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EXPONENTIAL TIME DIFFERENCING-PADÉ FINITE ELEMENT METHOD FOR NONLINEAR CONVECTION-DIFFUSION-REACTION EQUATIONS WITH TIME CONSTANT DELAY*

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Abstract

In this paper, ETD3-Padé and ETD4-Padé Galerkin finite element methods are proposed and analyzed for nonlinear delayed convection-diffusion-reaction equations with Dirichlet boundary conditions. An ETD-based RK is used for time integration of the corresponding equation. To overcome a well-known difficulty of numerical instability associated with the computation of the exponential operator, the Padé approach is used for such an exponential operator approximation, which in turn leads to the corresponding ETD-Padé schemes. An unconditional L^2 numerical stability is proved for the proposed numerical schemes, under a global Lipshitz continuity assumption. In addition, optimal rate error estimates are provided, which gives the convergence order of $O(k^3+h^r)$ (ETD3-Padé) or $O(k^4+h^r)$ (ETD4-Padé) in the L^2 norm, respectively. Numerical experiments are presented to demonstrate the robustness of the proposed numerical schemes.

Mathematics subject classification: 65N08, 65N12, 65N15.
Key words: Nonlinear delayed convection diffusion reaction equations, ETD-Padé scheme, Lipshitz continuity, L^2 stability analysis, Convergence analysis and error estimate.

1. Introduction

A delayed partial differential equation (DPDE) is formulated as

$$u_t + Au = f(t, u(\mathbf{x},t), u(\mathbf{x},t-\tau), \nabla u(\mathbf{x},t), \nabla u(\mathbf{x},t-\tau)), \tag{1.1}$$

in which τ is a fixed time quantity and A is a linear operator. This equation has played an important role in the simulation of many real-world problems, such as biological systems [1], epidemiology [32, 36], medicine, engineering control systems, climate models [9], etc. A variety of phenomenon in the natural and social sciences could be described by such a model, so that there have always been great significance and practical values to study it.

One distinguished feature of this model is associated with the fact that, the unknown quantity $u(\mathbf{x},t)$ depends not only on the solution at the present stage, but also on the solution at some past stage. For a survey of early results, we refer the readers to [39] and references therein.

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Since the 1950s, theoretical research of delayed differential equations has been attracted more and more attentions, with very rapid developments. The theoretical results on delay differential equations have already been quite rich [4, 19, 20, 22], while that for delayed partial differential equation is still an active developing branch [17,34,40,41,43]. In [34] the global asymptotic stability of traveling waves was studied for delayed reaction-diffusion equations; a similar analysis was carried out in [41] for Nicholson's blowflies equation with diffusion; more stability analysis have also been presented in [43]. A necessary and sufficient condition for oscillatory behavior of neutral hyperbolic delay partial differential equations was provided in [40]; the delayed reaction-diffusion model of the bacteriophage infection spread of bacteriophage infection was studied by Gourley and Kuang [17].

On the other hand, it is well-known that, an analytical solution for most DPDEs is not available [39]; even for some simple and specific equations, the theoretical solutions are often piecewise continuous or even cannot be expressed in analytical expression. In turn, the numerical investigation on DPDEs has attracted more attentions. For example, the Chebyshev spectral collocation method with waveform relaxation was considered in [21] for nonlinear D-PDEs. A local discontinuous Galerkin (LDG) method was proposed in [30] for reaction-diffusion dynamical systems with time delay. Also see [31, 33] and the related references.

Meanwhile, most existing numerical works have been focused on either first or second order accurate (in time) schemes; the numerical study of temporally third order or even higher-order numerical schemes for the delayed equation (1.1), as well as its theoretical analysis, has been very limited. In this article, we propose and analyze (temporally) third order and fourth order accurate numerical schemes for the delayed convection-diffusion-reaction equation, based on the exponential time differencing (ETD) temporal algorithm, combined with finite element spatial approximation. In general, an exact integration of the linear part of the PDE is involved in the ETD-based scheme. For the nonlinear terms, the multi-step approach has been based on explicit approximation of the temporal integral [2, 11, 18]. An application of such an idea to various gradient models has been reported in recent works [5–7,12,13,23–25,38,46]. On the other hand, in the extension of these ETD-based ideas to the delayed PDE, a higher order interpolation has to be involved in the numerical evaluation of the physical variables at the staggered temporal stencil mesh point around the delayed time instant. In addition, a direct computation of the exponential operator may lead to numerical instability, as discussed in an earlier work [26]. To overcome this well-known difficulty, the Padé approach is used for such an exponential operator approximation, which in turn leads to the corresponding ETD-Padé scheme, with the third or fourth order accuracy in the time discretization. The spatial discretization is the same as the ones to deal with the non-delay problem. In addition, the ETD-Padé schemes can reach the corresponding convergence order as the ETDRK schemes and the computational cost is less, also see the related derivations in [14, 27], etc.

A theoretical analysis of the third and fourth order accurate ETD-Padé schemes turns out to be highly challenging, due to the nonlinear and multi-step nature, combined with the delayed structure. An unconditional L^2 numerical stability is proved for the proposed schemes, under a global Lipshitz continuity assumption. With that assumption, the nonlinear difference could bounded by a linear growth term. In combination of the eigenvalue estimates for the operators involved in the ETD-Padé scheme, the L^2 numerical stability is derived with the help of discrete Gronwall inequality. In addition, an optimal rate convergence analysis and error estimate could be established in the same manner, in which the stability of the difference between the exact solution and numerical solution is analyzed. This in turn gives a convergence order of $O(\tau^3+h^r)$

or $O(\tau^4 + h^r)$ in the L^2 norm, respectively.

The rest of the article is organized as follows. In Section 2 we present the numerical schemes, including the third and fourth order ones. Subsequently, an unconditional L^2 numerical stability is established in Section 3. Moreover, the $\ell^\infty(0,T;L^2)$ convergence estimate is provided in Section 4. Some fast implementation techniques are discussed in Section 5 and numerical results are presented in Section 6. Finally, concluding remarks are given in Section 7.

2. The ETD-Padé Method for DPDE with Constant Delay

The following initial boundary value problem for the nonlinear delayed convection-diffusion-reaction equation is considered

$$\begin{array}{ll} u_t + Au = f(u(t-\tau), u(t), t), & \mathbf{x} \in \Omega,\ t \in (0,T] = J, \\ u(\mathbf{x},t) = 0, & \mathbf{x} \in \partial\Omega,\ t \in J, \\ u(\mathbf{x},t) = u_0(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [-\tau, 0], \end{array} \tag{2.1}$$

where $\Omega \subset R^d$ is a bounded domain with Lipschitz boundary and A is a uniformly elliptic operator

$$A := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{i,j}(x)\frac{\partial}{\partial x_j}) + \sum_{j=1}^{d} b_j(x)\frac{\partial}{\partial x_j} + b_0(x). \tag{2.2}$$

The coefficients $a_{i,j}$ and b_j are assumed to be C^∞ (or sufficiently smooth) functions on $\overline{\Omega} \times J$, $a_{i,j} = a_{j,i}$, and the following inequality is valid for some $c_0 > 0$

$$\sum_{i,j=1}^{d} a_{i,j}(.)\xi_i\xi_j \geq c_0|\xi|^2, \quad on\ \overline{\Omega}, \quad \forall\xi\ \in R^d. \tag{2.3}$$

The initial value problem (2.1) is formulated in a Hilbert space X. Let A be a linear, self-adjoint, positive definite closed operator with a compact inverse T, defined on a dense domain $D(A) \subset X$. For a general $\Omega \subset \mathbb{R}^d$ we denote below by $\|\cdot\|$ the norm in $L^2 = L^2(\Omega)$ and by $\|\cdot\|_r$ that in the Solobev space $H^r = H^r(\Omega) = W_2^r(\Omega)$, so that for real-valued function u,

$$\|u\| = \|u\|_{L_2} = \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}}, \tag{2.4}$$

and, for r a positive integer,

$$\|u\|_r = \|u\|_{H_r} = \left(\sum_{|\alpha|\leq r} \|D^\alpha u\|^2\right)^{\frac{1}{2}}, \tag{2.5}$$

where, with $\alpha = (\alpha_1, ...\alpha_d)$, $D^\alpha = (\partial/\partial x_1)^{\alpha_1}...(\partial/\partial x_d)^{\alpha_d}$ denotes an arbitrary derivative with respect to x of order $|\alpha| = \sum_{j=1}^{d} \alpha_j$. We assume there exists $M \geq 1$ such that

$$\|(zI - A)^{-1}\| \leq M|z|^{-1}, \qquad z \in \{z \in C : \gamma < |arg(z)| \leq \pi, z \neq 0\}, \tag{2.6}$$

in which γ is a complex angle in the first quadrant, i.e., $\gamma \in (0, \frac{\pi}{2})$. It follows that $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ which is the solution operator for (2.2) below, cf. [35]. A standard representation is given by

$$E(t) := e^{-tA} = \frac{1}{2\pi i}\int_\Lambda e^{-tz}(zI - A)^{-1}dz, \tag{2.7}$$

where $\Lambda := \{z \in C : |\arg(z)| = \Theta\}$, oriented so that $Im(z)$ decreases, for any $\Theta \in (\gamma, \frac{\pi}{2})$. By the Duhamel principle the exact solution can be rewritten as

$$u(t) = E(t)u_0(0) + \int_0^t E(t-s)f(u(s-\tau), u(s), s)ds. \tag{2.8}$$

The defining characteristic of DPDEs is that they admit solutions with derivative discontinuities at initial time and high order derivative discontinuties at subsequent points. For the constant delay τ, the potential discontinuous points are $t \in \{0, \tau, 2\tau, 3\tau, \cdots\}$. At these points the solution of a DPDE, regardless of how regular the given functions are, will in general exhibit a low degree of regularity. The simplest approach uses a constrained mesh including these discontinuous points. It is assumed that $k{=}\frac{\tau}{L}$, with L a positive integer and k a submultiple of τ. Over the time interval $J_n = [t_n, t_{n+1}]$, with $t_n = nk$, $0 \le n \le N$, the solution of DPDE is assumed to have high degree of regularity in J_n. Replacing t by $t+k$, using basic properties of E and by the change of variable $s - t = k\sigma$, we arrive at

$$u(t+k) = E(k)u(t) + k\int_0^1 E(k-k\sigma)f(u(t+k\sigma-\tau), u(t+k\sigma), t+\sigma k)d\sigma. \tag{2.9}$$

In turn, the following recurrence formula is available:

$$u(t_{n+1}) = e^{-kA}u(t_n) + k\int_0^1 e^{-kA(1-s)}f(u(t_n+sk-\tau), u(t_n+sk), t_n+sk)ds. \tag{2.10}$$

2.1. The ETDRK method for the DPDE

The recurrence formula (2.10) for the exact solution forms a basis of different time-stepping algorithms, depending upon how one could approximate matrix exponential function and integral. Various time-stepping schemes have been developed in [3, 11, 29] by using polynomial formulas, which in turn gives a multi-step or Runge-Kutta type higher-order approximations. This approach has an advantage of generating a family of high-order numerical schemes with potentially good performance, as long as a few computational difficulties could be resolved, as pointed out in [41]. Meanwhile, even the Kassam-Trefethen suggestion has left unresolved computational issues, especially related to the practical implementation of the numerical contour integration, as well as the contour choice as the numerical mesh is refined, since the spectrum of A will grow larger with more refined mesh and the location of the spectrum cannot be automatically calculated. Several Runge-Kutta-based time stepping schemes were also developed in [11, 26].

In fact, formula (2.10) is exact, and the essence of the ETD methods is associated with an approximation to the integral in this expression. We denote u^n as the numerical approximation to $u(t_n)$, u^{n-L} as the approximation to $u(t_n-\tau) = u(t_{n-L})$, and $\dddot{F} = \frac{\partial^3 f}{\partial t^3}$, $\ddddot{F} = \frac{\partial^4 f}{\partial t^4}$, respectively. The third-order accurate ETD-based scheme can be constructed in a similar way, analogous to the classical third-order RK method (see, for example [11, 45])

$$\begin{aligned} u(t_{n+1}) =& e^{-kA}u(t_n) \\ &+ k(kA)^{-3}(-k^2A^2e^{-kA} - 3kAe^{-kA} - 4e^{-kA} - kA + 4)f(u(t_n-\tau), u(t_n), t_n) \\ &+ k(kA)^{-3}(4kAe^{-kA} + 8e^{-kA} + 4kA - 8)f(u(t_n+\frac{k}{2}-\tau), u(t_n+\frac{k}{2}), t_n+\frac{k}{2}) \\ &+ k(kA)^{-3}(k^2A^2 - 3kA - kAe^{-kA} - 4e^{-kA} + 4)f(u(t_{n+1}-\tau), u(t_{n+1}), t_{n+1}) \\ &+ e^{-kA}\int_0^k e^{sA}\dddot{F}s^3ds. \end{aligned} \tag{2.11}$$

In fact, unconditionally stable arbitrarily high-order Runge-Kutta algorithms have been recently introduced for various gradient flows in [15]. Meanwhile, the following $O(k^4)$ interpolation formula is applied to approximate $u(t_n + \frac{k}{2} - \tau)$:

$$u(t_n + \frac{k}{2} - \tau) = -\frac{1}{16}(u^{n-L-1} + u^{n-L+2}) + \frac{9}{16}(u^{n-L} + u^{n-L+1}) + O(k^4). \tag{2.12}$$

Then, the following ETD3 numerical algorithm for (2.1) is available:

$$\begin{aligned} u^{n+1} =& e^{-kA}u^n \\ &+ k(kA)^{-3}(-k^2A^2e^{-kA} - 3kAe^{-kA} - 4e^{-kA} - kA + 4)f(u^{n-L}, u^n, t_n) \\ &+ k(kA)^{-3}(4kAe^{-kA} + 8e^{-kA} + 4kA - 8)f(a_n, b_n, t_n + \frac{k}{2}) \\ &+ k(kA)^{-3}(k^2A^2 - 3kA - kAe^{-kA} - 4e^{-kA} + 4)f(u^{n-L+1}, c_n, t_{n+1}), \end{aligned} \tag{2.13}$$

in which

$$\begin{aligned} a_n &= -\frac{1}{16}(u^{n-L-1} + u^{n-L+2}) + \frac{9}{16}(u^{n-L} + u^{n-L+1}), \\ b_n &= e^{-\frac{k}{2}A}u^n + k(kA)^{-1}(I - e^{-\frac{k}{2}A})f(u^{n-L}, u^n, t_n), \\ c_n &= e^{-kA}u^n + k(kA)^{-1}(I - e^{-kA})[2f(a_n, b_n, t_n + \frac{k}{2}) - f(u^{n-L}, u^n, t_n)]. \end{aligned} \tag{2.14}$$

The terms a_n, b_n and c_n are approximations of $u(t_n+\frac{k}{2}-\tau)$, $u(t_n+\frac{k}{2})$ and $u(t_n+k)$, respectively.

The fourth-order ETD (ETD4) scheme can be constructed in a similar way, analogous to the classical four-order RK method (see, for example, [11, 44]).

$$\begin{aligned} u^{n+1} =& e^{-kA}u^n \\ &+ k(kA)^{-3}(-k^2A^2e^{-kA} - 3kAe^{-kA} - 4e^{-kA} - kA + 4)f(u^{n-L}, u^n, t_n) \\ &+ k(kA)^{-3}(2kAe^{-kA} + 4e^{-kA} + 2kA - 4)f(a_n, b_n, t_n + \frac{k}{2}) \\ &+ k(kA)^{-3}(2kAe^{-kA} + 4e^{-kA} + 2kA - 4)f(a_n, c_n, t_n + \frac{k}{2}) \\ &+ k(kA)^{-3}(k^2A^2 - 3kA - kAe^{-kA} - 4e^{-kA} + 4)f(u^{n-L+1}, d_n, t_{n+1}), \end{aligned} \tag{2.15}$$

in which

$$\begin{aligned} a_n &= -\frac{1}{16}(u^{n-L-1} + u^{n-L+2}) + \frac{9}{16}(u^{n-L} + u^{n-L+1}), \\ b_n &= e^{-\frac{k}{2}A}u^n + k(kA)^{-1}(I - e^{-\frac{k}{2}A})f(u^{n-L}, u^n, t_n), \\ c_n &= e^{-\frac{k}{2}A}u^n + k(kA)^{-1}(I - e^{-\frac{k}{2}A})f(a_n, b_n, t_n + \tfrac{k}{2}), \\ d_n &= e^{-\frac{k}{2}A}b_n + k(kA)^{-1}(I - e^{-\frac{k}{2}A})[2f(a_n, c_n, t_n + \frac{k}{2}) - f(u^{n-L}, u^n, t_n)]. \end{aligned} \tag{2.16}$$

In the ETD3 and ETD4 schemes, there are many numerical challenges associated with the operator A, since the computations of $(-A)^{-1}$, $(-A)^{-2}$, $(-A)^{-3}$, as well as e^{-kA}, are needed. In fact, it may be accurate for small eigenvalues due to the cancellation error, while the above-mentioned polynomial expansion may lead to a loss of accuracy for large eigenvalues. In [27], it is pointed out that, these functions are higher-order matrix polynomials and cancellation errors can affect the computations perhaps even worse. To avoid this problem, we introduce the rational functions to approximate exponential operators.

2.2. ETD-Padé schemes for PDEs

Replaced e^{-z} and $e^{-\frac{z}{2}}$ in (2.13) and (2.14) by $R_{03}(z)$ and $\widetilde{R}_{03}(z)$, where $R_{03}(z)$ is $(0,3)$-Padé approximation to e^{-z} and $\widetilde{R}_{03}(z)$ the approximation to $e^{-\frac{z}{2}}$,

$$R_{03}(z) = (1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3)^{-1}, \ \widetilde{R}_{03}(z) = (1 + \frac{1}{2}z + \frac{1}{8}z^2 + \frac{1}{48}z^3)^{-1}. \tag{2.17}$$

Then we obtain the ETD3-Padé scheme:

$$\begin{aligned} u^{n+1} =& R_{03}(kA)u^n + kP_1(kA)f(u^{n-L}, u^n, t_n) + 2kP_2(kA)f(a_n, b_n, t_n + \frac{k}{2}) \\ &+ kP_3(kA)f(u^{n-L+1}, c_n, t_{n+1}), \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} a_n &= -\frac{1}{16}(u^{n-L-1} + u^{n-L+2}) + \frac{9}{16}(u^{n-L} + u^{n-L+1}), \\ b_n &= \widetilde{R}_{03}(kA)u^n + k\widetilde{P}(kA)f(u^{n-L}, u^n, t_n), \\ c_n &= R_{03}(kA)u^n + kP(kA)[2f(a_n, b_n, t_n + \frac{k}{2}) - f(u^{n-L}, u^n, t_n)]. \\ R_{03}(kA) &= 6(6I + 6kA + 3k^2A^2 + k^3A^3)^{-1}, \\ P(kA) &= (6 + 3kA + k^2A^2)(6I + 6kA + 3k^2A^2 + k^3A^3)^{-1}, \\ P_1(kA) &= (I - kA)(6I + 6kA + 3k^2A^2 + k^3A^3)^{-1}, \\ P_2(kA) &= 2(I + kA)(6I + 6kA + 3k^2A^2 + k^3A^3)^{-1}, \\ P_3(kA) &= (I + k^2A^2)(6I + 6kA + 3k^2A^2 + k^3A^3)^{-1}, \\ \widetilde{R}_{03}(kA) &= 48(48I + 24kA + 6k^2A^2 + k^3A^3)^{-1}, \\ \widetilde{P}(kA) &= (24I + 6kA + k^2A^2)(48I + 24kA + 6k^2A^2 + k^3A^3)^{-1}. \end{aligned} \tag{2.19}$$

In the ETD4 scheme, we denote $R_{22}(z)$ and $\widetilde{R}_{22}(z)$ for $(2,2)$-Padé approximations of e^{-z} and $e^{-\frac{z}{2}}$ respectively.

$$\begin{aligned} R_{22}(z) &= \left(1 - \frac{1}{2}z + \frac{1}{12}z^2\right)\left(1 + \frac{1}{2}z + \frac{1}{12}z^2\right)^{-1}, \\ \widetilde{R}_{22}(z) &= \left(1 - \frac{1}{4}z + \frac{1}{48}z^2\right)(1 + \frac{1}{4}z + \frac{1}{48}z^2\Big)^{-1}. \end{aligned} \tag{2.20}$$

Then the ETD4-Padé scheme for (2.1) becomes

$$\begin{aligned} u^{n+1} =& R_{22}(kA)u^n + kP_1(kA)f(u^{n-L}, u^n, t_n) + kP_2(kA)f(a_n, b_n, t_n + \frac{k}{2}) \\ &+ kP_2(kA)f(a_n, c_n, t_n + \frac{k}{2}) + kP_3(kA)f(u^{n-L+1}, d_n, t_{n+1}), \end{aligned} \tag{2.21}$$

where

$$
\begin{aligned}
&a_n = -\frac{1}{16}(u^{n-L-1} + u^{n-L+2}) + \frac{9}{16}(u^{n-L} + u^{n-L+1}),\\
&b_n = \widetilde{R}_{22}(kA)u^n + k\widetilde{P}(kA)f(u^{n-L}, u^n, t_n),\\
&c_n = \widetilde{R}_{22}(kA)u^n + k\widetilde{P}(kA)f(a_n, b_n, t_n + \frac{k}{2}),\\
&d_n = \widetilde{R}_{22}(kA)b_n + k\widetilde{P}(kA)[2f(a_n, c_n, t_n + \frac{k}{2}) - f(u^{n-L}, u^n, t_n)].\\
&R_{22}(kA) = (12I - 6kA + k^2A^2)(12I + 6kA + k^2A^2)^{-1},\\
&P(kA) = 12(12I + 6kA + k^2A^2)^{-1},\\
&P_1(kA) = (2I - kA)(12I + 6kA + k^2A^2)^{-1},\\
&P_2(kA) = 4(12I + 6kA + k^2A^2)^{-1},\\
&P_3(kA) = (2I + kA)(12I + 6kA + k^2A^2)^{-1},\\
&\widetilde{R}_{22}(kA) = (48I - 12kA + k^2A^2)(48I + 12kA + k^2A^2)^{-1},\\
&\widetilde{P}(kA) = 24(48I + 12kA + k^2A^2)^{-1}.
\end{aligned}
\tag{2.22}
$$

The following preliminary estimates will be useful in the later analysis.

Lemma 2.1 ([27, 28]). *The following estimates are available:*

$$
\begin{aligned}
&\|R_{03}(kA)\| \le 1, \|P_1(kA)\| \le 1,\ \ \|P_2(kA)\| \le \frac{1}{3},\ \ \|P_3(kA)\| \le \frac{1}{3},\\
&\|\widetilde{R}_{03}(kA)\| \le 1,\ \ \|P(kA)\| \le 1,\ \ \|\widetilde{P}(kA)\| \le \frac{1}{2}.
\end{aligned}
\tag{2.23}
$$

Lemma 2.2 ([42] Discrete Gronwall inequality). *Suppose $\alpha, \beta \ge 0$ are arbitrary constants, the sequence $\{\eta_n\}$ satisfies*

$$
\|\eta_n\| \le \beta + \alpha k \sum_{j=0}^{n-1} \|\eta_j\|,\ \ nk \le T, \tag{2.24}
$$

in which k is the step size, then

$$
\|\eta_n\| \le e^{\alpha T}(\beta + \alpha k\|\eta_0\|). \tag{2.25}
$$

Lemma 2.3 ([37] Continuous Gronwall inequality (Differential form)). *Suppose the nonnegative continuous function $\eta(t)$ satisfies*

$$
\eta'(t) \le \phi(t)\eta(t) + \varphi(t),\ \ t \in [0, T], \tag{2.26}
$$

where $\phi(t)$ and $\varphi(t)$ are non-negative integrable functions, then we have

$$
\eta(t) \le e^{\int_0^t \phi(s)ds}[\eta(0) + \int_0^t \varphi(s)ds],\ \ \forall t \in [0, T]. \tag{2.27}
$$

Lemma 2.4 ([14,28]). *The following estimates are available:*

$$
\begin{aligned}
&\|e^{-kA} - R_{03}(kA)\| \le Ck^4, \ \ \|e^{-\frac{k}{2}A} - \widetilde{R}_{03}(kA)\| \le Ck^4,\\
&\|\big(k(kA)^{-3}(-k^2A^2e^{-kA} - 3kAe^{-kA} - 4e^{-kA} - kA + 4) - kP_1(kA)\big)\| \le Ck^4,\\
&\|\big(k(kA)^{-3}(4kAe^{-kA} + 8e^{-kA} + 4kA - 8) - 2kP_2(kA)\big)\| \le Ck^4,\\
&\|(k(kA)^{-3}(k^2A^2 - 3kA - kAe^{-kA} - 4e^{-kA} + 4) - kP_3(kA))\| \le Ck^4,\\
&\|(k(kA)^{-1}(I - e^{-\frac{k}{2}A}) - k\widetilde{P}(kA))\| \le Ck^4,\\
&\|(k(kA)^{-1}(I - e^{-kA}) - kP(kA))\| \le Ck^4.
\end{aligned}
\tag{2.28}
$$

Remark 2.1. It is observed that, in the case of an elliptic operator A (formulated as (2.2)) with constant coefficients, an FFT-based fast solver could be efficiently applied in the numerical approximation of the Padé operators $R_{03}(kA)$, $R_{22}(kA)$, as well as $P_j(kA)$ $(1 \le j \le 3)$, $\widetilde{R}_{03}(kA)$, $\widetilde{R}_{22}(kA)$, $\widetilde{P}(kA)$. Of course, if variable-dependent coefficients are included in the elliptic operator A, such an FFT-based fast solver is not directly available. Meanwhile, even in this case, the ETD3-Padé and ETD4-Padé schemes have provided a great convenience in the numerical approximation, since the inverse of the polynomial operators, such as $(6I+6kA+3k^2A^2+k^3A^3)^{-1}$ in $R_{03}(kA)$, $(12I + 6kA + k^2A^2)^{-1}$ in $R_{22}(kA)$, could be efficiently obtained by careful finite element solvers. Even if the non-constant coefficient functions are included in the elliptic operator A, an FFT could be used as a preconditioning tool, which leads to an acceleration of the numerical solver.

Remark 2.2. Other than the ETD3 and ETD4 algorithms, there have been other numerical approaches of third and fourth order accuracy with a numerical stability preserved, such as the Adams-Moulton approximation to the diffusion term with a prolonged temporal stencil [8,16]. In fact, a careful analysis reveals that some higher order artificial diffusion has been included in these approaches. In comparison, the ETD3 and ETD4 methods do not contain artificial regularization in the diffusion part, and this feature is a prominent advantage of the ETD approach.

3. Stability Analysis of ETD-Padé Scheme

In this section, we analyze the stability of the ETD3-Padé scheme. The analysis for stability of ETD4-Padé scheme is similar and we omit here and leave it to the interested readers. Suppose Ω is a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary, $f(u_1,u_2,t)$ is a smooth function on $\mathbb{R}^3$, and the following Lipschitz continuity is assumed:

$$
|f(u_1,u_2,t) - f(v_1,v_2,t)| \le B(|u_1 - v_1| + |u_2 - v_2|), \quad \text{for } (u_1,u_2,t) \in \mathbb{R}^3. \tag{3.1}
$$

Theorem 3.1. *The ETD3-Padé scheme is stable. In more details, for two different initial data* $\widehat{u} = \{u^{-L}, u^{1-L}, u^{2-L}, \cdots, u^0\}$ *and* $\widehat{v} = \{v^{-L}, v^{1-L}, v^{2-L}, \cdots, v^0\}$*, the two different ETD3-Padé solution* $\{u^n\}$ *and* $\{v^n\}$ *satisfy*

$$
\|u^n - v^n\| \le C\sum_{j=0}^{L} \|u^{j-L} - v^{j-L}\|, \tag{3.2}
$$

where the constant C *is independent of the time step size* k*,* u^n *and* v^n*.*

Proof. We substitute the two different initial data $\widehat{u}$ and $\widehat{v}$ into the ETD3-Padé scheme (2.18), and obtain

$$\begin{aligned} u^{n+1} =& R_{03}(kA)u^n + kP_1(kA)f(u^{n-L}, u^n, t_n) + 2kP_2(kA)f(a_n^{\widehat{u}}, b_n^{\widehat{u}}, t_n + \frac{k}{2}) \\ &+ kP_3(kA)f(u^{n-L+1}, c_n^{\widehat{u}}, t_{n+1}), \\ v^{n+1} =& R_{03}(kA)v^n + kP_1(kA)f(v^{n-L}, v^n, t_n) + 2kP_2(kA)f(a_n^{\widehat{v}}, b_n^{\widehat{v}}, t_n + \frac{k}{2}) \\ &+ kP_3(kA)f(v^{n-L+1}, c_n^{\widehat{v}}, t_{n+1}). \end{aligned} \tag{3.3}$$

Then

$$\begin{aligned} u^{n+1} - v^{n+1} =& R_{03}(kA)(u^n - v^n) + kP_1(kA)(f(u^{n-L}, u^n, t_n) - f(v^{n-L}, v^n, t_n)) \\ &+ 2kP_2(kA)(f(a_n^{\widehat{u}}, b_n^{\widehat{u}}, t_n + \frac{k}{2}) - f(a_n^{\widehat{v}}, b_n^{\widehat{v}}, t_n + \frac{k}{2})) \\ &+ kP_3(kA)(f(u^{n-L+1}, c_n^{\widehat{u}}, t_{n+1}) - f(v^{n-L+1}, c_n^{\widehat{v}}, t_{n+1})). \end{aligned} \tag{3.4}$$

From (2.19) and Lemma 2.1, it is easy to verify that,

$$\begin{aligned} \|a_n^{\widehat{u}} - a_n^{\widehat{v}}\| \le& \frac{1}{16}(\|u^{n-L-1} - v^{n-L-1}\| + \|u^{n-L+2} - v^{n-L+2}\|) \\ &+ \frac{9}{16}(\|u^{n-L} - v^{n-L}\| + \|u^{n-L+1} - v^{n-L+1}\|), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \|b_n^{\widehat{u}} - b_n^{\widehat{v}}\| \le& \|\widetilde{R}_{03}(kA)\|\|u^n - v^n\| + k\|\widetilde{P}(kA)\|\|f(u^{n-L}, u^n, t_n) - f(v^{n-L}, v^n, t_n))\|, \\ \le& (1 + \frac{Bk}{2})\|u^n - v^n\| + \frac{Bk}{2}\|u^{n-L} - v^{n-L}\|, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \|c_n^{\widehat{u}} - c_n^{\widehat{v}}\| \le& \|R_{03}(kA)\|\|u^n - v^n\| + 2k\|P(kA)\|\|f(a_n^{\widehat{u}}, b_n^{\widehat{u}}, t_n + \frac{k}{2}) - f(a_n^{\widehat{v}}, b_n^{\widehat{v}}, t_n + \frac{k}{2}))\| \\ &+ k\|P(kA)\|\|f(u^{n-L}, u^n, t_n) - f(v^{n-L}, v^n, t_n)\| \\ \le& (1 + Bk)\|u^n - v^n\| + Bk\|u^{n-L} - v^{n-L}\| + 2Bk(\|a_n^{\widehat{u}} - a_n^{\widehat{v}}\| + \|b_n^{\widehat{u}} - b_n^{\widehat{v}}\|). \end{aligned} \tag{3.7}$$

From Lemma 2.1 and the Lipschitz continuity of f, we get

$$\begin{aligned} \|u^{n+1} - v^{n+1}\| \le& \|u^n - v^n\| + k\|f(u^{n-L}, u^n, t_n) - f(v^{n-L}, v^n, t_n)\| \\ &+ \frac{2}{3}k\|f(a_n^{\widehat{u}}, b_n^{\widehat{u}}, t_n + \frac{k}{2}) - f(a_n^{\widehat{v}}, b_n^{\widehat{v}}, t_n + \frac{k}{2})\| \\ &+ \frac{1}{3}k\|f(u^{n-L+1}, c_n^{\widehat{u}}, t_{n+1}) - f(v^{n-L+1}, c_n^{\widehat{v}}, t_{n+1})\| \\ \le& \|u^n - v^n\| + Bk(\|u^{n-L} - v^{n-L}\| + \|u^n - v^n\|) \\ &+ \frac{2}{3}Bk(\|a_n^{\widehat{u}} - a_n^{\widehat{v}}\| + \|b_n^{\widehat{u}} - b_n^{\widehat{v}}\|) \\ &+ \frac{1}{3}Bk(\|u^{n-L+1} - v^{n-L+1}\| + \|c_n^{\widehat{u}} - c_n^{\widehat{v}}\|). \end{aligned} \tag{3.8}$$

A substitution of (3.5), (3.6), and (3.7) into (3.8) results in

$$\begin{aligned}\|u^{n+1}-v^{n+1}\| \leq & \|u^n-v^n\| + Bk(\|u^{n-L}-v^{n-L}\| + \|u^n-v^n\|) + \frac{1}{3}Bk\|u^{n-L+1}-v^{n-L+1}\| \\ & + \frac{2}{3}Bk(\|a_n^{\widehat{u}}-a_n^{\widehat{v}}\| + \|b_n^{\widehat{u}}-b_n^{\widehat{v}}\|) + \frac{1}{3}Bk(1+Bk)\|u^n-v^n\| \\ & + \frac{1}{3}Bk\Big(Bk\|u^{n-L}-v^{n-L}\| + 2Bk(\|a_n^{\widehat{u}}-a_n^{\widehat{v}}\| + \|b_n^{\widehat{u}}-b_n^{\widehat{v}}\|)\Big) \\ \leq & (1+\frac{4}{3}Bk+\frac{1}{3}B^2k^2)\|u^n-v^n\| + (Bk+\frac{1}{3}B^2k^2)\|u^{n-L}-v^{n-L}\| \\ & + \frac{1}{3}Bk\|u^{n-L+1}-v^{n-L+1}\| + (\frac{2}{3}Bk+\frac{2}{3}B^2k^2)(\|a_n^{\widehat{u}}-a_n^{\widehat{v}}\| + \|b_n^{\widehat{u}}-b_n^{\widehat{v}}\|) \\ \leq & (1+2Bk+\frac{4}{3}B^2k^2+\frac{1}{3}B^3k^3)\|u^n-v^n\| \\ & + (\frac{11}{8}Bk+\frac{25}{24}B^2k^2+\frac{1}{3}B^3k^3)\|u^{n-L}-v^{n-L}\| \\ & + (\frac{17}{24}Bk+\frac{3}{8}B^2k^2)\|u^{n-L+1}-v^{n-L+1}\| \\ & + (\frac{1}{24}Bk+\frac{1}{24}B^2k^2)(\|u^{n-L-1}-v^{n-L-1}\| + \|u^{n-L+2}-v^{n-L+2}\|). \end{aligned} \tag{3.9}$$

Under a reasonable assumption $Bk \leq \frac{1}{2}$, (3.9) becomes

$$\begin{aligned}\|u^{n+1}-v^{n+1}\| \leq & (1+\frac{33}{12}Bk)\|u^n-v^n\| + (\frac{95}{48}Bk)\|u^{n-L}-v^{n-L}\| + (\frac{52}{48}Bk)\|u^{n-L+1}-v^{n-L+1}\| \\ & + (\frac{1}{16}Bk)\|u^{n-L-1}-v^{n-L-1}\| + (\frac{1}{16}Bk)\|u^{n-L+2}-v^{n-L+2}\|. \end{aligned} \tag{3.10}$$

In turn, an application of discrete Gronwall inequality (given by Lemma 2.2) leads to the following stability result of the ETD3-Padé scheme:

$$\|u^n-v^n\| \leq e^{CT}\Big(\|u^0-v^0\| + k\sum_{j=0}^{L}\|u^{j-L}-v^{j-L}\|\Big).$$

This completes the proof of Theorem 3.1. □

The numerical stability for the ETD4-Padé scheme can be obtained in similar ways. We omit here and leave the proof to the readers.

4. Convergence Analysis of ETD-Padé Finite Element Method

4.1. The semidiscrete ETD-Padé scheme

Theorem 4.1. *Let $u(t) \in C^4(J; C^2(\Omega))$ $(m \geq 3)$ be the exact solution for DPDE* (2.1)*, u^n be the semi-discrete ETD3-Padé approximation determined by* (2.18)*, then we have the following semi-discrete error estimate*

$$\|u(t_n)-u^n\| \leq Ck^3e^{CT}. \tag{4.1}$$

Proof. From (2.11), a careful Taylor expansion reveals the following $O(k^4)$ local truncation error of the ETD3RK scheme (see Theorem 3.5 in [45] for details):

$$\begin{aligned} u(t_{n+1}) =& e^{-kA}u(t_n) \\ &+ k(kA)^{-3}(-k^2A^2e^{-kA} - 3kAe^{-kA} - 4e^{-kA} - kA + 4)f(u(t_n-\tau), u(t_n), t_n) \\ &+ k(kA)^{-3}(4kAe^{-kA} + 8e^{-kA} + 4kA - 8)f(u(t_n + \frac{k}{2} - \tau), b_n^{u(t_n)}, t_n + \frac{k}{2}) \\ &+ k(kA)^{-3}(k^2A^2 - 3kA - kAe^{-kA} - 4e^{-kA} + 4)f(u(t_n + k - \tau), c_n^{u(t_n)}, t_{n+1}) \\ &+ Ck^4, \end{aligned} \tag{4.2}$$

in which

$$\begin{aligned} b_n^{u(t_n)} =& e^{-\frac{k}{2}A}u(t_n) + k(kA)^{-1}(I - e^{-\frac{k}{2}A})f(u(t_n-\tau), u(t_n), t_n), \\ c_n^{u(t_n)} =& e^{-kA}u(t_n) + k(kA)^{-1}(I - e^{-kA}) \\ & (2f(u(t_n + \frac{k}{2} - \tau, b_n^{u(t_n)}, t_n + \frac{k}{2}) - f(u(t_n-\tau), u(t_n), t_n)). \end{aligned} \tag{4.3}$$

Subtracting (4.2) from (2.18), we obtain the error evolutionary equation

$$\begin{aligned} u(t_{n+1}) - u^{n+1} =& R_{03}(kA)\big(u(t_n) - u^n\big) \\ &+ kP_1(kA)\big(f(u(t_n-\tau), u(t_n), t_n) - f(u^{n-L}, u^n, t_n)\big) \\ &+ 2kP_2(kA)\bigg(f(u(t_n + \frac{k}{2} - \tau), b_n^{u(t_n)}, t_n + \frac{k}{2}) - f(a_n, b_n, t_n + \frac{k}{2})\bigg) \\ &+ kP_3(kA)\big(f(u(t_{n+1}-\tau), c_n^{u(t_n)}, t_{n+1}) - f(u^{n-L+1}, c_n, t_{n+1})\big) + g(k)^n, \\ g(k)^n =& \big(e^{-kA} - R_{03}(kA)\big)u(t_n) \\ &+ \bigg(k(kA)^{-3}(-k^2A^2e^{-kA} - 3kAe^{-kA} - 4e^{-kA} - kA + 4) - kP_1(kA)\bigg) \\ & f\big(u(t_n-\tau), u(t_n), t_n\big) + \bigg(k(kA)^{-3}(4kAe^{-kA} + 8e^{-kA} + 4kA - 8) - 2kP_2(kA)\bigg) \\ & f(u(t_n + \frac{k}{2} - \tau), b_n^{u(t_n)}, t_n + \frac{k}{2}) + \bigg(k(kA)^{-3}(k^2A^2 - 3kA - kAe^{-kA} - 4e^{-kA} + 4) - kP_3(kA)\bigg) \\ & f\big(u(t_{n+1}-\tau), c_n^{u(t_n)}, t_{n+1}\big) + Ck^4. \end{aligned} \tag{4.4}$$

Let $e^n = u(t_n) - u^n$, then we get

$$\begin{aligned} \|e^{n+1}\| \le& \|R_{03}(kA)\|\|e^n\| + kB\|P_1(kA)\|(\|e^{n-L}\| + \|e^n\|) \\ &+ 2kB\|P_2(kA)\|(\|u(t_n + \frac{k}{2} - \tau) - a_n\| + \|b_n^{u(t_n)} - b_n\|) \\ &+ kB\|P_3(kA)\|(\|e^{n-L+1}\| + \|c_n^{u(t_n)} - c_n\|) + \|g(k)^n\|. \end{aligned} \tag{4.5}$$

By Lemma 2.4, it is easy to verify that $\|g(k)^n\| \le Ck^4$. From (2.12), we have $\|u(t_n + \frac{k}{2} - \tau) - a_n\| \le Ck^4$. Furthermore, from (2.19) and (4.3), we see that

$$\begin{aligned} b_n^{u(t_n)} - b_n =& \widetilde{R}_{03}(kA)\big(u(t_n) - u^n\big) + (e^{-\frac{k}{2}A} - \widetilde{R}_{03}(kA))u(t_n) \\ &+ k\widetilde{P}(kA)\big(f(u(t_n-\tau), u(t_n), t_n) - f(u^{n-L}, u^n, t_n)\big) \\ &+ (k(kA)^{-1}(I - e^{-\frac{kA}{2}}) - k\widetilde{P}(kA))f(u(t_n-\tau), u(t_n), t_n). \end{aligned} \tag{4.6}$$

By Lemmas 2.1, 2.4 and Lipschitz continuity property of f, the following inequality is available:

$$\begin{aligned}\|b_n^{u(t_n)} - b_n\| &\le \|\widetilde{R}_{03}(kA)\|\|e^n\| + kB\|\widetilde{P}(kA)\|(\|e^{n-L}\| + \|e^n\|) + Ck^4 \\ &\le (1 + \frac{k}{2}B)\|e^n\| + \frac{k}{2}B\|e^{n-L}\| + Ck^4.\end{aligned} \tag{4.7}$$

Similarly,

$$\begin{aligned}c_n^{u(t_n)} - c_n = &R_{03}(kA)(u(t_n) - u^n) + (e^{-kA} - R_{03}(kA))u(t_n) \\ &+ (k(kA)^{-1}(I - e^{-kA}) - kP(kA))\Big(2f(u(t_n + \frac{k}{2} - \tau), b_n^{u(t_n)}, t_n + \frac{k}{2}) - f(u(t_n - \tau), u(t_n), t_n)\Big) \\ &+ kP(kA)\Big(2f(u(t_n + \frac{k}{2} - \tau), b_n^{u(t_n)}, t_n + \frac{k}{2}) - 2f(a_n, b_n, t_n + \frac{k}{2})\Big) \\ &+ kP(kA)\Big(f(u^{n-L}, u^n, t_n) - f(u(t_n - \tau), u(t_n), t_n)\Big),\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\|c_n^{u(t_n)} - c_n\| \le &\|R_{03}(kA)\|\|e^n\| + 2kB\|P(kA)\|\Big(\|a_n - u(t_n + \frac{k}{2} - \tau)\| + \|b_n - b_n^{u(t_n)}\|\Big) \\ &+ kB\|P(kA)\|(\|e^{n-L}\| + \|e^n\|) + Ck^4 \\ \le &(1 + 3kB + k^2B^2)\|e^n\| + (kB + k^2B^2)\|e^{n-L}\| + Ck^4.\end{aligned} \tag{4.9}$$

Substituting (4.7) and (4.9) into (4.5), we obtain

$$\begin{aligned}\|e^{n+1}\| \le &(1 + 2kB + \frac{4k^2B^2}{3} + k^3B^3)\|e^n\| + (kB + \frac{2k^2B^2}{3} + \frac{k^2B^2}{3})\|e^{n-L}\| \\ &+ \frac{kB}{3}\|e^{n-L+1}\| + Ck^4.\end{aligned} \tag{4.10}$$

Under a reasonable assumption $Bk \le \frac{1}{2}$, (4.10) becomes

$$\|e^{n+1}\| \le (1 + \frac{35}{12}kB)\|e^n\| + (\frac{3}{2}kB)\|e^{n-L}\| + \frac{kB}{3}\|e^{n-L+1}\| + Ck^4. \tag{4.11}$$

An application of discrete Gronwall inequality (given by Lemma 2.2) leads to the desired error estimate for the ETD3-Padé method:

$$\|e^n\| \le Ck^3e^{CT}.$$

□

4.2. The semidiscrete Galerkin finite element method

Let $\{\sigma_j\}_{j=1}^{M_h}$ be a family of quasi-uniform subdivisions of Ω, and denote S_h as the finite-dimensional space of continuous functions on $\overline{\Omega}$ which reduce to polynomials of degree $\le r-1$ on each σ_j:

$$S_h = \{\chi \in C(\overline{\Omega}); \chi|_{\sigma_j} \in \Pi_{r-1}\}, \tag{4.12}$$

where Π_s denotes the set of polynomials of degree at most s. For $v \in H^r \cap H_0^1$, the following standard $O(h^r)$ approximation is satisfied (cf. [35] or [10]):

$$\inf_{\chi \in S_h}\{\|v - \chi\| + h\|\nabla(v - \chi)\|\} \le Ch^r\|v\|_r. \tag{4.13}$$

The semidiscrete finite element solution $u_h \in S_h \subset H^r \cap H_0^1$ is defined by

$$\begin{cases} (u_{ht}, \chi) + A(u_h, \chi) = (f(u_h(t-\tau), u_h(t), t), \chi), & \forall \chi \in S_h,\ t \in \bar{J}, \\ u_h(x,t) = u_{0h}(x,t), & (x,t) \in \Omega \times [-\tau, 0], \end{cases} \tag{4.14}$$

where $u_{0h}(t)$ stands for some approximation of $u_0(t)$ such as $u_{0h}(t) = P_h u_0(t)$, $\forall t \in [-\tau, 0]$, and P_h is denoted as the orthogonal projection of $u_0(t)$ onto S_h with respect to the inner product in L^2. With suitable regularity assumptions on u_0, the following error estimate is valid:

$$\|u_0(t) - P_h u_0(t)\| \le Ch^r \|u_0(t)\|_r,\ \ t \in [-\tau, 0]. \tag{4.15}$$

It is easy to verify that the semilinear system of delayed ordinary differential equations (4.14) has a unique solution. Let $R_h u$ be the elliptic Ritz projection in S_h of the exact solution u, i.e., $A(u - R_h u, \chi) = 0$, $\forall \chi \in S_h$, $t \ge 0$. For $u \in H^r(\Omega) \cap H_0^1(\Omega)$, the following error estimate is available (see [35] for details):

$$\|R_h u - u\| + h\|A(u - R_h u)\| \le Ch^r \|u\|_r. \tag{4.16}$$

Theorem 4.2. *Assume $u_h \in S_h$ is the solution of* (4.14)*, and $u \in H^r \cap H_0^1$ is the solution of* (2.1)*. Then we have*

$$\|u_h(t) - u(t)\| \le Ch^r \Big(e^{2Bt} \|u_0(0)\|_r + te^{2Bt} \int_0^t (\|u_t\| + \|u\|) ds + te^{2Bt} \int_{-\tau}^0 \|u_0(s)\|_r ds \Big). \tag{4.17}$$

Proof. We split $e(t) := u_h(t) - u(t)$ into two parts:

$$e(t) = u_h(t) - u(t) = u_h(t) - R_h u(t) + R_h u(t) - u(t) := \theta(t) + \rho(t), \tag{4.18}$$

where

$$\|\rho(t)\| = \|R_h u(t) - u(t)\| \le Ch^r \|u\|_r, \tag{4.19}$$

$$\|\rho_t\| = \|R_h u_t - u_t\| \le Ch^r \|u_t\|_r. \tag{4.20}$$

The exact solution u also satisfies the weak form:

$$(u_t, \chi) + (Au, \chi) = (f(u(t-\tau), u(t), t), \chi), \quad \forall \chi \in S_h. \tag{4.21}$$

The error equation is then obtained by subtracting (4.21) from (4.14):

$$(u_{ht} - u_t, \chi) + (A(u_h - u), \chi) = (f(u_h(t-\tau), u_h(t), t) - f(u(t-\tau), u(t), t), \chi). \tag{4.22}$$

By (4.18), we have

$$\begin{aligned} &(u_{ht} - R_h u_t, \chi) + (R_h u_t - u_t, \chi) + (A(u_h - R_h u), \chi) + (A(R_h u - u), \chi) \\ =&(f(u_h(t-\tau), u_h(t), t) - f(u(t-\tau), u(t), t), \chi), \end{aligned} \tag{4.23}$$

that is,

$$(\theta_t, \chi) + (A\theta, \chi) = -(\rho_t, \chi) + (f(u_h(t-\tau), u_h(t), t) - f(u(t-\tau), u(t), t), \chi). \tag{4.24}$$

Subsequently, we choose $\chi = \theta$ in (4.24) and get

$$(\theta_t, \theta) + (A\theta, \theta) = -(\rho_t, \theta) + (f(u_h(t-\tau), u_h(t), t) - f(u(t-\tau), u(t), t), \theta). \tag{4.25}$$

By the fact that $c\|\theta\|^2 \le (A\theta, \theta)$, $c > 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 \le \|\rho_t\|\|\theta\| + B\|u_h(t-\tau) - u(t-\tau)\|\|\theta\| + B\|\rho\|\|\theta\| + B\|\theta\|^2. \tag{4.26}$$

As a consequence, we arrive at

$$\frac{d}{dt}\|\theta\| \leq \|\rho_t\| + B\|u_h(t-\tau) - u(t-\tau)\| + B\|\rho\| + B\|\theta\|. \tag{4.27}$$

Integrating (4.27) from 0 to t leads to

$$\begin{aligned}\|\theta\| \leq & \|\theta(0)\| + \int_0^t \|\rho_t\|ds + B\int_0^t \|u_h(s-\tau) - u(s-\tau)\|ds + B\int_0^t \|\rho\|ds + B\int_0^t \|\theta\|ds \\ \leq & \|\theta(0)\| + \int_0^t \|\rho_t\|ds + B\int_{-\tau}^0 \|u_h(s) - u(s)\|ds + B\int_0^{t-\tau} \|u_h(s) - u(s)\|ds \\ & + B\int_0^t \|\rho\|ds + B\int_0^t \|\theta\|ds \\ \leq & \|\theta(0)\| + \int_0^t \|\rho_t\|ds + B\int_{-\tau}^0 \|u_h(s) - u_0(s)\|ds + 2B\int_0^t \|\rho\|ds + 2B\int_0^t \|\theta\|ds. \end{aligned} \tag{4.28}$$

Setting $\eta(t) = \int_0^t \|\theta\|ds$, we then have

$$\eta'(t) \leq 2B\eta(t) + \|\theta(0)\| + \int_0^t \|\rho_t\|ds + B\int_{-\tau}^0 \|u_h(s) - u_0(s)\|ds + 2B\int_0^t \|\rho\|ds. \tag{4.29}$$

An application of the continuous Gronwall inequality (given by Lemma 2.3) results in the following error estimate

$$\begin{aligned}\eta(t) \leq & te^{2Bt}\|\theta(0)\| + tBe^{2Bt}\int_{-\tau}^0 \|u_h(s) - u_0(s)\|ds \\ & + e^{2Bt}\int_0^t \Big(\int_0^s \|\rho_t\|dl\Big)ds + 2Be^{2Bt}\int_0^t \Big(\int_0^s \|\rho\|dl\Big)ds \\ \leq & te^{2Bt}\|\theta(0)\| + tBe^{2Bt}\int_{-\tau}^0 \|u_h(s) - u_0(s)\|ds + 2Bte^{2Bt}\int_0^t \|\rho_t\|ds \\ & + 4B^2te^{2Bt}\int_0^t \|\rho\|ds. \end{aligned} \tag{4.30}$$

Substituting (4.30) into (4.28), we conclude that

$$\begin{aligned}\|\theta\| \leq & (2Bte^{2Bt} + 1)\|\theta(0)\| + (2B^2te^{2Bt} + B)\int_{-\tau}^0 \|u_h(s) - u_0(s)\|ds \\ & + (1 + 2Bte^{2Bt})\int_0^t \|\rho_t(s)\|ds + (2B + 4B^2te^{2Bt})\int_0^t \|\rho(s)\|ds. \end{aligned} \tag{4.31}$$

Combing that fact that $u_h(t) = P_h u_0(t)$, $-\tau \leq t \leq 0$, $\|\theta(0)\| = \|u_h(0) - u(0) + u(0) - R_h u(0)\| \leq Ch^r\|u_0(0)\|_r$, (4.19), and (4.20), we see that the above estimate can be rewritten as

$$\|\theta\| \leq Ch^r\Big(e^{2Bt}\|u_0(0)\|_r + te^{2Bt}\int_0^t (\|u_t\| + \|u\|)ds + te^{2Bt}\int_{-\tau}^0 \|u_0(s)\|_r ds\Big). \tag{4.32}$$

Therefore, we arrive at

$$\|u_h(t) - u(t)\| \leq Ch^r\Big(e^{2Bt}\|u_0(0)\|_r + te^{2Bt}\int_0^t (\|u_t\| + \|u\|)ds + te^{2Bt}\int_{-\tau}^0 \|u_0(s)\|_r ds\Big). \tag{4.33}$$

□

4.3. The fully discrete ETD3-Padé finite element method

We now turn to the discussion of the fully discrete ETD3-Padé Galerkin finite element method for (2.1). Let A_h be the discrete elliptic operator, $\forall \chi \in S_h$, $t_n \in J$, U_h^n be the fully discrete ETD3-Padé Galerkin finite element approximation of $u(t_n)$, with $U_h^n = P_h u_0(t_n)$, $t_n \in [-\tau, 0]$, such that

$$\begin{aligned}\big(U_h^{n+1}, \chi\big) = &\big(R_{03}(kA_h)U_h^n, \chi\big) + k\big(P_1(kA_h)f(U_h^{n-L}, U_h^n, t_n), \chi\big) \\ &+ 2k\big(P_2(kA_h)f(a_n, b_n, t_n + \frac{k}{2}), \chi\big) + k\big(P_3(kA_h)f(U_h^{n-L+1}, c_n, t_{n+1}), \chi\big),\end{aligned} \tag{4.34}$$

in which a_n, b_n, and c_n are calculated by the following formula

$$\begin{aligned}a_n &= -\frac{1}{16}(U_h^{n-L-1} + U_h^{n-L+2}) + \frac{9}{16}(U_h^{n-L} + U_h^{n-L+1}), \\ b_n &= \widetilde{R}_{03}(kA_h)U_h^n + k\widetilde{P}(kA_h)f(U_h^{n-L}, U_h^n, t_n), \\ c_n &= R_{03}(kA_h)U_h^n + k(2P(kA_h)f(a_n, b_n, t_n + \frac{k}{2}) - P(kA_h)f(U^{n-L}, U^n, t_n)).\end{aligned} \tag{4.35}$$

Theorem 4.3. *Let U_h^n and u be the solution of (4.34) and (2.1), with $U_h^n = P_h u_0(t_n)$, $t_n \in [-\tau, 0]$. The following global error estimate is valid:*

$$\|U_h^n - u(t_n)\| \le Ck^3 e^{CT} + Ch^r\Big(\|u_0(0)\|_r + \int_0^{t_n}(\|u_t\|_r + \|u\|_r)ds + \int_{-\tau}^0 \|u_0(s)\|_r ds\Big).$$

Proof. We write $U_h^n - u(t_n) = (U_h^n - u_h(t_n)) + (u_h(t_n) - u(t_n))$.

By Theorem 4.2, the second term can be bounded. In fact,

$$\begin{aligned}\|u_h(t_n) - u(t_n)\| \le &Ch^r\Big(e^{2Bt_n}\|u_0(0)\|_r + t_n e^{2Bt_n}\int_0^{t_n}(\|u_t\| + \|u\|)ds \\ &+ t_n e^{2Bt_n}\int_{-\tau}^0 \|u_0(s)\|_r ds\Big) \\ \le &Ch^r\Big(\|u_0(0)\|_r + \int_0^{t_n}(\|u_t\|_r + \|u\|_r)ds + \int_{-\tau}^0 \|u_0(s)\|_r ds\Big).\end{aligned} \tag{4.36}$$

By Theorem 4.1, the first term can be bounded.

$$\|U_h^n - u_h(t_n)\| \le Ck^3 e^{CT}.$$

Then we get

$$\begin{aligned}\|U_h^n - u(t_n)\| &\le \|U_h^n - u_h(t_n)\| + \|u_h(t_n) - u(t_n)\| \\ &\le Ck^3 e^{CT} + Ch^r\Big(\|u_0(0)\|_r + \int_0^{t_n}(\|u_t\|_r + \|u\|_r)ds + \int_{-\tau}^0 \|u_0(s)\|_r ds\Big).\end{aligned}$$

This completes the convergence proof. □

5. Fast Numerical Implementation

For the high-order ETD3-Padé and ETD4-Padé schemes outlined above, the inverse of high-order matrix polynomials needs to be solved. In fact, the computation of b_n, c_n, d_n and u^{n+1} may lead to a loss of computational accuracy and roundoff errors. Motivated by the works [28], we deal with this problem by partial fraction decomposition and extend this fast algorithm technique to the nonlinear delayed convection-diffusion reaction equation.

Denote $\Re(x + iy)$ by the real part of a complex number $x + iy$.

In the detailed numerical implementation of the proposed ETD3-Padé method, the following partial decomposition can be applied to reduce the computational complexity (see [27] for more descriptions)

$$
\begin{aligned}
R_{03}(kA) &= w_1(kA - c_1 I)^{-1} + 2\Re(w_2(kA - c_2 I)^{-1}),\\
P_1(kA) &= w_{11}(kA - c_1 I)^{-1} + 2\Re(w_{12}(kA - c_2 I)^{-1}),\\
P_2(kA) &= w_{21}(kA - c_1 I)^{-1} + 2\Re(w_{22}(kA - c_2 I)^{-1}),\\
P_3(kA) &= w_{31}(kA - c_1 I)^{-1} + 2\Re(w_{32}(kA - c_2 I)^{-1}),\\
\widetilde{R}_{03}(kA) &= \widetilde{w}_1(kA - \widetilde{c}_1 I)^{-1} + 2\Re(\widetilde{w}_2(kA - \widetilde{c}_2 I)^{-1}),\\
\widetilde{P}(kA) &= \widetilde{\Omega}_1(kA - \widetilde{c}_1 I)^{-1} + 2\Re(\widetilde{\Omega}_2(kA - \widetilde{c}_2 I)^{-1}),\\
P(kA) &= w_{41}(kA - c_1 I)^{-1} + 2\Re(w_{42}(kA - c_2 I)^{-1}),
\end{aligned}
$$

in which the coefficients have been accurately computed as

$$
\begin{aligned}
&c_1 = -1.5960716379833215231, && c_2 = -0.7019641810083392384 - i1.8073394944520218535 7,\\
&w_1 = 1.4756865177957207165, && w_2 = -0.7378432588978603582 + i0.3650178408010284724 4,\\
&w_{11} = 0.6384979859006401044, && w_{12} = -0.3192489929503200522 - i0.1187143286748227393 7,\\
&w_{21} = -0.2932049599374663978, && w_{22} = 0.1466024799687331989 0 + i0.4807738845503311270 44,\\
&w_{31} = 0.8724860462367531838 8, && w_{32} = 0.0637569768816234080 5 - i0.4199375777501597149 6,\\
&w_{41} = 0.924574112, && w_{42} = 0.037712944 + i0.422895863,\\
&\widetilde{c}_1 = -3.1921432759666430462 2, && \widetilde{c}_2 = -1.4039283620166784768 8 - i3.6146789889040437071 5,\\
&\widetilde{w}_1 = 2.9513730355914414330 3, && \widetilde{w}_2 = -1.4756865177957207165 19 + i0.7300356816020569448 88,\\
&\widetilde{\Omega}_1 = 0.9245741122624604926 91, && \widetilde{\Omega}_2 = 0.0377129438687697536 54 + i0.4228958626756797997 442.
\end{aligned}
$$

1. The fast algorithm of computing b_n.

Substituting the partial fraction decomposition of $\widetilde{R}_{03}(kA)$ and $\widetilde{P}(kA)$ to obtain b_n:

$$
\begin{aligned}
b_n =& \widetilde{w}_1(kA - \widetilde{c}_1 I)^{-1}u^n + 2\Re(\widetilde{w}_2(kA - \widetilde{c}_2 I)^{-1})u^n + k\widetilde{\Omega}_1(kA - \widetilde{c}_1 I)^{-1}f(u^{n-L}, u^n, t_n)\\
&+ 2k\Re(\widetilde{\Omega}_2(kA - \widetilde{c}_2 I)^{-1})f(u^{n-L}, u^n, t_n).
\end{aligned}
$$

By setting

$$
\begin{aligned}
Nb_1 &= \widetilde{w}_1(kA - \widetilde{c}_1 I)^{-1}u^n + k\widetilde{\Omega}_1(kA - \widetilde{c}_1 I)^{-1}f(u^{n-L}, u^n, t_n),\\
Nb_2 &= \widetilde{w}_2(kA - \widetilde{c}_2 I)^{-1}u^n + k\widetilde{\Omega}_2(kA - \widetilde{c}_2 I)^{-1}f(u^{n-L}, u^n, t_n),
\end{aligned}
$$

we have

$$b_n = Nb_1 + 2\Re(Nb_2).$$

2. The fast algorithm of computing c_n.

Substituting the partial fraction decomposition of $R_{03}(kA)$ and $P(kA)$ to obtain c_n.

$$
\begin{aligned}
c_n =& 2\Re(w_2(kA - c_2 I)^{-1})u^n + 2k\Re(w_{42}(kA - c_2 I)^{-1})[2f(a_n, b_n, t_n + \frac{k}{2}) - f(u^{n-L}, u^n, t_n)]\\
&+ w_1(kA - c_1 I)^{-1}u^n + kw_{41}(kA - c_1 I)^{-1}[2f(a_n, b_n, t_n + \frac{k}{2}) - f(u^{n-L}, u^n, t_n)].
\end{aligned}
$$

By setting

$$
\begin{aligned}
Nc_1 &= w_1(kA - c_1 I)^{-1}u^n + kw_{41}(kA - c_1 I)^{-1}[2f(a_n, b_n, t_n + \frac{k}{2}) - f(u^{n-L}, u^n, t_n)],\\
Nc_2 &= w_2(kA - c_2 I)^{-1}u^n + kw_{42}(kA - c_2 I)^{-1}[2f(a_n, b_n, t_n + \frac{k}{2}) - f(u^{n-L}, u^n, t_n)],
\end{aligned}
$$

we get

$$c_n = Nc_1 + 2\Re(Nc_2).$$

3. The fast ETD3-Padé scheme:

Substituting the partial fraction decomposition of $R_{03}(kA)$, $P_1(kA)$, $P_2(kA)$, $P_3(kA)$ and a_n, b_n, c_n into (2.18), we get the following fast ETD3-Padé scheme:

$$\begin{aligned}
u^{n+1} = & w_1(kA - c_1I)^{-1}u^n + 2\Re(w_2(kA - c_2I)^{-1})u^n + kw_{11}(kA - c_1I)^{-1}f(u^{n-L}, u^n, t_n) \\
& +2k\Re(w_{12}(kA - c_2I)^{-1})f(u^{n-L}, u^n, t_n) + 2kw_{21}(kA - c_1I)^{-1}f(a_n, b_n, t_n + \frac{k}{2}) \\
& +4k\Re(w_{22}(kA - c_2I)^{-1})f(a_n, b_n, t_n + \frac{k}{2}) + kw_{31}(kA - c_1I)^{-1}f(u^{n-L+1}, c_n, t_{n+1}) \\
& +2k\Re(w_{32}(kA - c_2I)^{-1})f(u^{n-L+1}, c_n, t_{n+1}).
\end{aligned}$$

By setting

$$\begin{aligned}
Nu_1 = & w_1(kA - c_1I)^{-1}u^n + kw_{11}(kA - c_1I)^{-1}f(u^{n-L}, u^n, t_n) \\
& +2kw_{21}(kA - c_1I)^{-1}f(a_n, b_n, t_n + \frac{k}{2}) + kw_{31}(kA - c_1I)^{-1}f(u^{n-L+1}, c_n, t_{n+1}), \\
Nu_2 = & 2k\Re(w_{22}(kA - c_2I)^{-1})f(a_n, b_n, t_n + \frac{k}{2}) + k\Re(w_{32}(kA - c_2I)^{-1})f(u^{n-L+1}, c_n, t_{n+1}) \\
& +\Re(w_2(kA - c_2I)^{-1})u^n + k\Re(w_{12}(kA - c_2I)^{-1})f(u^{n-L}, u^n, t_n),
\end{aligned}$$

we get

$$u^{n+1} = Nu_1 + 2\Re(Nu_2).$$

Similarly, for ETD4-Padé method, the following decomposition is available

$$\begin{aligned}
&\widetilde{R}_{22}(kA) = 1 + 2\Re(\widetilde{w}_1(kA - \widetilde{c}_1I)^{-1}), && \widetilde{P}(kA) = 2\Re(\widetilde{\Omega}_1(kA - \widetilde{c}_1I)^{-1}), \\
&R_{22}(kA) = 1 + 2\Re(w_1(kA - c_1I)^{-1}), && P_1(kA) = 2\Re(w_{11}(kA - c_1I)^{-1}), \\
&P_2(kA) = 2\Re(w_{21}(kA - c_1I)^{-1}), && P_3(kA) = 2\Re(w_{31}(kA - c_1I)^{-1}),
\end{aligned}$$

with the corresponding coefficient values

$$\begin{aligned}
&c_1 = -3 + i1.73205080756887729352, && w_1 = -6 - i10.3923048454132637611, \\
&w_{11} = -0.5 - i1.44337567297406441127, && w_{21} = -i1.15470053837925152901, \\
&w_{31} = 0.5 + i0.28867513459481288225, && \widetilde{c}_1 = -6 + i3.4641016151377545870548, \\
&\widetilde{w}_1 = -12 - i20.78460969082652752232935 && \widetilde{\Omega}_1 = -i3.46410161513775458705.
\end{aligned}$$

1. The fast algorithm of computing b_n.

Substituting the partial fraction decomposition of $\widetilde{R}_{22}(kA)$ and $\widetilde{P}(kA)$ to obtain b_n:

$$b_n = u^n + 2\Re(\widetilde{w}_1(kA - \widetilde{c}_1I)^{-1})u^n + 2k\Re(\widetilde{\Omega}_1(kA - \widetilde{c}_1I)^{-1})f(u^{n-L}, u^n, t_n).$$

By setting

$$Nb_1 = \widetilde{w}_1(kA - \widetilde{c}_1I)^{-1}u^n + k\widetilde{\Omega}_1(kA - \widetilde{c}_1I)^{-1}f(u^{n-L}, u^n, t_n),$$

we obtain

$$b_n = u^n + 2\Re(Nb_1).$$

2. The fast algorithm of computing c_n.

Substituting the partial fraction decomposition of $\widetilde{R}_{22}(kA)$ and $\widetilde{P}(kA)$ to obtain c_n:

$$c_n = u^n + 2\Re(\widetilde{w}_1(kA - \widetilde{c}_1I)^{-1})u^n + 2k\Re(\widetilde{\Omega}_1(kA - \widetilde{c}_1I)^{-1})f(a_n, b_n, t_n + \frac{k}{2}).$$

By setting

$$Nc_1 = \widetilde{w}_1(kA - \widetilde{c}_1 I)^{-1}u^n + k\widetilde{\Omega}_1(kA - \widetilde{c}_1 I)^{-1}f(a_n, b_n, t_n + \frac{k}{2}),$$

we get

$$c_n = u^n + 2\Re(Nc_1).$$

3. The fast algorithm of computing d_n.

Substituting the partial fraction decomposition of $\widetilde{R}_{22}(kA)$ and $\widetilde{P}(kA)$ into d_n:

$$\begin{aligned} d_n =& b_n + 2\Re(\widetilde{w}_1(kA - \widetilde{c}_1 I)^{-1})b_n \\ &+ 2k\Re(\widetilde{\Omega}_1(kA - \widetilde{c}_1 I)^{-1})[2f(a_n, c_n, t_n + \frac{k}{2}) - f(u^{n-L}, u^n, t_n)]. \end{aligned}$$

By setting

$$Nd_1 = \widetilde{w}_1(kA - \widetilde{c}_1 I)^{-1}b_n + k\widetilde{\Omega}_1(kA - \widetilde{c}_1 I)^{-1}[2f(a_n, c_n, t_n + \frac{k}{2}) - f(u^{n-L}, u^n, t_n)],$$

we obtain

$$d_n = b_n + 2\Re(Nd_1).$$

4. The fast ETD4-Padé scheme.

Substituting the partial fraction decomposition of $R_{22}(kA)$, $P_1(kA)$, $P_2(kA)$, $P_3(kA)$ and a_n, b_n, c_n d_n into (2.21), we get the following fast ETD4-Padé scheme:

$$\begin{aligned} u^{n+1} = u^n &+ 2\Re(w_1(kA - c_1 I)^{-1})u^n + 2k\Re(w_{11}(kA - c_1 I)^{-1})f(u^{n-L}, u^n, t_n) \\ &+ 2k\Re(w_{21}(kA - c_1 I)^{-1})(f(a_n, b_n, t_n + \frac{k}{2}) + f(a_n, c_n, t_n + \frac{k}{2})) \\ &+ 2k\Re(w_{31}(kA - c_1 I)^{-1})f(u^{n-L+1}, d_n, t_{n+1}). \end{aligned}$$

By setting

$$\begin{aligned} Nu_1 = w_1(kA - c_1 I)^{-1}u^n &+ kw_{11}(kA - c_1 I)^{-1}f(u^{n-L}, u^n, t_n) \\ &+ kw_{21}(kA - c_1 I)^{-1}f(a_n, b_n, t_n + \frac{k}{2}) + kw_{21}(kA - c_1 I)^{-1}f(a_n, c_n, t_n + \frac{k}{2}) \\ &+ kw_{31}(kA - c_1 I)^{-1}f(u^{n-L+1}, d_n, t_{n+1}), \end{aligned}$$

we have

$$u^{n+1} = u^n + 2\Re(Nu_1).$$

6. Numerical Experiments

In this section, we use two examples to illustrate the theoretical results.

For simplicity, we introduce the notation:

$$\text{order} = \log_{\frac{k_1}{k_2}} \frac{error(k_1)}{error(k_2)},$$

where $error(k)$ denotes the error in the discrete L^2 or L^∞ norm with time step size k. The linear finite element (with $r = 2$) is taken as the spatial discretization.

Example 6.1. We consider the Mackey-Glass-type equation (see [45] for details), which simulates a single-species population with age-structure and diffusion

$$u_t = u_{xx} - au(x,t) + \frac{bu(x, t-\tau)}{1 + u^m(x, t-\tau)}. \tag{6.1}$$

Table 6.1: The temporal convergence order of ETD3-Padé FEM scheme for problem (6.2), with $N=2000$.

M	k	error(L^2)	order(L^2)	error(L^∞)	order(L^∞)
20	0.025	1.48e-3		3.1e-3	
40	0.0125	2.14e-4	2.79	4.665e-4	2.73
80	0.00625	3.01e-5	2.83	6.093e-5	2.92
160	0.003125	4.00e-6	2.91	7.796e-6	2.97
320	0.00156	4.89e-7	3.03	9.63e-7	3.01

Table 6.2: The spatial convergence order of ETD3-Padé FEM scheme for problem (6.2), with $L=32$.

N	h	error(L^2)	order(L^2)	error(L^∞)	order(L^∞)
10	0.1	1.76e-3		0.0021	
20	0.05	4.37e-4	2.01	5.28e-4	1.99
40	0.025	1.06e-4	2.03	1.32e-4	1.99
80	0.0125	2.42e-5	2.12	3.32e-5	1.99

Table 6.3: The temporal convergence order of ETD4-Padé FEM scheme for problem (6.2), with $N=2000$.

M	k	error(L^2)	order(L^2)	error(L^∞)	order(L^∞)
10	0.05	3.199e-3		6.500e-3	
20	0.025	1.609e-4	4.31	3.558e-4	4.19
40	0.0125	9.43e-6	4.09	2.147e-5	4.05
80	0.00625	5.53e-7	4.08	1.290e-6	4.05
160	0.003125	3.21e-8	4.10	6.097e-8	4.40

This problem is equipped with homogeneous Dirichlet boundary condition. We take $a=2$, $b=1$, $m=2$ and $\tau=0.1$ and solve the problem on $[0,1]\times[0,0.5]$. A source term $g(x,t)$ is added, so that the exact solution of the PDE

$$\begin{cases} u_t = u_{xx} - 2u(x,t) + \dfrac{u(x,t-0.1)}{1+u^2(x,t-0.1)} + g(x,t), \\ u(0)=u(1)=0, \\ u(x,t) = x(1-x)\sin(50t-x), x\in[0,1], t\in[-0.1,0], \end{cases} \tag{6.2}$$

is given by $u(x,t)=x(1-x)\sin(50t-x)$. We compute $u(x,t)$ using the ETD3-Padé and ETD4-Padé schemes, respectively, combined with linear finite element spatial discretization. The time step size is taken as $k=\frac{0.5}{M}=\frac{0.1}{L}$, and the spatial resolution is set as $h=\frac{1}{N}$.

The temporal and spatial numerical errors of the ETD3-Padé scheme (with linear finite element spatial approximation) are presented in Tables 6.1 and 6.2, respectively. A third order accuracy in time, and second order accuracy in space, have been clearly observed in the tables. The corresponding numerical results of ETD4-Padé Galerkin finite element method are presented in Tables 6.3 and 6.4. We also display the plots of the exact solution and the numerical solution in Fig. 6.1.

Next, we examine the conditional stability of the ETD-Padé method. For this purpose, the initial condition is replaced by its perturbation $\widehat{v}$:

$$\hat{v}(x,t) = x(1-x)(\sin(50t-x)+0.1), \quad (x,t)\in[0,1]\times[-0.1,0]. \tag{6.3}$$

Table 6.4: The spatial convergence order of ETD4-Padé FEM in space for problem (6.2) with $L = 32$.

N	h	error(L^2)	order(L^2)	error(L^∞)	order(L^∞)
10	0.1	1.76e-3		2.1e-3	
20	0.05	4.403e-4	2.00	5.28e-4	1.99
40	0.025	1.099e-4	2.14	1.32e-4	1.99
80	0.0125	2.74e-5	1.99	3.30e-5	1.99

Table 6.5: The temporal convergence order of ETD3-RK FEM scheme for problem (6.2), with $N = 2000$.

M	k	error(L^2)	order(L^2)	error(L^∞)	order(L^∞)
20	0.025	1.59e-4		2.28e-4	
40	0.0125	9.35e-6	4.09	1.28e-5	4.16
80	0.00625	5.80e-7	4.02	7.86e-7	4.03
160	0.003125	3.96e-8	3.87	5.33e-8	3.88
320	0.00156	6.04e-9	2.71	7.86e-9	2.76

Table 6.6: The spatial convergence order of ETD3-RK FEM scheme for problem (6.2), with $L = 32$.

N	h	error(L^2)	order(L^2)	error(L^∞)	order(L^∞)
10	0.1	1.57e-4		1.97e-04	
20	0.05	3.96e-5	1.99	4.99e-5	1.98
40	0.025	9.93e-6	2.00	1.25e-5	2.00
80	0.0125	2.50e-6	1.99	3.16e-6	1.98

We obtain the corresponding numerical solution v^n and then compute $\|u^n - v^n\|_{L^2}$ in different time step. Fig. 6.2 displays the time evolutionary curve of $\|u^n - v^n\|_{L^2}$, computed the ETD3-Padé and ETD4-Padé methods, with $N = 2000$. The numerical stability has been clearly observed.

Moreover, we present the results computed by the ETD3-RK numerical algorithm (2.13)-(2.14) in Tables 6.5 and 6.6. It is noticed that only constant coefficients have been included in the elliptic operator A. And also, the elliptic operator A is self-adjoint, since there is no convection term. Because of these two facts, we are able to apply FFT-based fast solvers. The third order temporal convergence order and second order spatial convergence order have also been observed for this numerical algorithm, as displayed in Tables 6.5 and 6.6. In fact, the corresponding results are even sharper than that of the ETD3-Padé method, which comes from the fact that e^{-kA} could be exactly evaluated, with the help of FFT-based fast solvers, in comparison with the Padé approximation, which has introduced an additional numerical discretization. In terms of the numerical efficiency, the ETD3-RK and ETD3-Padé methods are of a comparable level, because of the availability of the FFT-based fast solvers. Therefore, we conclude that, for a symmetric and constant coefficient elliptic operator A, the ETD3-RK has more advantages, with the help of FFT-based fast solvers. However, for a general elliptic operator A with either variable coefficients or with a convection term, a direct evaluation of e^{-kA} turns out to be very complicated and computationally very expensive. In this more general case, the ETD-Padé method becomes more preferred, in terms of numerical stability and efficiency.

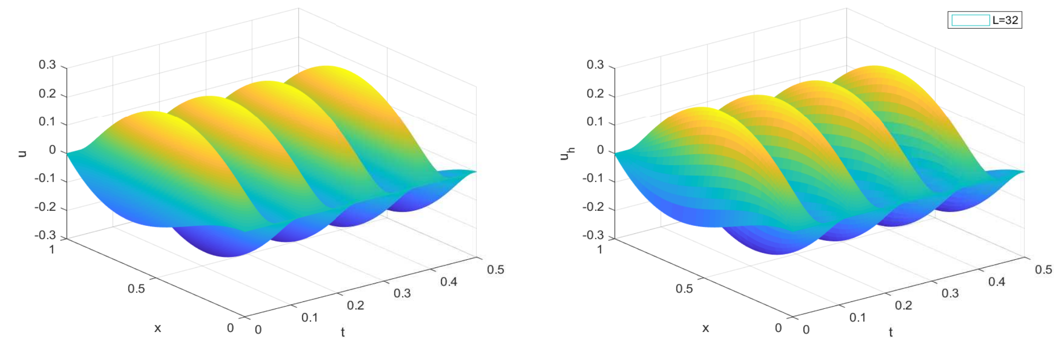

Fig. 6.1. The exact solution (left) and the ETD3-Padé finite element solution (right).

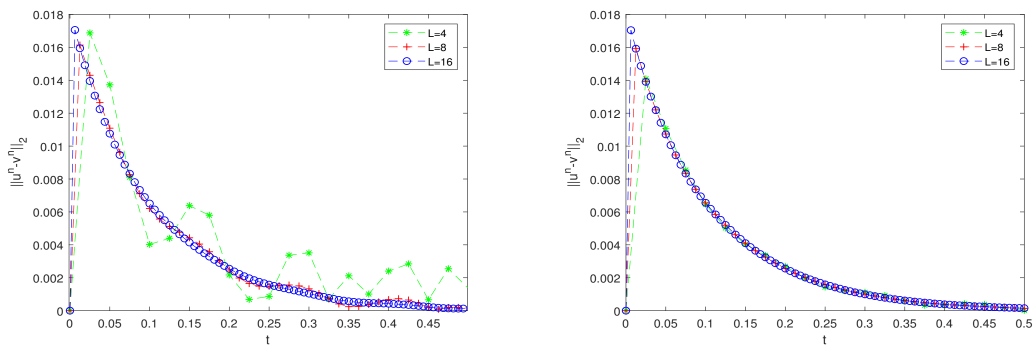

Fig. 6.2. Time evolution curve of $\|u^n - v^n\|_2$, computed by the ETD3-Padé (left) and ETD4-Padé (right) method.

Example 6.2. We consider the DPDE in [42],

$$\begin{cases} u_t = \triangle u + u_x + u_y + u^2(x,y,t-0.5) + u^2(x,y,t) + g(x,y,t), \ (x,y,t) \in [0,1]\times[0,1]\times[0,5], \\ u\mid_{\partial\Omega} = 0, \quad t \in [0,5], \\ u(x,y,t) = e^{-t}\sin(2\pi x)\sin(2\pi y) \quad (x,y) \in [0,1]\times[0,1], t \in [-0.5,0], \end{cases} \tag{6.4}$$

with exact solution $u(x,y,t) = e^{-t}\sin(2\pi x)\sin(2\pi y)$. In fact, this notation is consistent with (2.1), by taking $Au = -\Delta u - u_x - u_y$ (an elliptic operator), $f(u(t-\tau)), u(t), t) = u^2(t-\tau) + u^2(t)$, (with $\tau = 0.5$). And also, $g(x,y,t)$ is calculated as an external source term, since the given exact solution $u(x,y,t)$ does not satisfy the DPDE (2.1).

The temporal and spatial numerical errors of the ETD3-Padé scheme (with linear finite element spatial approximation) are presented in Tables 6.7 and 6.8, respectively. In the temporal accuracy check, we take $h = 2^{-\frac{1}{2}}k^{\frac{3}{2}}$ so that $k^3 = O(h^2)$. In the spatial accuracy check, the time step size is fixed as $k = 0.0005$, so that the numerical error is dominated by the spatial ones. A third order accuracy in time, and second order accuracy in space, have been clearly observed in the tables. The corresponding numerical results of ETD4-Padé Galerkin finite element method are presented in Tables 6.9 and 6.10. Similarly, in the temporal accuracy check, we take $h = 2k^2$ so that $k^4 = O(h^2)$. In the spatial accuracy check, the time step size is fixed as $k = 0.0005$, so that the numerical error is dominated by the spatial ones. A fourth order accuracy in time, and second order accuracy in space, have been clearly observed in the tables.

Table 6.7: The temporal convergence order of ETD3-Padé FEM scheme for problem (6.4), by taking $h = 2^{-\frac{1}{2}}k^{\frac{3}{2}}$.

k	h	error(L^2)	order(L^2)
0.5	2.5e-1	2.29e-3	
0.25	8.33e-2	3.60e-4	2.67
0.125	2.77e-2	4.17e-5	3.10
0.0625	9.25e-3	4.65e-6	3.16

Table 6.8: The spatial convergence order of ETD3-Padé FEM scheme for problem (6.4), by fixing $k = 0.0005$.

k	h	error(L^2)	order(L^2)
0.0005	2.5e-1	2.29e-3	
0.0005	1.25e-1	7.64e-4	1.58
0.0005	6.25e-2	2.06e-4	1.89
0.0005	3.12e-2	5.25e-5	1.97

Table 6.9: The temporal convergence order of ETD4-Padé FEM scheme for problem (6.4), by taking $h = 2k^2$.

k	h	error(L^2)	order(L^2)
0.5	0.5	5.02e-3	
0.25	1.25e-1	8.50e-4	2.56
0.125	3.12e-2	5.23e-5	4.02
0.0625	7.8e-3	3.30e-6	3.98

Table 6.10: The spatial convergence order of ETD4-Padé FEM scheme for problem (6.4), by fixing $k = 0.0005$.

k	h	error(L^2)	order(L^2)
0.0005	0.5	4.9e-3	
0.0005	2.5e-1	2.29e-3	1.11
0.0005	1.25e-1	7.65e-4	1.58
0.0005	6.25e-2	2.06e-4	1.89
0.0005	3.125e-2	5.25e-5	1.97

7. Concluding Remarks

In this paper, the ETD3-Padé and ETD4-Padé Galerkin finite element schemes are constructed and analyzed for nonlinear delayed diffusion-reaction equations with Dirichlet boundary conditions. This approach avoids a well-known difficulty of numerical instability caused by ETD-based RK method. To simplify the computation efforts, the nonlinear terms are approximated by explicit extrapolation formulas and a fast numerical algorithm is implemented. An unconditional L^2 numerical stability is proved for the proposed numerical schemes, under a global Lipshitz continuity assumption. Moreover, an optimal rate convergence analysis and error estimate are established in the same manner, which gives convergence order of $O(k^3 + h^r)$ and $O(k^4 + h^r)$ in the L^2 norm, respectively. Two numerical experiments have demonstrated the robustness of the proposed third and fourth order ETD-Padé methods.

In our future work, we will focus on the following topics:

- Extend the ETD-Padé method to partial differential equations with time dependent delay.
- The maximal principle of parabolic equations with time-dependent delay.

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