i,j}$ denote the beginning point of $\eta_{i,j}$ itself and $\mathbf{u}_{i,j}$ the unit tangent vector to l_{g_i} at the point x_i in $\mathcal{H}$.

Definition 4.8. We define three maps and two others slightly later.

- $g_{i,j}$: There is an element $g_{i,j} \in \Gamma$ so that $g_{i,j}(q_{i,j}) \in F$, and

$$g_{i,j}(\mathcal{H}_{i,j}) = \mathcal{H}_k \quad \text{for } k = 1, \dots, m_0 \quad \text{and} \quad g_{i,j}(q_{i,j}) \in F \cap \text{Cl}(\mathcal{H}_k).$$

- $\hat{h}_{i,j}$: Since $\{\mathcal{H}_1, \dots, \mathcal{H}_{m_0}\}$ is finite, we can put $\mathcal{H}_k$ to the standard horodisk D by a uniformly bounded sequence $h'_{i,j}$ of elements of $\text{SO}(2, 1)^0$. Since $g_{i,j}(q_{i,j})$ is in a compact set $F \cap \text{Cl}(\mathcal{H}_k)$, it follows that $h'_{i,j}(g_{i,j}(q_{i,j}))$ is in a uniformly bounded subset of $\text{U}\partial_h D$. Hence, we can put $h'_{i,j}(g_{i,j}(p_i))$ to be $\langle 0, -1, 1 \rangle$ by a bounded sequence $h''_{i,j}$ of parabolic elements fixing $\langle 0, 1, 1 \rangle$. Let $\hat{h}_{i,j} = h''_{i,j} \circ h'_{i,j}$. Then

$$\hat{h}_{i,j}(\mathcal{H}_{i,j}) = D, \quad \hat{h}_{i,j}(g_{i,j}(p_i)) = \langle 0, -1, 1 \rangle,$$

and $\hat{h}_{i,j}$ in a uniformly bounded set of elements of $\text{SO}(2, 1)^0$ not necessarily in Γ . This is called a *normalization map*. (There is a bound on the size of $\hat{h}_{i,j}$ depending only on F .)

- $h_{i,j}$: Let $h_{i,j} = \hat{h}_{i,j} \circ g_{i,j}$.

The image

$$\zeta_{i,j} = h_{i,j}(\eta_{i,j})$$

satisfies the premise of Lemma B.3. (See Figure 5.)

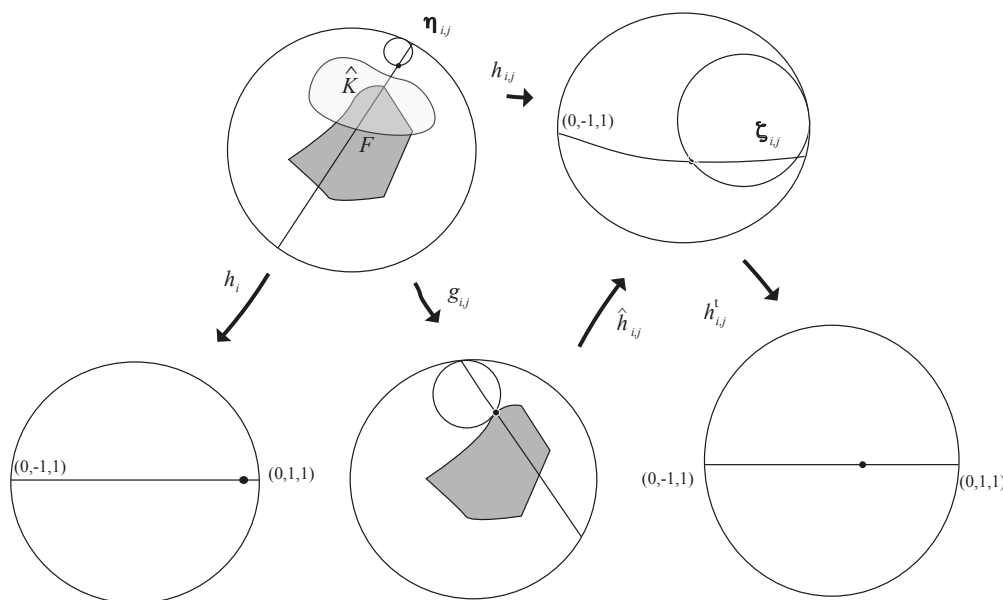


Figure 5: $g_{i,j} \in \Gamma$ moves $q_{i,j}$ to a point of F . $\hat{h}_{i,j}$ sends $\mathcal{H}_{i,j}$ to the standard horodisk D , h_i is a normalization map of l_{g_i} , $h'_{i,j}$ the normalization map for $h_{i,j}(l_{g_i})$, where $h_{ij} = \hat{h}_{ij} \circ g_{ij}$. See Definitions 4.8 and 4.9. The black dots indicate the images of $q_{i,j}$.

(a) We define

$$\mathbf{b}_{g_i, -}(\zeta_{i,j}) := \int_{I_{i,j}} \mathbb{D}\Psi(h_{i,j}(q_{i,j}), t - t_{i,j})^{-1} \left(h_{i,j}^{-1*} \tilde{\eta}_- \left(\frac{d\Psi(h_{i,j}(q_{i,j}), t - t_{i,j})}{dt} \right) \right) dt. \quad (4.15)$$

Proposition B.4 implies that

$$\|\mathbf{b}_{g_i, -}(\zeta_{i,j})\|_E \leq \mu_k r(\zeta_{i,j}).$$

Since there are only finitely many values of μ_k ,

$$\|\mathbf{b}_{g_i, -}(\zeta_{i,j})\|_E \leq \mu r(\zeta_{i,j}). \quad (4.16)$$

(b) We compute the actual contribution for η_i . We diagram the flow of the point $w \in \mathbb{U}\mathbb{S}_+$ and the action of the isometry g not necessarily in Γ :

$$\begin{array}{ccc} w & \xrightarrow{-t} & \Psi(w, -t) \\ \downarrow g & & \downarrow g \\ g(w) & \xrightarrow{-t} & \Psi(g(w), -t). \end{array} \quad (4.17)$$

Lemma 4.7. *We have*

$$\Pi_{\mathbb{R}^{2,1}}(\mathbb{D}h_{i,j}^{-1}(\mathbf{b}_{g_i, -}(\zeta_{i,j}))) = \Pi_{\mathbb{R}^{2,1}}(\mathbf{b}_{g_i, -}(\eta_{i,j})). \quad (4.18)$$

Proof. Since the flow commutes with isometry group action on $\mathbb{U}\mathbb{S}_+$, we have by considering (4.17) and the triviality of actions in the fibers

$$\begin{aligned} \mathbb{D}g(\mathbb{D}\Psi(w, -t)(\mathbf{v})) &= (\mathbb{D}g \circ \mathbb{D}\Psi(w, -t) \circ \mathbb{D}g^{-1}) \circ \mathbb{D}g(\mathbf{v}) \\ &= \mathbb{D}\Psi(g(w), -t) \circ \mathbb{D}g(\mathbf{v}) \quad \text{for } w \in \mathbb{U}\mathcal{H}, \mathbf{v} \in \mathbb{R}^{2,1}, g \in \mathrm{SO}(2, 1)^0. \end{aligned} \quad (4.19)$$

We apply $\mathbb{D}h_{i,j}^{-1}$ to (4.15). Since $\Psi(x, t)^{-1} = \Psi(x, -t)$, we obtain by (4.19)

$$\begin{aligned} &\mathbb{D}h_{i,j}^{-1} \left(\mathbb{D}\Psi(h_{i,j}(q_{i,j}, \mathbf{u}_{i,j}), t - t_{i,j})^{-1} \left(h_{i,j}^{-1*} \tilde{\eta}_- \left(\frac{d\Psi(h_{i,j}(q_{i,j}, \mathbf{u}_{i,j}), t - t_{i,j})}{dt} \right) \right) \right) \\ &= \mathbb{D}\Psi((q_{i,j}, \mathbf{u}_{i,j}), t - t_{i,j})^{-1} \mathbb{D}h_{i,j}^{-1} \left(h_{i,j}^{-1*} \tilde{\eta}_- \left(\frac{d\Psi(h_{i,j}(q_{i,j}, \mathbf{u}_{i,j}), t - t_{i,j})}{dt} \right) \right). \end{aligned} \quad (4.20)$$

The above (4.20) equals by (4.4)

$$\begin{aligned} &\mathbb{D}\Psi((q_{i,j}, \mathbf{u}_{i,j}), t - t_{i,j})^{-1} \mathbb{D}h_{i,j}^{-1} \left(h_{i,j}^{-1*} \tilde{\eta}_- \left(\frac{d\Psi(h_{i,j}(q_{i,j}, \mathbf{u}_{i,j}), t - t_{i,j})}{dt} \right) \right) \\ &= \mathbb{D}\Psi((q_{i,j}, \mathbf{u}_{i,j}), t - t_{i,j})^{-1} \left(\tilde{\eta}_- \left(\mathbb{D}h_{i,j}^{-1} \left(\frac{d\Psi(h_{i,j}(q_{i,j}, \mathbf{u}_{i,j}), t - t_{i,j})}{dt} \right) \right) \right). \end{aligned} \quad (4.21)$$

By the definition of differentials and (4.17), we obtain

$$\mathbb{D}h_{i,j}^{-1} \left(\frac{d\Psi(h_{i,j}(q_{i,j}, \mathbf{u}_{i,j}), t - t_{i,j})}{dt} \right) = \frac{d(h_{i,j}^{-1} \circ \Psi)((h_{i,j}(q_{i,j}, \mathbf{u}_{i,j}), t - t_{i,j}))}{dt} = \frac{d\Psi((q_{i,j}, \mathbf{u}_{i,j}), t - t_{i,j})}{dt}. \quad (4.22)$$

Above (4.21) equals by (4.22)

$$\mathbb{D}\Psi((q_{i,j}, \mathbf{u}_{i,j}), t - t_{i,j})^{-1} \left(\tilde{\eta}_- \left(\frac{d\Psi((q_{i,j}, \mathbf{u}_{i,j}), t - t_{i,j})}{dt} \right) \right). \quad (4.23)$$

Since $\Psi((x_i, \mathbf{u}_i), t_{i,j}) = (q_{i,j}, \mathbf{u}_{i,j})$, (4.23) equals

$$\begin{aligned} &\mathbb{D}\Psi(\Psi((x_i, \mathbf{u}_i), t_{i,j}), t - t_{i,j})^{-1} \left(\tilde{\eta}_- \left(\frac{d\Psi(\Psi((x_i, \mathbf{u}_i), t_{i,j}), t - t_{i,j})}{dt} \right) \right) \\ &= \mathbb{D}\Psi((x_i, \mathbf{u}_i), t)^{-1} \left(\tilde{\eta}_- \left(\frac{d\Psi((x_i, \mathbf{u}_i), t)}{dt} \right) \right) \quad \text{for every } t \in [t_{i,j}, \hat{t}_{i,j}], (x_i, \mathbf{u}_i) \in \mathbb{U}\mathcal{H}, \end{aligned} \quad (4.24)$$

where we multiplied by $\mathbb{D}\Psi((x_i, \mathbf{u}_i), t_{i,j})^{-1}$ which is I on the fibers to the left side. Integrating (4.20) and the last line of (4.24) for $[t_{i,j}, \hat{t}_{i,j}]$, we proved (4.18). $\square$

(c) Now, we compare the contributions of these arcs. Now, $h_{i,j}(q_{i,j}) \in \text{U}\partial_h D$ is in a uniformly bounded subset F' , $\text{U}F \subset F'$, independent of i, j , of $\text{U}\mathbb{S}_+$ since $h_{i,j}(p_i) = \langle 0, -1, 1 \rangle$ and the complete geodesic containing $h_{i,j}(\eta_{i,j})$ passes the standard horodisk D . Thus, $h_{i,j}(l_{g_i})$ is uniformly bounded from the line κ in $\mathbb{S}_+$ connecting $\langle 0, -1, 1 \rangle$ to $\langle \mathbf{j} + \mathbf{k} \rangle$, oriented towards $\langle 0, 1, 1 \rangle$. Let $\hat{\kappa}$ denote the lift of κ to $\text{U}\mathbb{S}_+$ taking the direction towards $\langle \mathbf{j} + \mathbf{k} \rangle$.

Definition 4.9. We define two additional normalization maps:

- $h_{i,j}^\dagger$: We take a uniformly bounded element $h_{i,j}^\dagger$ of $\text{SO}(2, 1)^o$ so that

$$h_{i,j}^\dagger(h_{i,j}(l_{g_i})) = \kappa \quad \text{and} \quad h_{i,j}^\dagger(h_{i,j}(q_{i,j})) = \langle 0, 0, 1 \rangle.$$

- h_i : Since l_{g_i} is a geodesic passing $\mathcal{K}$, we take a uniformly bounded element h_i of $\text{SO}(2, 1)^o$ so that $h_i(l_{g_i}) = \kappa$ and $h_i(x_i) = \langle 0, 0, 1 \rangle$ without changing the orientation. (The bound only depends on $\mathcal{K}$.)

Then

$$h_i \circ h_{i,j}^{-1} \circ h_{i,j}^{\dagger,-1}(h_{i,j}^\dagger(\zeta_{i,j})) = h_i(\eta_{i,j})$$

and $h_i \circ h_{i,j}^{-1} \circ h_{i,j}^{\dagger,-1}$ acts on κ .

- Under $h_i \circ h_{i,j}^{-1} \circ h_{i,j}^{\dagger,-1}$, $h_{i,j}^\dagger \circ h_{i,j}(q_{i,j})$ goes to a point $h_i(q_{i,j})$.
- We have

$$d_{\mathbb{S}_+}(h_{i,j}^\dagger \circ h_{i,j}(q_{i,j}), h_i(q_{i,j})) = t_{i,j} \quad (4.25)$$

since $h_i(x_i) = \langle 0, 0, 1 \rangle = h_{i,j}^\dagger \circ h_{i,j}(q_{i,j})$ and the $d_{\mathbb{S}_+}$ -length of the arc from x_i to $q_{i,j}$ is $t_{i,j}$ which is also the $d_{\mathbb{S}_+}$ -length of the arc from $h_i(x_i)$ to $h_i(q_{i,j})$.

By (4.1) and (4.25), the eigenvalue of $\mathcal{L}(h_i \circ h_{i,j}^{-1} \circ h_{i,j}^{\dagger,-1})$ at the eigenvector $(0, 1, -1)$ is $\exp(-t_{i,j}/2)$. Since

$$\Pi_{\mathbb{R}^{2,1}}(\mathbf{b}_{g_i, -}(\eta_{i,j})) = \Pi_{\mathbb{R}^{2,1}}(h_{i,j}^{-1*}(\mathbf{b}_{g_i, -}(\zeta_{i,j}))),$$

it follows that $\mathcal{L}(h_i \circ h_{i,j}^{-1} \circ h_{i,j}^{\dagger,-1})$ sends the $\mathbb{R}^{2,1}$ -vector

$$\Pi_{\mathbb{R}^{2,1}}(\mathcal{L}(h_{i,j}^\dagger)(\mathbf{b}_{g_i, -}(\zeta_{i,j}))) \in \langle (0, -1, 1) \rangle \quad \text{to} \quad \Pi_{\mathbb{R}^{2,1}}(\mathcal{L}(h_i)(\mathbf{b}_{g_i, -}(\eta_{i,j}))) \in \langle (0, -1, 1) \rangle$$

by multiplying by $\exp(-t_{i,j}/2)$. Since $h_{i,j}^\dagger$ and h_i are uniformly bounded depending only on $\mathcal{K}$ and F , we obtain

$$\tilde{C}(F, \mathcal{K}) \exp(-t_{i,j}/2) \|\mathbf{b}_{g_i, -}(\zeta_{i,j})\|_E \geq \|\mathbf{b}_{g_i, -}(\eta_{i,j})\|_E. \quad (4.26)$$

for a constant $\tilde{C}(F, \mathcal{K}) > 0$ depending only on $\mathcal{K}$ and F .

(IV) We sum up the contributions. Hence, $\frac{1}{R(\zeta_{i,j})} < 1$. By (4.13), (4.14), (4.16), (4.26) and Proposition B.4, we estimate the upper bound depending only on $E, \mathcal{K}, \eta|_{S_C \setminus E}$:

$$\begin{aligned} \|\mathbf{b}_{g_i, -}\|_E &\leq \tilde{C}(F, \mathcal{K}) \sum_j^{m_i} \exp\left(-\frac{t_{i,j}}{2}\right) \left(\mu \frac{r(\zeta_{i,j})(1 + 4R(\zeta_{i,j})^2)}{2\sqrt{2}R(\zeta_{i,j})^2} \right) + C_2(\mathcal{K}, N(S_C \setminus E)) + C_{\mathcal{K}} \\ &\leq \tilde{C}(F, \mathcal{K}) \sum_j^{m_i} \exp\left(-\frac{t_{i,j}}{2}\right) (4\mu r(\zeta_{i,j})) + C_2(\mathcal{K}, N(S_C \setminus E)) + C_{\mathcal{K}} \end{aligned} \quad (4.27)$$

since $R(\zeta_{ij}) \geq 1$.

4.5.3 $\alpha(g_i) \rightarrow \infty$ and $\frac{\alpha(g_i)}{\|\mathbf{b}_{g_i, -}\|} \rightarrow \infty$

In Step (V), we will prove that $\alpha(g_i) \rightarrow \infty$ and $\alpha(g_i)/\|\mathbf{b}_{g_i, -}\| \rightarrow +\infty$ provided $l_{\mathbb{S}_+}(g_i) \rightarrow \infty$ using the fact that we can absorb many negative uncertainties during perturbation into long edges in the cusps using Lemma 4.6.

We can do this by showing that every subsequence has a subsequence converging to $+\infty$. We give an outline of the step (V).

- First, we will choose some constants such as ϵ, δ, R_0 sufficiently small or large.
- Let g_i denote a closed geodesic. We replace the maximal segment ζ in a cusp neighborhood with $r(\zeta) > R_0 + h/2$ with one ζ' with the same endpoints but with $R_0 < r(\zeta') \leq R_0 + h/2$. We denote the result by $\tilde{g}_i$.

- (iii) Then we find a closed geodesic $\hat{g}_i$ freely homotopic to $\tilde{g}_i$. Then we estimate $|\alpha(\tilde{g}_i) - \alpha(\hat{g}_i)|$ in terms of the constant times the number of components of the above arcs in (4.32). This constant is bounded since $R_0\delta = 2$ by our choice below.
- (iv) This is the final step: $\alpha(g_i)$ is bounded below by $\alpha(\hat{g}_i)$ plus constant times the sum of $r(\zeta)^2$. Then we use the standard Schwartz inequalities.

Definition 4.10. Let g_i also denote the arclength-parameterized closed geodesic in S whose lift l_{g_i} passes a fixed compact set $\mathcal{K}$ in $\mathbb{S}_+$. Let J be the index set of mutually disjoint subintervals $I_i \subset I$ and $\alpha_i := g_i|I_i$. By $g_i \setminus \bigcup_{i \in J} \alpha_i$, we mean the map $g_i|I \setminus \bigcup_{i \in J} I_i$.

We denote by E_ϵ the set obtained by decreasing E inward by ϵ when $\epsilon > 0$ and the $(-\epsilon)$ -neighborhood of E when $\epsilon < 0$. We will assume that $E_{-1/2}$ is still a cusp-neighborhood, and $\eta|E_{-1/2}$ is still a standard cusp 1-form for each component by taking sufficiently smaller E if necessary.

We denote by $\mu_{i,j}$ the cusp coefficient for the cusp neighborhood that $\zeta_{i,j}$ goes into. There are only finitely many values. We assume that the horospherical lengths of all cusp neighborhood components of E equal h . Let $C_{S \setminus E}$ denote the neutral factor of the compact set $S \setminus E$. We remark that the following constants depend only on the two constants h and $C_{S \setminus E_1}$. There is no obstruction for the following choices.

- (i) The first step is to decide on constants to be used later:
- Choose $\delta > 0$ so that $0 < \delta < 1/40$ by Lemma 2.2 and let $\epsilon = 7\delta$.
 - We also require $\delta < \mu/(7C_{S \setminus E_1})$.
 - Also assume $6h\epsilon < 1$, $\epsilon < 1/8$, and $R_0 > 10$.
 - We require δ to be given by $\delta := 2/R_0$ by taking R_0 sufficiently large and δ sufficiently small. By Lemma 2.3, the angle that $\zeta_{i,j}$ with $r(\zeta_{i,j}) \geq R_0$ makes with the vertical line is $< \delta$ in the upper half-space model.
 - R_0 is a constant satisfying all conclusions for the variable R_1 in Lemma 4.6 for $C > 222 \frac{\mu}{\mu_{\min}}$. For simplicity, we assume $R_0 > 10$.
- (ii) We will replace very long $\zeta_{i,j}$ in g_i with ones that are outside some cusp neighborhood: We denote by $\zeta_{i,j}$ the sequence of maximal geodesics in g_i going into E . We denote by $J_{i,t}$ the set of $\zeta_{i,j}$ with $r(\zeta_{i,j}) > t$ for $t \geq 0$. For each $\zeta_{i,j}$ in $J_{i,R_0+h/2}$, we take a maximal geodesic $\tilde{\zeta}_{i,j}$ with the same endpoints but with

$$R_0 \leq r(\tilde{\zeta}_{i,j}) < R_0 + \frac{h}{2}$$

since we can decrease the $r(\zeta)$ -values by $h/2$ times integers by wrapping a smaller number of times around the cusps. Since the geodesics are unique up to homotopy classes relative to endpoints, the homotopy class of $\tilde{\zeta}_{i,j}$ is, of course, different from $\zeta_{i,j}$ relative to the endpoints. Thus, we obtain for $\zeta_{i,j} \in J_{i,R_0+h/2}$,

$$\alpha(\zeta_{i,j}) - \alpha(\tilde{\zeta}_{i,j}) \geq \mu_{i,j}\delta\left(R_0 + \frac{h}{2}\right), \quad \alpha(\zeta_{i,j}) - \alpha(\tilde{\zeta}_{i,j}) - \mu_{i,j}\eta_0 \geq C_{R_0+h/2,C'}^{(4,6)}\mu_{i,j}r(\zeta_{i,j})^2, \quad C_{R_0+h/2,C'}^{(4,6)} > 0$$

by Lemma 4.6 where $\eta_0 < C'$.

- (iii) The third step is to estimate the relationship between α -values for g_i and the closed curves $\hat{g}_i$ and $\tilde{g}_i$ to be constructed: Let $\hat{g}_i$ denote the closed curve obtained by g_i removing $\zeta_{i,j}$ and adding $\tilde{\zeta}_{i,j}$ for each $\zeta_{i,j} \in J_{i,R_0+h/2}$. By Lemma 2.3, $\hat{g}_i$ has turning angles $< \delta = 2/R_0$ at each endpoint of maximal geodesic segments by Lemma 2.3. We define

$$\hat{\alpha}(\hat{g}_i) := \alpha\left(g_i \setminus \bigcup_{\zeta \in J_{i,R_0+h/2}} \zeta\right) + \sum_{\zeta \in J_{i,R_0+h/2}} \alpha(\tilde{\zeta}).$$

There exists a closed geodesic $\tilde{g}_i$ homotopic to $\hat{g}_i$ which is in the ϵ -neighborhood of $\tilde{g}_i$ for $\epsilon = 7\delta$ by Lemma 2.2. Let $E_{R_0/2+h/4+\epsilon}$ denote the cusp neighborhood obtained by moving E inside by $R_0/2 + h/4 + \epsilon$. Then both $\hat{g}_i$ and $\tilde{g}_i$ are in $S \setminus E_{R_0/2+h/4+\epsilon}$.

Define $\hat{J}_{i,0}$ the subset of $J_{i,0}$ of consisting of arcs $\zeta_{i,j}$ where $d_{\mathbb{S}_+}$ -lengths are strictly bigger than $5/4$. For every arc in $J_{i,0} \setminus \hat{J}_{i,0}$, the arcs are in $S \setminus E_{5/8}$. We will not remove these from g_i in the following because of

this. By skipping these, we have

$$|\hat{t}_{i,j} - t_{i,j}| \geq \frac{5}{4}, \quad |t_{i,j+1} - \hat{t}_{i,j}| \geq C_E^{(4.10)} \geq \frac{5}{4} \quad \text{for every } \zeta_{i,j} \in \hat{J}_{i,0}, \quad (4.28)$$

where $C_E^{(4.10)}$ is from (4.10).

Each maximal geodesic $\zeta_{i,j} \in \hat{J}_{i,0} \setminus J_{i,R_0+h/2}$ in E of g_i goes to a geodesic $\tilde{\zeta}_{i,j}$ in $E_{-1/8}$ of $\tilde{g}_i$ by the perpendicular projection which moves points by distances $< \epsilon < 1/8$. We obtain two distances

$$d_{i,j,\pm} := d_{\mathbb{S}_+}(\partial_{\pm}\zeta_{i,j}, \partial_{\pm}\tilde{\zeta}_{i,j}).$$

These are less than ϵ by Lemma 2.1 since each endpoint of $\zeta_{i,j} \in J_{i,R_0+h/2}$ moves less than ϵ . The corresponding endpoints are at most distance $d_{i,j,\pm}$ apart, which are values of the divergence functions corresponding to $\hat{t}_{i,j}$ and $t_{i,j}$ respectively. Hence, their x -coordinate values differ by less than $1.1d_{i,j,\pm}$ respectively using (4.12) as $0 < \epsilon < 1/8$. By last parts of [12] and [13] of the differences in the α -values, we can estimate for $\zeta_{i,j} \in \hat{J}_{i,0} \setminus J_{i,R_0+h/2}$,

$$|\alpha(\tilde{\zeta}_{i,j}) - \alpha(\zeta_{i,j})| \leq 5\mu_{i,j}(R_0 + h/2)(d_{i,j,+}/2 + d_{i,j,-}/2) \quad (4.29)$$

since we can put in the new x -coordinates and take differences in $E_{-1/2}$ where η has the form of the standard cusp 1-form. Here, we need to use the fact that $r > 10$, $\epsilon < 1/8$, $\epsilon < r/80$, $r \mapsto \sqrt{r^2 + 1}$, $r > 0$, is distance decreasing, and estimates of differences of the inverses of radii of arcs using calculus.

We claim that the sum of $d_{i,j,+} + d_{i,j,-}$ for $\zeta_{i,j} \in \hat{J}_{i,0}$ in $\tilde{g}_i \setminus \bigcup_{\zeta_{i,j} \in J_{i,R_0+h/2}} \tilde{\zeta}_{i,j}$ is less than 2 times the sum of $d_{i,j,+}$ and $d_{i,j,-}$ over all $\zeta_{i,j} \in J_{i,R_0+h/2}$ which is less than $4\epsilon|J_{i,R_0+h/2}|$. We move

$$g_i \setminus \bigcup_{\zeta_{i,j} \in J_{i,R_0+h/2}} \zeta_{i,j} \quad \text{to} \quad \tilde{g}_i \setminus \bigcup_{\zeta_{i,j} \in J_{i,R_0+h/2}} \tilde{\zeta}_{i,j}$$

by perpendicular projections, and hence, the endpoints of $\zeta_{i,j}$ for $\zeta_{i,j} \in J_{i,R_0+h/2}$ moving to $\tilde{\zeta}_{i,j}$ gives us the divergence functions. The sum of the values of the divergence functions at $t_{i,j}, \hat{t}_{i,j}$ for the endpoints of $\zeta_{i,j} \in \hat{J}_{i,0} \setminus J_{i,R_0+h/2}$ in a component of $g_i \setminus \bigcup_{\zeta_{i,j} \in J_{i,R_0+h/2}} \zeta_{i,j}$, is less than 2 times the sum of the values of its endpoints by (4.28) and Lemma 2.1.

Since each endpoint of $\zeta_{i,j} \in J_{i,R_0+h/2}$ moves less than ϵ , we have by (4.29)

$$\sum_{\zeta_{i,j} \in \hat{J}_{i,0} \setminus J_{i,R_0+h/2}} |\alpha(\tilde{\zeta}_{i,j}) - \alpha(\zeta_{i,j})| \leq 10\mu(R_0 + h/2)\epsilon|J_{i,R_0+h/2}|. \quad (4.30)$$

As in the third paragraph above, for arcs $\hat{\zeta}_{i,j} \in J_{i,R_0+h/2}$, we have

$$|\alpha(\tilde{\zeta}_{i,j}) - \alpha(\hat{\zeta}_{i,j})| \leq 5\mu(R_0 + h/2)\epsilon.$$

Hence,

$$\sum_{\zeta_{i,j} \in J_{i,R_0+h/2}} |\alpha(\tilde{\zeta}_{i,j}) - \alpha(\hat{\zeta}_{i,j})| \leq 5\mu(R_0 + h/2)\epsilon|J_{i,R_0+h/2}|. \quad (4.31)$$

For α -values outside these, we integrate η projected to the neutral bundle over

$$g_i \setminus \bigcup_{\zeta_{i,j} \in \hat{J}_{i,0}} \zeta_{i,j} \quad \text{and} \quad \tilde{g}_i \setminus \bigcup_{\zeta_{i,j} \in \hat{J}_{i,0}} \tilde{\zeta}_{i,j},$$

they all happen inside $S \setminus E_{5/8+\epsilon}$. By Lemmas 2.1 and 2.2, the absolute value of the α -value difference is bounded above by the neutral factor $C_{S \setminus E_{5/8+\epsilon}}$ times 2 times the sum of perpendicular distances at the endpoints of the corresponding arcs. These values are from endpoints of arcs in $\hat{J}_{i,0}$ considered by a paragraph above (4.29) or endpoints of arcs in $J_{R_0+h/2}$. Hence the absolute value of the α -value difference is bounded above by $4\epsilon C_{S \setminus E_{5/8+\epsilon}}|J_{i,R_0+h}|$.

Hence, we obtain by (4.30) and (4.31) and the assumptions in (i).

$$\begin{aligned} |\alpha(\tilde{g}_i) - \hat{\alpha}(\tilde{g}_i)| &\leq |J_{i,R_0+h/2}|(4C_{S \setminus E_{5/8+\epsilon}}\epsilon + 15\mu(R_0 + h/2)\epsilon), \\ 15(R_0\epsilon + h\epsilon/2) &= 15 \times 2 \times 7 + 15h\epsilon/2 < 218. \end{aligned} \quad (4.32)$$

(iv) Lastly, we apply the above to complete the convergences to ∞ . At (i), we chose above a sufficiently small ϵ so that $C_{S \setminus E_1} \epsilon < \mu$. Since $C_{S \setminus E_1} \geq C_{S \setminus E_{5/8+\epsilon}}$, we obtain

$$\begin{aligned} \alpha(g_i) &= \hat{\alpha}(\tilde{g}_i) + \sum_{\zeta \in J_{i,R_0+h/2}} (\alpha(\zeta) - \alpha(\tilde{\zeta})) \\ &\geq \alpha(\tilde{g}_i) + \sum_{\zeta \in J_{i,R_0+h/2}} (\alpha(\zeta) - \alpha(\tilde{\zeta}) - (4C_{S \setminus E_1} \epsilon + 218\mu)) \\ &\geq \alpha(\tilde{g}_i) + \sum_{\zeta \in J_{i,R_0+h/2}} \mu_{\min} C_{R_0+h/2, C'}^{(4.6)} r(\zeta)^2 \quad \text{for } C' := \frac{222\mu}{\mu_{\min}}, \end{aligned} \quad (4.33)$$

by Lemma 4.6 and (4.32).

Now we can show that $\alpha(g_i) \rightarrow \infty$ provided $l_{\mathbb{S}_+}(g_i) \rightarrow \infty$: Suppose that $l_{\mathbb{S}_+}(g_i) \rightarrow \infty$. If $\alpha(\tilde{g}_i) \rightarrow \infty$, then $\alpha(g_i) \rightarrow \infty$ by (4.33), and we are done. Suppose that $\alpha(\tilde{g}_i)$ is bounded above. Then $l_{\mathbb{S}_+}(\tilde{g}_i)$ is also bounded above by Lemma 1.4. Since $r(\tilde{\zeta}_{i,j}) \geq R_0$, we have $l_{\mathbb{S}_+}(\tilde{\zeta}_{i,j}) \geq l_{\mathbb{S}_+}(\zeta_{i,j}) - 2\epsilon = 2\operatorname{arcsinh}(R_0) - 2\epsilon$ by (4.12). Since $R_0 > 10$, $\operatorname{arcsinh}(10) > 2.99$, $1/8 > \epsilon$ by assumptions in (i), it follows that $|J_{i,R_0+h/2}|$ is bounded above. Only possibility is $r(\zeta_{i,j}) \rightarrow \infty$ for some members $\zeta_{i,j}$ of $J_{i,R_0+h/2}$ in order that $l_{\mathbb{S}_+}(g_i) \rightarrow \infty$. This also implies $\alpha(g_i) \rightarrow \infty$ by (4.33).

Now we go to the ratio limit. Notice that

$$\sum_{\zeta_{i,j} \in J_{i,0}} \exp\left(\frac{-t_{i,j}}{2}\right) r(\zeta_{i,j}) \leq \sum_{\zeta_{i,j} \in J_{i,R_0+h/2}} \exp\left(\frac{-t_{i,j}}{2}\right) r(\zeta_{i,j}) + \sum_{\zeta_{i,j} \in J_{i,0} \setminus J_{i,R_0+h/2}} \exp\left(\frac{-t_{i,j}}{2}\right) r(\zeta_{i,j}). \quad (4.34)$$

The second term is bounded above by a constant since each term is bounded above. This term can be absorbed into $C_{\mathcal{K}}$ in (4.35).

We obtain by (4.27), (4.33), and (4.34) that

$$\frac{\alpha(g_i)}{\mathbf{b}_{g_i,-}} \geq \frac{\alpha(\tilde{g}_i) + \sum_{\zeta \in J_{i,R_0+h/2}} \mu_{\min} C_{R_0+h/2, C'}^{(4.6)} r(\zeta)^2}{\tilde{C}(F, \mathcal{K}) \sum_{\zeta_{i,j} \in J_{i,R_0+h/2}} \exp\left(\frac{-t_{i,j}}{2}\right) r(\zeta_{i,j}) + C_2(\mathcal{K}, N(S_C \setminus E)) + C_{\mathcal{K}}}. \quad (4.35)$$

If $J_{i,R_0+h/2} = \emptyset$ for infinitely many i , then $\alpha(g_i) \rightarrow \infty$ up to a choice of a subsequence by Lemma 1.4. Since the nominator is a sum of bounded constants, we are done. Suppose not and that we have a sequence such that $\sum_{\zeta_{i,j} \in J_{i,R_0+h/2}} \exp(-t_{i,j}/2) r(\zeta_{i,j}) \rightarrow 0$ as $i \rightarrow \infty$. Define $\hat{t}_i$ to be the first $t_{i,j}$, where $r(\zeta_{i,j}) > R_0 + h/2$. This means that $\hat{t}_i \rightarrow \infty$ and $l_{\mathbb{S}_+}(\tilde{g}_i) \rightarrow \infty$ and $\tilde{g}_i \in S \setminus E_{R_0+h/2+\epsilon}$. By Lemma 1.4, $\alpha(\tilde{g}_i) \rightarrow \infty$, and we are done for the purpose of Section 4.5.3.

Since we need to show the result for subsequences only, we may assume that

$$C_0 \tilde{C}(F, \mathcal{K}) \sum_{\zeta_{i,j} \in J_{i,R_0+h/2}} \exp\left(\frac{-t_{i,j}}{2}\right) r(\zeta_{i,j}) \geq (C_2(\mathcal{K}, N(S_C \setminus E)) + C_{\mathcal{K}})$$

for a constant $C_0 > 0$. Hence, we obtain

$$\begin{aligned} \frac{\alpha(g_i)}{\|\mathbf{b}_{g_i,-}\|} &\geq \frac{\alpha(\tilde{g}_i)}{\tilde{C}(F, \mathcal{K})(1 + C_0) \sum_{\zeta_{i,j} \in J_{i,R_0+h/2}} \exp\left(\frac{-t_{i,j}}{2}\right) r(\zeta_{i,j})} \\ &\quad + \frac{\mu_{\min} C_{R_0+h/2, b, C'}^{(4.6)} \sum_{\zeta \in J_{i,R_0+h/2}} r(\zeta)^2}{\tilde{C}(F, \mathcal{K})(1 + C_0) \sum_{\zeta_{i,j} \in J_{i,R_0+h/2}} \exp\left(\frac{-t_{i,j}}{2}\right) r(\zeta_{i,j})}. \end{aligned}$$

We define

$$\begin{aligned} \tilde{e}_i &:= \left(\exp\left(\frac{t_{i,j}}{2}\right) \right)_{\zeta_{i,j} \in J_{i,R_0+h/2}}, \\ \tilde{r}_i &:= (r(\zeta_{i,j}))_{\zeta_{i,j} \in J_{i,R_0+h/2}} \in \mathbb{R}^{|J_{i,R_0+h/2}|}, \\ \|\tilde{v}\|_{i,R_0+h/2} &:= \sqrt{\tilde{v} \cdot \tilde{v}}, \tilde{v} \in \mathbb{R}^{|J_{i,R_0+h/2}|}. \end{aligned}$$

Using the Schwarz inequality

$$\left| \sum_{\zeta_{i,j} \in J_{i,R_0+h/2}} \left(\frac{-t_{i,j}}{2} \right) r(\zeta_{i,j}) \right| \leq \|\tilde{e}_i\|_{i,R_0+h/2} \|\tilde{r}_i\|_{i,R_0+h/2},$$

we obtain that (4.35) is bigger than or equal to $\frac{1}{\tilde{C}(F, \mathcal{K})}$ times

$$\frac{\alpha(\tilde{g}_i)}{(1 + C_0)\|\tilde{e}_i\|_{i, R_0+h/2}\|\tilde{r}_i\|_{i, R_0+h/2}} + \frac{\mu_{\min} C_{R_0+h/2, b, C'}^{(4.6)} \|\tilde{r}_i\|_{i, R_0+h/2}}{(1 + C_0)\|\tilde{e}_i\|_{i, R_0+h/2}}.$$

For each arc ζ in $J_{i, R_0+h/2}$, there is a corresponding maximal geodesic ζ_f in $\tilde{g}_i$ given by the perpendicular projection and extending to a maximal geodesic arc in E .

Using the perpendicular projection paths at the endpoints and the triangle inequalities, we obtain

$$l_{\mathbb{S}_+}(\tilde{g}_i) \geq l_S\left(\tilde{g}_i \setminus \bigcup_{\zeta \in J_{i, R_0+h/2}} \tilde{\zeta}\right) + \sum_{\zeta \in J_{i, R_0+h/2}} l_{\mathbb{S}_+}(\tilde{\zeta}) \geq l_{S \setminus E}(g_i) + \sum_{\zeta \in J_{i, R_0+h/2}} (l_{\mathbb{S}_+}(\tilde{\zeta}) - 2\epsilon_+(\zeta) - 2\epsilon_-(\zeta))$$

where $\epsilon_+(\zeta)$ and $\epsilon_-(\zeta)$ respectively are the vertical projection path lengths from the forward and backward endpoints of $\zeta \in J_{i, R_0+h/2}$ to the corresponding ones of the arc $\tilde{\zeta}$ in $\tilde{g}_i$. We obtain $l_{\mathbb{S}_+}(\tilde{\zeta}) \geq 2\operatorname{arcsinh}(R_0) > 5.8$ by (4.12) as $R_0 > 10$ by the assumption in (i). Since $\epsilon_{\pm}(\zeta) \leq \epsilon < 1/8$, the positivity of the later terms follows.

By Lemma 1.4, $\alpha(\tilde{g}_i) \geq c_{S \setminus E, R_0/2+h/2+\epsilon}^{(1.4)} l_{\mathbb{S}_+}(\tilde{g}_i)$. We obtain (4.35) is bigger than or equal to $\frac{1}{\tilde{C}(F, \mathcal{K})}$ times

$$\frac{c_{S \setminus E, R_0/2+h/2+\epsilon}^{(1.4)} l_{S \setminus E}(g_i)}{(1 + C_0)\|\tilde{e}_i\|_{i, R_0+h/2}\|\tilde{r}_i\|_{i, R_0+h/2}} + \frac{\mu_{\min} C_{R_0+h/2, b, C'}^{(4.6)} \|\tilde{r}_i\|_{i, R_0+h/2}}{(1 + C_0)\|\tilde{e}_i\|_{i, R_0+h/2}}.$$

Now, this is a function converging to ∞ as

$$\max\{l_{S \setminus E}(g_i), \|\tilde{r}_i\|_{i, R_0+h/2}\} \rightarrow \infty.$$

Suppose that $l_{\mathbb{S}_+}(g_i) \rightarrow \infty$. Then we claim that $\max\{l_{S \setminus E}(g_i), \|\tilde{r}_i\|_{i, R_0+h/2}\} \rightarrow \infty$:

Suppose that $l_{S \setminus E}(g_i)$ is bounded. Then the number of maximal geodesic arcs of g_i going into $S \setminus E$ is finite by (4.10). Then $r(\zeta_{i,j}) \rightarrow \infty$ for some index (i, j) as $i \rightarrow \infty$ since otherwise we will have $l_{\mathbb{S}_+}(g_i)$ bounded by (4.12). Hence, $\|\tilde{r}_i\|_{i, R_0+h/2} \rightarrow \infty$.

Conversely, suppose that $\{\|\tilde{r}_i\|_{i, R_0+h/2}\}$ is bounded above. If $|J_{i, R_0+h/2}| \rightarrow \infty$, then $l_{S \setminus E}(g_i) \rightarrow \infty$ by (4.28). Otherwise, if $|J_{i, R_0+h/2}|$ is bounded, there is an upper bound to the absolute values of the coordinates of $\tilde{r}_i$ and $l_{\mathbb{S}_+}(\zeta)$ for $\zeta \in J_{i, R_0+h/2}$, implying the absurdity that $l_{\mathbb{S}_+}(g_i)$ is bounded above.

We are done proving the main aim of Section 4.5.3.

4.5.4 The direction result

(VI) We come to the last step.

Theorem 4.8. Assume Criterion 1.1 and $\mathcal{L}(\Gamma) \subset \operatorname{SO}(2, 1)^0$. Let η be a $\mathcal{V}$ -valued 1-form corresponding to the boundary cocycle for Γ . Let $\mathcal{K}$ be a compact subset of $\mathcal{CH}(\Lambda_{\Gamma, \mathbb{S}_+}) \setminus \mathcal{H}$. For every sequence $\{g_i\}$ with $\{l_{\mathbb{S}_+}(g_i)\} \rightarrow \infty$ of elements of $\Gamma_{\mathcal{K}}$, the following hold:

- $\{\|\mathbf{b}_{g_i}\|_E\} \rightarrow \infty$.
- $\alpha(g_i) \rightarrow \infty$ and $\alpha(g_i)/\|\mathbf{b}_{g_i, -}\|_E \rightarrow \infty$.
- $\{\mathbf{d}((\mathbf{b}_{g_i}), \operatorname{Cl}(\zeta_{a_{g_i}}))\} \rightarrow 0$.

Proof. The first item follows since otherwise $g_i(O)$ is in a bounded set contradicting the properness of the Γ -action.

By Lemma 4.5, we may also assume that

$$\{\mathbf{v}_{+, (x_i, \mathbf{u}_i)}\} \rightarrow \mathbf{v}_+, \quad v_{g_i} \rightarrow v, \quad \text{and} \quad \{\mathbf{v}_{-, (x_i, \mathbf{u}_i)}\} \rightarrow \mathbf{v}_- \quad (4.36)$$

for an independent set of vectors $\mathbf{v}_+$, v , $\mathbf{v}_-$ by choosing subsequences if necessary. These are all positively oriented in E . Let $\mathcal{C}_{\infty}$ denote the matrix with columns $\mathbf{v}_+$, v , and $\mathbf{v}_-$.

We showed in Section 4.5.3 that

$$\alpha(g_i) \rightarrow \infty \quad \text{and} \quad \frac{\alpha(g_i)}{\|\mathbf{b}_{g_i, -}\|} \rightarrow \infty \quad \text{as} \quad l_{\mathbb{S}_+}(g_i) \rightarrow \infty.$$

Hence, for $i \rightarrow \infty$

$$\{(\|\mathbf{b}_{g_i,+}\|_E : \|\mathbf{b}_{g_i,0}\|_E : \|\mathbf{b}_{g_i,-}\|_E)\} \rightarrow (\pm 1 : 0 : 0) \text{ or } (*_1 : *_2 : 0),$$

where $*_2 \geq 0$ since $\alpha(g_i) > 0$ by Criterion 1.1. Then (4.36) implies

$$\{\mathbf{d}(\langle \mathbf{b}_{g_i} \rangle), \text{Cl}(\zeta_{a_{g_i}})\} \rightarrow 0 \quad \text{as } l_{\mathbb{S}_+}(g_i) \rightarrow \infty$$

by the above conclusion. $\square$

4.6 Accumulation points of Γ -orbits

Recall $N_{\mathbf{d},\epsilon}(\cdot)$ from Section 2.3. We again use $\mathbf{d}_H$ in $\mathbb{S}^3$. We say that $\gamma_i(K)$ for a compact set K and a sequence γ_i *accumulates only to* a set A if $\gamma_i(z_i)$, $z_i \in K$ has accumulation points only in A . Of course, the same definition extends to the case when K is a point. It is easy to see that this condition is equivalent to the condition that

$$\text{for every } \epsilon > 0, \text{ there is } I \text{ so that } \gamma_i(K) \subset N_{\mathbf{d},\epsilon}(A) \text{ for } i > I.$$

(For the point case, we need to change the symbol $\subset$ to the symbol $\in$.)

Corollary 4.9. *Assume Criterion 1.1 and $\mathcal{L}(\Gamma) \subset \text{SO}(2, 1)^0$. Let $K \subset E$ be a compact subset. Let $y \in \mathbb{S}_+$, and let $\gamma_i \in \Gamma$ be a sequence such that $\{\gamma_i(y)\} \rightarrow y_\infty$ for $y_\infty \in \partial\mathbb{S}_+$. Then for every $\epsilon > 0$, there exists I_0 such that*

$$\gamma_i(K) \subset N_{\mathbf{d},\epsilon}(\text{Cl}(\zeta_{y_\infty})) \quad \text{for } i > I_0. \quad (4.37)$$

Equivalently, any sequence $\{\gamma_i(z_i) : z_i \in K\}$ accumulates only to $\text{Cl}(\zeta_{y_\infty})$.

Proof. It is enough to prove for subsequences of every subsequence that the conclusion holds. To obtain all limit points of $\{\gamma_i(K)\}$, we will use the fact that Γ acts as a convergence group on $\partial\mathbb{S}_+$ from Section 4.1. Up to choosing subsequences, we assume that $\{\gamma_i\}$ is a convergence sequence with the attracting point a and the repelling point r .

We first consider the case $a \neq r$. Then γ_i acts on a geodesic l_i in $\mathbb{S}_+$ passing a compact set $\mathcal{K}$ for sufficiently large i . Let $x_i \in \mathcal{K} \cap l_i$, where l_i is given the direction $\mathbf{u}_i$ so that γ_i acts in the forward direction. Using the notation of the proof of Theorem 4.8, we have

$$a(\gamma_i) = (\mathbf{v}_{+, (x_i, \mathbf{u}_i)}), \quad r(\gamma_i) = (\mathbf{v}_{-, (x_i, \mathbf{u}_i)}).$$

We only need to consider subsequences $\{\gamma_i\}$, $\gamma_i \in \Gamma$, where the sequence $a(\gamma_i) \in \partial\mathbb{S}_+$ of attracting fixed points and the sequence $r(\gamma_i) \in \partial\mathbb{S}_+$ of repelling fixed points are both convergent. Here,

$$\{a(\gamma_i)\} \rightarrow a \quad \text{and} \quad \{r(\gamma_i)\} \rightarrow r \quad \text{in } \partial\mathbb{S}_+.$$

Since $\{\gamma_i(y)\} \rightarrow y_\infty \in \partial\mathbb{S}_+$, it follows that γ_i is unbounded in Γ and hence $\{l_{\mathbb{S}_+}(\gamma_i)\} \rightarrow \infty$, and $\{\lambda(\gamma_i)\} \rightarrow \infty$ for the largest eigenvalue $\lambda(\gamma_i)$ of γ_i .

The convergences are uniform on the compact set $K \subset E$. To explain, we recall (4.36). We introduce the $(x^{(i)}, y^{(i)}, z^{(i)})$ -coordinate system where

$$\mathbf{v}_{+, (x_i, \mathbf{u}_i)}, \quad \mathbf{v}_{\gamma_i}, \quad \text{and} \quad \mathbf{v}_{-, (x_i, \mathbf{u}_i)}$$

form a coordinate basis parallel to the $x^{(i)}$ -, $y^{(i)}$ -, and $z^{(i)}$ -axes respectively. We let x, y, z denote the coordinate functions, where $(\mathbf{v}_+, \mathbf{v}_-, \mathbf{v}_\gamma)$ forms a coordinate basis.

Note that K is in a region R_i given by

$$[-C_1, C_1] \times [-C_2, C_2] \times [-C_3, C_3]$$

in the $(x^{(i)}, y^{(i)}, z^{(i)})$ -coordinate system. We may assume C_1, C_2, C_3 are independent of i since the coordinate functions $x^{(i)}$, $y^{(i)}$, and $z^{(i)}$ converge respectively to coordinate functions x, y , and z on E . We write $\gamma_i(x) = A_{\gamma_i}x + \mathbf{b}_{\gamma_i}$. Since the sequence of largest eigenvalues of the linear parts of γ_i goes to $+\infty$, it follows

that $A_{\gamma_i}(R_i)$ is given under the $(x^{(i)}, y^{(i)}, z^{(i)})$ -coordinate system by

$$[-D_i, D_i] \times [-E_i, E_i] \times [-F_i, F_i]$$

where $\{D_i\} \rightarrow \infty$, $E_i = C_2$, $\{F_i\} \rightarrow 0$ for $F_i > 0$. By Definition 4.4, $\gamma_i(R_i)$ is in

$$S_i := [-\infty, \infty] \times [-E_i + \alpha(\gamma_i), E_i + \alpha(\gamma_i)] \times \left[-F_i - \frac{\|\mathbf{b}_{\gamma_i, -}\|_E}{\|\mathbf{v}_{-, (x_i, \mathbf{u}_i)}\|_E}, F_i + \frac{\|\mathbf{b}_{\gamma_i, -}\|_E}{\|\mathbf{v}_{-, (x_i, \mathbf{u}_i)}\|_E} \right] \quad (4.38)$$

in the $(x^{(i)}, y^{(i)}, z^{(i)})$ -coordinate system. Recall coordinate change maps $\mathcal{C}_i$ and $\mathcal{C}_\infty$ near (4.36). For sufficiently large i , we deduce that $\gamma_i(R_i)$ is a subset of $N_{\mathbf{d}, \epsilon}(\text{Cl}(\zeta_a))$ as follows: There is a sequence of coordinate change maps $h_i : E \rightarrow E$ with a uniformly bounded matrix $\mathcal{C}_\infty \mathcal{C}_i^{-1}$ such that

$$x \circ h_i = x^{(i)}, \quad y \circ h_i = y^{(i)}, \quad z \circ h_i = z^{(i)}.$$

Since

$$x^{(i)} \rightarrow x, \quad y^{(i)} \rightarrow y, \quad z^{(i)} \rightarrow z$$

by (4.36), we obtain $h_i \rightarrow I_{\mathbb{S}^3}$ as $i \rightarrow \infty$. What h_i does is to send a box in the $(x^{(i)}, y^{(i)}, z^{(i)})$ -coordinate system to the box of the same coordinates in the (x, y, z) -coordinate system.

Since $\alpha(\gamma_i) \rightarrow \infty$ and $\alpha(\gamma_i)/\|\mathbf{b}_{\gamma_i, -}\|_E \rightarrow \infty$ by Theorem 4.8, equation (4.38) implies that $h_i(S_i) \rightarrow \text{Cl}(\zeta_a)$ geometrically. Since $h_i \rightarrow I_{\mathbb{S}^3}$, we deduce that $S_i \rightarrow \text{Cl}(\zeta_a)$ by Corollary 2.5. Hence, for every $\epsilon > 0$, we have

$$\gamma_i(R_i) \subset S_i \subset N_{\mathbf{d}, \epsilon}(\text{Cl}(\zeta_a))$$

for sufficiently large i , and (4.37) holds.

Finally, suppose that $a = r$. We choose γ so that

$$a\left(\lim_{i \rightarrow \infty} \gamma \gamma_i\right) = \gamma(a) \neq r = \lim_{i \rightarrow \infty} r(\gamma_i)$$

and use the sequence $\gamma \gamma_i$ as our convergence sequence. Then $\{\gamma \gamma_i(K)\}$ accumulates only on $\text{Cl}(\zeta_{\gamma(a)}) = \gamma(\text{Cl}(\zeta_a))$. Therefore, $\{\gamma_i(K)\}$ accumulates only on $\text{Cl}(\zeta_a)$. $\square$

We end with the following:

Converse part of Theorem 1.5. Suppose that $\Gamma \subset \text{SO}(2, 1)^o$. To show the proper action of Γ , we show that for any sequence $\{g_i\}$ of infinite elements, $g_i(K) \cap K \neq \emptyset$ for only finitely many elements. Suppose not. Then by taking a subsequence, we may assume that $\gamma_i(y) \rightarrow y_\infty$ for $y_\infty \in \partial \mathbb{S}_+$. By Corollary 4.9, we showed that this cannot happen.

If Γ is not in $\text{SO}(2, 1)^o$, then we use the index 2 subgroup $\Gamma' \subset \text{SO}(2, 1)^o$ and it acts properly on E and so does Γ . $\square$

5 The topology of Margulis space-times with parabolics

We first give an outline of this long section. We discuss the classical theory of Scott and Tucker [43] on open 3-manifolds homotopy equivalent to compact ones. Next, we will construct parabolic regions in $\tilde{M}$.

In Section 5.2, we will find a fundamental region for Γ in E using the work of Epstein and Petronio [24]. By Proposition 5.1, we obtain an exhausting sequence

$$M_{(1)} \subset M_{(2)} \subset M_{(3)} \subset \cdots \subset E/\Gamma.$$

In Section 5.2.2, we discuss some boundedness properties of the inverse image $\tilde{M}_{(J)}$ of $M_{(J)}$ for some J meeting with disks and topological polytopes. First, we construct the candidate disks to bound a candidate fundamental domain. The key step is Proposition 5.5 that the universal cover $\tilde{M}_{(J)}$ of an element $M_{(J)}$ of the exhausting sequence meets the candidate disks and parabolic regions in bounded sets. This implies Corollary 5.7 that $\tilde{M}_{(J)}$ meets a candidate fundamental domain $\mathbf{F}$ in a compact submanifold and hence $\mathbf{F} \setminus \tilde{M}_{(J)}$ is

a compact finite-sided topological polytope. In Section 5.2.3, we choose our candidate disks $\mathcal{D}_j$, $j = 1, \dots, \mathbf{g}$, and the candidate fundamental domain $\mathbf{F}$. Then we divide $\tilde{M}$ into $\tilde{M}_{(J)}$ and $\tilde{M} \setminus \tilde{M}_{(J)}$. We show $\mathbf{F} \setminus \tilde{M}_{(J)}$ for sufficiently large J is the fundamental domain of $\tilde{M} \setminus \tilde{M}_{(J)}$ using Proposition 2.6 (the Poincaré fundamental domain theorem). Candidate disks in $\tilde{M}$ are replaced by ones mapping to embedded disks in M by replacing the parts in $\tilde{M}_{(J)}$ by Theorem 5.3, i.e., Dehn's lemma. We obtain the fundamental domain of $\tilde{M}_{(J)}$, proving the tameness of M , and the first part of Theorem 1.1.

In Section 5.3, we will show that for a choice of parabolic regions sufficiently far from $\tilde{M}_{(J)}$, their images under Γ are mutually disjoint. To show this, we use the tessellations by the images of a fundamental domain, and we explain how they intersect with the parabolic regions. Then we can account for every image by its relationship with the images of the fundamental domain.

In Section 5.4, we will discuss the relative compactification of $\tilde{M}$. We will prove the final part of Theorem 1.1 and Corollary 1.2. (See Marden [36–38] for many aspects of ideas in this section.)

5.1 Handlebody exhaustion of the Margulis space-times

The ends of $\mathbb{S}_+/\mathcal{L}(\Gamma)$ are finitely many, and some of these are cusps. A *peripheral* element of Γ is an element corresponding to a closed loop in the complete hyperbolic surface freely homotopic to one in an end neighborhood homeomorphic to an annulus. Let $\mathcal{J}'$ denote the collection of the maximal peripheral cyclic subgroups of Γ , and let $\mathcal{J}$ denote the ones with hyperbolic holonomy. Each peripheral element of Γ acts on a point of $\partial\mathbb{S}_+$ as a parabolic element or on a connected arc $a_i \subset \partial\mathbb{S}_+$, $i \in \mathcal{J}$ with the hyperbolic cyclic group $\langle \mathcal{G}_i \rangle$ acting on it. Here,

$$\Sigma_+ := \left(\mathbb{S}_+ \cup \bigcup_{i \in \mathcal{J}} a_i \right) / \Gamma$$

is a finite-type surface with finitely many punctures and boundary components covered by arcs of the form a_i .

We define $A_i := \bigcup_{x \in a_i} \zeta_x$, $i \in \mathcal{J}$, an open domain where ζ_x is the accordant great segment for x . We define

$$\tilde{\Sigma} := \mathbb{S}_+ \cup \mathbb{S}_- \cup \bigcup_{i \in \mathcal{J}} (A_i \cup a_i \cup \mathcal{A}(a_i)). \quad (5.1)$$

Then Γ acts properly on $\tilde{\Sigma}$, and $\Sigma := \tilde{\Sigma}/\Gamma$ is a real projective surface. This follows by the same proof as [16, Theorem 5.3] without change. Again, Σ has twice the number of punctures as Σ_+ and $\chi(\Sigma) = 2\chi(\Sigma_+)$.

We define $\tilde{N} := E \cup \tilde{\Sigma}$. We will show below that Γ acts properly on $\tilde{N}$ to give us a manifold quotient $\tilde{N}/\Gamma$:

Let N be a manifold. A sequence N_i of submanifolds of N is *exhausting* if $N_i \subset N_{i+1}$ for all i and every compact subset of N is a subset of N_i for some i . We obtain $N = \bigcup_{i=1}^{\infty} N_i$ necessarily.

The following is essentially due to Scott and Tucker [43], which we learned from some talks by Ohshika [42] in this form (See also Canary and Minsky [5, p. 5]).

Proposition 5.1. *Let E/Γ be a Margulis space-time with parabolics. Then E/Γ has a sequence of handlebodies*

$$M_{(1)} \subset M_{(2)} \subset \dots \subset M_{(i)} \subset M_{(i+1)} \subset \dots$$

so that $M_0 = \bigcup_{i=1}^{\infty} M_{(i)}$. They have the following properties:

- $\pi_1(M_{(1)}) \rightarrow \pi_1(M)$ is an isomorphism.
- The inverse image $\tilde{M}_{(i)}$ of $M_{(i)}$ in $\tilde{M}$ is connected.
- $\pi_1(M_{(i)}) \rightarrow \pi_1(M)$ is surjective.
- For each compact subset $K \subset E/\Gamma$, there exists an integer I so that for $i > I$, $K \subset M_{(i)}$.

Proof. The existence of exhaustion is clear. We choose $M_{(1)}$ by using the 1-complex homotopy equivalent to M . $\pi_1(M_{(i)}) \rightarrow \pi_1(M)$ is surjective since $\pi_1(M_{(1)}) \rightarrow \pi_1(M)$ factors into this map and $\pi_1(M_{(1)}) \rightarrow \pi_1(M_{(i)})$. Choose a base point x_0 of $M_{(1)}$. Any closed loop in M with a basepoint in M_1 is homotopic to a closed loop in $M_{(i)}$. Hence, any two points of the inverse image x_0 in $\tilde{M}$ is connected by a path in $\tilde{M}_{(i)}$ by the homotopy path-lifting theorem of Poincaré. Thus, $\tilde{M}_{(i)}$ is connected. $\square$

5.1.1 Parabolic solid-torus regions

Let $S := S_+/\mathcal{L}(\Gamma)$. It has finitely many ends. Some of these are cusp ends, and some are hyperbolic ends. Note that Γ has parabolics

$$\mathbf{g}_1, \dots, \mathbf{g}_{m_0},$$

each of which represents a generator of the fundamental group of a cusp neighborhood of S . We let each of

$$\mathbf{g}_{m_0+1}, \dots, \mathbf{g}_{m_0+h_0}$$

represent the generator of each of the fundamental groups of the hyperbolic end neighborhoods of S . We choose the generators along the boundary orientation of S .

Recall the notations from Section 2.2. We take components $\mathcal{H}_i \subset S_+$, $i \in \mathcal{I}' \setminus \mathcal{J}$ in S_+ of $\mathcal{H}$. A parabolic primitive element $\mathbf{g}_i$ conjugate to $\mathbf{g}_j$ for some j acts on $\mathcal{H}_i$. We also note for every $g \in \Gamma$,

- either $g(\mathcal{H}_i) = \mathcal{H}_i$ and $g = \mathbf{g}_i^n$ for $n \in \mathbb{Z}$, or else
- $g(\mathcal{H}_i) \cap \mathcal{H}_i = \emptyset$.

We define $\mathcal{H}_{i,-} = A(\mathcal{H}_i)$. There is a fixed point p_i of $\mathbf{g}_i$ in $\text{bd}_{S_+} \mathcal{H}_i \cap \partial S_+$ for each $i \in \mathcal{I}' \setminus \mathcal{J}$.

For each $i \in 1, \dots, m_0$, Theorem 3.14 gives us a properly embedded ruled surface $S_i := S_{f_i, r_0} \subset E$ for some fixed function $f_i : (0, 1) \rightarrow \mathbb{R}$ and

$$\text{Cl}(S_i) \setminus S_i = \text{Cl}(\zeta_{a(\mathbf{g}_i)}) \cup \partial_h \mathcal{H}_i \cup \partial_h \mathcal{H}_{i,-},$$

where $a(\mathbf{g}_i)$ is the parabolic fixed point of $\mathbf{g}_i$ in ∂S_+ . The set S_i is called a *parabolic ruled surface*. The component of $E \setminus S_i$ whose closure contains $\mathcal{H}_i^o$ is called a *parabolic region*, denoted by $\mathcal{P}_i$, which is homeomorphic to a 3-cell by Theorem 3.13. These are distinct from parabolic cylinders. Here, f_i is fixed for each conjugacy class of parabolic elements. (See Section 3.1 for detail.)

For each $i \in \mathcal{I}' \setminus \mathcal{J}$, we define $S_i = \gamma(S_j)$ and $\mathcal{P}_i = \gamma(\mathcal{P}_j)$ for any $j, j = 1, \dots, m_0$, and γ so that $\gamma(\mathcal{H}_j) = \mathcal{H}_i$. This surface S_i is well-defined since any element acting on $\mathcal{H}_j$ acts on S_i and $\mathcal{P}_i$. We have the Γ -equivariant choice of parabolic ruled surfaces and parabolic regions.

Theorem 3.14 gives us a foliation $\mathcal{S}_{f_i, r_i}$ with leaves that are parabolic ruled surfaces and a transversal foliation $\mathcal{D}_{f_i, r_i}$ for each $\mathcal{P}_i$ for each $i = 1, \dots, m_0$. For other $\mathcal{P}_i$, we use the induced ones from $\mathcal{P}_j$ such that $\mathcal{P}_i = \gamma(\mathcal{P}_j)$ for $j = 1, \dots, m_0$.

Finally, we will make these S_i and $\mathcal{P}_i$ sufficiently far whenever it is necessary to do so in this paper. (See Definition 3.5.) We may do so without acknowledging.

5.2 Finding the fundamental domain

A *topological polytope* in E is a 3-manifold closed as a subset of E and whose closure in $\text{Cl}(E)$ is a compact manifold with boundary that is a union of finitely many smoothly and properly embedded compact submanifold. In [16], we defined a *crooked circle* to be a simple closed curve in S of the form

$$d \cup A(d) \cup \bigcup_{x \in \partial d} \text{Cl}(\zeta_x)$$

for a complete geodesic d in S_+ with boundary in a parabolic fixed point or in a boundary component of $\tilde{S}_+$. We may refer to them as being *positively oriented* since the definition depends on the orientations of E .

Recall parabolic regions from Section 5.1.1.

Definition 5.1. A *crooked-circle disk* D is a properly embedded open disk in E whose boundary ∂D is a crooked circle satisfying the condition: If x is a parabolic fixed point in ∂d and $\mathcal{P}_i$ is a sufficiently far away parabolic region for x , $\mathcal{P}_i \cap D$ is a ruled surface in a leaf of the transversal foliation $\mathcal{D}_{f_i, r_i}$ obtained as in Theorem 3.14.

A disk D in E is *separating* if it is properly embedded and $E \setminus D$ has two components. Crooked-circle disks and parabolic ruled surfaces are separating.

5.2.1 The simple case of the properly acting parabolic cyclic group

Theorem 5.2 is a much easier version of that of Theorem 1.1 presented analogously.

Theorem 5.2 (Small tameness). *Assume as in Theorem 1.1. Suppose that D is a crooked-circle disk in E with a point $p \in \partial D$ fixed by a parabolic element γ with a positive Charette–Drumm invariant. Then we can modify D inside a compact set in E so that $D \cap \gamma(D) = \emptyset$. If we denote F_p to be the connected domain in E bounded by D and $\gamma(D)$, then F_p is a fundamental domain of $\langle \gamma \rangle$ in E . Furthermore, $E/\langle \gamma \rangle$ is homeomorphic to a solid torus.*

Proof. We take an arbitrary compact set K in E . Then there exists a sufficiently far away parabolic region R'_p where γ acts so that $K \cap R'_p = \emptyset$. We have

$$\bigcup_{n \in \mathbb{Z}} \gamma^n(K) \cap R'_p = \emptyset \quad (5.2)$$

since R'_p is γ -invariant.

By taking sufficiently large K , we may assume that $\tilde{T} := \bigcup_{n \in \mathbb{Z}} \gamma^n(K)$ is connected. By the proper discontinuity of the action of $\langle \gamma \rangle$, K meets only finitely many $\gamma^n(K)$. Choose K as a generic 3-ball so that $T := \tilde{T}/\langle \gamma \rangle$ is a compact manifold.

We take a sequence of generic compact 3-balls K_i exhausting E . Then the corresponding $T_i, i = 1, 2, \dots$, form an exhausting sequence of compact 3-manifolds of $E/\langle \gamma \rangle$. We denote $\tilde{T}_i := \bigcup_{n \in \mathbb{Z}} \gamma^n(K_i)$.

(I) We first show that $\tilde{T}_i$ meet with D in a compact set and find a candidate fundamental domain F bounded by two disks in a compact set.

By Theorem 1.5 and Corollary 4.9, $\gamma^n(K_i)$ as $n \rightarrow \pm\infty$ can have accumulation points only in $\text{Cl}(\zeta_p)$. We have $D \cap R'_p \cap \tilde{T}_i = \emptyset$ by (5.2) for sufficiently far choice of R'_p . Since $\{\gamma^n(K_i) : n \in \mathbb{Z}\}$ is a locally finite collection of sets in E accumulating only to $\text{Cl}(\zeta_p)$ by Corollary 4.9, and $D \setminus R'_p$ is $\mathbf{d}$ -bounded away from $\text{Cl}(\zeta_p)$, it follows that $(D \setminus R'_p) \cap \tilde{T}_i$ is compact. Hence, $D \cap \tilde{T}_i$ is compact for each i . Similarly, so is $\gamma(D) \cap \tilde{T}_i$.

By construction in Definition 5.1 and Theorem 3.14,

$$D \cap \gamma(D) \cap R'_p = \emptyset \quad \text{and} \quad (\partial D \cap \gamma(\partial D)) \setminus \text{Cl}(R'_p) = \emptyset.$$

Since $D \cap R'_p$ is a ruled disk so that $\gamma(D \cap R'_p) \cap D \cap R'_p = \emptyset$, we can find a thin tubular neighborhood T'' in $\text{Cl}(D \setminus R'_p)$ of $\partial \text{Cl}(D \setminus R'_p)$ so that $T'' \cap \gamma(T'') = \emptyset$. We add the disk $D \cap R'_p$ to T'' to obtain T' . Hence, we have $\gamma(T') \cap T' = \emptyset$.

We modify the disk $\gamma(D) \setminus \gamma(T')$ to another disk D_1 to be disjoint from D . Then D and D_1 bound a topological polytope F closed in E .

Choose a sufficiently large i so that

$$D \setminus T', D_1 \setminus \gamma(T'), D \cap D_1 \subset \tilde{T}_i,$$

and we choose sufficiently far R'_p so that $R'_p \cap \gamma^n(K_i) = \emptyset$ for every $n \in \mathbb{Z}$. We obtain that

$$T' \setminus \tilde{T}_i = D \setminus \tilde{T}_i \quad \text{and} \quad \gamma(T') \setminus \tilde{T}_i = \gamma(D) \setminus \tilde{T}_i$$

is a matching set under $\{\gamma, \gamma^{-1}\}$. They are also in $\text{bd}F \cap E$.

Also, $(F \setminus R'_p) \cap \tilde{T}_i$ is again compact in E since $F \setminus R'_p$ is $\mathbf{d}$ -bounded away from $\text{Cl}(\zeta_p)$ and $\gamma^n(K_i)$ accumulates only to $\text{Cl}(\zeta_p)$ by Corollary 4.9. Since $R'_p \cap \tilde{T}_i = \emptyset$, it follows that $F \cap \tilde{T}_i$ is compact, and $F \setminus \tilde{T}_i^o$ is a topological polytope.

(II) We find a fundamental domain that is a topological polytope.

Since $\{T_j \setminus \tilde{T}_i^o : j > i\}$ is an exhausting sequence of $E/\langle \gamma \rangle \setminus \tilde{T}_i^o$, Proposition 2.6 implies that

$$T' \cap (F \setminus \tilde{T}_i^o) \quad \text{and} \quad \gamma(T') \cap (F \setminus \tilde{T}_i^o)$$

bound a topological polytope $F \setminus \tilde{T}_i^o$ that is a fundamental domain of $E \setminus \tilde{T}_i^o$ under $\langle \gamma \rangle$.

We choose a generic set denoted by T_i so that $D \cap \tilde{T}_i$ is a union of simple closed curves. The image in $E/\langle \gamma \rangle$ of the bounded component of $D \setminus \tilde{T}_i^o$ is embedded since $F \setminus \tilde{T}_i^o$ is a fundamental domain of $E \setminus \tilde{T}_i^o$ under $\langle \gamma \rangle$. We take mutually disjoint tubular neighborhoods of the images of these bounded components

in $E/\langle \gamma \rangle \setminus T_i^o$ whose lifts in $E \setminus \tilde{T}_i^o$ are disjoint. We add these tubular neighborhoods to $\tilde{T}_i$, and now each component of $D \cap \tilde{T}_i$ is a disk.

Theorem 5.3 (Dehn's lemma, see Hempel [32]). *Let M' be a 3-manifold M and let $f : B \rightarrow M$ be a map from a disk B such that for some neighborhood A of the boundary ∂B in B . If $f|_A$ is an embedding and $f^{-1}(f(A)) = A$, then $f|_{\partial B}$ extends to an embedding $g : B \rightarrow M$.*

By Theorem 5.3, we replace the images in T_i of disk components of $D \cap \tilde{T}_i$ by embedded disks in T_i . We lift these disks to $\tilde{T}_i$ and attach the adjacent ones to $D \setminus \tilde{T}_i^o$. We obtain a disk D'' , and it is clear that $D'' \cap \gamma(D'') = \emptyset$.

We rename D'' by D . Let F_p denote the region in E bounded by D and $\gamma(D)$. Since $\text{Cl}(F_p) \setminus \text{Cl}(R'_p)$ is bounded away from $\text{Cl}(\zeta_p)$ under $\mathbf{d}$, and $\{\gamma^n(K) | n \in \mathbb{Z}\}$ is a locally finite collection of sets in E accumulating only to $\text{Cl}(\zeta_p)$ by Theorem 1.5 and Corollary 4.9, we obtain that $(F_p \setminus R'_p) \cap \tilde{T}_i$ is a compact set. Since $\tilde{T}_i \cap R'_p = \emptyset$, $F_p \cap \tilde{T}_i$ is also compact.

Also, $F_p \cap \tilde{T}_i$ is compact for each i . By Proposition 2.6, F_p is a fundamental domain in E of $\langle \gamma \rangle$. The existence of the fundamental domain tells us that $E/\langle \gamma \rangle$ is tame and hence is homeomorphic to a solid torus. $\square$

Remark 5.1. Using the closure of the fundamental domain F_p and identifying D and $\gamma(D)$, we deduce that

$$\left(E \cup \mathbb{S}_+ \cup \mathbb{S}_- \cup \bigcup_{x \in \partial \mathbb{S}_+ \setminus \{p\}} \text{Cl}(\zeta_x) \right) / \langle \gamma \rangle$$

is homeomorphic to $A \times [0, 1)$ for a compact annulus $A = \bigcup_{x \in \partial \mathbb{S}_+ \setminus \{p\}} \text{Cl}(\zeta_x) / \langle \gamma \rangle$, forming a relative compactification.

As an alternative proof of Theorem 5.2, we may use a γ -invariant foliation of E by crooked planes from the results of Charette and Kim [7] to prove the relative compactification. A fairly simple computation shows that there exists such a foliation in $E/\langle \gamma \rangle$.

5.2.2 The boundedness of $\tilde{M}_{(J)} \cap F$ for some polytope F

We choose an exhausting sequence $M_{(J)}$, $J = 1, 2, 3, \dots$, by Proposition 5.1. We aim to prove Corollary 5.7 showing that the $\tilde{M}_{(J)}$ meets a “candidate” fundamental domain in a bounded set.

Lemma 5.4. *Let R be a conical region in $\mathbb{S}_+$ that is a fundamental domain of a parabolic element γ with p as the fixed point in $\partial \mathbb{S}_+$, and let F_p be a fundamental domain in E of γ bounded by two embedded disjoint crooked-circle disks D_1 and $\gamma(D_1)$ in E , where*

$$\text{Cl}(R) \cap \mathbb{S}_+ = \text{Cl}(F_p) \cap \mathbb{S}_+ \quad \text{and} \quad \text{Cl}(D_1) \cap \gamma(\text{Cl}(D_1)) = \text{Cl}(\zeta_p).$$

Let L be a fundamental domain of $\tilde{M}_{(J)}$. Suppose that

- *the sequence $\{\eta_j\}$, $\eta_j \in \Gamma$, takes infinitely many values, and*
- *$\{\eta_j(y)\}$, $y \in \mathbb{S}_+$, accumulates only to*

$$\text{Cl}(R) \cap \partial \mathbb{S}_+ \setminus \{p\}.$$

Then

$$\bigcup_{j=1}^{\infty} \eta_j(L) \subset \bigcup_{i=-m_0}^{m_0} \gamma^i(F_p) \quad \text{for some finite } m_0.$$

Proof. Since $\text{Cl}(F_p) \cap \mathbb{S}$ is bounded by two crooked circles $\text{Cl}(D_1) \cap \mathbb{S}$ and $\text{Cl}(D_2) \cap \mathbb{S}$, we obtain

$$(\text{Cl}(F_p) \setminus \text{Cl}(\zeta_p)) \cap \mathbb{S}_0 = \bigcup_{z \in \text{Cl}(R) \cap \partial \mathbb{S}_+ \setminus \{p\}} \text{Cl}(\zeta_z).$$

Since $\eta_j(y)$ accumulates only to $\text{Cl}(R) \cap \partial \mathbb{S}_+ \setminus \{p\}$, it follows that $\eta_j(L)$ accumulates only to

$$(\text{Cl}(F_p) \setminus \text{Cl}(\zeta_p)) \cap \mathbb{S}_0 = \bigcup_{z \in \text{Cl}(R) \cap \partial \mathbb{S}_+ \setminus \{p\}} \text{Cl}(\zeta_z) \quad (5.3)$$

by Theorem 1.5 and Corollary 4.9. The relative boundary $\text{bd}_{\mathcal{H}}\text{Cl}(F_P)$ in the 3-hemisphere $\mathcal{H}$ is a union of two disks $\text{Cl}(D_1)$ and $\gamma(\text{Cl}(D_1))$ with boundary in $\mathbb{S}$.

Since $\text{bd}_{\mathcal{H}}F_P$ has two components $\text{Cl}(D_1)$ and $\gamma(\text{Cl}(D_1))$ which coincide with a component of $\text{bd}_{\mathcal{H}}\gamma^{-1}(F_P)$ and one of $\text{bd}_{\mathcal{H}}\gamma(F_P)$ respectively,

$$F'' := \text{Cl}(F_P) \cup \gamma(\text{Cl}(F_P)) \cup \gamma^{-1}(\text{Cl}(F_P))$$

has the boundary set

$$\text{bd}_{\mathcal{H}}F'' = \gamma^2(\text{Cl}(D_1)) \cup \gamma^{-1}(\text{Cl}(D_1)) \subset \mathcal{H},$$

and it follows that $F'' - \text{Cl}(\zeta_p)$ contains a neighborhood of $(\text{Cl}(F_P) \setminus \text{Cl}(\zeta_p)) \cap \mathbb{S}_0$ in $\mathcal{H}$. Hence, we obtain by (5.3) that except for finitely many $\eta_j(L)$,

$$\eta_j(L) \subset (\text{Cl}(F_P) \cup \gamma(\text{Cl}(F_P)) \cup \gamma^{-1}(\text{Cl}(F_P))) \setminus \text{Cl}(\zeta_p).$$

Since F_P is a fundamental domain of γ , we obtain $E \subset \bigcup_{i \in \mathbb{Z}} \gamma^i(F_P)$. By the paragraph above, we obtain

$$\bigcup_{j=1}^{\infty} \eta_j(L) \subset \bigcup_{i=-m_0}^{m_0} \gamma^i(F_P) \quad \text{for some finite } m_0. \quad \square$$

The following is a crucial step in this paper:

Proposition 5.5 (Boundedness of $\tilde{M}_{(J)}$ in disks). *Let J be an arbitrary positive integer. For any crooked-circle disk D , $D \cap \tilde{M}_{(J)}$ is compact, i.e., bounded, and has only finitely many components.*

Proof. Suppose not. Then we can find a compact fundamental domain L of $\tilde{M}_{(J)}$ and an unbounded sequence $g_j \in \Gamma$, $g_j(L) \cap D \neq \emptyset$ for infinitely many j . Again, we may assume without loss of generality that g_j is a convergence sequence acting on $\partial\mathbb{S}_+$ with a as an attractor and r as a repeller. (See Section 4.1.) Hence, we can find a sequence $x_j \in L$ with $g_j(x_j) \in D$, and $\{g_j(x_j)\}$ accumulates to a point x of

$$\mathbb{S} \cap \partial D.$$

If $x \in \mathbb{S}_+ \cup \mathbb{S}_-$, then Theorem 1.5 and Corollary 4.9 contradict this. In fact, we have $x \in \text{Cl}(\zeta_\gamma)$ for some $\gamma \in \Lambda_{\Gamma, \mathbb{S}_+}$.

If $\text{Cl}(D)$ is disjoint from $\Lambda_{\Gamma, \mathbb{S}_+}$, then $D \cap \tilde{M}_{(J)}$ is compact by the above paragraph. We are finished in this case.

Now assume $D \cap \text{Cl}(\mathbb{S}_+) \cap \Lambda_{\Gamma, \mathbb{S}_+}$ is a finite set of parabolic fixed points or is empty. Suppose that there exists a sequence

$$\{g_j(x_j) \in D : x_j \in L\} \rightarrow x \in \text{Cl}(\zeta_p) \quad (5.4)$$

for a fixed point p , $p \in \partial D$, of a parabolic element $\gamma \in \Gamma$ (see Definition 3.3). Let y be a point of $\mathbb{S}_+$. If $\{g_j(y)\}$ converges to $q \neq p$, then

$$x \in \text{Cl}(\zeta_q) \neq \text{Cl}(\zeta_p) \quad \text{with } \text{Cl}(\zeta_q) \cap \text{Cl}(\zeta_p) = \emptyset$$

by Theorem 1.5 and Corollary 4.9. Since this is a contradiction, we obtain $g_j(y) \rightarrow p$.

We obtain $g_j(y) \rightarrow p$ for a point $y \in \mathbb{S}_+$. We can choose a sequence $\gamma^{k(j)} \in \Gamma$, $k(j) \in \mathbb{Z}$, so that $\gamma^{k(j)}g_j(y)$ is in a conical region R closed in $\mathbb{S}_+$ bounded by two complete geodesics l , $\gamma(l)$ with the common endpoint p in $\partial\mathbb{S}_+$.

Since p is not a conical limit point by Tukia [48], $\gamma^{k(j)}g_j(y)$ is bounded away from p in R . Therefore, $\eta_j := \gamma^{k(j)}g_j$ is a sequence so that $\eta_j(y) = \gamma^{k(j)}g_j(y)$ has accumulation points only in $(\text{Cl}(R) \cap \partial\mathbb{S}_+) \setminus \{p\}$.

Here, $k(j)$ is an unbounded sequence since $\gamma^{k(j)}g_j(y)$ still converges to p otherwise. By choosing a subsequence and the choice of γ , we may assume without loss of generality that $k(j) \rightarrow \infty$.

We now modify the disk D in a compact set in E by Theorem 5.2. Hence, the new disk D does not violate the existence of a sequence as in (5.4).

By Theorem 5.2, we find a fundamental domain F_P closed in E of $\langle \gamma \rangle$ bounded by a crooked-circle disk D and its image $\gamma(D)$ disjoint from D . Here, $\text{Cl}(F_P) \cap \mathbb{S}_+ = R$.

Suppose that η_j takes infinitely many values. Since ∂D is a crooked circle, $\text{Cl}(D)$ and $\gamma(\text{Cl}(D))$ meet only in $\text{Cl}(\zeta_p)$, Lemma 5.4 shows that

$$\bigcup_{j=1}^{\infty} \eta_j(L) \subset \bigcup_{i=-m_0}^{m_0} \gamma^i(F_P) \quad \text{for some finite } m_0.$$

When η_j takes only finitely many values, this is also obvious.

Since $k(j) \rightarrow \infty$, the finiteness of m_1 and the nature of the parabolic action of $\gamma^{-k(j)}$ show

$$g_j(x_j) = \gamma^{-k(j)} \eta_j(x_j), \quad x_j \in L,$$

cannot lie on the fixed disk D containing ζ_p . $\square$

Proposition 5.6. *Let $\eta \in \Gamma$ be a parabolic element acting on a parabolic region R_η . Let p_η denote the parabolic fixed point of η in $\partial \mathbb{S}_+$. Let $\hat{R}_\eta$ denote the closure in R_η of a component of $R_\eta - D_1 - D_2$ for two crooked-circle disks D_1 and D_2 whose closures contain $\text{Cl}(\zeta_{p_\eta})$. Assume that $D_i \cap R_\eta$, $i = 1, 2$, is a ruled disk of the form of Theorem 3.14. Suppose that $D_1 \cap R_\eta$ and $\eta^\delta(D_1) \cap R_\eta$ for $\delta = 1$ or -1 bound a region in R_η containing $\hat{R}_\eta$. Then $\hat{R}_\eta \cap \tilde{M}_{(J)}$ is compact for each J . Furthermore, we may assume that*

$$\tilde{M}_{(J)} \cap R_\eta = \emptyset \quad \text{for } j = 1, \dots, m_0,$$

by choosing R_η sufficiently far away. (See Definition 3.5.)

Proof. Suppose that $\hat{R}_\eta \cap \tilde{M}_{(J)}$ is not compact. Then again, we can find a compact fundamental domain L of $M_{(J)}$ so that $g_k(L)$ meets $\hat{R}_\eta$ for infinitely many k . Then $\{g_k(L)\}$ has limit points in $\text{Cl}(\zeta_x)$ for $x \in \Lambda_{\Gamma, \mathbb{S}_+}$ by Theorem 1.5 and Corollary 4.9. Since we have a sequence

$$\{x_k\}, x_k \in \text{Cl}(\hat{R}_\eta) \cap g_k(L) \quad \text{and} \quad \text{Cl}(\hat{R}_\eta) \cap \mathbb{S}_0 \subset \text{Cl}(\zeta_{p_\eta})$$

for the parabolic fixed point p_η on $\partial \mathbb{S}_+$ fixed by η , it follows that $\{g_k(L)\}$ has limit points in $\text{Cl}(\zeta_{p_\eta})$

Let $y \in \mathbb{S}_+$. We again write $\eta_i = \gamma^{k(i)} g_i$ so that $\eta_i(y)$ is in a conical region R as in the proof of Proposition 5.5. The sequence $\{\eta_i(y)\}$ accumulates only to $(\text{Cl}(R) \cap \partial \mathbb{S}_+) \setminus \{p_\eta\}$. Again $k(i) \rightarrow \pm\infty$ since $g_i(y) \rightarrow p_\eta$. Now,

$$g_i(x_i) = \gamma^{-k(i)} \eta_i(x_i), \quad x_i \in L,$$

cannot lie on $\hat{R}_\eta$ by Lemma 5.4 since $k(i) \rightarrow \pm\infty$.

For the final item, we can choose a new parabolic region R'_η sufficiently far away so that $R'_\eta \cap \hat{R}_\eta \cap \tilde{M}_{(J)} = \emptyset$. Then $R'_\eta \cap \tilde{M}_{(J)} = \emptyset$ by the parabolic action of $\langle \eta \rangle$. $\square$

Recall $\tilde{\Sigma}$ from (5.1).

Corollary 5.7 (Finiteness). *Let F be a topological polytope in E bounded by finitely many crooked-circle disks. Suppose that every pair of these disks the closures of which contain ζ_p for a parabolic fixed point p satisfy the properties of D_1 and D_2 in Proposition 5.6. Assume*

$$\text{Cl}(F) \cap \mathbb{S} \subset \text{Cl}(F) \cap \left(\tilde{\Sigma} \cup \bigcup_{k \in \mathcal{J}_F} \text{Cl}(\zeta_{p_k}) \right)$$

for a finite subset $\mathcal{J}_F \subset \mathcal{J}' \setminus \mathcal{J}$. Then the subspaces $F \cap \tilde{M}_{(J)}$ and $\text{Cl}(F) \setminus \tilde{M}_{(J)}^0$ are both compact topological polytopes for each J .

Proof. The premise says that F is disjoint from Λ_Γ except at $\bigcup_{k \in \mathcal{J}_F} \text{Cl}(\zeta_{p_k})$. Propositions 5.5 and 5.6 imply that $\bigcup_{y \in \Gamma} \gamma(L) \cap F = \tilde{M}_{(J)} \cap F$ can have accumulation points outside itself only in the compact surface

$$(\tilde{\Sigma} \cap \text{Cl}(F)) \setminus \bigcup_{k \in \mathcal{J}_F} \mathcal{H}_k \subset \text{Cl}(F) \cap \left(\mathbb{S}_+ \cup \mathbb{S}_- \cup \bigcup_{i \in \mathcal{J}} (A_i \cup a_i \cup \mathcal{A}(a_i)) \right)$$

by (5.1). This set is disjoint from $\bigcup_{x \in \Lambda_{\Gamma, \mathbb{S}_+}} \text{Cl}(\zeta_x)$. The existence of the accumulation points in here contradicts Theorem 1.5 and Corollary 4.9. Hence, $\tilde{M}_{(J)} \cap F$ is a bounded subset of F .

Also, $\text{Cl}(F) \setminus \tilde{M}_{(J)}^0$ is bounded by a union of finitely many smooth finite-type surfaces. Hence, it is a compact topological polytope. (See Figure 6.) $\square$

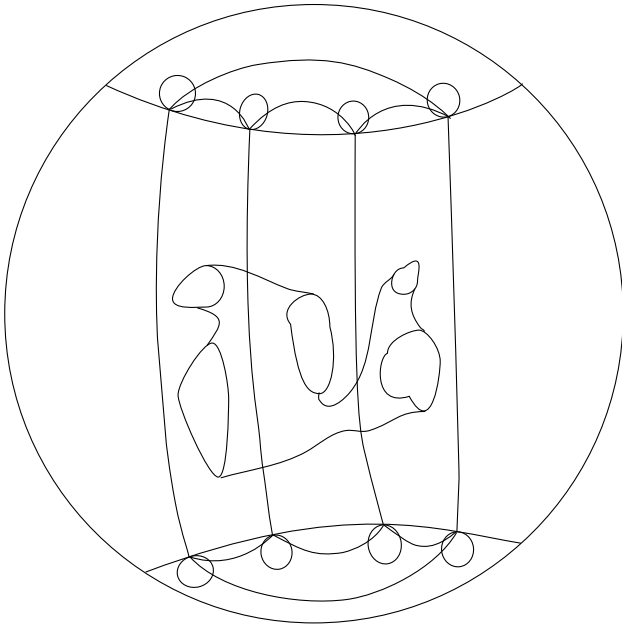


Figure 6: The fundamental domain bounded by disks $\mathcal{D}_j$, $j = 1, \dots, 2\mathbf{g}$, and some horodisks drawn topologically.

5.2.3 Choosing the candidate fundamental domain and side-pairing disks

Suppose that $\mathbf{g}$ is the rank of Γ . We recall from [16, Section 7]. Now Σ_+ denotes the surface

$$((\mathbb{S}_+ \cup \partial\mathbb{S}_+) \setminus \Lambda_{\Gamma, \mathbb{S}_+})/\Gamma,$$

where $\mathbb{S}$ is a dense subset of Σ_+ with $\chi(\mathbb{S}) = 1 - \mathbf{g}$. Also, $\Sigma_+ = (\mathbb{S}_+ \cup \bigcup_{i \in \mathcal{J}} a_i)/\Gamma$. We add to $(\mathbb{S}_+ \cup \partial\mathbb{S}_+) \setminus \Lambda_{\Gamma, \mathbb{S}_+}$ the set of ideal parabolic fixed points a_i , $i \in \mathcal{J}' \setminus \mathcal{J}$. The topology is given by a basis consisting of horodisks with fixed points added or the open disks in $(\mathbb{S}_+ \cup \partial\mathbb{S}_+) \setminus \Lambda_{\Gamma, \mathbb{S}_+}$. We obtain a new surface

$$\hat{\Sigma}_+ := \mathbb{S}_+ \cup \bigcup_{i \in \mathcal{J}'} a_i/\Gamma.$$

We choose a collection $\{\hat{d}_j : j = 1, \dots, m_0\}$ of disjoint geodesics ending at one of the ideal vertices or the boundary arc of $\hat{\Sigma}_+$ so that the complement of their union is the union of mutually disjoint open regions, each of which is homeomorphic to one of the following:

- a hexagon where three alternate edges are arcs in $\partial\hat{\Sigma}_+$,
- a pentagon with one ideal vertex (collapsed from a boundary component) and two alternate edges in $\partial\hat{\Sigma}_+$,
- a quadrilateral with two ideal vertices (collapsed from two boundary components) and one edge in $\partial\hat{\Sigma}_+$,
- or
- a triangle with three ideal vertices.

We may choose a set $\hat{d}_{i_j}$, $j = 1, \dots, 2\mathbf{g}$, where the complement of their union is a connected cell. We relabel these to be $\hat{d}_1, \dots, \hat{d}_{2\mathbf{g}}$.

The lifts of the geodesics are geodesics in $\mathbb{S}_+$ ending at points of $\bigcup_{i \in \mathcal{J}} a_i$.

Lemma 5.8. *We can choose the mutually disjoint collection $\mathcal{D}_j \subset E$ of properly embedded open disks and a tubular neighborhood $T_j \subset \text{Cl}(\mathcal{D}_j)$ of $\partial\mathcal{D}_j$ for each j , $j = 1, \dots, 2\mathbf{g}$, that form a matching set $\{T_j | j = 1, \dots, 2\mathbf{g}\}$ for a collection $\mathcal{S}_0$ of generators of Γ . Finally, $\partial\mathcal{D}_j = d_j \cup \mathcal{A}(d_j) \cup \bigcup_{x \in \partial d_j} \text{Cl}(\zeta_x)$ for a lift d_j of $\hat{d}_j$.*

Proof. We choose lifts $d_1, \dots, d_{2\mathbf{g}}$ of $\hat{d}_1, \dots, \hat{d}_{2\mathbf{g}}$ bounding a connected fundamental domain in $\text{Cl}(\mathbb{S}_+)$. Since a component L of

$$\mathbb{S}_+ \setminus \bigcup_{g \in \Gamma} \bigcup_{i=1}^{2\mathbf{g}} g(d_i)$$

is the fundamental domain of the Γ -action on $\mathbb{S}_+$, we obtain $\gamma_1, \dots, \gamma_{\mathbf{g}}$ generating Γ forming a matching col-

lection $\mathcal{S}_0$ by adding $\gamma_1^{-1}, \dots, \gamma_{\mathbf{g}}^{-1}$. Label $\gamma_j^{-1} = \gamma_{\mathbf{g}+j}$ for $j = 1, \dots, \mathbf{g}$. Hence, we may assume that $\gamma_j(d_j) = d_{\mathbf{g}+j}$ for $j = 1, \dots, 2\mathbf{g}$, mod $2\mathbf{g}$ and $\{d_1, \dots, d_{2\mathbf{g}}\}$ is a matching set for $\mathcal{S}_0$.

Let $p_1, \dots, p_{m_1}$ denote the set of parabolic fixed points on any of d_j . By choosing the parabolic regions $R_{p_1}, \dots, R_{p_{m_1}}$ sufficiently far away, we may assume that these are mutually disjoint. (See Section 3.3. Temporarily, we are not using the terminology of Section 5.1.1.)

We remove the interior of R_{p_j} , $j = 1, \dots, m_1$ from E . Let S_{p_j} denote the ruled surface boundary in E of R_{p_j} , where $\mathbf{g}_j^t$, $t \in \mathbb{R}$ acts on. Note that R_{p_j} meets $\mathbb{S}_+$ in a closed horodisk $\mathcal{H}_{p_j}$. Then we define

$$\tilde{\Sigma}^* := \left(\tilde{\Sigma} \cup \bigcup_{j=1}^{m_1} S_{p_j} \right) \setminus \bigcup_{j=1}^{m_1} \mathcal{H}_{p_j}^o \setminus \bigcup_{j=1}^{m_1} \mathcal{H}_{p_j,-}^o.$$

We assume that d_j to be disjoint from $\mathcal{H}_{p_k}$ if p_k is not an endpoint of d_j by taking the cusp neighborhood sufficiently small.

- For each geodesic segment d_j passing $\mathcal{H}_{p_k}$ for some k , we let $d_j' := d_j \cap \tilde{\Sigma}^*$. For each point of $x \in \partial d_j' \cap \mathbb{S}_+$, we obtain a line L_x in the ruled surface S_{p_j} . (See Appendix A.) We denote it by ζ_x .
- For the endpoint of d_j' in $\partial \mathbb{S}_+$, i.e., in $\bigcup_{i \in \mathcal{I}} a_i$, we already defined ζ_x in Definition 3.3.

We define

$$\tilde{d}_j = d_j' \cup A(d_j') \cup \bigcup_{x \in \partial d_j'} \text{Cl}(\zeta_x).$$

Then $\tilde{d}_j \cap \tilde{d}_k = \emptyset$ for $j \neq k$, $j, k = 1, \dots, 2\mathbf{g}$ since $\{d_j' | j = 1, \dots, 2\mathbf{g}\}$ is a mutually disjoint collection of simple closed curves. Since $d_j \cap \mathcal{H}_{p_k}$ is a geodesic ending in p_k or is empty for all j, k by our choice, $d_j \cap \partial_h \mathcal{H}_{p_k}$ is the unique point or is empty. Also,

$$d_j \cap \partial_h \mathcal{H}_{p_k}, \quad j = 1, \dots, 2\mathbf{g},$$

are distinct for a fixed k as d_j are mutually disjoint. Thus, $\{\text{Cl}(\zeta_x), x \in \partial d_j'\}$ is a mutually disjoint collection.

Furthermore, $\tilde{d}_1, \dots, \tilde{d}_{2\mathbf{g}}$ form a matching set for $\mathcal{S}_0$.

For each $x \in \partial d_j'$, $j = 1, \dots, \mathbf{g}$, we take

- for $x \in \partial \mathbb{S}_+$, a disk Z_x of the form

$$\bigcup_{y \in b} \text{Cl}(\zeta_y) \subset A_k = \bigcup_{y \in a_k} \text{Cl}(\zeta_y)$$

where b is a small open interval in a_k and $x \in a_k \cap \partial d_j'$, and

- for $x \in \partial \mathcal{H}_{p_j}$, a ruled tubular open neighborhood Z_x of $\text{Cl}(\zeta_x)$ in the ruled surface $\text{Cl}(S_{p_j})$.

Here, each Z_x , $x \in \partial d_j'$, $j = 1, \dots, \mathbf{g}$, is chosen sufficiently thin so that under elements of $\mathcal{S}_0$, the collection of Z_x and their images is a collection of mutually disjoint sets. We take a union of all of these disks with

$$(\mathbb{S}_+ \cup \mathbb{S}_-) \setminus \bigcup_{j=1}^m \mathcal{H}_{p_j} \setminus \bigcup_{j=1}^m \mathcal{H}_{p_j,-}$$

to $E \setminus \bigcup_{j=1}^{2\mathbf{g}} R_{p_j}$ to obtain a 3-manifold with boundary. Then we can apply Theorem 5.3 for the simple closed curve $\tilde{d}_j$ to obtain open disks $\mathcal{D}_j'$ so that $\tilde{d}_j = \partial \mathcal{D}_j'$ for each $j = 1, \dots, \mathbf{g}$. These are chosen to be mutually disjoint by the same theorem.

Then we obtain $\mathcal{D}_{j+\mathbf{g}}'$ as the image $\gamma_j(\mathcal{D}_j')$ for $\gamma_j \in \mathcal{S}_0$. Since the boundary components of $\mathcal{D}_j'$, $j = 1, \dots, 2\mathbf{g}$, are mutually disjoint, using Theorem 5.3 again, we may do disk exchanges to obtain mutually disjoint disks $\mathcal{D}_j''$, $j = 1, \dots, 2\mathbf{g}$.

Let p_k be a parabolic fixed point where d_j ends. Now for each ζ_x , $x \in \partial d_j'$, is in the boundary of a leaf of the foliation $\mathcal{D}_{f,r}$ in R_{p_k} obtained by Theorem 3.14. For each such p_k and d_j , we add the disk to $\mathcal{D}_j'$ by joining them at each ζ_x , $x \in \partial d_j'$. We call the results $\mathcal{D}_j$, $j = 1, \dots, 2\mathbf{g}$. These are mutually disjoint.

Now,

$$\text{Cl}(\mathcal{D}_j) \cap \text{Cl}(R_{p_k}) \cap \mathbb{S}_+ = \mathcal{H}_{p_k} \cap d_j = \mathcal{H}_{p_k} \cap \tilde{d}_j.$$

Hence, by adding these arcs back, we obtain

$$\partial \mathcal{D}_j = d_j \cup A(d_j) \cup \bigcup_{x \in \partial d_j} \text{Cl}(\zeta_x).$$

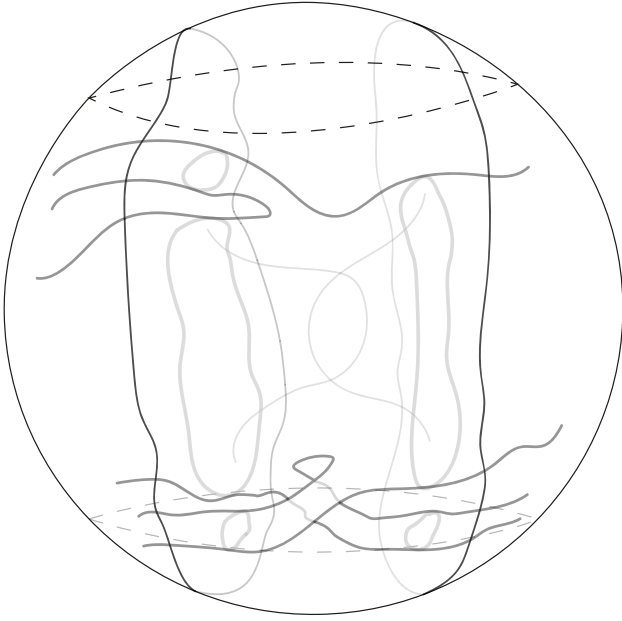


Figure 7: $\tilde{M}_{(J)}$ meeting with disks.

Since we do not change the sufficiently thin tubular neighborhoods of $\partial\mathcal{D}_j$ under the above disk exchanges, there exists a matching collection of tubular neighborhoods $\{T_j : j = 1, \dots, 2g\}$ of $\{\partial\mathcal{D}_j : j = 1, \dots, 2g\}$ under S_0 . $\square$

Here, of course, the disk collection is not yet a matching set under S_0 . By Lemma 5.8, the collection $\{\text{Cl}(\mathcal{D}_j)\}$ are mutually disjoint in $\text{Cl}(E)$. The collection $\mathcal{D}_j, j = 1, 2, \dots, 2g$, bound a region $\mathbf{F}$ closed in E with a compact closure in $\text{Cl}(E)$, a finite-sided polytope in the topological sense.

Now we consider K_0 to be the set

$$\bigcup_{j=1}^{2g} (\mathcal{D}_j \setminus T_j^o) \cup \bigcup_{1 \leq j < k \leq 2g} (\mathcal{D}_j \cap \mathcal{D}_k).$$

By Proposition 5.1, we choose $M_{(J)}$ in our exhaustion sequence of M so that

$$\tilde{M}_{(J)} \supset N_{\mathbf{d}, \epsilon}(K_0) \quad (5.5)$$

for an ϵ -neighborhood, $\epsilon > 0$. (See Figure 7.)

5.2.4 Outside tameness

The following is enough to prove tameness.

Proposition 5.9 (Outside tameness). *Let M denote a Margulis space-time E/Γ , where Γ is an isometry group with $\mathcal{L}(\Gamma) \subset \text{SO}(2, 1)^o$. Let $\mathbf{F}$ be the domain bounded by $\bigcup_{i=1}^{2g} \mathcal{D}_i$. Suppose that $M_{(J)}$ satisfies (5.5). Then $\mathbf{F} \setminus \tilde{M}_{(J)}$ is a fundamental domain of $M \setminus M_{(J)}$, and M is tame. Furthermore, $\bigcup_{i=1}^{2g} \mathcal{D}_i \setminus \tilde{M}_{(J)}$ embeds to a union of mutually disjoint properly embedded surfaces in M .*

Proof. By Corollary 5.7, $\mathbf{F} \setminus \tilde{M}_{(J)}^o$ is a tame 3-manifold. Let X denote $\mathbf{F} \setminus \tilde{M}_{(J)}^o$, a tame 3-manifold bounded by a union of finitely many compact surfaces. Note that

$$M_{(J+1)} \setminus M_{(J)}^o \subset M_{(J+2)} \setminus M_{(J)}^o \subset \dots$$

is an exhausting sequence of compact submanifolds in $M \setminus M_{(J)}^o$. Since $\mathcal{D}_i \setminus \tilde{M}_{(J)}^o \subset T_i$, it follows that

$$\{\mathcal{D}_i \setminus \tilde{M}_{(J)}^o \mid i = 1, \dots, 2g\} \subset \text{bd}\mathbf{F} \cap E$$

is a matching collection under S_0 by Lemma 5.8. Also, $\mathbf{F} \cap (\tilde{M}_{(j+n)} \setminus \tilde{M}_{(j)}^o)$ for each n is a compact topological polytope by Corollary 5.7. By Proposition 2.6, X is the fundamental domain of $E \setminus \tilde{M}_{(j)}^o$. Hence, $M \setminus M_{(j)}^o$ is tame.

The tameness of M follows since $M \setminus M_{(j)}^o$ and $M_{(j)}$ are tame. The last statement follows since $\bigsqcup_{i=1}^{2\mathbf{g}} \mathcal{D}_i \setminus \tilde{M}_{(j)}^o$ is the boundary of a fundamental domain in $E \setminus \tilde{M}_{(j)}^o$. $\square$

5.2.5 Considering the whole disks $\mathcal{D}_i \cap \tilde{M}_{(j)}$

We consider bounded components of $\mathcal{D}_i \setminus \tilde{M}_{(j)}^o$ for $i = 1, \dots, 2\mathbf{g}$. By Proposition 5.9, the union of these planar surfaces embeds to the union of disjoint ones in M . We take the mutually disjoint thin tubular neighborhoods of the images of compact planar components of $\mathcal{D}_i \setminus \tilde{M}_{(j)}^o$ and take the inverse image to E . We add these to $\tilde{M}_{(j)}$. Let us call the result $\tilde{M}_{(j)} \subset \tilde{M}$ again. Since Γ acts on $\tilde{M}_{(j)}$, we obtain a compact submanifold $M_{(j)}$ in M .

Thus, by Theorem 5.3 applied to $M_{(j)}$, each component of $\mathcal{D}_i \cap \partial \tilde{M}_{(j)}$ bounds a disk mapping to a mutually disjoint collection of embedded disks in $M_{(j)}$. We modify $\mathcal{D}_i$ by replacing each component of $\mathcal{D}_i \cap \tilde{M}_{(j)}$ with lifts of these disks. (See [29] and [32] for some details.)

The results are still embedded in $\tilde{M}$ since we modify only inside $\tilde{M}_{(j)}$ where the disks are also disjoint. Hence, we conclude

$$\begin{aligned} \gamma_j(\mathcal{D}_j) &= \mathcal{D}_{j+\mathbf{g}} & \text{for } \gamma_j \in S, \\ \gamma_j(\mathcal{D}_l) \cap \mathcal{D}_m &= \emptyset & \text{for } (j, l, m) \neq (j, j, j + \mathbf{g}) \pmod{2\mathbf{g}}. \end{aligned} \quad (5.6)$$

We summarize:

Proposition 5.10. *Let M denote a Margulis space-time E/Γ where Γ is an isometry group with $\mathcal{L}(\Gamma) \subset \mathrm{SO}(2, 1)^o$. Then there exists a fundamental domain $\mathcal{R}$ closed in E bounded by finitely many crooked-circle disks $\mathcal{D}_j$, $j = 1, \dots, 2\mathbf{g}$. Moreover, $\mathrm{Cl}(\mathcal{R}) \cap (E \cup \tilde{\Sigma})$ is the fundamental domain of a manifold $(E \cup \tilde{\Sigma})/\Gamma$ with boundary Σ . Here, $\mathcal{R}^o$ and $\mathrm{Cl}(\mathcal{R})$ are 3-cells, and E/Γ is homeomorphic to the interior of a handlebody of genus $\mathbf{g}$.*

Proof. Let $\mathcal{R}$ be the region in E with boundary equal to $\bigcup_{j=1}^{2\mathbf{g}} \mathcal{D}_j$. Since $\mathcal{D}_j$ is a properly embedded separating disk in a cell, repeated applications of Lemma 1.12 of [36] imply that $\mathcal{R}^o$ is a cell. Since $\mathrm{Cl}(\mathcal{R})$ is a polyhedral manifold whose interior is a 3-cell, it is a 3-cell.

Since by (5.6),

$$\{\mathcal{D}_j : j = 1, \dots, 2\mathbf{g}\}$$

is a matched set under S_0 , it follows that $\mathcal{R}$ is the fundamental domain by Proposition 2.6. The quotient space is homeomorphic to the interior of a handlebody since we can find a homeomorphism of $\mathcal{R}$ to the standard 3-ball, where $\mathcal{D}_j$, $j = 1, \dots, 2\mathbf{g}$, correspond to disjoint open disks with piecewise smooth boundary.

Also,

$$\{\mathrm{Cl}(\mathcal{D}_j) \cap (E \cup \tilde{\Sigma}) : j = 1, \dots, 2\mathbf{g}\}$$

is a matched set under S_0 . Also, every point in $\tilde{N} := E \cup \tilde{\Sigma}$ is equivalent to a point of $\mathcal{R}' := \mathrm{Cl}(\mathcal{R}) \cap \tilde{N}$ by the action of Γ . Hence, $\mathcal{R}'$ is a fundamental domain giving us the properness of the action of Γ on $\tilde{N} := E \cup \tilde{\Sigma}$. Thus, $\tilde{N}/\Gamma$ is a manifold with boundary Σ . $\square$

This proves the first part of Theorem 1.1. The remaining part of Theorem 1.1 will be completed in Section 5.4.

5.3 Parabolic regions and the intersection properties

We will now choose the parabolic regions so that their images under the deck transformation groups are mutually disjoint. We will need this in Proposition 5.13.

The basic idea used is that disks are separating E into two components. We choose the parabolic regions for each parabolic point in the closure of the fundamental domain so that they meet the fundamental domain $\mathcal{R}$ in a nice manner. Using this, we can show that each image of a parabolic region meets an image of $\mathcal{R}$

the fundamental domain in finitely many manners. This will essentially give us the needed intersection properties.

The above fundamental domain $\mathcal{R}$ is bounded by a union of disks $\mathcal{D}_i$, $i = 1, \dots, 2\mathbf{g}$, in the boundary of $\mathcal{R}$. We call $\mathcal{D}_i$ the *facial disks* of $\mathcal{R}$. By construction, the closure of $\mathcal{D}_i$ is disjoint from $\Lambda_{\Gamma, \mathbb{S}_+} \subset \partial\mathbb{S}_+$ except for parabolic points. The set of parabolic points meeting at least one $\text{Cl}(\mathcal{D}_i)$ is a finite set $\{p_1, \dots, p_{m_1}\}$. (Possibly two or more of the points p_i may be in the same orbit of Γ .)

We use the notations of Section 5.1.1. For each p_i , $i = 1, \dots, m_1$, we have a parabolic region of the form $\gamma(\mathcal{P}_j)$ for some j , $j = 1, \dots, m_0$, and $\gamma \in \Gamma$ whose closure contains p_i . For each i , we denote by $\mathcal{P}_i$ this region $\gamma(\mathcal{P}_j)$. We denote by η_i the parabolic element fixing p_i , following the boundary orientation if we remove E , and hence $\eta_i = \gamma\eta_j\gamma^{-1}$ for some $j = 1, \dots, m_0$.

Now, we choose $\mathcal{P}_j$, $j = 1, \dots, m_0$, so that $\text{Cl}(\mathcal{P}_i) \cap \mathbb{S}_+$ for each i equals a component of the inverse image of E . These $\mathcal{P}_j$, $j = 1, \dots, m_0$, form a mutually disjoint collection of closed cusp neighborhoods of $\mathbb{S}$ as given in Section 2.2. Note that $\text{Cl}(\mathcal{P}_i) \cap \mathbb{S}_+$ for each i equals a component of the inverse image of E . Also, by our construction in Theorem 3.13, we have $\text{Cl}(\mathcal{P}_i) \cap \mathbb{S}_- = \mathcal{A}(\text{Cl}(\mathcal{P}_i) \cap \mathbb{S}_+)$.

Definition 5.2. Let p_i and $\mathcal{P}_i$ be as above for $i \in \mathcal{I}' \setminus \mathcal{I}$, and let η_i be the parabolic primitive element fixing p_i . Let $D_{f_i, r_i, t}$ denote the *canonically defined* properly embedded disks by Theorem 3.14. We say that an image $\gamma(\mathcal{R})$, $\gamma \in \Gamma$, of the fundamental domain $\mathcal{R}$ bounded by crooked-circle disks meets *nicely* with $\eta(\mathcal{P}_i)$, $\eta \in \Gamma$, if

$$\eta(\mathcal{P}_i) \cap \gamma(\mathcal{R}) = \bigcup_{t \in [t_1, t_2]} D_{f_i, r_i, t} \quad \text{for some } t_1, t_2 \in \mathbb{R}, t_1 < t_2 \quad (5.7)$$

and

$$\text{Cl}(\zeta_{\eta(\mathcal{P}_i)}) \subset \text{Cl}(\gamma(\mathcal{R})), \quad \eta(\mathcal{P}_i) \subset \left(\bigcup_{k \in \mathbb{Z}} \gamma \eta_i^k \left(\bigcup_{j=1}^{k_0} \kappa_j(\mathcal{R}) \right) \right)^o$$

for a finite collection of $\{\kappa_j \in \Gamma\}$, where $\text{Cl}(\zeta_{\eta(\mathcal{P}_i)}) \subset \text{Cl}(\gamma(\kappa_j(\mathcal{R})))$.

Of course, by the definition $\gamma \circ \eta_i^k \circ \kappa_j(\mathcal{R})$, $k \in \mathbb{Z}$, meets with $\eta(\mathcal{P}_i)$ nicely as well.

Lemma 5.11. Let Γ satisfy Criterion 1.1. Let $\mathcal{R}$ be the fundamental domain of E/Γ bounded by crooked-circle disks as constructed by Proposition 5.10. Let q be a parabolic fixed point in $\text{Cl}(\mathcal{R})$. Then $q = p_i$ for some i , $i = 1, \dots, m_1$. Moreover, the following hold:

- $\text{Cl}(\zeta_{p_i}) \subset \text{Cl}(\mathcal{R})$,
- $\text{Cl}(\zeta_{p_i})$ is a subset of the closures of exactly two facial disks $\mathcal{D}_l$ and $\mathcal{D}_m$ among the facial disks of $\mathcal{R}$, and
- the corresponding parabolic region $\mathcal{P}_i$ meets nicely with $\mathcal{R}$ provided we choose $\mathcal{P}_i$, $i = 1, \dots, m_0$, sufficiently far away.

Proof. Since $q \in \text{Cl}(\mathcal{R})$, we obtain $q = p_i$ by the construction in Lemma 5.8 of $\partial\mathcal{D}_j$, $j = 1, \dots, 2\mathbf{g}$. Since the closure of $\mathcal{D}_j$ is compact, either $\mathcal{D}_j$ contains ζ_{p_i} in its boundary, or there is an ϵ -**d**-neighborhood of $\text{Cl}(\zeta_{p_i})$ disjoint from it for some $\epsilon > 0$. We can choose the boundary ruled surface of $\mathcal{P}_i$ sufficiently far so that (5.7) holds and hence only facial disks of $\mathcal{R}$ that meet $\mathcal{P}_i$ are the two facial disks whose closures contain ζ_{p_i} . (See Definition 3.5.) Let us call these $\mathcal{D}_j$ and $\mathcal{D}_k$.

Now, $\partial\mathcal{P}_i \cap E = S_i$ is an open disk separating $\mathcal{P}_i$ from $E \setminus \text{Cl}(\mathcal{P}_i)$. The set $\partial\mathcal{P}_i \cap \mathcal{R}$ has the boundary formed by two lines respectively in $\mathcal{D}_j$ and $\mathcal{D}_k$. We take finitely many images $\kappa_l(\mathcal{R})$, $l = 1, \dots, k_0$ of $\mathcal{R}$ with $\kappa_1 = \text{I}$ so that $\kappa_l(\mathcal{R}) \cap \kappa_{l+1}(\mathcal{R})$ is a copy of $\mathcal{D}_{i_0}$ for some i_0 whose closure contains ζ_{p_i} . Since the collection $\{\gamma(\mathcal{R}) : \gamma \in \Gamma\}$ tessellates E , we can choose enough of κ_j so that $\kappa_{k_0+1}(\mathcal{R}) = \eta_i^{\pm 1}(\kappa_1(\mathcal{R}))$ for either $+$ or $-$ sign.

Except for the closures of facial disks of $\{\kappa_j(\mathcal{R}) : j = 1, \dots, k_0\}$ containing $\text{Cl}(\zeta_{p_i})$, the closures of other facial disks contained in the boundary of $\{\kappa_j(\mathcal{R}) : j = 1, \dots, k_0\}$ are disjoint from $\text{Cl}(\zeta_{p_i})$. Let $\hat{K}$ denote the union of the closures of these images of facial disks of $\{\kappa_j(\mathcal{R}) : j = 1, \dots, k_0\}$ disjoint from $\text{Cl}(\zeta_{p_i})$. Since these are separating disks in $\mathcal{H}$, we may choose $\mathcal{P}_i$ sufficiently far so that $\text{Cl}(\mathcal{P}_i) \cap \hat{K} = \emptyset$.

Since $\text{Cl}(\mathcal{P}_i)$ is η_i -invariant, it follows that $\text{Cl}(\mathcal{P}_i)$ is disjoint from $\bigcup_{m \in \mathbb{Z}} \eta_i^m(\hat{K})$. Now, $\bigcup_{m \in \mathbb{Z}} \eta_i^m(\hat{K})$ is a separating set in E . A component of $E \setminus \bigcup_{m \in \mathbb{Z}} \eta_i^m(\hat{K})$ equals

$$\left(\bigcup_{k \in \mathbb{Z}} \eta_i^k \left(\bigcup_{j=1}^{k_0} \kappa_j(\mathcal{R}) \right) \right)^o.$$

Since $\mathcal{P}_i$ is a connected set, we obtain

$$\mathcal{P}_i \subset \left(\bigcup_{k \in \mathbb{Z}} \eta_i^k \left(\bigcup_{j=1}^{k_0} \kappa_j(\mathcal{R}) \right) \right)^o, \quad (5.8)$$

as desired $\square$

Notice that (5.8) gives us the conditions of Definition 5.2.

Proposition 5.12. *Let $\mathcal{P}_i$ be a parabolic region for the parabolic fixed point p_i , $i = 1, \dots, m_0$, as we chose at the beginning of Section 5.3. We can always choose $\mathcal{P}_i$, $i = 1, \dots, m_0$, so that for every pair $\eta, \gamma \in \Gamma$, so that exclusively one of the following holds:*

- $\eta(p_i) \notin \gamma(\text{Cl}(\mathcal{R}))$, or else
- $\gamma(\mathcal{R})$ meets $\eta(\mathcal{P}_i)$ nicely, and $\gamma(\mathcal{P}_j) = \eta(\mathcal{P}_i)$ for some $j = 1, \dots, m_1$.

Proof. We may assume $\gamma = I$ since we can change η to $\gamma^{-1}\eta$. Then the result follows by Lemma 5.11. $\square$

We choose $\mathcal{P}_j$ far away for $j = 1, \dots, m_0$ so that the conclusions of Proposition 5.12 are satisfied.

Let $P = \bigcup_{\gamma \in \Gamma} \bigcup_{i=1, \dots, m_0} \gamma(\mathcal{P}_i)$, and let $\mathcal{P}_{\mathcal{R}} := (\mathcal{P}_1 \cup \dots \cup \mathcal{P}_{m_1}) \cap \mathcal{R}$.

Proposition 5.13. *We can choose the sufficiently far away parabolic regions*

$$\mathcal{P}_1, \dots, \mathcal{P}_{m_0}$$

meeting $\mathcal{R}$ nicely so that they are disjoint in E . Then the following hold:

- *The following are equivalent:*
 - (1) $\gamma(\mathcal{P}_i)$ meets $\mathcal{R}$ nicely.
 - (2) $\gamma(\mathcal{P}_i) = \mathcal{P}_j$ for some j , $j = 1, \dots, m_1$.
 - (3) $\gamma(\mathcal{P}_i) \cap \mathcal{R} \neq \emptyset$.
- $\mathcal{R}$ meets only $\mathcal{P}_1, \dots, \mathcal{P}_{m_1}$ among all images $\gamma(\mathcal{P}_r)$ for $\gamma \in \Gamma$, $r = 1, \dots, m_0$.
- Moreover, for every pair $\gamma, \eta \in \Gamma$,

$$\gamma(\mathcal{P}_j) \cap \eta(\mathcal{P}_k) = \emptyset \quad \text{or} \quad \gamma(\mathcal{P}_j) = \eta(\mathcal{P}_k), \quad j, k = 1, \dots, m_0.$$

Proof. We first choose $\mathcal{P}_i$, $i = 1, \dots, m_0$, sufficiently far so that $\mathcal{P}_i \cap \mathcal{P}_j \cap \mathcal{R} = \emptyset$ for $i \neq j$, $i, j = 1, \dots, m_1$, and every $\mathcal{P}_j$, $j = 1, \dots, m_1$, meets $\mathcal{R}$ nicely by Proposition 5.12.

Obviously, (2) implies (1). For the first item, we show that (1) implies (2): Suppose $\gamma(\mathcal{P}_j)$ meets $\mathcal{R}$ nicely. Then $\gamma(p_j)$ is in $\text{Cl}(\mathcal{R})$. Since $\gamma(p_j)$ is a parabolic fixed point, it equals p_l for some $l = 1, \dots, m_1$. Only elements of Γ fixing p_l are of the form η_l^m for some integer $m \in \mathbb{Z}$. We have

$$\gamma(\text{Cl}(\mathcal{P}_j)) \cap \mathbb{S}_+ = \gamma(\mathcal{H}_k) \cup \gamma(\partial_h \mathcal{H}_k)$$

for a horodisk $\mathcal{H}_k$. Now, $\gamma(\mathcal{H}_k) = \mathcal{H}_l$. Hence, the parabolic group acting on $\gamma(\mathcal{P}_j)$ is the same one acting on $\mathcal{P}_l$. By our choice of $\mathcal{P}_j$ in Section 5.3 from choosing orbit representatives of parabolic fixed points, we obtain

$$\gamma(\mathcal{P}_j) = \mathcal{P}_l \quad \text{for some } l = 1, \dots, m_1.$$

Clearly, (2) implies (3) by Lemma 5.11.

Now we show that (3) implies (2): Suppose that $\gamma(\mathcal{P}_j) \cap \mathcal{R} \neq \emptyset$. Now, $\gamma(p_j) \in \eta(\text{Cl}(\mathcal{R}))$ for some $\eta \in \Gamma$, and $\gamma(\mathcal{P}_j)$ meets $\eta(\mathcal{R})$ nicely and $\gamma(\mathcal{P}_j) = \eta(\mathcal{P}_k)$ for some k by Proposition 5.12. Moreover,

$$\eta(\mathcal{P}_k) \subset \left(\bigcup_{l \in \mathbb{Z}} \eta \eta_k^l \left(\bigcup_{r=1}^{k_0} \kappa_r(\mathcal{R}) \right) \right)^o \quad (5.9)$$

by Proposition 5.12. Hence, $\gamma(\mathcal{P}_j)$ meets with only the images of $\mathcal{R}$ of the form $\eta \eta_k^l \kappa_j(\mathcal{R})$. If

$$\eta \eta_k^l (\kappa_j(\mathcal{R})) = \mathcal{R} \quad \text{for some } k, l, j,$$

then $\gamma(\mathcal{P}_j)$ meets $\mathcal{R}$ nicely since we can check Definition 5.2. If $\eta \eta_k^l (\kappa_j(\mathcal{R})) \neq \mathcal{R}$ for all k, l, j , then $\gamma(\mathcal{P}_j)$ does not meet $\mathcal{R}$ by (5.9). By Lemma 5.11, (2) implies (1). We proved the first item.

The second item follows from it.

Suppose that two respective images P'_i and P'_j of some $\mathcal{P}_k$ and $\mathcal{P}_l$ for $k, l = 1, \dots, m_0$ meets in a nonempty set. Hence, they meet in $\gamma(\mathcal{R})$ for some $\gamma \in \Gamma$. Thus,

$$\gamma^{-1}(P'_i) \cap \gamma^{-1}(P'_j) \cap \mathcal{R} \neq \emptyset.$$

The first item implies that

$$\mathcal{P}_j = \gamma^{-1}(P'_i) \quad \text{and} \quad \mathcal{P}_l = \gamma^{-1}(P'_j) \quad \text{for some } l, m = 1, \dots, m_1$$

However, $\mathcal{P}_j \cap \mathcal{P}_l \cap \mathcal{R} = \emptyset$ or $P'_i = P'_j$ by our construction of parabolic regions. $\square$

5.4 Relative compactification

5.4.1 Proof of Theorem 1.1

Proposition 5.10 proves the first part of the theorem. First, we recall our bordifying surface as defined by (5.1):

$$\tilde{\Sigma}_0 := \mathbb{S}_+ \cup \mathbb{S}_- \cup \bigcup_{i \in \mathcal{I}} (A_i \cup a_i \cup \mathcal{A}(a_i)).$$

We set $\Sigma := \tilde{\Sigma}_0/\Gamma$ and $N := (E \cup \tilde{\Sigma})/\Gamma$, which is a manifold by Proposition 5.10.

By Proposition 5.13, we define P to be a union of mutually disjoint parabolic regions of the form $\gamma(\mathcal{P}_i)$ for $\gamma \in \Gamma$, $i = 1, \dots, m_0$. Since the boundary of their union in $\mathbb{S}$ is the union of mutually disjoint closed horodisks, their closures in $\mathcal{H} = \text{Cl}(E)$ are mutually disjoint. Now, we take the closure $\text{Cl}(P)$ of P and take the relative interior P' in the closed hemisphere $\mathcal{H}$. Let $\partial_E P'$ denote $\text{bd} P' \cap E$. Then define $\tilde{N}' := (E \cup \tilde{\Sigma}) \setminus P'$. Note that Γ acts properly discontinuously on $\tilde{N}'$ since $\tilde{N}'$ is a Γ -invariant proper subspace of $\tilde{N}$. We note that $\partial_E P'$ is transversal to $\mathbb{S}$. Thus, $N' := \tilde{N}'/\Gamma$ is a manifold.

The manifold boundary $\partial N'$ of N' is

$$((\tilde{\Sigma} \setminus P') \cup \partial_E P')/\Gamma.$$

Define $P'' = P'/\Gamma$. Also, $(\partial_E P')/\Gamma$ is a union of a finite number of disjoint annuli. Note that $\partial N'$ is homeomorphic to $(\Sigma \setminus P'') \cup (\partial_E P')/\Gamma$.

Recall that the union of facial disks $\mathcal{D}_i$, $i = 1, \dots, 2g$, bounds the fundamental domain $\mathcal{R}$ in $\mathcal{H}$. Then

$$\bigcup_{i=1}^{2g} \text{Cl}(\mathcal{D}_i) \cap ((E \cup \tilde{\Sigma}) \setminus P')$$

bounds a fundamental domain

$$\text{Cl}(\mathcal{R}) \cap ((E \cup \tilde{\Sigma}) \setminus P').$$

The boundary is homeomorphic to a 2-sphere and, hence, the fundamental domain is homeomorphic to a compact 3-cell. Since this fundamental domain is compact, N' is compact.

Since we pasted disjoint disks on a cell, N' is homotopy equivalent to a bouquet of circles. Now, N' has no fake-cell since $\tilde{N}'$ is a subset of E . It follows that N' is homeomorphic to a compact handlebody of genus g by Theorem 5.2 of [32].

Let $\hat{P}$ be the closure of P' in $\tilde{N}$. We realize that N' is a deformation retract of N by collapsing $\hat{P}/\Gamma$, homeomorphic to a disjoint union of copies of $A^2 \times [0, 1)$, to its boundary in N homeomorphic to a disjoint union of embedded images of A^2 for a compact annulus A^2 with boundary. This completes the proof of Theorem 1.1.

5.4.2 Proof of Corollary 1.2

If $\mathcal{L}(\Gamma) \subset \text{SO}(2, 1)^o$, we are done by Theorem 1.1.

Suppose not. We have an index-two subgroup Γ' of Γ acting on $\mathbb{S}_+$ with $\mathcal{L}(\Gamma') \subset \mathrm{SO}(2, 1)^o$. Then Γ' acts on $(E \cup \tilde{\Sigma}) \setminus P'$, where we construct $\tilde{\Sigma}$ and P' as above for Γ' . There exists an element ϕ of $\Gamma - \Gamma'$ so that $\phi(\mathbb{S}_+) = \mathbb{S}_-$ and $\phi^2 \in \Gamma'$ and ϕ normalizes Γ' . Since ϕ acts as an orientation-preserving map of $\mathbb{S}$, and

$$\mathcal{L}(\phi) \circ \mathcal{L}(\Gamma') \circ \mathcal{L}(\phi)^{-1} = \mathcal{L}(\Gamma'),$$

it follows that ϕ induces a diffeomorphism $\mathbb{S}_+/\Gamma'$ with $\mathbb{S}_-/\Gamma'$ preserving orientations. Since $\mathbb{S}_-$ is a Klein model also, we can define a limit set $\Lambda_{\Gamma', \mathbb{S}_-}$. Hence, for the limit sets, we have

$$\phi(\Lambda_{\Gamma', \mathbb{S}_+}) = \Lambda_{\Gamma', \mathbb{S}_-} \quad \text{and} \quad \phi(\partial \mathbb{S}_+ \setminus \Lambda_{\Gamma', \mathbb{S}_+}) = \partial \mathbb{S}_- \setminus \Lambda_{\Gamma', \mathbb{S}_-}.$$

Since each element of $\mathcal{L}(\Gamma')$ commutes with $\mathcal{A}$, we obtain

$$\mathcal{A}(\Lambda_{\Gamma', \mathbb{S}_+}) = \Lambda_{\Gamma', \mathbb{S}_-} \quad \text{and} \quad \mathcal{A}(\partial \mathbb{S}_+ \setminus \Lambda_{\Gamma', \mathbb{S}_+}) = \partial \mathbb{S}_- \setminus \Lambda_{\Gamma', \mathbb{S}_-}.$$

Let $\mathcal{I}$ denote the collection of open intervals of $\partial \mathbb{S}_+ \setminus \Lambda_{\Gamma', \mathbb{S}_+}$. We define $\tilde{\Sigma}$ for Γ' as in (5.1),

$$\mathbb{S}_+ \cup \mathbb{S}_- \cup \bigcup_{a \in \mathcal{I}} \left(a \cup \mathcal{A}(a) \cup \bigcup_{x \in a} \zeta_x \right).$$

Since ϕ is orientation-preserving, it follows that ϕ sends the disk $A_a = \bigcup_{x \in a} \zeta_x$, $a \in \mathcal{I}$, to $A_{\mathcal{A}(\phi(a))}$. Since $a \mapsto \mathcal{A}\phi(a)$ gives us an automorphism of $\mathcal{I}$, ϕ acts on $\tilde{\Sigma}$.

Given a component P_1 of P' , there is a parabolic primitive element γ_1 acting on it. Then $\gamma_2 := \phi \circ \gamma_1 \circ \phi^{-1}$ acts on $\phi(P_1)$. Since $\gamma_2 \in \Gamma'$ also, it follows that γ_2 acts on a component P_2 of P' . We denote $\gamma_2^*(P_1) = P_2$, where we may not yet have $\gamma(P_1) = P_2$.

Let $\tilde{\mathcal{P}}$ denote the set of parabolic fixed points of $\partial \mathbb{S}_+$. Then let a finite $\hat{\mathcal{P}}$ denote the collection of the Γ' -orbit classes of $\tilde{\mathcal{P}}$. The above action of ϕ induces an automorphism of $\hat{\mathcal{P}}$.

Lemma 5.14. *There is no fixed point in $\hat{\mathcal{P}}$ under this action of ϕ on $\hat{\mathcal{P}}$.*

Proof. Suppose not. Then using orbit equivalence under Γ , there exists an isometry $\psi \in \Gamma \setminus \Gamma'$ so that $\mathcal{A} \circ \mathcal{L}(\psi)(q) = q$ for a parabolic fixed point q . Note that $\mathcal{A} \circ \mathcal{L}(\psi)$ acts on $\mathbb{S}_+$ acting on a component $\mathcal{H}_i$ for some i . Since $\mathcal{A} \circ \mathcal{L}(\psi)$ acts as an orientation reversing isometry on $\mathbb{S}_+$, it follows that $\mathcal{A} \circ \mathcal{L}(\psi)$ acts on a complete geodesic l_q ending at q . Since it must fix the point $\partial_h \mathcal{H}_i \cap l_q$, it fixes each point of l_q . Hence, $\mathcal{L}(\psi)$ acts as $-I$ on a time-like vector subspace P_{l_q} corresponding to l_q , and is the identity on a space-like vector subspace. Since ψ cannot have a fixed point on E , it follows that ψ^2 cannot be the identity on E , and it is a Lorentzian translation on a space-like geodesic l orthogonal to P_{l_q} , and $\psi^2 \in \Gamma'$ since $[\Gamma : \Gamma'] = 2$. However, Γ' does not have a translation element as it is an affine deformation of $\mathcal{L}(\Gamma')$. $\square$

Since there is no fixed point of the action, we divide the collection $\hat{\mathcal{P}}$ of components of P' into equivalence classes of orbits under Γ' . This is a finite set $\hat{P}_1, \dots, \hat{P}_{2m}$. Now ϕ acts on this set. We may assume that ϕ sends $\hat{P}_i$ to $\hat{P}_{m+i}$.

We replace each element of $\hat{P}_{m+i}$ with $\phi(P'')$ for the corresponding element P'' of $\hat{P}_i$ for $i = 1, \dots, m$. We obtain a new set P' . Here, for the parabolic element γ''' corresponding to $\phi(P'')$, we have

$$\gamma''' = \phi \circ \gamma'' \circ \phi^{-1}$$

for a parabolic element γ'' acting on P'' . Since γ''' is in the unique one-parameter subgroup γ'''^t , $t \in \mathbb{R}$, of parabolic isometries, it follows that γ'''^t , $t \in \mathbb{R}$, acts on $\phi(P'')$. Therefore, the boundary $\partial \phi(P'') \cap E$ is a parabolic ruled surface for γ''' as defined by Definition 3.5.

Obviously, Γ acts on P' . Also, we may assume that elements of P' are mutually disjoint: we take a finite set of components of P' that meets the fundamental domain $\mathcal{R}$. We can make these disjoint by taking them sufficiently far away. Proposition 5.13 shows that these are mutually disjoint.

Therefore, $N' := ((E \cup \tilde{\Sigma}) \setminus P')/\Gamma$ is compact and is homeomorphic to a handlebody of genus g by [32, Theorem 5.2] as in Section 5.4.1. Since ϕ does not act on any component of P' , we can show that N deformation retracts to N' as above.

A Parabolic ruled surfaces

We will be using the parabolic coordinate system obtained in Section 3.1. These constructions are canonical except for the ambiguity in the x -coordinates up to translations. (See Remark 3.1.)

A.1 Proper embedding of ruled surfaces

Only prerequisites are Sections 2 and 3.1. Our purpose is to prove Theorem 3.13 using Lemma A.1, Proposition A.2, and Lemma A.3.

Lemma A.1. *Assume as in Theorem 3.13. Every g^t -orbit in $\mathcal{H}_{\kappa_1, \kappa_2}$ starts and ends at $\zeta_{(1,0,0,0)}$.*

Proof. Let l be a segment so that $\text{Cl}(l) \in \mathcal{H}_{s_0, \kappa_1, \kappa_2}$. An endpoint of l_∞ must be $(1, 0, 0, 0)$ since

$$g^{t_i}(q) \rightarrow (1, 0, 0, 0)$$

for each point $q \in l \cap P_T$. Since $\text{Cl}(l)$ has a pair of antipodal points, the other endpoint of $\text{Cl}(l_\infty)$ is $(-1, 0, 0, 0)$. We compute the intersection of an arbitrary image of $g^t(l)$ at the plane given by $x = 0$

$$\begin{aligned} & \left(\left(0, -\frac{ct(\mu t^3 + 6ty_0)}{6a + 3ct^2} + \frac{\mu t^2}{2} + y_0, \mu t - \frac{c(\mu t^3 + 6ty_0)}{6a + 3ct^2}, 1 \right) \right) \\ &= \left(\left(0, -\frac{\mu t^2 + 6y_0}{\frac{6a}{ct^2} + 3} + \frac{\mu t^2}{2} + y_0, \mu t - \frac{\mu t + 6y_0/t}{\frac{6a}{ct^2} + 3}, 1 \right) \right) \rightarrow (0, 1, 0, 0) \in \mathbb{S}^3 \end{aligned} \quad (\text{A.1})$$

as $t \rightarrow \infty$ or $t \rightarrow -\infty$. See [9]. Since this point is in $\text{Cl}(l_\infty)$, we showed that $l_\infty = \zeta_{(1,0,0,0)}$. $\square$

Proposition A.2. *Assume as in Theorem 3.13. Choose κ_1 and κ_2 satisfying $0 < \kappa_1 \leq \kappa_2 < 1$. Then the closure of $\mathcal{H}_{s_0, \kappa_1, \kappa_2}$ under $\mathbf{d}_H$ is a compact set $\mathcal{H}_{s_0, \kappa_1, \kappa_2} \cup \{\zeta_{(1,0,0,0)}\}$.*

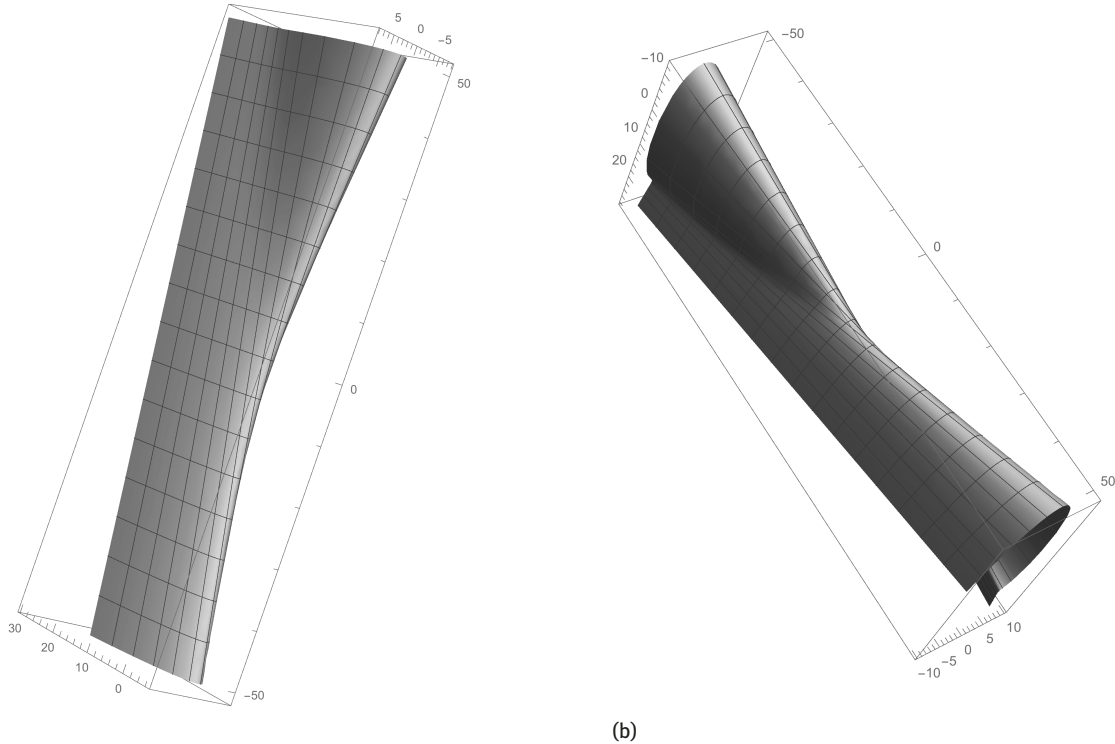


Figure 8: Two parabolic ruled surfaces. See [9].

Proof. The space of open geodesic segments of $\mathbf{d}$ -length π in the 3-hemisphere $\mathcal{H}$ forms a compact metric space under the Hausdorff metric $\mathbf{d}_H$. We show this by showing that every sequence of elements of $\mathcal{H}_{s_0, \kappa_1, \kappa_2}$ has an accumulation point in $\mathcal{H}_{s_0, \kappa_1, \kappa_2}$ or accumulates to $\text{Cl}(\zeta_{(1,0,0,0)})$.

Given a sequence of segments $\{u_i\}$ in $\mathcal{H}_{s_0, \kappa_1, \kappa_2}$, $u_i = g^{t_i}(l_i)$, where $l_i \cap E$ is given by

$$l_i(s) = (sa_i, y_{0,i}, sc_i) \quad \text{for } y_{0,i} \geq s_0, \quad a_i, c_i > 0, \quad \frac{\kappa_1 a_i}{c_i} \leq \frac{y_{0,i}}{\mu} \leq \frac{\kappa_2 a_i}{c_i}, \quad a_i^2 + c_i^2 = 1.$$

The boundedness of one of $y_{0,i}$ or $\frac{a_i}{c_i}$ implies that of the other. If $y_{0,i}$ or $\frac{a_i}{c_i}$ is bounded above, then l_i geometrically converges to an element of $\mathcal{H}_{s_0, \kappa_1, \kappa_2}$ up to a choice of a subsequence. If $t_i \rightarrow \pm\infty$, then $u_i \rightarrow \zeta_{(1,0,0,0)}$ since the estimates in (A.1) in the proof of Lemma A.1 hold in this case. If t_i is bounded, then $u_i \rightarrow u_0 \in \mathcal{H}_{s_0, \kappa_1, \kappa_2}$.

Hence, we are left with the case where

$$y_{0,i} \rightarrow \infty, \quad \frac{a_i}{c_i} \rightarrow \infty, \quad \text{and} \quad t_i \rightarrow \pm\infty.$$

We will show that $u_i \rightarrow \zeta_{(1,0,0,0)}$: Suppose not. Then u_i converges to a line u_∞ passing E under the metric $\mathbf{d}_H$. Then u_∞ has the direction $(1, 0, 0)$ since $(\mathcal{L}(\Phi_t)(\mathbf{v})) \rightarrow (1, 0, 0)$ for a generic vector $\mathbf{v}$.

By applying an element of g^t to u_∞ and the sequence u_i , we may assume that $u_\infty \cap E$ is given as the line

$$x = s, \quad y = C, \quad z = 0, \quad s \in \mathbb{R}.$$

Since u_i geometrically converges to u_∞ , it follows that u_i intersected with $x = 0$ is near $(0, C, 0)$. By changing u_i by a bounded g^{s_i} with $s_i \rightarrow 0$, we may assume without loss of generality that u_i passes $(0, C_i, 0)$ while we still have $u_i \rightarrow u$ under $\mathbf{d}_H$. Here, $C_i \rightarrow C$.

By our construction, u_i is contained in a hyperplane P_i tangent to a parabolic cylinder S_i given by the equation $2\mu y = z^2 + 2\mu C_{1,i}$ for some $C_{1,i} \in \mathbb{R}$. The line u_i meets S_i at the unique point (x_i^*, y_i^*, z_i^*) . Project P_i and S_i to the yz -plane. Then the image of P_i passes $(C_i, 0)$ and tangent to the parabola $2\mu y = z^2 + 2\mu C_{1,i}$. We compute by elementary geometry

$$z_i^* = \pm \sqrt{2\mu C_{1,i} - 2\mu C_i} \quad \text{and} \quad y_i^* = 2C_{1,i} - C_i.$$

Now, we wish to compute t_i so that $g^{t_i}(l) = u_i$ as in (3.9) where $l(s)$ passes $(0, C_{1,i}, 0)$. We compute t satisfying

$$\Phi_t(0, C_{1,i}, 0) = \left(tC_{1,i} + \frac{\mu t^3}{3}, C_{1,i} + \frac{\mu t^2}{2}, \mu t \right) = (x_i^*, y_i^*, z_i^*)$$

recalling (3.5). We let t_i denote the answer

$$y_i^* = 2C_{1,i} - C_i = C_{1,i} + \frac{\mu t_i^2}{2} \quad \text{and} \quad t_i = \pm \sqrt{\frac{2(C_{1,i} - C_i)}{\mu}}. \quad (\text{A.2})$$

The vector α_i tangent to u_i is given by

$$\left(t_i C_{1,i} + \frac{\mu t_i^3}{6}, C_{1,i} + \frac{\mu t_i^2}{2}, \mu t_i \right) - (0, C_i, 0) = \left(t_i C_{1,i} + \frac{\mu t_i^3}{6}, C_{1,i} - C_i + \frac{\mu t_i^2}{2}, \mu t_i \right).$$

Since the sequence of the directions of the vectors converges to $(1, 0, 0)$ by our assumption on u_i , we obtain $t_i \rightarrow \pm\infty$.

Recall

$$\mathcal{L}(\Phi_{t_i})^{-1} = \begin{pmatrix} 1 & -t_i & \frac{t_i^2}{2} \\ 0 & 1 & -t_i \\ 0 & 0 & 1 \end{pmatrix}.$$

We compute $\mathcal{L}(\Phi_{t_i}^{-1})(\alpha_i)$ to be

$$\left(t_i C_{1,i} + \frac{\mu t_i^3}{6} - t_i(C_{1,i} - C_i) - \frac{\mu t_i^3}{2} + \frac{\mu t_i^3}{2}, C_{1,i} - C_i + \frac{\mu t_i^2}{2} - \mu t_i^2, \mu t_i \right) = \left(\frac{\mu t_i^3}{6} + t_i C_i, 0, \mu t_i \right).$$

Recall the condition (3.9) to $g^{-t_i}(u_i) = l$, which yields $C_{1,i}/\mu \leq \kappa_2(t_i^2/6 + C_i/\mu)$, and we obtain

$$C_{1,i} \leq \kappa_2 \frac{1}{3} |C_{1,i} - C_i| + \kappa_2 C_i \quad \text{and} \quad \kappa_2 < 1$$

by (A.2). Since $t_i \rightarrow \pm\infty$, we obtain $C_{1,i} \rightarrow +\infty$ and $C_i \rightarrow C$. This contradicts the above inequality.

We conclude that u_i can converge only to points of $\mathcal{H}_{s_0, \kappa_1, \kappa_2}$ or $\zeta_{(1,0,0,0)}$. This gives us a sequential convergence property. The closure of $\mathcal{H}_{s_0, \kappa_1, \kappa_2}$ is a compact metric space $\mathcal{H}_{s_0, \kappa_1, \kappa_2} \cup \{\zeta_{(1,0,0,0)}\}$. $\square$

Lemma A.3. *Let M be a compact metric space. Suppose that there exists a 1-dimensional flow $\phi_t : M \rightarrow M$, $t \in \mathbb{R}$, with a fixed point p . Suppose that the orbit of every point starts and ends at p , and the orbit space $(M \setminus \{p\})/\sim$ is not compact. Then every ϵ -ball of p contains an orbit starting and ending at p .*

Proof. Choose a compact set $K = M \setminus B_\epsilon(p)$ for an open ϵ -ball of p . Since

$$\left(\bigcup_{t \in \mathbb{R}} \phi_t(K) \right) / \sim = K / \sim$$

is compact, and $(M \setminus \{p\})/\sim$ is not compact, it follows that

$$M \setminus \bigcup_{t \in \mathbb{R}} \phi_t(K) = \bigcap_{t \in \mathbb{R}} \phi_t(B_\epsilon(p)) \subset B_\epsilon(p)$$

is not empty. Then a point here gives us an example of the closed orbit. $\square$

Proof of Theorem 3.13. We will first show that $\Psi : \mathbb{R}^2 \rightarrow \mathbb{E}$ is a proper injective map.

Since g^t acts on $P_{T'}$ for each T' , we have a self-intersection of Ψ if

$$g^t(l(s) \cap P_{T'}) = l(s') \cap P_{T'} \quad \text{for some } t > 0, s, s' \in \mathbb{R} \text{ and } T' \in \mathbb{R}.$$

The following hold:

- $l(s) \cap P_{T'}$ is a pair of points provided $T' > -2\mu y_0$, or
- $T' = -2\mu y_0$ and $l(s) \cap P_{T'}$ is $(0, y_0, 0)$, or else
- $l(s) \cap P_{T'}$ is empty for $T' < -2\mu y_0$.

Thus, only in the first case, we can have a self-intersection of the image of Ψ under the quotient space $\mathbb{E}/\langle g \rangle$.

Now $l(s) \cap P_{T'}$ can be computed as follows:

$$(sc)^2 - 2\mu y_0 = T' \quad \text{and} \quad s_0 = \frac{\sqrt{T' + 2\mu y_0}}{c}$$

and the points are $(\pm s_0 a, y_0, \pm s_0 c)$. And we obtain

$$F_3(\pm s_0 a, y_0, \pm s_0 c) = \pm(s_0^3 c^3 - 3\mu y_0 s_0 c + 3\mu^2 s_0 a).$$

These are distinct unless the value is 0. Since F_3 is invariant under g , it follows that if F_3 -values of two points are distinct, then they cannot be in the same orbit of $\langle g \rangle$. If $F_3 = 0$, we must have

$$\frac{a}{c} = -\frac{T'}{3\mu^2} + \frac{y_0}{3\mu}.$$

Since $-T' < 2\mu y_0$, we obtain

$$\frac{a}{c} < \frac{2\mu y_0}{3\mu^2} + \frac{y_0}{3\mu} = \frac{y_0}{\mu}.$$

Thus, if we choose $y_0 < \mu \frac{a}{c}$, the self-intersection of Ψ never happens. For example, choosing y_0 sufficiently small or choosing $\frac{a}{c}$ sufficiently large would satisfy the condition. This proves the injectivity of Ψ . By (3.6), g^t acts properly on each parabolic cylinder P_T since F_1 and F_2 are invariants of the vector field on ϕ , and each intersection of $F_1 \cap F_2$ is a complete flow line.

We now prove the properness of Ψ . Suppose that there is a compact set $K \subset \mathbb{E}$ and $g^{t_i}(l) \cap K$ is not empty for a sequence $\{t_i\}$ of real numbers such that $t_i \rightarrow \infty$. (The case when $t_i \rightarrow -\infty$ is entirely similar.) However, K is in the region B in $\mathbb{E}$ bounded by two parabolic cylinders P_{T_1} and P_{T_2} for some pair T_1 and T_2 . Then l meets P_{T_j} , $j = 1$ at most two points. If l does not meet B , then $g^t(l)$, $t \in \mathbb{R}$ is disjoint from K since the region bounded by P_{T_j} is g^t -invariant. If $l \cap B \neq \emptyset$, then

$$g^{t_i}(l \cap B) \rightarrow \{(1, 0, 0, 0)\} \text{ or } \{(-1, 0, 0, 0)\} \quad \text{as } t_i \rightarrow \pm\infty$$

by convexity since the endpoints of $l \cap B$ do this. This proves the properness of $\Psi : \mathbb{R}^2 \rightarrow \mathbb{E}$ and that $g^{t_i}(l)$ can have limit points only in $\mathbb{S}$.

The first item is proved by Lemma A.1.

Choose κ_1 and κ_2 satisfying $0 < \kappa_1 \leq \kappa_2 < 1$. There is a continuous map $\iota_R : \mathcal{H}_{s_0, \kappa_1, \kappa_2} \rightarrow \mathbb{S}_+$ by taking the endpoints in $\mathbb{S}_+$. The image is a horodisk $\mathcal{E}$. Since $\mathcal{E}/\sim$ is not compact, it follows that $\mathcal{H}_{s_0, \kappa_1, \kappa_2}/\sim$ is not compact under the orbit equivalence relation under g^t , $t \in \mathbb{R}$.

By Proposition A.2, $\mathcal{H}_{s_0, \kappa_1, \kappa_2} \cup \{\zeta_{(1,0,0,0)}\}$ is compact. In any ϵ - $\mathbf{d}_H$ -neighborhood $\hat{N}$, $\epsilon > 0$, of $\text{Cl}(\zeta_{(1,0,0,0)})$ in $\mathcal{H}_{s_0, \kappa_1, \kappa_2}$, we can find a g^t -orbit in $\hat{N}$ by Lemma A.3. Take any neighborhood N of $\text{Cl}(\zeta_{(1,0,0,0)})$ in $\mathbb{S}^3$. Since we are using the Hausdorff metric, we can find an ϵ - $\mathbf{d}_H$ -neighborhood $\hat{N}$ in $\mathcal{H}_{s_0, \kappa_1, \kappa_2} \cup \{\zeta_{(1,0,0,0)}\}$ so that any segment in $\hat{N}$ is a segment in N . Then the g^t -orbit as above will give us the desired ruled surface in N . This proves the second item.

The first and second items imply the fact on the boundary of $S_{f,r}$. Clearly, $S_{f,r}$ bounds a domain in E with boundary $\text{Cl}(S_{f,r})$. This domain is homeomorphic to a 3-cell by [36, Lemma 1.12]. Also, g sends the disk leaves of the foliation $\mathcal{D}_{f,r_0}$ of the domain to a disjoint disk leaf in Theorem 3.14. Hence, the quotient space is homeomorphic to a solid torus. $\square$

A.2 Two transversal foliations

Proof of Theorem 3.14. The fact that $S_{f,r}$ is a properly embedded surface is proved in Theorem 3.13. We defined $l_{f,r}(s) = (sr, f(\rho), s\sqrt{1-r^2})$. We define

$$l_f : [r_0, 1) \times \mathbb{R} \rightarrow E \quad \text{given by} \quad l_f(r, s) = (sr, f(\rho), s\sqrt{1-r^2}).$$

Let $u_{l_{f,r}}$ denote the vector field $(r, 0, \sqrt{1-r^2})$ tangent to $l_{f,r}(s)$. Also, the vector field ϕ generating g^t is given by (y, z, μ) .

$$\frac{\partial l_f}{\partial r} = Y_f = \left(s, f'(\rho), \frac{-sr}{\sqrt{1-r^2}} \right)$$

is tangent to $D_{f,r_0,0}$ obtained by taking a tangent vector along the direction of $\frac{\partial}{\partial r}$. A *triple product of three vectors* is the volume of the span of three vectors in E . We compute the triple product on the line $l_{f,r}$

$$(u_{l_{f,r}}, Y_f, \phi) = \sqrt{1-r^2} \left(\frac{\mu r}{\sqrt{1-r^2}} - f(\rho) \right) f'(\rho) + s^2 > 0, \quad (\text{A.3})$$

which follows by our condition on f and r . It follows that $u_{l_{f,r}}, Y_f, \phi$ form always an independent frame in the standard orientation on $l_{f,r}$, and so are their images under g^t since g^t is volume-preserving. Thus,

$$Dg^t(u_{l_{f,r}}), \quad Dg^t(Y_f), \quad Dg^t(\phi)$$

form an independent frame at each point of $S_{f,r}$.

We claim that $S_{f,r}$ is disjoint from $S_{f,r'}$ for $r_0 \leq r < r' < 1$: By (A.3), Y_f is transversal to $S_{f,r}$ on $l_{f,r}$. We define the vector field Y_f on $S_{f,r}$ so that

$$Y_f(g^t(sr, f(\rho), s\sqrt{1-r^2})) = Dg^t(Y_f(sr, f(\rho), s\sqrt{1-r^2})).$$

The extended Y_f is transversal to $S_{f,r}$ since the triple product is invariant under the Lorentzian isometries. Define $\Xi_f(r, t, s) = g^t(l_f(r, s))$, which gives us a parametrization of $S_{f,r}$. We obtain the partial derivative with respect to r by chain-rules:

$$\frac{\partial \Xi_f(r, t, s)}{\partial r} = Dg^t(Y_f(l_f(r, s))) = Y_f(\Xi_f(r, t, s)).$$

Solving the following ordinary differential equation with respect to the variable r

$$\frac{\partial \Xi_f(r, t, s)}{\partial r} = Y_f(\Xi_f(r, t, s))$$

gives us a flow $\Xi_f(r, t, s)$ for r in some interval with fixed t, s . Using the quasi-linear Cauchy theorem ([35, Theorem 9.52]) and the transversality, we obtain the disjointness.

Also, for each point x of R_{f,r_0} , there is a leaf $S_{f,r'}$ containing it: Let x_i be a sequence converging to x and $x_i \in S_{f,r_i}$, $r_i > r_0$. Then let L_i be the line in S_{f,r_i} containing x_i . Since we showed that $\mathcal{H}_{S_0, \kappa_1, \kappa_2} \cup \text{Cl}(\zeta_{(1,0,0,0)})$ is compact by Proposition A.2, $\text{Cl}(L_i)$ geometrically converges to an element of $\mathcal{H}_{S_0, \kappa_1, \kappa_2}$ or to $\text{Cl}(\zeta_{(1,0,0,0)})$ by choosing a subsequence if necessary. Proposition A.2 shows that $L_i = g^{t_i}(l_f(r_i))$ for bounded t_i in the first case. Hence, x is in $S_{f, \lim_i r_i}$. In the other case, x_i does not have x as a limit. This proves the closedness of the foliated subset in R_{f,r_0} .

Using the flows, we can prove the openness of the set $\bigcup_{r_0 \leq r < 1} S_{f,r'}$. Hence, R_{f,r_0} is foliated by leaves $S_{f,r}$, $r \geq r_0$.

Since each line in $D_{f,r_0,0}$ lies on a different plane given by equations of the form $y = c$, it follows that $D_{f,r_0,0}$ is an embedded surface, and so are $D_{f,r_0,t}$. Proposition A.2 implies that $D_{f,r_0,0}$ is properly embedded since $\text{Cl}(l_{f,r_i})$ geometrically converges to $\text{Cl}(\zeta_{(1,0,0,0)})$ as $r_i \rightarrow 1$. Hence, $D_{f,r_0,t}$ is properly embedded for all t .

Since g^{t_0} is generated by a vector field ϕ transversal to $D_{f,r_0,t}$ for every t by the above paragraph, the images under the flows of $D_{f,r_0,t}$ are disjoint from $D_{f,r_0,t}$. Also, $g^{t_0}(D_{f,r_0,t}) = D_{f,r_0,t+t_0}$ follows by our above definition of $D_{f,r_0,t}$.

Also, R_{f,r_0} is foliated by leaves of the form $D_{f,r_0,t}$ as follows: $\bigcup_{t \in \mathbb{R}} D_{f,r_0,t}$ is open since we can use the flow generated by g^t . The closedness follows by Proposition A.2 again as above.

Now, we have a foliation by leaves of the form $S_{f,r}$ for $r \in [r_0, 1)$. Then $D_{f,r_0,t} \cap S_{f,r}$ contains a geodesic given by $g^t(l(s))$, $s \in \mathbb{R}$. At $t = 0$, the tangent space of $D_{f,r_0,0}$ is generated by u_l , Y_f , and that of $S_{f,r}$ is generated by u_l , ϕ . The independence above implies the transversality of $D_{f,r_0,0}$ and $S_{f,r}$. Thus, the transversality of $D_{f,r_0,t}$ and $S_{f,r}$ follows. $\square$

Remark A.1. There seems to be a vast literature on ruled surfaces on which a one-parameter Lorentzian isometry group acts; however, there seems to be no article on the topological properties. See Dillen and Kühlen [21] for a survey of geometric aspects.

B The flat $\mathbb{R}^{2,1}$ -bundle valued 1-forms on a cusp neighborhood

Only prerequisites are Sections 2 and 3.1 and the notation in Section 4, in particular Definition 4.1.

B.1 Replacing forms by standard cusp 1-forms in the cusp neighborhoods

Suppose that Γ is a discrete Lorentzian isometry group so that Γ is a Fuchsian group acting on $\mathbb{S}_+$ with a parabolic element g fixing $p \in \partial \mathbb{S}_+$. Let $S := \mathbb{S}_+/\Gamma$ be a complete hyperbolic surface with a cusp neighborhood E . Note that E is covered by a horodisk $P \subset \mathbb{S}_+$ with $p \in \text{bd}_{\mathbb{S}} P$. Then $P/\langle g \rangle$ is isometric to E .

We recall the vector bundle $\mathcal{V}$ given as the quotient of $\widetilde{\mathcal{V}} = \mathbb{S}_+ \times \mathbb{R}^{2,1}$ with action given by

$$\gamma(x, \mathbf{v}) = (\gamma(x), \mathcal{L}(\gamma)(\mathbf{v})) \quad \text{for } \gamma \in \Gamma, \mathbf{v} \in \mathbb{R}^{2,1}.$$

Recall (4.4) that for $\widetilde{\mathcal{V}}$ -valued 1-forms on $\mathbb{S}_+$, the action is given by

$$\gamma^*(\mathbf{v} \otimes dx) = \mathcal{L}(\gamma)^{-1}(\mathbf{v}) \otimes dx \circ \gamma.$$

Proposition B.1. *Let S, Γ, P, E , and γ be as above in Section B.1. Let η be a closed $\mathcal{V}$ -valued 1-form representing a class in $H^1(S, \mathcal{V})$. Let ζ be a closed $\mathcal{V}$ -valued 1-form in E so that ζ is cohomologous to $\eta|_E$ in $H^1(E, \mathcal{V})$. Then we can find a closed $\mathcal{V}$ -valued 1-form η' on S cohomologous to η and a cusp neighborhood $E' \subset E$ so that $\eta'|_{E'} = \zeta|_{E'}$.*

Proof. Let $E' \subset E$ be a smaller cusp neighborhood so that $\text{Cl}(E') \subset E$. Consider $\eta - \zeta$ on E' . Then $\eta - \zeta = df$ for a section $f: E' \rightarrow \mathcal{V}$. We can extend f to a smooth section $f: S \rightarrow \mathcal{V}$ by a partition of unity so that $f = 0$ on $S \setminus E$. Then define $\eta' = \eta$ on $S \setminus E$ and $\eta' = \zeta$ on E' and $\eta' = \eta - df$ on $E \setminus E'$. $\square$

Proposition B.2. *We have $H^1(E, \mathcal{V}) = \mathbb{R}$.*

Proof. Recall that E is homotopy equivalent to $\mathbb{S}^1$. Thus, $\pi_1(E)$ is an infinite cyclic group. Note that $\mathcal{V}|_E$ is $P \times \mathbb{R}^{2,1}/\langle g \rangle$. Recall that $\mathcal{L}(g) = I + N(g) + N(g)^2/2$ for a nilpotent matrix $N(g)$ of rank 2 from (3.5). We conclude using the knowledge of [28, Section 3]:

$$\begin{aligned} Z^1(\langle g \rangle, \mathcal{V}) &= \{\mathbf{v} : \mathbb{Z} \rightarrow \mathbb{R}^{2,1} \mid \mathcal{L}(g^i)\mathbf{v}(g^j) - \mathbf{v}(g^{i+j}) = 0 \text{ for all } i, j \in \mathbb{Z}\} \cong \mathbb{R}^{2,1}, \\ B^1(\langle g \rangle, \mathcal{V}) &= \{\mathbf{v} : \mathbb{Z} \rightarrow \mathbb{R}^{2,1} \mid \mathbf{v}(g) = \mathcal{L}(g)\mathbf{v}_0 - \mathbf{v}_0, \mathbf{v}_0 \in \mathbb{R}^{2,1}\} \\ &= N(g)\left(\left(I + \frac{1}{2}N(g)\right)(\mathbb{R}^{2,1})\right) = N(g)(\mathbb{R}^{2,1}), \\ H^1(\langle g \rangle, \mathcal{V}) &= \mathbb{R}^{2,1}/N(g)(\mathbb{R}^{2,1}) \cong \mathbb{R}. \end{aligned}$$

The second to last equation follows since $I + N(g)/2$ is invertible. The last follows since $N(g)$ is of rank two by Section 3.1. $\square$

B.2 The integral of the standard cusp 1-form

We will use the notation of Section 4.4. Let P be the standard horodisk in $\mathbb{S}_+$ with $\mathbf{p}$ as a null vector in the direction of $\text{Cl}(P) \cap \partial\mathbb{S}_+$. The standard cusp 1-form for P is given where P is given by $y > 1$ in the upper half-space model $\mathbb{U}^2$ of the hyperbolic plane. A geodesic in P is given by equation $(x \pm R)^2 + y^2 = R^2$ in $\mathbb{U}^2$ and parameterized by $\zeta(\theta) := (R \cos \theta \mp R, R \sin \theta)$. The starting point and the endpoint are given by $R \sin \theta = 1$. Thus, the beginning θ_0 and ending $\theta_1 = \pi - \theta_0$ is one of the values of $\sin^{-1}(\frac{1}{R})$.

We assume that a complete geodesic l passes a cusp region with the cusp point $p = \langle \mathbf{p} \rangle$ and the standard cusp 1-form. We assume that by choice of the coordinates of $\mathbb{U}^2$, $p = \infty$ and the geodesic starts at $(0, 0)$ and ends at $(2R, 0)$ or at $(-2R, 0)$. We say that l and any horizontal translation of l in the upper half model have radius R .

There is an isometry $H : \mathbb{U}^2 \rightarrow \mathbb{S}_+$ to the Klein model $\mathbb{S}_+$:

$$H(x, y) := \left(\overrightarrow{H(x, y)} \right), \quad \text{where } \overrightarrow{H(x, y)} := \left(\frac{2x}{x^2 + y^2 + 1}, 1 - \frac{2}{x^2 + y^2 + 1}, 1 \right),$$

(See Hongchan Kim [33, Theorem 7.1].) This extends to the boundary $y = 0$ and induces a homeomorphism from $\mathbb{U}^2 \cup \{\infty\}$ to the closure of the unit disk where ∞ goes to $\langle \mathbf{j} + \mathbf{k} \rangle$.

The standard horodisk in $\mathbb{U}^2$ is given by $y > 1$. The image of this under H is the standard horodisk Q of the Klein model. The standard horodisk has the point $\langle \mathbf{j} + \mathbf{k} \rangle$ of $\mathbb{S}_+$ in the boundary and $\partial_h Q \ni \langle \mathbf{k} \rangle$.

This makes things simpler.

Lemma B.3. *Let g , l , and $\|\cdot\|_E$ be as above. Let D be the standard horodisk. Set*

$$\mathbf{v}(x) = \left(x, -\frac{x^2}{2\sqrt{2}} + \frac{1}{\sqrt{2}}, \frac{x^2}{2} + \frac{1}{\sqrt{2}} \right) \quad \text{for } x \in \mathbb{R}.$$

Let l be a complete geodesic passing D of radius R and starts at $H(0, 0)$ and ends at $H(\pm 2R, 0)$. Assume $R > 1$. Suppose that l corresponds in $\mathbb{S}_+$ to the geodesic passing a point of $\partial_h D$ in the direction of a unit vector $\mathbf{u}$ away from $H(0, 0) \in \partial\mathbb{U}^2$. Then the following hold:

- *for any point z on l with coordinate x in the upper half-space model,*

$$\|\Pi_{\tilde{\mathbf{v}}_0(z, \mathbf{u})}(\mu\mathbf{v}(x))\|_E = \left| \mu \left(x - \frac{\pm\sqrt{2}}{R} \right) \right|$$

for the cusp coefficient μ , and

- *we have*

$$\|\Pi_{\tilde{\mathbf{v}}_-(z, \mathbf{u})}(\mu\mathbf{v})\|_E = \left| \frac{\mu(4R^2 + 1)}{4\sqrt{2}R^2} \right| \quad \text{and} \quad \|\Pi_{\tilde{\mathbf{v}}_+(z, \mathbf{u})}(\mu\mathbf{v})\|_E = \left| \mu \left(\frac{1}{4\sqrt{2}R^2} - \frac{\pm x}{2R} + \frac{x^2}{2\sqrt{2}} \right) \right|.$$

Proof. These are simple computations using H , and the frames used there form uniformly bounded matrices in $\text{GL}(3, \mathbb{R})$. Hence, the estimations are uniformly compatible with the standard Euclidean metric results. (See [12] and [13].) $\square$

Let $\zeta = l \cap D$ be a geodesic segment with both endpoints in $\partial_h D$. Suppose that l is in the form of Lemma B.3 parameterized by the angle θ from the center of the semicircle in the upper-half-space model containing l . Also, recall the geodesic flow Ψ_t acting on $\mathbb{R}^{2,1} \times \mathbb{U}\mathbb{S}_+$ from Section 3.2. We reparametrize l by $\Psi(z, \theta)$ for z the beginning point of ζ and $\theta \in (0, \pi)$ with $z = \Psi(z, \theta_0)$. Let η denote the standard 1-form defined on D . We define

$$\mathbf{b}(\zeta) := \int_{\theta_0}^{\pi-\theta_0} \mathbb{D}\Psi(z, \theta - \theta_0)^{-1} \left(\eta \left(\frac{d\Psi(z, \theta)}{d\theta} \right) \right) d\theta,$$

where θ_0 and $\pi - \theta_0$ are the start and the end angles of the semicircle l parameterized by angles.

We recall $\mathbf{b}_{\pm}(\zeta)$ from (4.15) as the $\tilde{V}_{\pm}$ -component of $\mathbf{b}(\zeta)$: that is,

$$\mathbf{b}_{\pm}(\zeta) := \int_{\theta_0}^{\pi-\theta_0} \Pi_{\tilde{V}_{\pm}} \left(\mathbb{D}\Psi(z, \theta - \theta_0)^{-1} \left(\eta \left(\frac{d\Psi(z, \theta)}{d\theta} \right) \right) \right) d\theta.$$

We define

$$\alpha(\zeta) := \int \mathbf{B} \left(v, \eta \left(\frac{d\Psi(z, \theta)}{d\theta} \right) \right) d\theta.$$

Proposition B.4. *Let g , l , and $\|\cdot\|_E$ be as above. Let D be the standard horodisk. Let η be a standard cusp 1-form for a cusp constant $\mu > 0$. (See (4.9).) Suppose that a complement geodesic l of radius R is in the form of Lemma B.3. Let $\zeta = l \cap D$ be a geodesic segment with both endpoints in $\partial_h D$. Then we obtain*

$$\begin{aligned} \|\mathbf{b}_{-}(\zeta)\|_E &= \mu \frac{\sqrt{-1+R^2}(1+4R^2)}{2\sqrt{2}R^2} \leq \frac{5}{2\sqrt{2}}\mu R, \\ \mu(-\sqrt{2}+2R^2) \frac{\sqrt{-1+R^2}}{R} &\leq \alpha(\zeta) = \mu \frac{\sqrt{-1+R^2}}{R} (\pm\sqrt{2}+2R^2) \leq \mu(\sqrt{2}+2R^2), \end{aligned}$$

where $R \geq 1$.

Proof. In this case, we may regard $\mathbb{D}\Psi(z, \theta)^{-1}$ as the identity since we will work directly over $\mathbb{S}_+$ (see Remark 4.1): Since the projection $\Pi_{\tilde{V}_{-}}$ to $\tilde{V}_{-}$ commutes with $\mathbb{D}\Psi(z, \theta)^{-1}$, we have

$$\mathbf{b}_{-}(\zeta) := \int_{\theta_0}^{\pi-\theta_0} \mathbb{D}\Psi(z, \theta)^{-1} (\Pi_{\tilde{V}_{-}}(\eta)) dx \left(\frac{d\Psi(z, \theta)}{d\theta} \right) d\theta.$$

By computations in [12] or [13], we obtain

$$\|\mathbf{b}_{-}(\zeta)\| = \mu \sqrt{-1+R^2} \frac{(1+4R^2)}{2\sqrt{2}R^2}.$$

And we evaluate the contribution of $l \cap P$ to $\mathbf{b}_0$:

$$\alpha(\zeta) = \frac{\mu(\sqrt{-1+R^2})(\pm\sqrt{2}+2R^2)}{R}.$$

□

Note. During the preparation of this manuscript, our coauthor Todd Drumm tragically passed away. Todd pioneered the field by developing the geometric approach to Margulis's breakthrough discovery [39] and [40] of proper affine actions of nonabelian free groups. We miss him dearly and dedicate this work to his lasting memory.

Acknowledgment: We thank Virginie Charette, Jeffrey Danciger, Michael Kapovich, and Fanny Kassel for helpful comments. We thank Richard Canary for the idea to pursue the proof here similar to ones of Marden [36] and Thurston [47] in the Kleinian group theory where they separate the parabolic regions. We thank the MSRI for the hospitality where this work was partially carried out during the program “Dynamics on Moduli Spaces of Geometric Structures” in 2015. Also, we initiated the work during the conference “Exotic Geometric Structures” at the ICERM, Brown University, September 16–20, 2013. We do apologize for the length of the article. We felt that the shortening might confuse the readers since we use many ideas in a novel way. Also, dividing the paper seemed a bit unethical and to be a disservice to the mathematical community.

Funding: Choi was supported by the Mid-career Researcher Program through the National Research Foundation of Korea grant NRF-2013R1A1A2056698 funded by the MEST. Goldman was partially supported by the National Science Foundation Grant DMS-1709791.

References

- [1] J. W. Anderson, P. Bonfert-Taylor and E. C. Taylor, Convergence groups, Hausdorff dimension, and a theorem of Sullivan and Tukia, *Geom. Dedicata* **103** (2004), 51–67.
- [2] A. F. Beardon, *The Geometry of Discrete Groups*, Grad. Texts in Math. 91, Springer, New York, 1983.
- [3] A. F. Beardon and B. Maskit, Limit points of Kleinian groups and finite sided fundamental polyhedra, *Acta Math.* **132** (1974), 1–12.
- [4] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Universitext, Springer, Berlin, 1992.
- [5] R. D. Canary and Y. N. Minsky, On limits of tame hyperbolic 3-manifolds, *J. Differential Geom.* **43** (1996), no. 1, 1–41.
- [6] V. Charette and T. A. Drumm, The Margulis invariant for parabolic transformations, *Proc. Amer. Math. Soc.* **133** (2005), no. 8, 2439–2447.
- [7] V. Charette and Y. Kim, Foliations of Minkowski $2 + 1$ spacetime by crooked planes, *Internat. J. Math.* **25** (2014), no. 9, Article ID 1450088.
- [8] S. Choi, Calculus-v2.nb, Mathematica™, <https://mathsci.kaist.ac.kr/~schoi/research.html>.
- [9] S. Choi, Convergence-v2.nb, Mathematica™, <https://mathsci.kaist.ac.kr/~schoi/research.html>.
- [10] S. Choi, FoliationsV2.nb, Mathematica™, <https://mathsci.kaist.ac.kr/~schoi/research.html>.
- [11] S. Choi, GedesicFinal-v2.nb, Mathematica™, <https://mathsci.kaist.ac.kr/~schoi/research.html>.
- [12] S. Choi, NewNormalComp-v3.nb, Mathematica™, <https://mathsci.kaist.ac.kr/~schoi/research.html>.
- [13] S. Choi, NewNormalComp2-v3.nb, Mathematica™, <https://mathsci.kaist.ac.kr/~schoi/research.html>.
- [14] S. Choi, StandardFormII-v4.nb, Mathematica™, <https://mathsci.kaist.ac.kr/~schoi/research.html>.
- [15] S. Choi, Vanalysis-v2.nb, Mathematica™, <https://mathsci.kaist.ac.kr/~schoi/research.html>.
- [16] S. Choi and W. Goldman, Topological tameness of Margulis spacetimes, *Amer. J. Math.* **139** (2017), no. 2, 297–345.
- [17] J. Danciger, F. Guéritaud and F. Kassel, Fundamental domains for free groups acting on anti-de Sitter 3-space, *Math. Res. Lett.* **23** (2016), no. 3, 735–770.
- [18] J. Danciger, F. Guéritaud and F. Kassel, Geometry and topology of complete Lorentz spacetimes of constant curvature, *Ann. Sci. Éc. Norm. Supér. (4)* **49** (2016), no. 1, 1–56.
- [19] J. Danciger, F. Guéritaud and F. Kassel, Margulis spacetimes via the arc complex, *Invent. Math.* **204** (2016), no. 1, 133–193.
- [20] J. Danciger, F. Guéritaud and F. Kassel, Margulis spacetimes with parabolic elements, in preparation.
- [21] F. Dillen and W. Kühnel, Curvature of ruled surfaces and groups of Lorentzian motions, in: *Geometry and Topology of Submanifolds, IX* (Valenciennes/Lyon/Leuven 1997), World Scientific, River Edge (1999), 135–138.
- [22] T. A. Drumm, Fundamental polyhedra for Margulis space-times, *Topology* **31** (1992), no. 4, 677–683.
- [23] D. B. A. Epstein and A. Marden, Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces [mr0903852], in: *Fundamentals of Hyperbolic Geometry: Selected Expositions*, London Math. Soc. Lecture Note Ser. 328, Cambridge University, Cambridge (2006), 117–266.
- [24] D. B. A. Epstein and C. Petronio, An exposition of Poincaré’s polyhedron theorem, *Enseign. Math. (2)* **40** (1994), no. 1–2, 113–170.
- [25] W. Goldman and M. W. Hirsch, The radiance obstruction and parallel forms on affine manifolds, *Trans. Amer. Math. Soc.* **286** (1984), no. 2, 629–649.
- [26] W. M. Goldman, Projective structures with Fuchsian holonomy, *J. Differential Geom.* **25** (1987), no. 3, 297–326.
- [27] W. M. Goldman and F. Labourie, Geodesics in Margulis spacetimes, *Ergodic Theory Dynam. Systems* **32** (2012), no. 2, 643–651.
- [28] W. M. Goldman, F. Labourie and G. Margulis, Proper affine actions and geodesic flows of hyperbolic surfaces, *Ann. of Math. (2)* **170** (2009), no. 3, 1051–1083.
- [29] C. M. Gordon and R. A. Litherland, Incompressible surfaces in branched coverings, in: *The Smith Conjecture* (New York 1979), Pure Appl. Math. 112, Academic Press, Orlando (1984), 139–152.
- [30] S. Ghosh and N. Treib, Affine anosov representations and proper actions, preprint (2017), <https://arxiv.org/abs/1711.09712>.
- [31] E. Heintze and H.-C. Im Hof, Geometry of horospheres, *J. Differential Geom.* **12** (1977), no. 4, 481–491.
- [32] J. Hempel, *3-Manifolds*, AMS Chelsea, Providence, 2004.
- [33] H. C. Kim, Embedding of the Teichmüller space into the Goldman space, *J. Korean Math. Soc.* **43** (2006), no. 6, 1231–1252.
- [34] F. Labourie, *Lectures on Representations of Surface Groups*, Zur. Lect. Adv. Math., European Mathematical Society, Zürich, 2013.
- [35] J. M. Lee, *Introduction to Riemannian Manifolds*, Grad. Texts in Math. 176, Springer, Cham, 2018.

- [36] A. Marden, The geometry of finitely generated Kleinian groups, *Ann. of Math. (2)* **99** (1974), 383–462.
- [37] A. Marden, *Outer Circles. An Introduction to Hyperbolic 3-Manifolds*, Cambridge University, Cambridge, 2007.
- [38] A. Marden, *Hyperbolic Manifolds*, Cambridge University, Cambridge, 2016.
- [39] G. A. Margulis, Free completely discontinuous groups of affine transformations, *Dokl. Akad. Nauk SSSR* **272** (1983), no. 4, 785–788.
- [40] G. A. Margulis, Complete affine locally flat manifolds with a free fundamental group. Automorphic functions and number theory. II, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **134** (1984), 190–205.
- [41] J. R. Munkres, *Topology: A First Course*, Prentice-Hall, Englewood Cliffs, 1975.
- [42] K. Ohishka, Oral communications, 1997.
- [43] P. Scott and T. Tucker, Some examples of exotic noncompact 3-manifolds, *Quart. J. Math. Oxford Ser. (2)* **40** (1989), no. 160, 481–499.
- [44] I. Smilga, Proper affine actions on semisimple Lie algebras, *Ann. Inst. Fourier (Grenoble)* **66** (2016), no. 2, 785–831.
- [45] I. Smilga, Proper affine actions in non-swinging representations, *Groups Geom. Dyn.* **12** (2018), no. 2, 449–528.
- [46] I. Smilga, Proper affine actions: A sufficient criterion, *Math. Ann.* **382** (2022), no. 1–2, 513–605.
- [47] W. Thurston, *Geometry and topology of 3-manifolds*, Lecture notes, Princeton University, 1984, <http://www.msri.org/publications/books/gt3m/>.
- [48] P. Tukia, Conical limit points and uniform convergence groups, *J. Reine Angew. Math.* **501** (1998), 71–98.